1. Elastic Waves from a Point in an Isotropic Heterogeneous Sphere. Part I.

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1. In 1921, E. Meissner[1] found that a kind of surface waves resembling Love wave can be propagated along a semi-infinite elastic body, in which rigidity and density vary with depth. This is perhaps the first paper concerning the wave propagated in an isotropic heterogeneous medium. But the equation of motion, on which his treatments are based, lacks a term depending on the gradient of rigidity, and, because of being based on this equation, even the detailed calculations executed in the next year by K. Aiti[2] left something to be added. The correction of the equation was made by E. Meissner[3] himself in 1926, and the problem was reduced to a boundary value problem and numbers of dispersion-curves were obtained, though full discussion of this equation was not executed.

The investigations by these two authors were confined to a one-dimensional problem, but, in 1931, K. Sezawa[4] obtained a rigorous solution in cylindrical co-ordinates of the surface wave of the kind first mentioned, which diverges from a point, has no components of displacement perpendicular to the surface and involves no dilatation. The effect of heterogenuity in the lower medium was also discussed by H. Jeffreys[5] and T. Matuzawa.[6] Rayleigh-wave propagated in a heterogenous medium was also discussed by H. Honda[7]. As to the bodily wave through a heterogeneous medium, the solution has been obtained by T. Matuzawa only for a plane wave propagated in the direction in which the elastic constants vary.

1) E. MEISSNER, *Vierteljahr. Nat. Forsch. Ges.*, Zurich, **67** (1921), 181.
2) K. AITI, *Proc. Phys.-Math. Soc.*, Japan, [3], 4 (1922), 137–142.
3) E. MEISSNER, *Verh. 2. Int. Kongr. f. Tech. Mech.*, (Zurich, 1926), 3–11.
4) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **9** (1931), 310–315.
5) H. JEFFREYS, *Geophys. Suppl. Month. Not. Astr. Soc.*, London, **2** (1928), 110.
6) T. MATUZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929), 225–228.
7) H. HONDA, *Kisyô-syûsi*, [9], **6** (1931), 237.

It is hoped by the author to obtain general solutions of waves divergeing from a point in an isotropic heterogeneous sphere in which elastic constants and density are certain functions of a radial distance only from the centre of the sphere. And one of the special cases is solved in this paper.

2. K. Uller[8] investigated the velocity of propagation of wave-front in an isotropic heterogeneous medium, and gave an expression, essentially equall to the following, as an equation of motion,

$$\left.\begin{aligned}\rho\frac{\partial^2 \boldsymbol{D}}{\partial T^2}=(\lambda+2\mu)\,\mathrm{grad}\,\Delta-2\mu\,\mathrm{rot}\,\boldsymbol{W}+\Delta\cdot\mathrm{grad}\,\lambda \\ -2[\boldsymbol{W},\mathrm{grad}\,\mu]+2(\mathrm{grad}\,\mu\ \nabla)\boldsymbol{D}, \\ \Delta=\mathrm{div}\,\boldsymbol{D},\ \ \boldsymbol{W}=\frac{1}{2}\,\mathrm{rot}\,\boldsymbol{D},\end{aligned}\right\}\ \ldots\ldots \quad (1)$$

where $\boldsymbol{D}$ and T stand for a vector of displacement and time-co-ordinate respectively, and the density and two Lame's constants in the medium are respectively denoted by ρ, λ and μ. The vector $(\mathrm{grad}\,\mu\ \nabla)\boldsymbol{D}$ can be considered as an inner product of grad μ and an asymmetrical tensor τ, where

$$\tau=\begin{bmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z}\end{bmatrix}$$

in Cartesian co-ordinate, in which u, v and w denote respectively x-, y- and z-component of displacement $\boldsymbol{D}$. This tensor (τ) can be decomposed into two parts, the one symmetrical and the other skew symmetrical, and the former is so-called strain-tensor and the latter consists of components equal to those of rotation.

A system of orthogonal curvilinear co-ordinates α, β and γ being now considered, three components of displacement $\boldsymbol{D}$ are expressed with u_α, u_β and u_γ; three components of rotation $\boldsymbol{W}$ with ϖ_α, ϖ_β and ϖ_γ; six componenents of strain with $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\gamma\gamma}$, $e_{\beta\gamma}$, $e_{\gamma\alpha}$ and $e_{\alpha\beta}$. Then the two tensors, into which the asymmetrical tensor, mentioned above, can be decomposed, are expressed in the following form,

8) K. ULLER, *Beitr. z. Geophys.*, **15** (1926), 219-238.

$$\begin{bmatrix} e_{\alpha\alpha} & \frac{1}{2}e_{\alpha\beta} & \frac{1}{2}e_{\gamma\alpha} \\ \frac{1}{2}e_{\alpha\beta} & e_{\beta\beta} & \frac{1}{2}e_{\beta\gamma} \\ \frac{1}{2}e_{\gamma\alpha} & \frac{1}{2}e_{\beta\gamma} & e_{\gamma\gamma} \end{bmatrix}, \qquad \begin{bmatrix} 0 & -\varpi_\gamma & \varpi_\beta \\ \varpi_\gamma & 0 & -\varpi_\alpha \\ -\varpi_\beta & \varpi_\alpha & 0 \end{bmatrix}$$

and, in what follows, the former, symmetrical tensor, shall be denoted by $\mathfrak{E}$.

Thus the equation of motion (1) is transformed as follows[9]:

$$\rho\frac{\partial^2 \boldsymbol{D}}{\partial T^2} = \operatorname{grad}\{(\lambda+2\mu)\Delta\} - 2\operatorname{rot}\mu\boldsymbol{W} + 2[\operatorname{grad}\mu, \boldsymbol{W}]$$
$$-2\Delta\operatorname{grad}\mu + 2(\mathfrak{E}\operatorname{grad}\mu). \qquad \text{(2)}$$

In order to discuss a wave propagated from a point in a heterogeneous medium which has a spherical symmetry, we first take an auxiliary co-ordinates system r, θ, φ taking its origin at the centre (O) of the spherical symmetry, and let us introduce a new system of curvilinear co-ordinates t, β, φ; t and β being defined by the following relations, and φ remaining the same in both systems. The origin of the new co-ordinates is at $r=h$, $\theta=\varphi=0$, denoted by O'.

$$t = \int^r \frac{g^2(r)dr}{r\sqrt{g^2(r)-\kappa^2}}, \qquad \text{(3)}$$

$$\theta = \int^r \frac{\kappa dr}{r\sqrt{g^2(r)-\kappa^2}}, \qquad \text{(4)}$$

$$g(r) = r\cdot\nu(r), \quad \kappa = g(h)\sin\beta, \qquad \text{(5)}$$

where $\nu(r)$, consequently $g(r)$ also, is a given function of r only.

Then we have,

$$\frac{\partial r}{\partial t} = \frac{r\sqrt{g^2(r)-\kappa^2}}{g^2(r)}, \qquad \text{(6)}$$

$$\frac{\partial r}{\partial \beta} = -\kappa^2\cot\beta\cdot\frac{r\sqrt{g^2(r)-\kappa^2}}{g^2(r)}\int^r \frac{g^2(r)dr}{r\{g^2(r)-\kappa^2\}^{\frac{3}{2}}}, \qquad \text{(7)}$$

$$\frac{\partial \theta}{\partial t} = \frac{\kappa}{g^2(r)}, \qquad \text{(8)}$$

$$\frac{\partial \theta}{\partial \beta} = \kappa\cdot\cot\beta\,\frac{g^2(r)-\kappa^2}{g^2(r)}\int^r \frac{g^2(r)dr}{r\{g^2(r)-\kappa^2\}^{\frac{3}{2}}}, \qquad \text{(9)}$$

9) This equation of motion was independently derived by the author without knowing the investigations by K. Uller.

$$\frac{\partial r}{\partial \varphi}=\frac{\partial \theta}{\partial \varphi}=\frac{\partial \varphi}{\partial t}=\frac{\partial \varphi}{\partial \beta}=0, \quad \frac{\partial \varphi}{\partial \varphi}=1, \quad \ldots\ldots\ldots\ldots (10)$$

and, since Cartesian co-ordinates,

$$x=r \sin \theta \cos \varphi, \quad y=r \sin \theta \sin \varphi, \quad z=r \cos \theta, \quad \ldots\ldots\ldots (11)$$

we obtain the following quantities necessary to the transformation of co-ordinates,

$$\frac{1}{h_1^2}=\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2=\left(\frac{\partial r}{\partial t}\right)^2+r^2\left(\frac{\partial \theta}{\partial t}\right)^2=\frac{1}{\nu^2(r)}, \quad \ldots\ldots (12)$$

$$\frac{1}{h_2^2}=\left(\frac{\partial x}{\partial \beta}\right)^2+\left(\frac{\partial y}{\partial \beta}\right)^2+\left(\frac{\partial z}{\partial \beta}\right)^2=\left(\frac{\partial r}{\partial \beta}\right)^2+r^2\left(\frac{\partial \theta}{\partial \beta}\right)^2$$

$$=\kappa^2 \cot^2 \beta \, \frac{g^2(r)-\kappa^2}{\nu^2(r)}\left\{\int^r \frac{g^2(r)dr}{r\{g^2(r)-\kappa^2\}^{\frac{3}{2}}}\right\}^2, \quad (13)$$

$$\frac{1}{h_3^2}=\left(\frac{\partial x}{\partial \varphi}\right)^2+\left(\frac{\partial y}{\partial \varphi}\right)^2+\left(\frac{\partial z}{\partial \varphi}\right)^2=r^2 \sin^2 \theta, \quad \ldots\ldots\ldots\ldots\ldots (14)$$

and, since we have nine formulae for direction cosines as follows:

$$\left.\begin{array}{lll} \cos(t,x)=h_1\dfrac{\partial x}{\partial t}, & \cos(t,y)=h_1\dfrac{\partial y}{\partial t}, & \cos(t,z)=h_1\dfrac{\partial z}{\partial t}, \\ \cos(\beta,x)=h_2\dfrac{\partial x}{\partial \beta}, & \cos(\beta,y)=h_2\dfrac{\partial y}{\partial \beta}, & \cos(\beta,z)=h_2\dfrac{\partial z}{\partial \beta}, \\ \cos(\varphi,x)=h_3\dfrac{\partial x}{\partial \varphi}, & \cos(\varphi,y)=h_3\dfrac{\partial y}{\partial \varphi}, & \cos(\varphi,z)=h_3\dfrac{\partial z}{\partial \varphi}, \end{array}\right\} \ldots (15)$$

we can easily prove that the families of surfaces (t=const., β=const. and φ=const.) cut one another at right angles everywhere. Thus t, β and φ form a system of orthogonal curvilinear co-ordinates.

The curve, determined by

$$\beta=\text{const.}, \quad \varphi=\text{const.}, \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

has notable characteristics, that

$$r \cdot \nu(r) \cdot \sin i=h \cdot \nu(h) \cdot \sin\beta=\kappa, \quad \ldots\ldots\ldots\ldots (17)$$

where i is an angle considered at any point between radius vector r and tangent at the point to the curve, and β is equal to the value of i at the origin of the co-ordinates system. (See Fig. 1).

Though nothing has been assumed to the physical meaning of the above quantities, we can easily prove by the Fermat's Principle that t and θ become equal to travel time and epicentral distance of an earthquake originating at the origin O' of co-ordinates, provided $\nu(r)$

represents reciprocal of velocity of wave. Therefore the surface, $t=$const., respresents wave-front and the curve given by (16) gives a path of the wave, which is one of orthogonal trajectories to the wave-front. From such a point of view, it seems quite natural that t, β and φ form a system of orthogonal curvilinaer co-ordinates, and we can easily see that one set of values of t, β and φ corresponds to a point in space, provided

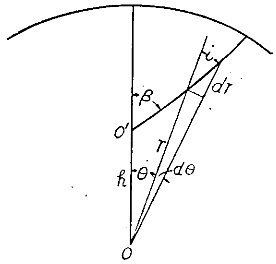

Fig. 1.

$$t>0, \quad 0\leqq\beta\leqq\pi, \quad 0\leqq\varphi\leqq 2\pi. \qquad (18)$$

The strain components refered to such a co-ordinate system are expressed as follows:

$$e_{tt}=\nu\frac{\partial u_t}{\partial t}-\frac{\kappa}{r}\frac{d}{dr}\left(\frac{1}{\nu}\right)\cdot u_\beta,$$

$$e_{\beta\beta}=\frac{1}{\kappa\cot\beta\cdot r\cos i\cdot f}\frac{\partial u_\beta}{\partial\beta}+\frac{1}{r\cos i}\left\{1-\frac{\kappa^2}{g(r)}\frac{d}{dr}\left(\frac{1}{\nu}\right)+\frac{1}{f\cdot g(r)\cos i}\right\}u_t,$$

$$e_{\varphi\varphi}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\sin(\theta+i)}{r\sin\theta}u_t+\frac{\cos(\theta+i)}{r\sin\theta}u_\beta,$$

$$e_{\beta\varphi}=\frac{1}{\kappa\cot\beta\cdot r\cos i\cdot f}\frac{\partial u_\varphi}{\partial\beta}-\frac{\cos(\theta+i)}{r\sin\theta}u_\varphi+\frac{1}{r\sin\theta}\frac{\partial u_\beta}{\partial\varphi},$$

$$e_{\varphi t}=\frac{1}{r\sin\theta}\frac{\partial u_t}{\partial\varphi}+\nu\frac{\partial u_\varphi}{\partial t}-\frac{\sin(\theta+i)}{r\sin\theta}u_\varphi,$$

$$e_{t\beta}=\nu\frac{\partial u_\beta}{\partial t}-\frac{1}{r\cos i}\left\{1-\frac{\kappa^2}{g(r)}\frac{d}{dr}\left(\frac{1}{\nu}\right)+\frac{1}{f\cdot g(r)\cos i}\right\}u_\beta+\frac{1}{\kappa\cot\beta\cdot r\cos i\cdot f}\frac{\partial u_t}{\partial\beta}+\frac{\kappa}{r}\frac{d}{dr}\left(\frac{1}{\nu}\right)u_t,$$

where
$$f\equiv\int^r\frac{g^2(r)dr}{r\{g^2(r)-\kappa^2\}^{\frac{3}{2}}}.$$

3. Now in this paper, the discussion shall be confined to a special case, where the density (ρ) and the rigidity (μ) in the medium are constant, and λ alone is a function of r only,

$$\sqrt{\frac{\lambda+2\mu}{\rho}}=a-br^2, \qquad (19)$$

in which a and b are constants. The author hopes to treat the more general cases in future occasions.

Then the equation of motion (1) or (2) reduces to

$$\left.\begin{aligned}\rho\,\frac{\partial^2 \boldsymbol{D}}{\partial T^2}&=\operatorname{grad}\{(\lambda+2\mu)\Delta\}-2\mu\operatorname{rot}\boldsymbol{W},\\ \Delta=\operatorname{div}\boldsymbol{D},\quad \boldsymbol{W}&=\frac{1}{2}\operatorname{rot}\boldsymbol{D},\end{aligned}\right\}\quad\cdots\cdots\cdots \quad (20)$$

and evidently from this equation it follows,

$$\frac{\partial^2\Delta}{\partial T^2}=\nabla^2\left(\frac{\lambda+2\mu}{\rho}\cdot\Delta\right), \quad\cdots\cdots\cdots \quad (21)$$

and

$$\frac{\partial^2 \boldsymbol{W}}{\partial T^2}=\frac{\mu}{\rho}\nabla^2\boldsymbol{W}. \quad\cdots\cdots\cdots \quad (22)$$

Thus we see that dilatational wave and distortional wave can be propagated independently each other, never exciting the other on the way of propagation. And it is remarkable that the distortional wave given by (22) is not affected by the heterogenuity in this case. Since the distortional wave given by (22) has been fully discussed by many senior authorities, the author confines himself in this paper to the study of the dilatational wave.

From (19) and (21) we have

$$\frac{\partial^2\Delta}{\partial T^2}=\nabla^2\{(a-br^2)^2\Delta\}. \quad\cdots\cdots\cdots \quad (23)$$

Putting now

$$\nu(r)=\frac{1}{a-br^2} \quad\cdots\cdots\cdots \quad (24)$$

in the equations of co-ordinates transformation (3) and (4), we get

$$t=\frac{1}{2\sqrt{ab}}\left\{\sinh^{-1}\frac{2\sqrt{ab}\cdot r\cos i}{\sqrt{1+4ab\kappa^2}\cdot(a-br^2)}-\sinh^{-1}\frac{2\sqrt{ab}\cdot h\cos\beta}{\sqrt{1+4ab\kappa^2}\cdot(a-bh^2)}\right\}, \quad (25)$$

$$\theta=\sin^{-1}\frac{\cos i}{\sqrt{1+4ab\kappa^2}}-\sin^{-1}\frac{\cos\beta}{\sqrt{1+4ab\kappa^2}}, \quad\cdots\cdots\cdots \quad (26)$$

$$\frac{1}{h_1}=a-br^2, \quad\cdots\cdots\cdots \quad (27)$$

$$\frac{1}{h_2}=\frac{\sinh 2\sqrt{ab}t}{2\sqrt{ab}}(a-br^2), \quad\cdots\cdots\cdots \quad (28)$$

$$\frac{1}{h_3}=\frac{\sinh 2\sqrt{ab}t}{2\sqrt{ab}}(a-br^2)\sin\beta. \quad\cdots\cdots\cdots \quad (29)$$

Then by the relation,

$$\nabla^2\phi=h_1h_2h_3\left\{\frac{\partial}{\partial t}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial t}\right)+\frac{\partial}{\partial\beta}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial\gamma}\right)\right\}, \quad\cdots \quad (30)$$

and the substitution,

$$\Delta=\frac{2\sqrt{ab}}{(a-br^2)^{\frac{5}{2}}\sinh 2\sqrt{ab}t}\cdot\Delta'\cdot e^{ipT}, \quad\text{..........}\quad (31)$$

the following equation for Δ' is obtained from (23),

$$\frac{\partial^2\Delta'}{\partial t^2}+\frac{4ab}{\sinh^2 2\sqrt{ab}t}\frac{\partial^2\Delta'}{\partial\beta^2}+\frac{4ab}{\sinh^2 2\sqrt{ab}t\cdot\sin^2\beta}\frac{\partial^2\Delta'}{\partial\varphi^2}$$
$$+\frac{4ab\cot\beta}{\sinh^2 2\sqrt{ab}t}\frac{\partial\Delta'}{\partial\beta}+(p^2-ab)\Delta'=0. \quad\text{........}\quad (32)$$

This equation can be satisfied by a function of the form

$$\Delta'=X(t)\,Y(\beta)\,Z(\varphi), \quad\text{..................}\quad (33)$$

if

$$\frac{\partial^2 Z}{\partial\varphi^2}+m^2 Z=0, \quad\text{..............................}\quad (34)$$

$$\frac{\partial^2 Y}{\partial\beta^2}+\cot\beta\frac{\partial Y}{\partial\beta}+\left\{n(n+1)-\frac{m^2}{\sin^2\beta}\right\}Y=0, \quad\text{........}\quad (35)$$

$$\frac{\sinh^2 2\sqrt{ab}t}{4ab}\frac{\partial^2 X}{\partial t^2}+\left\{\frac{\sinh^2 2\sqrt{ab}t}{4ab}(p^2-ab)-n(n+1)\right\}X=0 \qquad (36)$$

where m and n are positive integers.

Thus (34) is satisfied by

$$Z=\begin{cases}\sin m\varphi\\ \cos m\varphi,\end{cases} \quad\text{..................}\quad (37)$$

and (35) by

$$Y=\begin{cases}P_n^m(\cos\beta)\\ Q_n^m(\cos\beta),\end{cases} \quad\text{..................}\quad (38)$$

where $P_n^m(\cos\beta)$ and $Q_n^m(\cos\beta)$ are the associated Legendre's functions.

Since the function X depends not only on t, but also on n, the letter n is hereafter suffixed to X in the equation (36),

$$\frac{\sinh^2 2\sqrt{ab}t}{4ab}\frac{d^2 X_n}{dt^2}+\left\{\frac{\sinh^2 2\sqrt{ab}t}{4ab}(p^2-ab)-n(n+1)\right\}X_n=0. \quad (36')$$

Assuming

$$X_n=A\sin\sqrt{p^2-ab}\,t+B\cos\sqrt{p^2-ab}\,t, \quad\text{..........}\quad (39)$$

substitute in (36'), and we obtain the following simultaneous differential equation to determine A and B.

$$\frac{d^2 A}{dt^2}-2\sqrt{p^2-ab}\frac{dB}{dt}-\frac{4abn(n+1)}{\sinh^2 2\sqrt{ab}t}A=0, \quad\text{.......}\quad (40)$$

$$\frac{d^2B}{dt^2}+2\sqrt{p^2-ab}\,\frac{dA}{dt}-\frac{4abn(n+1)}{\sinh^2 2\sqrt{ab}t}B=0, \quad \ldots\ldots\ldots (41)$$

As the equation (36′) is of the second degree, two particular solutions are necessary, one is analytic at $t=0$, and the other is convergent for a large absolute value of t. The former shall be denoted by $X_n^{(1)}$ and the latter by $X_n^{(2)}$.

When $n=0$, 1 and 2, the solutions of (40) and (41) are easily obtained.

$$\left.\begin{aligned}
n=0&\begin{cases} X_0^{(1)}=\sin\sqrt{p^2-ab}t, \\ X_0^{(2)}=\cos\sqrt{p^2-ab}t, \end{cases}\\
n=1&\begin{cases} X_1^{(1)}=\dfrac{u}{\sqrt{p^2-ab}}\sin\sqrt{p^2-ab}t-\cos\sqrt{p^2-ab}t, \\ X_1^{(2)}=-\sin\sqrt{p^2-ab}t-\dfrac{u}{\sqrt{p^2-ab}}\cos\sqrt{p^2-ab}t, \end{cases}\\
n=2&\begin{cases} X_2^{(1)}=\left\{3\dfrac{u^2}{\sqrt{p^2-ab}}-\dfrac{(p^2-ab)+4ab}{p^2-ab}\right\}\sin\sqrt{p^2-ab}t \\ \qquad\qquad -3\dfrac{u}{\sqrt{p^2-ab}}\cos\sqrt{p^2-ab}t, \\ X_2^{(2)}=3\dfrac{u}{\sqrt{p^2-ab}}\sin\sqrt{p^2-ab}t+\left\{3\dfrac{u^2}{p^2-ab}\right. \\ \qquad\qquad \left.-\dfrac{(p^2-ab)+4ab}{p^2-ab}\right\}\cos\sqrt{p^2-ab}t, \end{cases}
\end{aligned}\right\}\cdots (42)$$

where
$$u\equiv 2\sqrt{ab}\coth 2\sqrt{ab}t. \quad \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

If we introduce a function $R_n^{(i)}$ defined by

$$\sqrt{\frac{2}{\pi p}}\sqrt{2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}t}\,X_n^{(i)}=R_n^{(i)}, \quad (i=1,2), \quad \ldots\ldots\ldots (44)$$

following recurrence formulae are obtained by means of mathematical induction from (36′), (42) and (44).

$$\frac{(p^2-ab)+4n^2ab}{p^2-ab}R_{n-1}^{(i)}+R_{n+1}^{(i)}=(-1)^{i+1}\frac{(2n+1)2\sqrt{ab}\coth 2\sqrt{ab}t}{\sqrt{p^2-ab}}R_n^{(i)}, \quad \ldots\ldots (45)$$

$$\frac{(p^2-ab)+4n^2ab}{p^2-ab}R_{n-1}^{(i)}-R_{n+1}^{(i)}=(-1)^{i+1}\frac{2}{\sqrt{p^2-ab}}\frac{dR_n^{(i)}}{dt}, \ (i=1,2). \quad \ldots\ldots\ldots (46)$$

The solution are therefore,

$$R_n^{(1)}=\sqrt{\frac{2}{\pi p}}(2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}t)^{-n-\frac{1}{2}}\left(\frac{-2ab}{\sqrt{p^2-ab}}\right)^n$$

$$\cdot\left(\frac{d}{d(\sinh^2\sqrt{ab}\,t)}\right)^n\left(\frac{2\sqrt{ab}\,\sin\sqrt{p^2-ab}\,t}{\sinh 2\sqrt{ab}\,t}\right), \quad\ldots\ldots (47)$$

$$R_n^{(2)}=\sqrt{\frac{2}{\pi p}}\,(2\sqrt{ab}\,\operatorname{cosech} 2\sqrt{ab}\,t)^{-n-\frac{1}{2}}\left(\frac{2ab}{\sqrt{p^2-ab}}\right)^n$$

$$\cdot\left(\frac{d}{d(\sinh^2\sqrt{ab}\,t)}\right)^n\left(\frac{2\sqrt{ab}\,\cos\sqrt{p^2-ab}\,t}{\sinh 2\sqrt{ab}\,t}\right)\ldots\ldots (48)$$

It is remarkable that the R-functions depend not only on t and n, but also on p and ab. And, when b approaches zero, R-functions tend to the Bessel's functions of the first and the second kind. The differences between R-functions and the Bessel's functions increase with n and the difference between t and $\frac{\sinh 2\sqrt{ab}\,t}{2\sqrt{ab}}$. Both of the functions are oscillatory, and their phase-difference depends on the ratio of p to ab.

Though the earth may be assumed to be isotropic sphere having a spherical symmetry with respect to its centre, the assumption that the density and rigidity are uniform, as is adopted in this section, is evidently different from what it is. But the author dared to use the values of a and b, obtained by H. Kawasumi[10] by means of analysis of time-distance curves in earthquakes, and calculated the values of R-functions and compared them with the Bessel's functions by which the R-functions are to be replaced when the earth is considered homogeneous.

According to the investigations by H. Kawasumi,

$$\sqrt{ab}\fallingdotseq 2{\cdot}82.10^{-3}, \qquad b\fallingdotseq 2{\cdot}35.10^{-3},$$

the radius of the earth being taken as unit.

Remark: A precise investigation of velocity of wave, especially, of shocks, which is not executed by the author, is necessary to use the value of a and b obtained by H. Kawasumi. But, if the origin of

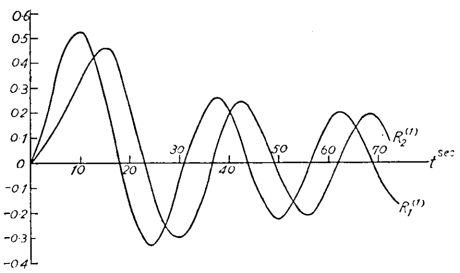

Fig. 2. $R_1^{(1)}, R_2^{(1)}\left(p=\frac{1}{4}\right)$

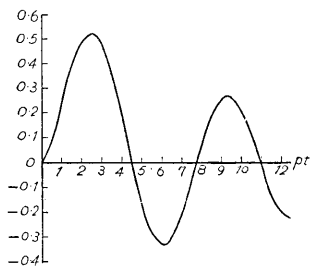

Fig. 3.

10) H. KAWASUMI, *Bull. Earthq. Res. Inst.*, **10** (1932), 94–129.

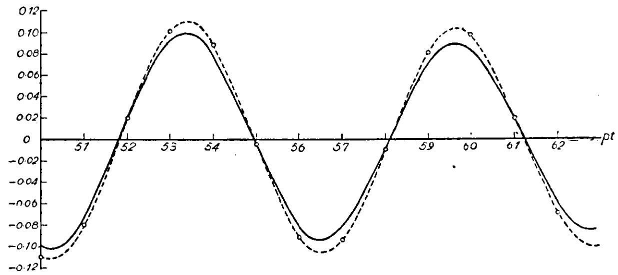

Fig. 4. ------- $J_{\frac{3}{2}}$
∘ $R_1^{(1)}(p=3)$
—— $R_1^{(1)}\left(p=\frac{1}{4}\right)$

earthquake is considered as a point, we can deduce from the discussion of the velocity of surface of discontinuity in "Mathematical Theory of Elasticity" by A. E. H. Love that a and b can be calculated by means of analysis of time-distance curves of earthquakes.

$R_1^{(1)}$ and $R_2^{(1)}$ as functions of t, when $p=\frac{1}{4}$, are shown in Fig. 2. As functions of pt, $J_{\frac{3}{2}}$ and $R_1^{(1)}$, when $p=\frac{1}{4}$ and also $p=3$, are compared in Fig. 3 and 4. The former is for smaller values of t while the latter is for larger ones and is given in larger scale than the former. So far as pt is less than 15, the differences between these functions are negligible, as we see from Fig. 3, but when pt become greater than 50, as shown in Fig. 4, $(R_1^{(1)})_{p=\frac{1}{4}}$ is noticeably small than any of the other two, while the differences between $J_{\frac{3}{2}}$ and $(R_1^{(1)})_{p=3}$ are still negligible.

$R_n^{(t)}\cdot e^{ipT}$, as well as $J_n\cdot e^{ipT}$, expresses a standing wave. For the progressive waves, the following functions are introduced.

For a converging wave,

$$U_n^{(1)}=R_n^{(1)}-(-1)^n iR_n^{(2)} \quad \ldots\ldots\ldots\ldots\ldots (49)$$

For a diverging wave,

$$U_n^{(2)}=R_n^{(1)}+(-1)^n iR_n^{(1)} \quad \ldots\ldots\ldots\ldots\ldots (50)$$

The relation of the U-function to the R-function is just that of the Hankel's function to the Bessel's function, and

$$U_n^{(1)}=\sqrt{\frac{2}{\pi p}}\left(\frac{2ab}{\sqrt{p^2-ab}}\right)^n(2\sqrt{ab}\operatorname{cosech} 2\sqrt{ab}t)^{-n-\frac{1}{2}}$$
$$\cdot\left(\frac{d}{d(\sinh^2\sqrt{ab}t)}\right)^n\left(\frac{2\sqrt{ab}\,e^{i\left(\sqrt{p^2-ab}t-\frac{2n+1}{2}\pi\right)}}{\sinh 2\sqrt{ab}t}\right) \quad \ldots\ldots (51)$$

$$U_n^{(2)}=\sqrt{\frac{2}{\pi p}}\left(\frac{2ab}{\sqrt{p^2-ab}}\right)^n (2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t)^{-n-\frac{1}{2}}$$

$$\cdot\left(\frac{d}{d(\sinh^2\sqrt{ab}t)}\right)^n\left(\frac{2\sqrt{ab}\,e^{-i\left(\sqrt{p^2-ab}t-\frac{2n+1}{2}\pi\right)}}{\sinh 2\sqrt{ab}t}\right) \quad\ldots\ldots (52)$$

Recurrence formulae are obtained for U-functions as follows:

$$\frac{(p^2-ab)+4n^2ab}{p^2-ab}U_{n-1}^{(i)}+U_{n+1}^{(i)}=\frac{(2n+1)2\sqrt{ab}\coth 2\sqrt{ab}t}{\sqrt{p^2-ab}}U_n^{(i)}, \quad\ldots (53)$$

$$\frac{(p^2-ab)+4n^2ab}{p^2-ab}U_{n-1}^{(i)}-U_{n+1}^{(i)}=\frac{2}{\sqrt{p^2-ab}}\frac{dU_n^{(i)}}{dt}. \quad\ldots\ldots\ldots\ldots (54)$$

$$(i=1,\ 2.)$$

From what has been stated, follows the solution of the equation (21) for a diverging wave,

$$\Delta=\frac{\sqrt{2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t}}{(a-br^2)^{\frac{5}{2}}}U_n^{(2)}\cdot P_n^m(\cos\beta)\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}\,e^{ipT} \quad\ldots\ldots (55)$$

Roughly speaking, the surface given by $t=$const. is a wave-front, and, in such case as adopted in this section, this surface is a sphere, though the proof is omitted here. And it is worthy of note that the solution of the equation of motion involves a factor $P_n^m(\cos\beta)$, as in the similar case when the medium is homogeneous, quite concordantly with the fact the surface expressed by $t=$const. is a sphere.

Thus we can discuss the distribution of "pull and push" of the motion on any surface quite similarly to the case when the medium is homogeneous.

The components of displacement derived from Δ under the condition that rotation vanishes are,

$$u_t=-\frac{1}{p^2}\frac{1}{(a-br^2)^{\frac{3}{2}}}P_n^m(\cos\beta)\begin{matrix}\sin\\ \cos\end{matrix}m\varphi\cdot e^{ipT}\left\{\frac{d}{dt}\left(\sqrt{2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t}\,U_n^{(2)}\right)\right.$$

$$\left.+br\cos i\sqrt{2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t}\,U_n^{(2)}\right\},$$

$$u_\beta=-\frac{1}{p^2}\frac{(2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t)^{\frac{3}{2}}}{(a-br^2)^{\frac{1}{2}}}U_n^{(2)}\begin{matrix}\sin\\ \cos\end{matrix}m\varphi\cdot e^{ipT}\left\{\frac{1}{a-br^2}\frac{dP_n^m(\cos\beta)}{d\beta}\right.$$

$$\left.-\frac{bh\sin\beta}{a-bh^2}\frac{P_n^m(\cos\beta)}{2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t}\right\},$$

$$u_\varphi=-\frac{m}{p^2}\frac{(2\sqrt{ab}\ \mathrm{cosech}\ 2\sqrt{ab}t)^{\frac{3}{2}}}{(a-br^2)^{\frac{3}{2}}}U_n^{(2)}\cdot\frac{P_n^m(\cos\beta)}{\sin\beta}\begin{matrix}\cos\\ -\sin\end{matrix}m\varphi e^{ipT}.$$

Asymptotic expansions of U-functions being respectively,

$$U_n^{(1)} \approx \frac{(p^2-ab)+4(n-1)^2 ab}{p^2-ab}$$

$$\cdot \sqrt{\frac{2}{\pi p}} (2\sqrt{ab} \operatorname{cosech} 2\sqrt{ab}t)^{\frac{1}{2}} e^{i\left(\sqrt{p^2-ab}t - \frac{n+1}{2}\pi\right)},$$

$$U_n^{(2)} \approx \frac{(p^2-ab)+4(n-1)^2 ab}{p^2-ab}$$

$$\cdot \sqrt{\frac{2}{\pi p}} (2\sqrt{ab} \operatorname{cosech} 2\sqrt{ab}t)^{\frac{1}{2}} e^{-i\left(\sqrt{p^2-ab}t - \frac{n+1}{2}\pi\right)},$$

if amplitude of displacement be observed on any surface expressed by $r=$const., the component parallel to the direction of propagation, or, to the direction of the tangent at the point of observation to the path of wave determined by the Fermat's Principle of trajectory, is proportional to the reciprocal of $\frac{\sinh 2\sqrt{ab}t}{2\sqrt{ab}}$, and the other components transversal to the same direction are to the square of the same factor.

And we have from the equations (3) and (4),

$$t = \int^r \frac{ds}{a-br^2},$$

provided ds is elementary distance along the curve determined by (16). For a large value of t, the phase-velocity V is determined by

$$\frac{\partial}{\partial T}(pT - \sqrt{p^2-ab}t) = 0,$$

therefore

$$V = \frac{\partial S}{\partial T} = (a-br^2)\frac{p}{\sqrt{p^2-ab}}.$$

As the ratio of p to ab is a number of a order of 10^{-6}, the dispersion may be negligible compared with that due to the effect of gravity, which was investigated by Bromwich and others.

In conclusion, the author desires to express his sincere thanks to Professor T. Matuzawa and Dr. H. Kawasumi for their kind Guidance.

1. 等方不均一球内の一點より起る弾性波（第一報）

地震學教室 吉 山 良 一

密度及び彈性的性質が一點からの距離によりて定つてゐる樣な等方彈性體內を傳る波動の研究への緒として先づ密度及び剛性率が一樣であつて唯ラーメの彈性常數の中λのみが a, b を二つの常數とし r を球の中心からの距離とする時

$$\sqrt{\frac{\lambda+2\mu}{\rho}}=a-br^2$$

で表はさるゝ場合を少しく計算した.

斯る場合に著しいことは先づ密度と剛性率が變はらなければ所謂縱波と橫波とは各獨立に傳播することである。更に橫波に就いてはこれまで均質な媒質の中の問題として研究された結果に何等附け加ふべきことがない.

縱波に就いては,

(1) 震原に於いて，振幅の分布が其の點に於ける震波線の射出角の餘弦を變數とする表面球函數で與へられた場合には其振幅分布は震波線に沿つて其のまゝ保存される.

(2) 距離による減衰程度は今迄ベッセル乃至ハンケルの函數がこれを表はしてゐたが此處に述べる樣な場合には少しく異つた新しい函數が必要である．筆者はこれを R-函數又は U-函數と名附けたがこれが今迄果して研究されなかつた函數であるか否かについては確言することが出來ない．二三の場合について此の函數の値を計算して圖示した．又斯る物理學的の問題に必要なる範圍に於いて此等の函數の一般の形及び漸近級數の最も重要なる項を求めた

斯る媒質の中では波は一般に分散するが地球がもし斯るものであるとすればその分散程度は非常に小さいものである.

震原から相當離れた所で大體の所を考ふれば振動の成分の中傳播の方向への成分は $(a-br^2)^{\frac{3}{2}}$ と $\dfrac{\sinh 2\sqrt{ab}t}{2\sqrt{ab}}$ に逆比例する．これに垂直な成分は二つ共 $(a-br^2)^{\frac{3}{2}}$ と $\left(\dfrac{\sinh 2\sqrt{ab}t}{2\sqrt{ab}}\right)^2$ とに逆比例する．但し地震の樣に勝手な形の衝擊の傳播に就いては上述の諸函數の性質が未だ究めてないので嚴密なことは論じ得られない.