50. *Reflection and Refraction of Seismic Waves in a Stratified Body.*

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Recently Dr. T. Suzuki[1] found from records of seismic observation the fact that the apparent angle of incidence of longitudinal waves of short vibration periods is relatively small while that angle corresponding to large periods is not small. He concluded that this fact may be due to the refractive character of surface layer of the earth. His result suggested us to solve the problem of reflection and refraction of obliquely incident seismic waves at the boundary surfaces of a stratified semi-infinitely body.

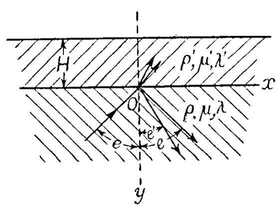

Fig. 1.

Let ϕ, ψ, ϕ', ψ' be wave functions of longitudinal and transverse waves in the bottom medium and the stratified layer. If the primary incident waves are longitudinal and expressed by

$$\phi = Ae^{if(x-\alpha y-nt)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where α is cotangent of the angle of incidence and fn is circular frequency of the vibratory motion of waves, the general expressions of ϕ, ψ, ϕ', ψ' are denoted by

$$\left.\begin{aligned} \phi &= Ae^{i(fx-\alpha y-nt)} + A_1 e^{if(x+\alpha y-nt)}, \\ \psi &= B_1 e^{if(x+\beta y-nt)}, \\ \phi' &= (Ce^{\gamma y} + De^{-\gamma y})\, e^{if(x-nt)}, \\ \psi' &= (Ee^{\delta y} + Fe^{-\delta y})\, e^{if(x-nt)}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

in which β is cotangent of the reflective angle of the reflected transverse waves; and A_1, B_1, C, D, E, F are arbitrary constants to be determined.

From the equations of the vibratory motion of elastic bodies:

1) T. SUZUKI, *Bull. Earthq. Res. Inst.*, 10 (1932), 517-530.

$$\left.\begin{aligned}&\rho\frac{\partial^2\phi}{\partial t^2}=(\lambda+2\mu)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right),\quad \rho\frac{\partial^2\psi}{\partial t^2}=\mu\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right),\\&\rho'\frac{\partial^2\phi'}{\partial t^2}=(\lambda'2\mu')\left(\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2}\right),\quad \rho'\frac{\partial^2\psi'}{\partial t^2}=\mu'\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right),\end{aligned}\right\}\quad\ldots(3)$$

we find

$$\left.\begin{aligned}&n^2=\frac{\lambda+2\mu}{\rho}(1+\alpha^2)=\frac{\mu}{\rho}(1+\beta^2),\\&r^2=f^2\left(1-\frac{\rho' n^2}{\lambda'+2\mu'}\right),\quad s^2=f^2\left(1-\frac{\rho' n^2}{\mu'}\right).\end{aligned}\right\}\quad\ldots\ldots\ldots(4)$$

The horizontal and vertical components of the displacements in the bottom medium and in the layer are determined from

$$\left.\begin{aligned}&u_1=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y},\qquad v_1=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},\\&u_2=\frac{\partial\phi'}{\partial x}+\frac{\partial\psi'}{\partial y},\qquad v_2=\frac{\partial\phi'}{\partial y}-\frac{\partial\psi'}{\partial x}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(5)$$

The boundary conditions are such that

$$y=0;\quad u_1=u_2,\quad v_1=v_2,$$

$$\left.\begin{aligned}&\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\mu\frac{\partial v_1}{\partial y}=\lambda'\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)+2\mu'\frac{\partial v_2}{\partial y},\\&\mu\left(\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y}\right)=\mu'\left(\frac{\partial v_2}{\partial x}+\frac{\partial u_2}{\partial y}\right),\end{aligned}\right\}\quad(6)$$

$$y=-H;\quad\left.\begin{aligned}&\lambda'\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)+2\mu'\frac{\partial v_2}{\partial y}=0,\\&\frac{\partial v_2}{\partial x}+\frac{\partial u_2}{\partial y}=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Substituting from (2), (5) in (6), (7), we get

$$A_1+\beta B_1-C-D+i\frac{s}{f}E-i\frac{s}{f}F=-A,$$

$$\alpha A_1-B_1+i\frac{r}{f}C-i\frac{r}{f}D+E+F=\alpha A,$$

$$-\left\{\frac{\lambda}{\mu}(1+\alpha^2)+2\alpha^2\right\}A_1+2\beta B_1-\left\{\frac{\lambda'}{\mu}\left(\frac{r^2}{f^2}-1\right)+\frac{2\mu'}{\mu}\cdot\frac{r^2}{f^2}\right\}C-\left\{\frac{\lambda'}{\mu}\left(\frac{r^2}{f^2}-1\right)\right.$$

$$\left.+\frac{2\mu'}{\mu}\cdot\frac{r^2}{f^2}\right\}D+2i\frac{\mu'}{\mu}\cdot\frac{s}{f}E-2i\frac{\mu'}{\mu}\cdot\frac{s}{f}F=\left\{\frac{\lambda}{\mu}(1+\alpha^2)+2\alpha^2\right\}A,$$

$$2\alpha A_1+(\beta^2-1)B_1+2i\frac{\mu'}{\mu}.\frac{r}{f}C-2i\frac{\mu'}{\mu}.\frac{r}{f}D+\frac{\mu'}{\mu}\left(\frac{s^2}{f^2}+1\right)E$$
$$+\frac{\mu'}{\mu}\left(\frac{s^2}{f^2}+1\right)F=2\alpha A,$$
$$\left\{\frac{\lambda'}{\mu'}\left(\frac{r^2}{f^2}-1\right)+2\frac{r^2}{f^2}\right\}e^{-rH}C+\left\{\frac{\lambda'}{\mu'}\left(\frac{r^2}{f^2}-1\right)+2\frac{r^2}{f^2}\right\}e^{rH}D-2i\frac{s}{f}e^{-sH}E$$
$$+2i\frac{s}{f}e^{sH}F=0,$$
$$2i\frac{r}{f}e^{-rH}C-2i\frac{r}{f}e^{rH}D+\left(\frac{s^2}{f^2}+1\right)e^{-sH}E+\left(\frac{s^2}{f^2}+1\right)e^{sH}F=0. \qquad (8)$$

From these equations the values of A_1, B_1, C, D, E, F are determined as functions of A, fH, α. For the sake of simplicity we put $\rho=\rho'$, $\lambda=\mu$, $\lambda'=\mu'$, $\mu=2\mu'$, $\alpha=\cot\frac{\pi}{4}$. Then we get

$$A_1=[\{1186{\cdot}354-341{\cdot}320\cos(a+b)-2304{\cdot}171\cos(a-b)\}$$
$$+i\{307{\cdot}890\sin(a+b)-2491{\cdot}920\sin(a-b)\}]\frac{A}{G},$$
$$B_1=2\{976{\cdot}565+333{\cdot}954\cos(a+b)+343{\cdot}954\cos(a-b)\}\frac{A}{G},$$
$$C=3[\{161{\cdot}618-28{\cdot}625\cos(a+b)-620{\cdot}688\cos(a-b)\}$$
$$+i\{28{\cdot}625\sin(a+b)-620{\cdot}688\sin(a-b)\}]\frac{A}{G},$$
$$D=3[\{95{\cdot}286+991{\cdot}055\cos(a+b)+17{\cdot}932\cos(a-b)\}$$
$$+i\{-991{\cdot}055\sin(a+b)-17{\cdot}932\sin(a-b)\}]\frac{A}{G},$$
$$E=3[\{558{\cdot}342+88{\cdot}330\cos(a+b)+93{\cdot}829\cos(a-b)\}$$
$$+i\{88{\cdot}330\sin(a+b)-93{\cdot}829\sin(a-b)\}]\frac{A}{G},$$
$$F=3[\{16{\cdot}126-149{\cdot}820\cos(a+b)-55{\cdot}320\cos(a-b)\}$$
$$+i\{149{\cdot}820\sin(a+b)-55{\cdot}320\sin(a-b)\}]\frac{A}{G},$$
$$\ldots\ldots\ldots(9)$$

where

$$G=\{611{\cdot}904+4104{\cdot}738\cos(a+b)+441{\cdot}887\cos(a-b)\}$$
$$+i\{-3978{\cdot}738\sin(a+b)+192{\cdot}852\sin(a-b)\},\ldots.(10)$$

and $a=rH$, $b=sH$.

The displacement of a point ($x=0$) on the surface $y=H$ is obtained by using (2) and (5). We are to take real parts of complex forms of expressions of u_2 and v_2. These real parts correspond with the primary waves:

$$\left.\begin{aligned} u_{10} &= -Af\sin f(x-\alpha y-nt), \\ v_{10} &= Af\alpha\sin f(x-\alpha y-nt). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (11)$$

As the length (L) of the primary waves is $2\pi f/\sqrt{1+\alpha^2}$, we find $fH=\frac{2\pi}{\sqrt{1+\alpha^2}}\frac{H}{L}$. We have calculated three cases of fH, namely $fH=1$, $fH=100$, $fH=\frac{1}{100}$, the result being shown in Figs. 2, 3, 4. In these figures, the respective upper curves indicate the variation of u- and v-components with respect to time, while the lower ones give the orbits of a certain point on the surface of the body, U_0 being the amplitude of the primary waves.

It will be seen that when the primary waves are long compared with the thickness of the layer, the motion of a point on the surface is approximately similar to that of these primary waves. The apparent incidence angle on the surface is somewhat different from the incidence angle of the primary waves. When the length of primary waves is

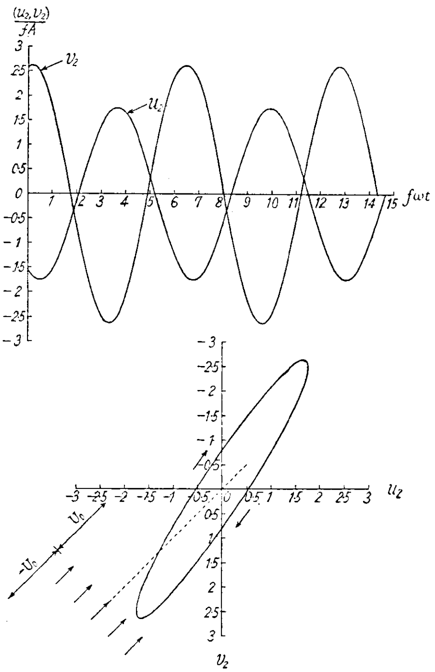

Fig. 2.

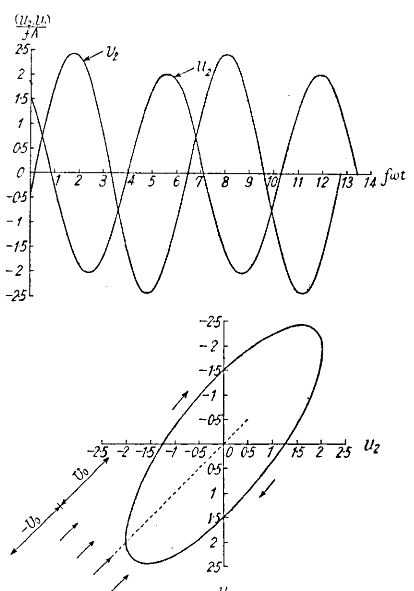

Fig. 3.

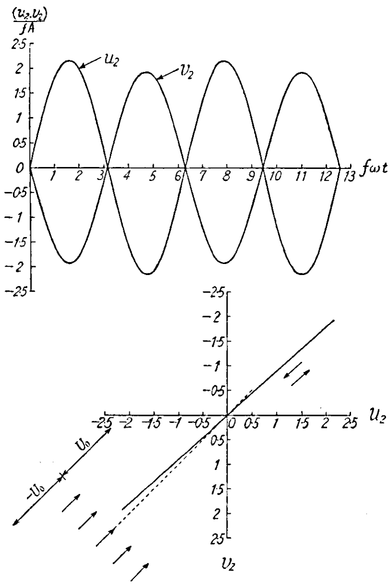

Fig. 4.

small in comparison with the thickness of the surface layer, the apparent angle of incidence becomes small as Dr. Suzuki pointed out. It is remarkable fact that the ratio of length of the minor axis to that of the major axis is relartively great for small wave length. For very large wave length the length of the minor axis is approximately zero.

The movement of ground due to an arbitrary disturbance may be found by applying Fourier's integral expression, namely

$$F(x,y)=\frac{1}{\pi^2}\int_0^\infty dp\int_0^\infty dq\int_{-\infty}^\infty d\xi\int_{-\infty}^\infty d\eta F(\xi,\eta)\cos p(x-\xi)\cos q(y-\eta)$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^\infty dp\int_{-\infty}^\infty dq\int_{-\infty}^\infty d\xi\int_{-\infty}^\infty d\eta F(\xi,\eta)\,e^{ip(x-\xi)}\,e^{-iq(y-\eta)}. \quad \ldots.(12)$$

We find thus

$$\phi_1=F(x,y,t)=\frac{1}{4\pi^2}\int_{-\infty}^\infty df\int_{-\infty}^\infty d(f\alpha)\int_{-\infty}^\infty d\xi\int_{-\infty}^\infty d\eta F(\xi,\eta)\,e^{if(x-\xi)}\,e^{-i(f\alpha)(y-\eta)}\,e^{-if\Omega t}$$

$$(f\alpha=g)$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^\infty e^{-if\Omega t}df\int_{-\infty}^\infty dg\int_{-\infty}^\infty d\xi\int_{-\infty}^\infty d\eta F(\xi,\eta)\,e^{i\{f(x-\xi)-g(y-\eta)\}},$$

$$\phi=\phi_1+\frac{1}{4\pi^2}\int_{-\infty}^\infty\frac{A_1}{A}e^{-if\Omega t}df\int_{-\infty}^\infty dg\int_{-\infty}^\infty d\xi\int_{-\infty}^\infty d\eta F(\xi,\eta)\,e^{i\{f(x-\xi)+g(y+\eta)\}},$$

$$\psi=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{B_1}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta F(\xi,\eta)e^{i\left\{f(x-\xi)+g\left(\frac{\beta}{\alpha}y+\eta\right)\right\}},$$

$$\phi'=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{C}{A}e^{-ifnt}df\int_{\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta F(\xi,\eta)e^{i\{f(x-\xi)+g\eta\}+\frac{g}{\alpha}\sqrt{1-\frac{\rho'n^2}{\lambda'+2\mu'}}y}$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{D}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta F(\xi,\eta)e^{i\{f(x-\xi)+g\eta\}-\frac{g}{\alpha}\sqrt{1-\frac{\rho'n^2}{\lambda'+2\mu'}}y},$$

$$\psi'=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{E}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta F(\xi,\eta)e^{i\{f(x-\xi)+g\eta\}+\frac{g}{\alpha}\sqrt{1-\frac{\rho'n^2}{\mu'}}y}$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{E}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta F(\xi,\eta)e^{i\{f(x-\xi)+g\eta\}-\frac{g}{\alpha}\sqrt{1-\frac{\rho'n^2}{\mu'}}y}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots(13)$$

Put

$$F(\xi,\eta)=e^{-\frac{(\xi-\alpha\eta)^2}{c^2}},\ \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

then we get

$$\phi_1=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)-g(y-\eta)\}}$$

$$=\frac{c}{4\pi^{3/2}\alpha}\int_{-\infty}^{\infty}e^{-if(x-nt)}df\int_{-\infty}^{\infty}e^{-\frac{c^2}{4\alpha^2}g^2+ig\left(\frac{\xi}{\alpha}-y\right)}dg\int_{-\infty}^{\infty}e^{-if\xi}d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-if(x-nt)}df\int_{-\infty}^{\infty}e^{-\frac{(\xi-\alpha\eta)^2}{c^2}-if\xi}d\xi$$

$$=\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{c^2}{4}f^2-if(x-\alpha y-nt)}df$$

$$=e^{-\frac{(x-\alpha y-nt)^2}{c^2}}.$$

Similarly we obtain

$$\phi=\phi_1+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{A_1}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)+g(y+\eta)\}}d\eta$$

$$=\phi_1+\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}F(f)e^{-\frac{c^2}{4}f^2+if(x+\alpha y-nt)}df$$

$$=e^{-\frac{(x-\alpha y-nt)^4}{c^2}}$$

$$+a_1e^{-\frac{(x+\alpha y-nt)^2}{c^2}}+\frac{cb_1}{\sqrt{c^2+4h_1H^2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4h_1H^2}}+\frac{cd_1}{\sqrt{c^2+4k_1H^2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4k_1H^2}},$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots(16)$$

where use is made of the approximate formula

$$\frac{A_1}{A}=F(f)=a_1+b_1e^{-h_1(fH)^2}+d_1e^{-k_1(fH)^2}.\ \ \ldots\ldots\ldots\ldots(17)$$

In the same manner of treatment we get

$$\psi=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{B_1}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\left\{f(x-\xi)+g\left(\frac{\beta}{\alpha}y+\eta\right)\right\}}$$

$$=\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\chi(f)\,e^{-\frac{c^2}{4}f^2+if(x+\beta y-nt)}df$$

$$=a_2 e^{-\frac{(x+\beta y-nt)^2}{c^2}}+\frac{cb_2}{\sqrt{c^2+4h_2H^2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4h_2H^2}}+\frac{cd_2}{\sqrt{c^2+4k_2H^2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4k_2H^2}},$$

............(18)

where

$$\frac{B_1}{A}=\chi(f)=a_2+b_2e^{-h_2(fH)^2}+d_2e^{-k_2(fH)^2},$$

$$\phi'=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{C}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)+g\eta\}+\frac{g}{\alpha}\sqrt{1-\frac{\rho' n^2}{\lambda'+2\mu'}}y}$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{D}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)+g\eta\}-\frac{g}{\alpha}\sqrt{1-\frac{\rho' n^2}{\lambda'+2\mu'}}y}$$

$$=\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\Phi(f)\,e^{-\frac{c^2}{4}f^2+if\left(x+\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)}df$$

$$+\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\Phi'(f)\,e^{-\frac{c^2}{4}f^2+if\left(x-\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)}df$$

$$=a_3e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}+\frac{cb_3}{\sqrt{c^2+4h_3H^2}}e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_3H^2}}$$

$$+\frac{cd_3}{\sqrt{c^2+4k_3H^2}}e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_3H^2}}$$

$$+a_4e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}+\frac{cb_4}{\sqrt{c^2+4h_4H^2}}e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_4H^2}}$$

$$+\frac{cd_4}{\sqrt{c^2+4k_4H^2}}e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_4H^2}},\quad \text{......(19)}$$

where

$$\left.\begin{aligned}\frac{C}{A}&=\Phi(f)=a_3+b_3e^{-h_3(fH)^2}+d_3e^{-k_3(fH)^2},\\ \frac{D}{A}&=\Phi'(f)=a_4+b_4e^{-h_4(fH)^2}+d_4e^{-k_4(fH)^2},\end{aligned}\right\}\text{..........(20)}$$

$$\psi'=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{E}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)+g\eta\}+\frac{g}{\alpha}\sqrt{1-\frac{\rho' n^2}{\mu'}}y}$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{F}{A}e^{-ifnt}df\int_{-\infty}^{\infty}dg\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\, e^{-\frac{(\xi-\alpha\eta)^2}{c^2}}e^{i\{f(x-\xi)+gy\}-\frac{g}{\alpha}\sqrt{1-\frac{\rho' n^2}{\mu'}}y}$$

$$=\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\Psi(f)\,e^{-\frac{c^2}{4}f^2+if\left(x+\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)}df$$

$$+\frac{c}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\Psi'(f)\,e^{-\frac{c^2}{4}f^2+if\left(x-\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)}df$$

$$=a_5 e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}+\frac{cb_5}{\sqrt{c^2+4h_5H^2}}e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_5H^2}}$$

$$+\frac{cd_5}{\sqrt{c^2+4k_5H^2}}e^{-\frac{\left(x+\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_5H^2}}$$

$$=a_6 e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}+\frac{cb_6}{\sqrt{c^2+4h_6H^2}}e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_6H^2}}$$

$$+\frac{cd_6}{\sqrt{c^2+4k_6H^2}}e^{-\frac{\left(x-\sqrt{\frac{\rho' n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_6H^2}}, \quad \text{........(21)}$$

where

$$\left.\begin{aligned}\frac{E}{A}&=\Psi(f)=a_5+b_5e^{-h_5(fH)^2}+d_5e^{-k_5(fH)^2},\\ \frac{F}{A}&=\Psi'(f)=a_6+b_6e^{-h_6(fH)^2}+d_6e^{-k_6(fH)^2}.\end{aligned}\right\} \quad \text{.........(22)}$$

By means of the formula:

$$\left.\begin{aligned}u_1&=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}, & v_1&=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},\\ u_2&=\frac{\partial\phi'}{\partial x}+\frac{\partial\psi'}{\partial y}, & v_2&=\frac{\partial\phi'}{\partial y}-\frac{\partial\psi'}{\partial x},\end{aligned}\right\} \quad \text{.............(5')}$$

the expressions of displacements become

$$u_1=-\frac{2(x-\alpha y-nt)}{c^2}e^{-\frac{(x-\alpha y-nt)^2}{c^2}}$$

$$-2(x+\alpha y-nt)\left\{\frac{a_1}{c^2}e^{-\frac{(x+\alpha y-nt)^2}{c^2}}+\frac{cb_1}{(c^2+4h_1H^2)^{3/2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4h_1H^2}}\right.$$

$$\left.+\frac{cd_1}{(c^2+4k_1H^2)^{3/2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4k_1H^2}}\right\}$$

$$-2\beta(x+\beta y-nt)\left\{\frac{a^2}{c^2}e^{-\frac{(x+\beta y-nt)^2}{c^2}}+\frac{cb_2}{(c^2+4h_2H^2)^{3/2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4h_2H^2}}\right.$$

$$\left.+\frac{cd_2}{(c^2+4k_2H^2)^{3/2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4k_2H^2}}\right\},$$

$$\ldots\ldots\ldots\ldots(23)$$

$$v_1=\frac{2\alpha(x-\alpha y-nt)}{c^2}e^{-\frac{(x-\alpha y-nt)^2}{c^2}}$$

$$-2\alpha(x+\alpha y-nt)\left\{\frac{a_1}{c^2}e^{-\frac{(x+\alpha y-nt)^2}{c^2}}+\frac{cb_1}{(c^2+4h_1H^2)^{3/2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4h_2H^2}}\right.$$

$$\left.+\frac{cd_1}{(c^2+4k_1H^2)^{3/2}}e^{-\frac{(x+\alpha y-nt)^2}{c^2+4k_2H^2}}\right\}$$

$$+2(x+\beta y-nt)\left\{\frac{a_2}{c^2}e^{-\frac{(x+\beta y-nt)^2}{c^2}}+\frac{cb_2}{(c^2+4h_2H^2)^{3/2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4h_2H^2}}\right.$$

$$\left.+\frac{cd_2}{(c^2+4k_2H^2)^{3/2}}e^{-\frac{(x+\beta y-nt)^2}{c^2+4k_2H^2}}\right\},$$

$$\ldots\ldots\ldots\ldots(24)$$

$$u_2=-2\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)\left\{\frac{a_3}{c^2}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$\left.+\frac{cb_3}{(c^2+4h_3H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_3H^2}}+\frac{cd_3}{(c^2+4k_3H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_3H^2}}\right\}$$

$$-2\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)\left\{\frac{a_4}{c^2}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$\left.+\frac{cb_4}{(c^2+4h_4H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_4H^2}}+\frac{cd_4}{(c^2+4k_4H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda^2+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_4H^2}}\right\}$$

$$-2\sqrt{\frac{\rho'n^2}{\mu'}-1}\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)\left\{\frac{a_5}{c^2}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$\left.+\frac{cb_5}{(c^2+4h_5H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_5H^2}}+\frac{cd_5}{(c^2+4k_5H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_5H^2}}\right\}$$

$$+2\sqrt{\frac{\rho'n^2}{\mu'}-1}\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)\left\{\frac{a_6}{c^2}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_6}{(c^2+4h_6H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_6H^2}}+\frac{cd_6}{(c^2+4k_6H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_6H^2}}\Bigg\},$$

$$\ldots\ldots\ldots\ldots(25)$$

$$v_2=-2\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)\left\{\frac{a_3}{c^2}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_3}{(c^2+4h_3H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_3H^2}}+\frac{cd_3}{(c^2+4k_3H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_3H^2}}\Bigg\}$$

$$+2\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)\left\{\frac{a_4}{c^2}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_4}{(c^2+4h_4H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4h_4H^2}}\quad\frac{cd_4}{(c^2+4k_4H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,y-nt\right)^2}{c^2+4k_4H^2}}\Bigg\}$$

$$+2\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)\left\{\frac{a_5}{c^2}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_5}{(c^2+4h_5H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_5H^2}}+\frac{cd_5}{(c^2+4k_5H^2)^{3/2}}e^{-\frac{\left(x+\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_5H^2}}\Bigg\}$$

$$+2\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)\left\{\frac{a_6}{c^2}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_6}{(c^2+4h_6H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4h_6H^2}}+\frac{cd_6}{(c^2+4k_6H^2)^{3/2}}e^{-\frac{\left(x-\sqrt{\frac{\rho'n^2}{\mu'}-1}\,y-nt\right)^2}{c^2+4k_6H^2}}\Bigg\}.$$

$$\ldots\ldots\ldots\ldots(26)$$

The movement of the ground surface may be obtained by putting $x=0$, $y=-H$ in the expressions of u_2, v_2. The result is shown as follows:

$$u_2=2\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H+nt\right)\left\{\frac{a_3}{c^2}e^{-\frac{\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2}}\right.$$

$$+\frac{cb_3}{(c^2+4h_3H^2)^{3/2}}e^{-\frac{\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2+4h_3H^2}}+\frac{cd_3}{(c^2+4k_3H^2)^{3/2}}e^{-\frac{\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2+4k_3H^2}}\Bigg\}$$

$$-2\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H-nt\right)\left\{\frac{a_4}{c^2}e^{-\frac{\left(\sqrt{\frac{\rho'n^2}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2}}\right.$$

$$
\begin{aligned}
&+\frac{cb_4}{(c^2+4h_4H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho n'4}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2+4h_4H^2}}+\frac{cd_4}{(c^2+4k_4H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2+4k_4H^2}}\Bigg\}\\
&+2\sqrt{\frac{\rho' n^2}{\mu'}-1}\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)\Bigg\{\frac{a_5}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2}}\\
&+\frac{cb_5}{(c^2+4h_5H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2+4h_5H^2}}+\frac{cd_5}{(c^2+4k_5H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2+4k_5H^2}}\Bigg\}\\
&+2\sqrt{\frac{\rho' n^2}{\mu'}-1}\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)\Bigg\{\frac{a_6}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2}}\\
&+\frac{cb_6}{(c^2+4h_6H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2+4h_6H^2}}+\frac{cd_6}{(c^2+4k_6H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2+4k_6H^2}}\Bigg\},
\end{aligned}
\qquad \ldots\ldots\ldots\ldots(27)
$$

$$
\begin{aligned}
v_2=\;&2\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H+nt\right)\Bigg\{\frac{a_3}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2}}\\
&+\frac{cb_3}{(c^2+4h_3H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2+4h_3H^2}}+\frac{cd_3}{(c^2+4k_3H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H+nt\right)^2}{c^2+4k_3H^2}}\Bigg\}\\
&+2\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H-nt\right)\Bigg\{\frac{a_4}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2}}\\
&+\frac{cb_4}{(c^2+4h_4H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2+4h_4H^4}}+\frac{cd_4}{(c^2+4k_4H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\lambda'+2\mu'}-1}\,H-nt\right)^2}{c^2+4k_4H^2}}\Bigg\}\\
&-2\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)\Bigg\{\frac{a_5}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2}}\\
&+\frac{cb_5}{(c^2+4h_5H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2+4h_5H^2}}+\frac{cd_5}{(c^2+4k_5H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H+nt\right)^2}{c^2+4k_5H^2}}\Bigg\}\\
&+2\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)\Bigg\{\frac{a_6}{c^2}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2}}\\
&+\frac{cb_6}{(c^2+4h_6H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2+4h_6H^2}}+\frac{cd_6}{(c^2+4k_6H^2)^{3/2}}\,e^{-\frac{\left(\sqrt{\frac{\rho' n^2}{\mu'}-1}\,H-nt\right)^2}{c^2+4k_6H^2}}\Bigg\}.
\end{aligned}
\qquad \ldots\ldots\ldots\ldots(28)
$$

It seems that the movement of the ground due to arbitrary waves is quite similar to that due to the waves of harmonic type of infinite extent.

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50. 地表に一つの層があるときの地震波の反射及屈折

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地表に一つの層があるときに水平面に對して斜の方向から來た地震波が層の境界面で反射屈折する模樣を彈性力學的に正確なる方法でしらべた。その結果

1. 地表の層の厚さに比して地震波の波長が極めて長いときには地表上の點の運動は Knott などがやつたやうに振動方向が大體に於て入射波の方向に向く事がわかつた。
2. 地震の層の厚さに比して地震波の波長が短く、且つ入射波が無限につづく正弦波のときには、地表の運動は、若し上層の彈性が低いときには振動方向が垂直の方へ向いてくるものであつて、この結果は鈴木理學士の實驗結果と大體に於て一致してゐる。
3. 地震波が層の厚さに比して短いとき程、表面上の點の振動軌跡の楕圓形に於て短軸の方が零よりも少しづつ大なる値となることがわかつた。