49. *Vibrations of a Singled-storyed Framed Structure.*

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The problem of vibrations of a chimney and similar constructions due to seismic waves was studied theoretically by Professor Mononobe[1] about twelve years ago and by Byerly and others[2] very recently. The same problem was again dealt with by Professor Suyehiro[3] taking into account of the initial condition of the vibratory motion. The case of framed structure was also treated of by many authors: Reissner[4] Sano[5], Taniguchi[6], Muto[7], Majima[8], Mizuhara[9], Prager[10], Martel[11], Hohenemser[12] and others. But the standpoints, on which these latter authors are based, are not theoretically rigorous, as their methods of treatment are mainly to apply approximately the principle of energy or to neglect some of boundary conditions. Indeed, the exact calculation of the complicated framed structure is very difficult in general. We have obtained the solution of the vibration problem of a single-storyed framed structure.

We have calculated the natural periods of a framed structure with endless extent of the series of successive spans, the distribution of the

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1) N. MONONOBE, *Jour. Soc. Civil Eng.*, Tokyo, **5** (1919), 561–647 ; **6** (1920), 581–690 ; *ZAMM* (1921), 445–451.

2) P. BYERLY, J. HESTER and K. MARSHALL, *Bull. Seism. Soc.*, America, **21** (1931), 268–276.

3) K. SUYEHIRO, *Jour. Soc. Arch.*, Tokyo, **40** (1926), 531–559.

4) H. REISSNER, *ZS. f. Bauwesen*, **8** (1903), 135–162.

5) R. SANO, *Rep. Earthq. Inv. Comm.*, No. 83 (1916), A, 1–141 ; B, 1–137.

6) T. TANIGUCHI, *Jour. Soc. Arch.*, Tokyo, **39** (1925), 43–62, 91–99, 177–206, 487–498 ; **40** (1926), No. 480.

7) K. MUTO, *Jour. Soc. Arch.*, Tokyo, **39** (1925), No. 476 ; **43** (1929), 67–106, 49–66.

8) K. MAJIMA, *Jour. Soc. Civil Eng.*, **12** (1926), 229–304 : *Disin to Kentiku.*

9) A. MIZUHARA, *Jour. Soc. Arch.*, Tokyo, **39** (1925), No. 475 ; **40** (1926), No. 487 ; **41** (1927), No. 493, No. 476, No. 501 ; **43** (1929), No. 522 ; **44** (1930), No. 534.

10) W. PRAGER, *Bauing.*, **8** (1927), 129 ; *ZS. f. tech. Physik*, **9** (1928), 223–227 ; *ZS. f. tech. Physik*, **10** (1929), 275–280.

11) R. R. MARTEL, *Proc. WEC*, **3** (1929, Tokyo).

12) K. HOHENEMSER, *Ing. Arch.*, **1** (1930), 271–292.

deflections as well as bending moments in the structure due to forcep oscillations, and the natural periods and the forced oscillation of the case of two spans.

I. Natural Periods of Framed Structure of an Infinite Number of Spans.

In this paper the following notations are employed.

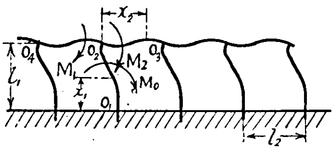

Fig. 1.

ρ_1, ρ_2 = densities of pillar and beam respectively,

E_1, E_2 = Young's Moduli of ....,

a_1, a_2 = effective areas of ....,

k_1, k_2 = radii of gyration of the sections of.......................,

l_1, l_2 = effective lengths of pillar and the intercepted portion of the beam respectively,

y_1, y_2 = transverse deflections of pillar and beam at x_1 and x_2 respectively,

M_0, M_1, M_2 = bending moments of pillar and beam at joint O_2.

The equation of motion of the bar $0_1 0_2$ is expressed by

$$E_1 k_1^2 \frac{\partial^4 y_1}{\partial x_1^4} + \rho_1 \frac{\partial^2 y_1}{\partial t^2} = 0. \tag{1}$$

If we write $y_1 = u_1 \cos pt$, then

$$\frac{d^4 u_1}{dx_1^4} = m_1^4 u_1, \tag{2}$$

where

$$m_1^4 = \frac{\rho_1 p^2}{E_1 k_1^2}. \tag{3}$$

The solution of (2) is

$$u_1 = A_1 \cos m_1 x_1 + B_1 \sin m_1 x_1 + C_1 \cosh m_1 x_1 + D_1 \sinh m_1 x_1. \tag{4}$$

The boundary conditions at 0_1, which is the clamped end of the pillar, are written by

$$x_1 = 0\,; \quad u_1 = 0, \quad \frac{du_1}{dx_1} = 0. \tag{5}$$

Substituting from (4) in (5), we get

$$u_1 = A_1 (\cos m_1 x_1 - \cosh m_1 x_1) + B_1 (\sin m_1 x_1 - \sinh m_1 x_1). \tag{6}$$

The boundary conditions at the other ends 0_2 are

$$x_1=l_1\,;\quad -E_1a_1k_1^2\frac{d^2u_1}{dx_1^2}=M_0,\quad -E_1a_1k_1^2\frac{d^3u_1}{dx_1^3}=\rho_2a_2l_2p^2u_1. \qquad (7),\ (8)$$

The equation of the motion of the member 0_20_3 (or 0_40_2) is expressed by

$$E_2k_2^2\frac{\partial^4y_2}{\partial x_2^4}+\rho_2\frac{\partial^2y_2}{\partial x_2^2}=0. \qquad (9)$$

If we write $y_2=u_2\cos pt$, then

$$\frac{d^4u_2}{dx_2^4}=m_2^4u_2, \qquad (10)$$

where

$$m_2^4=\frac{\rho_2p^2}{E_2k_2^2}. \qquad (11)$$

The solution of (10) is

$$u_2=A_2\cos m_2x_2+B_2\sin m_2x_2+C_2\cosh m_2x_2+D_2\sinh m_2x_2. \qquad (12)$$

The conditions at the clamped joint, $x_2=0$, $x_1=l_1$, are

$$u_2=0,\quad \frac{du_2}{dx_2}=\frac{du_1}{dx_1},\quad -E_2a_2k_2^2\frac{d^2u_2}{dx_2^2}=M_2\,; \qquad (13),\ (14),\ (15)$$

and the condition at the clamped joint $x_2=l_2$, $x_1=l_1$, are

$$u_2=0,\quad \frac{du_2}{dx_2}=\frac{du_1}{dx_1},\quad -E_2a_2k_2^2\frac{d^2u_2}{dx_2^2}=M_1. \qquad (16),\ (17),\ (18)$$

The condition of the fixing moment at 0_2 (or 0_3) is written by

$$M_0+M_1-M_2=0. \qquad (19)$$

Applying the conditions and the relations in (7) (8), (13), (14), (15), (16), (17), (18), (19) to the equations (6) and (12), it is possible to find the ratios of the constants A_1, B_1, A_2, B_2, C_2, D_2 besides M_0, M_1, M_2, and to determine the values of m_1 and m_2 which give us the frequency p.

Write

$$m_1l_1\equiv\alpha,\quad m_2l_2\equiv\beta,\quad \frac{\rho_2a_2l_2p^2}{E_1a_1k_1^2m_1^3}\equiv\gamma,\quad \frac{m_2}{m_1}=\xi,\quad \frac{E_2a_2k_2^2m_2^2}{E_1a_1k_1^2m_1^2}\equiv\eta\,; \qquad (20)$$

then, from the conditions (8) we get

$$A_1[(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)]$$
$$=\beta_1[(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)]. \qquad (21)$$

From (13), (14) we find

$$(B_2+D_2)\,\xi=-A_1(\sin\alpha+\sinh\alpha)+B_1(\cos\alpha-\cosh\alpha), \qquad (22)$$

while (16) gives us

$$A_2(\cos\beta-\cosh\beta)+B_2\sin\beta+D_2\sinh\beta=0. \qquad (23)$$

By means of (17) and (22) we obtain

$$A_2(\sin\beta+\sinh\beta)+B_2(1-\cos\beta)+D_2(1-\cosh\beta)=0. \qquad (24)$$

Finally, using the relations (7), (15), (18) and the condition (19), we get

$$A_1(\cos\alpha+\cosh\alpha)+B_1(\sin\alpha+\sinh\alpha)+A_2\eta(-2+\cos\beta+\cosh\beta)$$
$$+B_2\eta\sin\beta-D_2\eta\sinh\beta=0. \qquad (25)$$

Eliminating A_1, B_1, A_2, B_2, D_2 between (21), (22), (23), (24), (25), we arrive at the relation,

$$2\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma(\cos\alpha\cosh\alpha-1\}$$
$$\times\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}=\frac{\xi}{\eta}\{(\cos\alpha\cosh\alpha+1)$$
$$+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}(\cos\beta\cosh\beta-1). \qquad (26)$$

From this equation and the relations in (20), we can find the frequency p or the period of vibrations $\frac{2\pi}{p}$.

When the stiffness of the pillar is very high compared with that of the beam, the condition, $\frac{\eta}{\xi}=0$ holds, so that

$$\cos\beta\cosh\beta-1=0. \qquad (27)$$

This is the frequency equation of the beam, whose both ends are clamped in position and inclination. In a special case, where there is no beam, we know that $\frac{\eta}{\xi}=0$ and $\gamma=0$, so that we get

$$\cos\alpha\cosh\alpha+1=0. \qquad (28)$$

This is the frequency equation of a clamped-free bar and gives the equation of a clamped-free bar.

When the stiffness of the beam is very high in comparison with that of the pillar and the mass of the beam is large, we may put $\frac{\xi}{\eta}=0$, $\frac{1}{\gamma}=0$, so that

$$\cos\alpha\cosh\alpha-1=0. \qquad (29)$$

This is the frequency equation of a bar whose both ends are clamped in position and in slope.

As it is very difficult to solve (26) in general, we have employed the method of trial and error to get frequencies of various cases. Again, we have limited the calculation to the determination of the fundamental free vibration of the cases, $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\frac{1}{2}$, 1, besides the conditions that $l_1=l_2$, $E_1=E_2$, and $a_1=a_2$. The result is shown as follows.

$$\frac{\rho_1}{\rho_2}=1, \qquad \frac{1}{2}, \qquad \frac{1}{4}.$$

$$\left(\frac{\rho_1 p^2}{E_1 k_1^2}\right)^{\frac{1}{4}} l_1 = 1{\cdot}645 \qquad 1{\cdot}429 \qquad 1{\cdot}224$$

II. Forced Vibrations of Framed Structure of an Infinite Number of Spans.

When the bottom of each pillar is oscillating horizontally with amplitude b and with period $\frac{2\pi}{p'}$, the boundary conditions at the end 0_1, which is the clamped end of the pillar, are written by

$$x_1=0\,; \quad u_1=b, \quad \frac{du_1}{dx_1}=0\,; \qquad (5')$$

and other conditions are the same as those in the preceding case.

As the types of the solutions we write $y_1=u_1 \cos p't$, $y_2=u_2 \cos p't$, so that m_1^4, m_2^4 should have the meanings such that

$$m_1^4=\frac{\rho_1 p'^2}{E_1 k_1^2}, \quad m_2^4=\frac{\rho_2 p'^2}{E_2 k_2^2}. \qquad (3'), (11')$$

Substituting from (4) in (5′), we get

$$u_1=A_1(\cos m_1x_1-\cosh m_1x_1)+B_1(\sin m_1x_1-\sinh m_1x_1) + b\cosh m_1x_1. \qquad (6')$$

Applying the conditions and relations in (7), (8), (9), (13), (14), (15), (16), (17), (18), (19) to the equations (6′) and (12), it is possible to find the values of the constants A_1, B_1, A_2, B_2, C_2, D_2 besides M_0, M_1, M_2.

Write

$$m_1 l_1 \equiv \alpha, \quad m_2 l_2 \equiv \beta, \quad \frac{\rho_2 a_2 l_2 p'^2}{E_1 a_1 k_1^2 m_1^3} \equiv \gamma, \quad \frac{m_2}{m_1} \equiv \xi, \quad \frac{E_2 a_2 k_2^2 m_2^2}{E_1 a_1 k_1^2 m_1^2} \equiv \eta\,; \qquad (20')$$

then, from (6′) and (8) we get

$$A_1\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}-B_1\{(\cos\alpha+\cosh\alpha) -\gamma(\sin\alpha-\sinh\alpha)\}=-b(\sinh\alpha+\gamma\cosh\alpha). \qquad (21')$$

From (12) and (13) we find

$$C_2+A_2=0. \qquad (21'')$$

By means of (12) and (14) we obtain

$$(B_2+D_2)+A_1(\sin\alpha+\sinh\alpha)-B_1(\cos\alpha-\cosh\alpha)=b\sinh\alpha. \qquad (22')$$

From (16) we get

$$A_2(\cos\beta-\cosh\beta)+B_2\sin\beta+D_2\sinh\beta=0. \qquad (23')$$

Again, from (17) and (22)

$$A_2(\sin\beta+\sinh\beta)+B_2(1-\cos\beta)+D_2(1-\cosh\beta)=0. \qquad (24')$$

(7), (15), (18), (19) give us

$$A_1(\cos\alpha+\cosh\alpha)+B_1(\sin\alpha+\sinh\alpha)+A_2\eta(-2+\cos\beta+\cosh\beta)$$
$$+B_2\eta\sin\beta-D_2\eta\sinh\beta=b\cosh\alpha. \qquad (25')$$

Thus we find from (21′), (22′), (23′), (24′), (25′)

$$A_1=\frac{\begin{aligned}b[\xi(1-\cos\beta\cosh\beta)\{(\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha+1)-2\gamma\sin\alpha\cosh\alpha\}\\+2\eta\{\sin\beta(\cosh\beta-1)\sinh\beta(1-\cos\beta)\}\\\times\{2\cos\alpha\sinh\alpha+\gamma(\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha-1)\}]\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}, \qquad (26')$$

$$B_1=\frac{\begin{aligned}b[\xi(1-\cos\beta\cosh\beta)\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+2\gamma\cos\alpha\cosh\alpha\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}]\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}, \qquad (27')$$

$$A_2=\frac{\begin{aligned}b\{\sin\beta(1-\cosh\beta)-(1-\cos\beta)\sinh\beta\}\\\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}, \qquad (28')$$

$$B_2=\frac{\begin{aligned}b\{\sinh\beta(\sin\beta+\sinh\beta)-(1-\cosh\beta)(\cos\beta-\cosh\beta)\}\\\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}, \qquad (29')$$

$$D_2=\frac{\begin{aligned}b\{(1-\cos\beta)(\cos\beta-\cosh\beta)-\sin\beta(\sin\beta+\sinh\beta)\}\\\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}, \qquad (30')$$

The distributions of the deflections of pillar and the beam are shown by the equations.

$$y_1=\frac{\begin{aligned}b\cos p't[\xi(1-\cos\beta\cosh\beta)\{(\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha+1)-2\gamma\sin\alpha\cosh\alpha\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{2\cos\alpha\sinh\alpha+\gamma(\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha-1)\}]\\\times(\cos m_1x_1-\cosh m_1x_1)\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}$$

$$+\frac{\begin{aligned}b\cos p't[\xi(1-\cos\beta\cosh\beta)\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+2\gamma\cos\alpha\cosh\alpha\}\\+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}]\\\times(\sin m_1x_1-\sinh m_1x_1)\end{aligned}}{\begin{aligned}2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\\times+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\}\\\times\{(\sin\alpha\cosh\alpha+\cos\alpha\sinh\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}]\end{aligned}}$$

$$+ b\cos p't \cosh m_1x_1, \tag{31'}$$

$$y_2 = \frac{\begin{array}{l} b\cos p't\,\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \\ \quad\times[\{\sin\beta(1-\cosh\beta)-(1-\cos\beta)\sinh\beta\}(\cos m_2x_2-\cosh m_2x_2) \\ \quad+\{\sinh\beta(\sin\beta+\sinh\beta)-(1-\cosh\beta)(\cos\beta-\cosh\beta)\}\sin m_2x_2 \\ \quad+\{(1-\cos\beta)(\cos\beta-\cosh\beta)-\sin\beta(\sin\beta-\sinh\beta)\}\sinh m_2x_2] \end{array}}{\begin{array}{l} 2[\xi(1-\cos\beta\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\} \\ \quad+2\eta\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\} \\ \quad\times\{(\sin\alpha\cosh\alpha+\cos\alpha\sin\alpha)+\gamma(\cos\alpha\cosh\alpha-1)\}] \end{array}}. \tag{32'}$$

When the condition,

$$\begin{aligned} &2\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma(\cos\alpha\cosh\alpha)\} \\ &\quad\times\{\sin\beta(\cosh\beta-1)+\sinh\beta(1-\cos\beta)\} \\ &\quad=\frac{\xi}{\eta}\{(\cos\alpha\cosh\alpha+1)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\} \\ &\qquad\qquad\times(\cos\beta\cosh\beta-1) \end{aligned} \tag{30}$$

is satisfied, each denominator of the expressions of y_1 and y_2 vanishes. As the above expression gives the frequency equation of the natural vibrations of framed structure, this gives the condition of resonance vibration of that structure.

The distributions of the displacements in the cases, where $E_1=E_2$; $l_1=l_2$; $a_1=a_2$; $\frac{\rho_1}{\rho_2}=1, \frac{1}{2}, \frac{1}{4}$; and frequencies of vibrations nearly satisfy

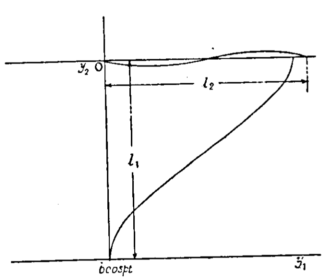

Fig. 2. $\frac{\rho_1}{\rho_2}=1$, $\alpha=1{\cdot}645$

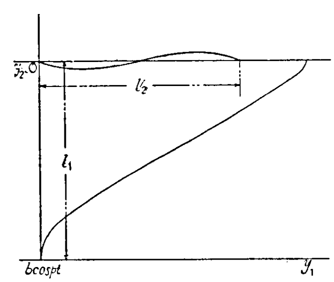

Fig. 3. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1{\cdot}429$

$\left(\frac{\rho_1 p'^2}{E_1k_1^2}\right)^{\frac{1}{4}} l_1=1{\cdot}645,\ 1{\cdot}429,\ 1{\cdot}224$ for respective values of $\frac{\rho_1}{\rho_2}$ given above, are shown in Figs. 2, 3, 4. These figures indicate merely the proportions of displacements, but not their absolute magnitudes.

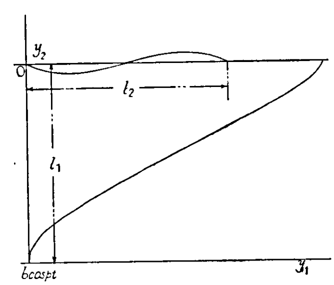

Fig. 4. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1{\cdot}224$

The distributions of displacements and bending moments in the cases, $\frac{\rho_1}{\rho_2}=1$, $\alpha=1$, 2, 2·5, 3, 4, 10; $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1$, 1·5, 1·8, 4, 8; $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1$, 1·5, 4, 8 are shown in Figs. 5—34 and in Table I—XII. In these y_1, y_2, M_1, M_2 are the deflections and bending moments in pillar and beam respectively, and $\alpha=\left(\frac{\rho_1 p'^2}{E_1 k_1^2}\right)^{\frac{1}{4}} l_1$.

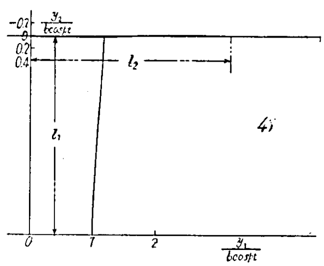

Fig. 5. $\frac{\rho_1}{\rho_2}=1$, $\alpha=1$.

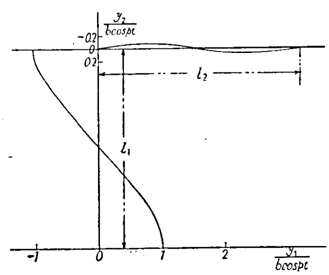

Fig. 6. $\frac{\rho_1}{\rho_2}=1$, $\alpha=2$.

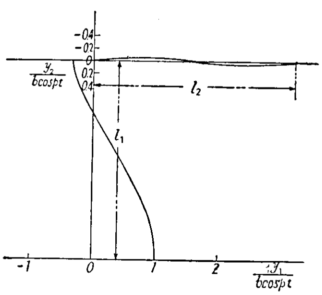

Fig. 7. $\frac{\rho_1}{\rho_2}=1$, $\alpha=2·5$

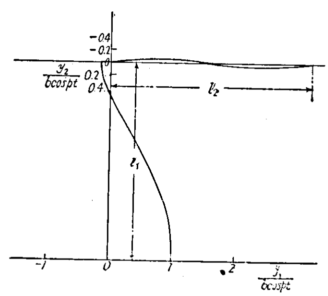

Fig. 8. $\frac{\rho_1}{\rho_2}=1$, $\alpha=3$.

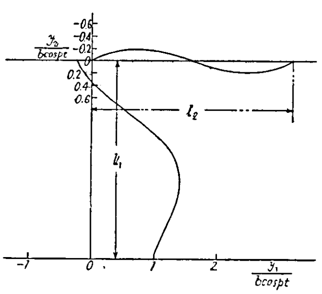

Fig. 9. $\frac{\rho_1}{\rho_2}=1$, $\alpha=4$.

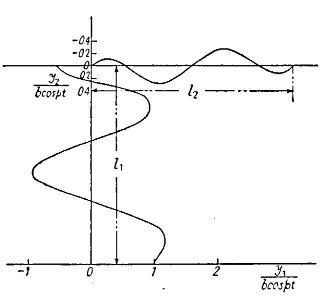

Fig. 10. $\frac{\rho_1}{\rho_2}=1$, $\alpha=10$.

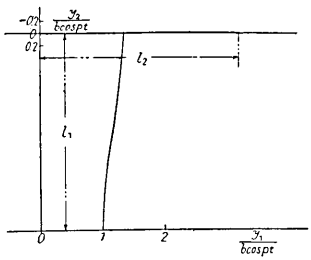

Fig. 11. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1$.

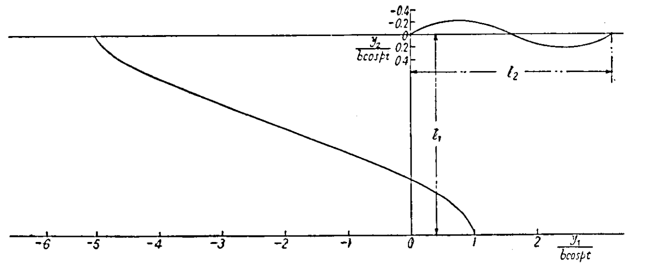

Fig. 12. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1\cdot5$.

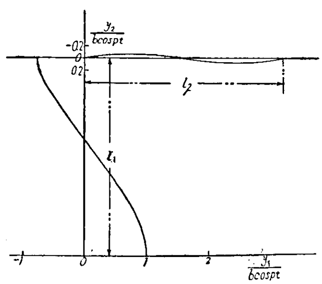

Fig. 13. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1\cdot8$.

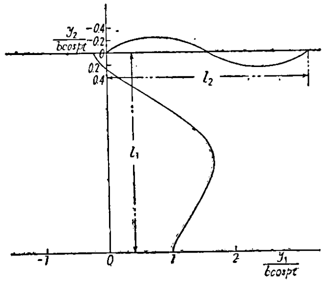

Fig. 14. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=4$.

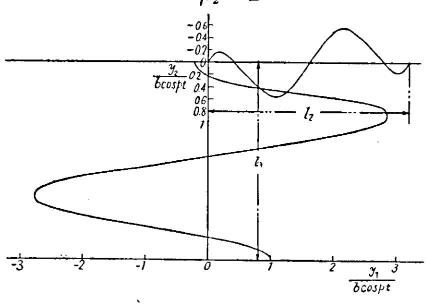

Fig. 15. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=8$.

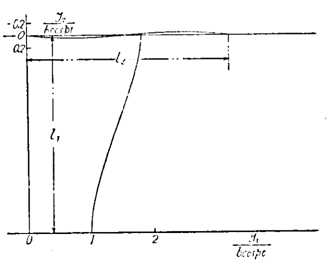

Fig. 16. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1$.

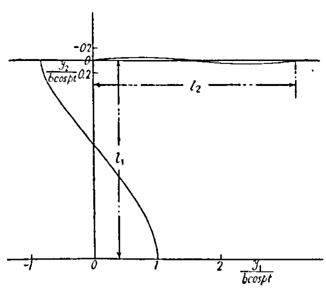

Fig. 17. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1\cdot5$.

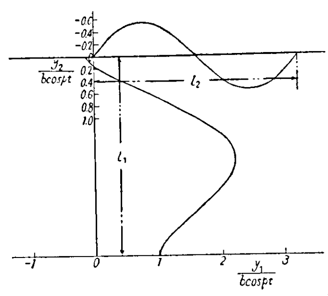

Fig. 18. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=4$

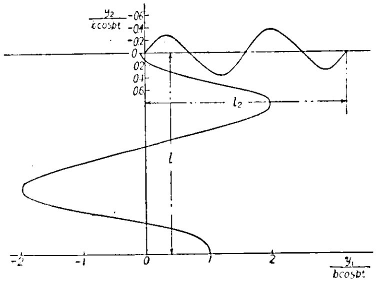

Fig. 19. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=8$.

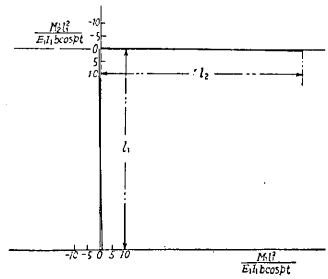

Fig. 20. $\frac{\rho_1}{\rho_2}=1$, $\alpha=1$.

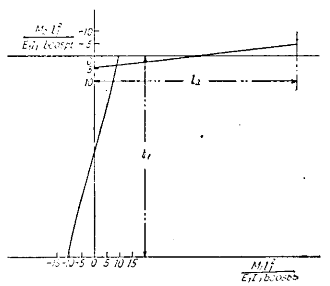

Fig. 21. $\frac{\rho_1}{\rho_2}=1$, $\alpha=2$.

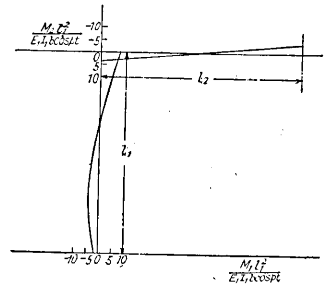

Fig. 22. $\frac{\rho}{\rho_2}=1$, $\alpha=2{\cdot}5$.

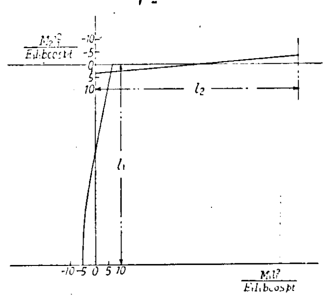

Fig. 23. $\frac{\rho_1}{\rho_2}=1$, $\alpha=3$.

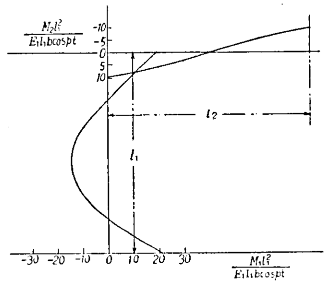

Fig. 24. $\frac{\rho_1}{\rho_2}=1$, $\alpha=4$.

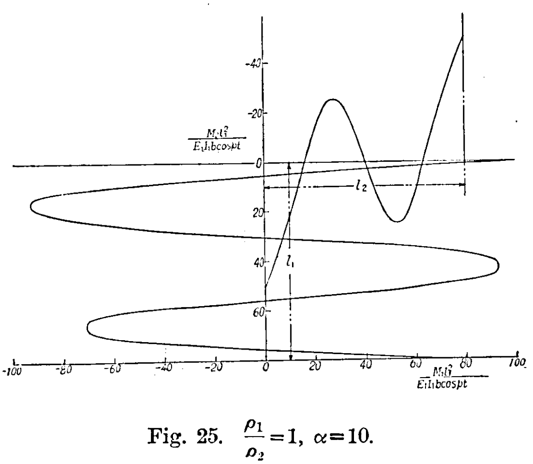

Fig. 25. $\frac{\rho_1}{\rho_2}=1$, $\alpha=10$.

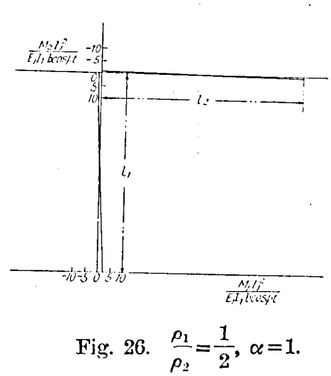

Fig. 26. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1$.

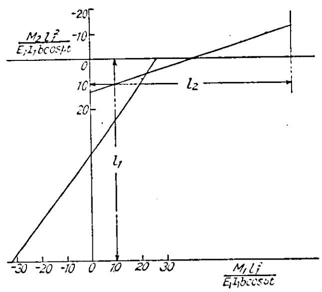

Fig. 27. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1{\cdot}5$.

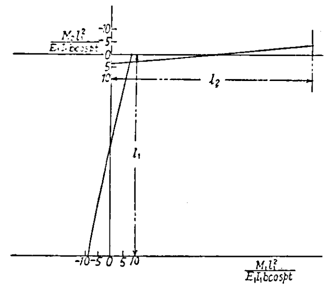

Fig. 28. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=1{\cdot}8$.

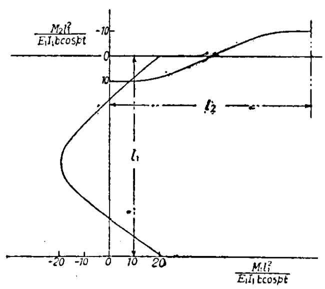

Fig. 29. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=4$.

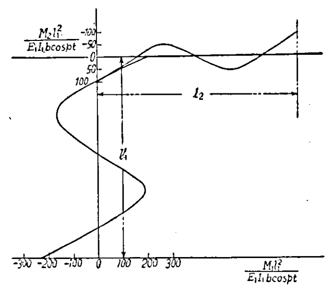

Fig. 30. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, $\alpha=8$.

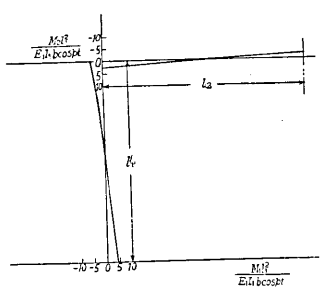

Fig. 31. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1$.

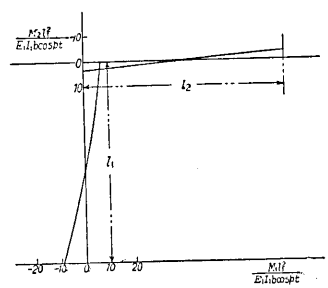

Fig. 32. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=1{\cdot}5$.

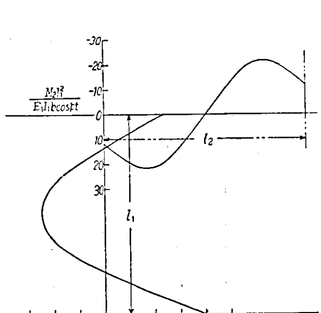

Fig. 33. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=4$.

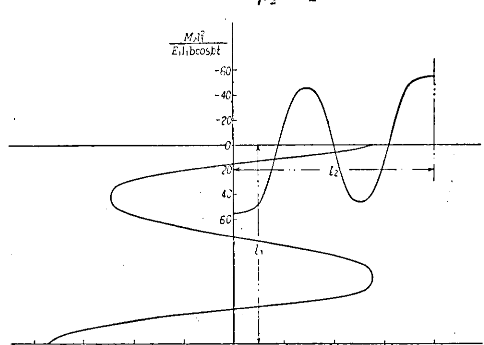

Fig. 34. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, $\alpha=8$.

Table I. $\frac{\rho_1}{\rho_2}=1$, y_1.

x_1	$\alpha=1$	$\alpha=2$	$\alpha=2{\cdot}5$	$\alpha=3$	$\alpha=4$	$\alpha=10$
0	1·1	1·	1·	1·	1·	1·
$\frac{1}{10}\,l_1$						1·17
$\frac{1}{5}\,l_1$	1·0174	0·811	0·91	0·948	1·242	0·9722
$\frac{3}{10}\,l_1$						0·127
$\frac{2}{5}\,l_1$	1·05835	0·342	0·623	0·75	1·412	−0·73
$\frac{1}{2}\,l_1$		0·05		0·597		−0·88
$\frac{3}{5}\,l_1$	1·10717	−0·249	0·247	0·418	1·077	−0·2
$\frac{7}{10}\,l_1$						0·667
$\frac{4}{5}\,l_1$	1·15	−0·75	−0·118	0·042	0·33	0·93
$\frac{9}{10}\,l_1$						0·34
l_1	1·175	−1·04	−0·34	−0·162	−0·24	−0·56

Table II. $\frac{\rho_1}{\rho_2}=1$, y_2.

x_2	$\alpha=1$	$\alpha=2$	$\alpha=2{\cdot}5$	$\alpha=3$	$\alpha=4$	$\alpha=10$
0	0·	0·	0·	0·	0·	0·
$\frac{1}{16}\,l_2$					−0·0943	−0·112
$\frac{1}{8}\,l_2$			−0·0465	−0·05306	−0·154	−0·077
$\frac{3}{16}\,l_2$					−0·182	0·043
$\frac{1}{5}$ 〃	0·00555	−0·078	−0·0545			
$\frac{1}{4}$ 〃		−0·08	−0·0537	−0·0615	−0·182	0·176
$\frac{5}{16}$ 〃					−0·158	0·26
$\frac{3}{8}$ 〃				−0·0389	−0·12	0·253
$\frac{2}{5}$ 〃	0·00357	−0·04				
$\frac{7}{16}$ 〃					−0·0615	0·15
$\frac{1}{2}$ 〃	0·	0·	0·	0·	0·	0·
$\frac{3}{5}$ 〃	−0·00357	0·04				
$\frac{5}{8}$ 〃				0·0389	0·12	−0·253
$\frac{3}{4}$ 〃		0·08	0·0537	0·0615	0·182	−0·176
$\frac{4}{5}$ 〃	−0·00554	0·078	0·0545			
$\frac{7}{8}$ 〃			0·0465	0·05306	0·154	0·077
l_2	0·	0·	0·	0·	0·	0·

Table III. $\frac{\rho_1}{\rho_2}=1$, M_1.

x_1	$\alpha=1$	$\alpha=2$	$\alpha=2.5$	$\alpha=3$	$\alpha=4$	$\alpha=10$
0	0·984	−10·6	−5·125	−1·395	20·848	89·1
$\frac{1}{10}l_1$						−47·8
$\frac{1}{5}l_1$	0·585	−7·0	−4·54	−4·0	−2·848	−70·9
$\frac{3}{10}l_1$				$(\frac{1}{3}l_1)$ −4·05		−3·2
$\frac{2}{5}l_1$	0·196	−2·88	−2·565	−3·546	−13·888	77·
$\frac{1}{2}l_1$		−0·72				90·
$\frac{3}{5}l_1$	−0·152	1·364	0·3625	−0·9	−10·976	21·4
$\frac{7}{10}l_1$						−66·1
$\frac{4}{5}l_1$	−0·453	5·628	3·67	3·24	2·56	−92·7
$\frac{9}{10}l_1$						−34·
l_1	−0·714	9·308	6·631	7·38	19·328	99·7

Table IV. $\frac{\rho_1}{\rho_2}=1$, M_2.

x_2	$\alpha=1$	$\alpha=2$	$\alpha=2.5$	$\alpha=3$	$\alpha=4$	$\alpha=10$
0	−0·358	4·66	3·315	3·69	9·664	50·
$\frac{1}{16}l_2$					8·944	38·
$\frac{1}{8}$ "			2·519	2·864	8·128	22·1
$\frac{3}{16}$ "						3·4
$\frac{1}{5}$ "	−0·215	2·808				
$\frac{1}{4}$ "			1·705	1·978	6·032	−13·6
$\frac{5}{16}$ "						−24·
$\frac{3}{8}$ "			0·867	1·021	3·280	−24·2
$\frac{2}{5}$ "	0·072	0·976				
$\frac{7}{16}$ "						−15·
$\frac{1}{2}$ "	0·	0·	0·	0·	0·	0·
$\frac{3}{5}$ "	0·072	−0·976				
$\frac{5}{8}$ "			−0·867	−1·021	−3·280	24·2
$\frac{3}{4}$ "			−1·705	−1·978	−6·032	13·6
$\frac{4}{5}$ "	0·215	−2·808				
$\frac{7}{8}$ "			−2·519	−2·864	−8·128	−22·1
l_2	0·358	−4·66	−3·315	−3·69	−9·664	−50·

Table V. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, y_1.

x_1	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=1{\cdot}8$	$\alpha=4$	$\alpha=8$
0	1·	1·	1·	1·	1·
$\frac{1}{10}l_1$					
$\frac{1}{5}l_1$	1·0332	0·437	0·8415	1·326	−1·668
$\frac{3}{10}l_1$					−2·726
$\frac{2}{5}$ ″	1·1077	−0·9245	0·448	1·637	−2·288
$\frac{1}{2}l_1$					
$\frac{3}{5}$ ″	1·2002	−2·6	−0·043	1·361	1·573
$\frac{7}{10}l_1$					2·77
$\frac{4}{5}$ ″	1·2905	−4·11	−0·501	0·555	2·41
$\frac{9}{10}$ ″					
l_1	1·349	−5·	−0·77	−0·204	−0·224

Table VI. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, y_2.

x_2	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=1{\cdot}8$	$\alpha=4$	$\alpha=8$
0	0·	·0	0·	0·	0·
$\frac{1}{16}l_2$					−0·165
$\frac{1}{8}$ ″		−0·182	−0·0532	0·208	−0·006
$\frac{3}{16}$ ″				0·25	
$\frac{1}{5}$ ″	0·01505				
$\frac{1}{4}$ ″	0·01042	−0·208	−0·0637	−0·253	0·426
$\frac{5}{16}$ ″				−0·2225	0·56
$\frac{3}{8}$ ″				−0·165	0·521
$\frac{2}{5}$ ″	0·00543				
$\frac{7}{16}$ ″					
$\frac{1}{2}$ ″	0·	0·	0·	0·	0·
$\frac{3}{5}$ ″	−0·00543				
$\frac{5}{8}$ ″				0·165	−0·521
$\frac{3}{4}$ ″	−0·01042	0·208	0·0637	0·253	−0·426
$\frac{4}{5}$ ″	−0·01505				
$\frac{7}{8}$ ″		0·182	0·0532	0·208	0·006
l_2	0·	0·	0·	0·	0·

Table VII. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, M_1.

x_1	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=1{\cdot}8$	$\alpha=4$	$\alpha=8$
0	1·84	−32·175	−8·91	21·312	−233·856
$\frac{1}{10}l_1$					
$\frac{1}{5}l_1$	1·1228	−20·036	−5·911	−1·52	72·32
$\frac{3}{10}$ 〃					181·376
$\frac{2}{5}$ 〃	0·448	−7·796	−2·56	−16·064	141·056
$\frac{1}{2}$ 〃					
$\frac{3}{5}$ 〃	−0·192	4·277	1·001	−14·4	−94·656
$\frac{7}{10}$ 〃					−161·152
$\frac{4}{5}$ 〃	−0·773	15·775	4·484	0·72	−121·344
$\frac{9}{10}$ 〃					
l_1	−1·382	26·26	7·90	21·568	197·184

Table VIII. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$, M_2.

x_2	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=1{\cdot}8$	$\alpha=4$	$\alpha=8$
0	−0·693	13·137	3·957	10·80	98·656
$\frac{1}{16}l_2$					69·421
$\frac{1}{8}$ 〃		9·908	2·991	10·544	35·48
$\frac{3}{16}$ 〃				9·888	
$\frac{1}{5}$ 〃	−0·414				
$\frac{1}{4}$ 〃	−0·351	6·616	2·029	8·745	−29·506
$\frac{5}{16}$ 〃				7·105	−45·888
$\frac{3}{8}$ 〃				5·001	−47·156
$\frac{2}{5}$ 〃	−0·109				
$\frac{7}{16}$ 〃					
$\frac{1}{2}$ 〃	0·	0·	0·	0·	0·
$\frac{3}{5}$ 〃	0·109				
$\frac{5}{8}$ 〃				−5·001	47·156
$\frac{3}{4}$ 〃	0·351	−6·616	−2·029	−8·745	29·506
$\frac{4}{5}$ 〃	0·414				
$\frac{7}{8}$ 〃		−9·908	−2·991	−10·544	−35·48
l_2	0·693	−13·137	−3·957	−10·80	−98·656

Table IX. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, y_1.

x_1	$\alpha=1$	$\alpha=1\cdot5$	$\alpha=4$	$\alpha=8$
0	1·	1·	1·	1·
$\frac{1}{10}l_1$				
$\frac{1}{5}l_1$	1·077	0·832	1·52	-0·844
$\frac{3}{10}$ 〃				
$\frac{2}{5}$ 〃	1·263	0·42	2·148	-1·64
$\frac{1}{2}$ 〃		0·165	2·194	
$\frac{3}{5}$ 〃	1·492	−0·09	1·997	0·833
$\frac{7}{10}$ 〃				
$\frac{4}{5}$ 〃	1·699	−0·564	1·	1·725
$\frac{9}{10}$ 〃				
l_1	1·816	0·843	−0·144	−0·076

Table X. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, y_2.

x_2	$\alpha=1$	$\alpha=1\cdot5$	$\alpha=4$	$\alpha=8$
0	0·	0·	0·	0·
$\frac{1}{16}l_2$				
$\frac{1}{8}$ 〃	0·0345	−0·0477	−0·424	-0·275
$\frac{3}{16}$ 〃				
$\frac{1}{5}$ 〃				
$\frac{1}{4}$ 〃	0·0398	−0·0546	−0·554	0·126
$\frac{5}{16}$ 〃				
$\frac{3}{8}$ 〃			−0·376	0·366
$\frac{2}{5}$ 〃				
$\frac{7}{16}$ 〃				
$\frac{1}{2}$ 〃	0·	0·	0·	0·
$\frac{3}{5}$ 〃				
$\frac{5}{8}$ 〃			0·376	−0·366
$\frac{3}{4}$ 〃	−0·0398	0·0546	0·554	−0·126
$\frac{4}{5}$ 〃				
$\frac{7}{8}$ 〃	−0·0398	0·0477	0·424	0·275
l_2	0·	0·	0·	0·

Table XI. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, M_1.

x_1	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=4$	$\alpha=8$
0	4·425	−9·518	39·793	−148·698
$\frac{1}{10}l_1$				
$\frac{1}{5}l_1$	2·72	−6·109	1·4144	37·011
$\frac{3}{10}$ 〃				
$\frac{2}{5}$ 〃	1·0535	−2·52	−21·333	102·406
$\frac{1}{2}$ 〃			−24·84	
$\frac{3}{5}$ 〃	0·56	1·125	−22·779	−49·709
$\frac{7}{10}$ 〃				
$\frac{4}{5}$ 〃	2·113	4·77	−4·536	−90·893
$\frac{9}{10}$ 〃				
l_1	4.95	7·135	23·806	110·432

Table XII. $\frac{\rho_1}{\rho_2}=\frac{1}{4}$, M_2.

x_2	$\alpha=1$	$\alpha=1{\cdot}5$	$\alpha=4$	$\alpha=8$
0	−2·49	3·51	11·903	55·05
$\frac{1}{16}l_2$				
$\frac{1}{8}$ 〃	−1·87	2·66	19·392	48·563
$\frac{3}{16}$ 〃			20·974	
$\frac{1}{5}$ 〃				
$\frac{1}{4}$ 〃	−1·27	1·791	20·472	−12·826
$\frac{5}{16}$ 〃				
$\frac{3}{8}$ 〃			13·109	−45·99
$\frac{2}{5}$ 〃				
$\frac{7}{16}$ 〃				
$\frac{1}{2}$ 〃	0·	0·	0·	0·
$\frac{3}{5}$ 〃				
$\frac{5}{8}$ 〃			−13·109	45·99
$\frac{3}{4}$ 〃	1·27	−1·791	−20·472	12·826
$\frac{4}{5}$ 〃				
$\frac{7}{8}$ 〃	1·87	−2·66	−19·392	−48·563
l_2	2·49	−3·51	−11·903	−55·05

The important facts, which may be found from these results, are enumerated as follows.

i) The greatest amplitude of horizontal vibrations of the pillar occurs at its top in resonance condition.

ii) When the frequency of vibrations of seismic waves is higher than those of natural vibration of the pillar, the amplitudes of the top of the pillar and hence those of the horizontal movement of the beam is not so great as the horizontal movement of the ground. This nature is more distinctly revealed when ρ_2 is greater than ρ_1.

iii) In the forced vibrations of higher frequency the deflection curves of pillars as well as those of beams become wavy. The amplitudes of the horizontal vibrations of the beams in this case, too, are not large.

iv) The bending moment at low frequency is small and uniformly increasing along its length, so that the moment is very similar to that obtained in the manner of the statical problem. This fact was also cited in Professor Suyehiro's paper[10] concerning the vibration of a chimney.

v) At the frequency which is higher than the natural one the bending moment varies in an oscillatory type along the beam and the pillar. The moments take greater values as ρ_2/ρ_1 increases. The moments increase too as the increase of the frequency of the forced vibrations.

III. Natural Periods of Framed Structure of Two Spans.

Let u_1, u_1', u_1'', u_2, u_2' be the amplitudes of pillars and beams at x_1, x_1', x_1'', x_2, x_2' respectively, M_0, M_0', M_0'', M_2, M_1, M_2', M_1' be the bending moments at 0_2, 0_4, 0_6 respectively, x_1, x_1', x_1'', x_2, x_2' be the coordinates along 0_10_2, 0_30_4, 0_50_6, 0_20_4, 0_40_6 respectively, then the solution of the vibrations of the member, 0_10_2, fulfilling the conditions at the bottom end of this bar is written by

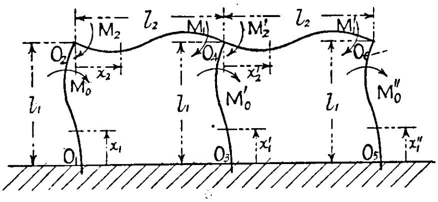

Fig. 35.

$$u_1 = A_1(\cos m_1x_1 - \cosh m_1x_1) + B_1(\sin m_1x_1 - \sinh m_1x_1). \tag{31}$$

The conditions at the other end are

$$x_1 = l_1, \quad -E_1a_1k_1^2\frac{d^2u_1}{dx_1^2} = M_0, \quad -E_1a_1k_1^2\frac{d^3u_1}{dx_1^3} = \frac{\rho_2a_2l_2p^2u_1}{2}. \qquad (32), (33)$$

10) K. SUYEHIRO, *loc. cit.*

The solution of the vibrations of the beam, $0_2 0_4$, is expressed by

$$u_2 = A_2 \cos m_2 x_2 + B_2 \sin m_2 x_2 + C_2 \cosh m_2 x_2 + D_2 \sinh m_2 x_2. \tag{34}$$

The boundary conditions at one end, 0_2, are denoted by

$$x_2=0,\ x_1=l_1\ ;\ u_2=0,\ \frac{du_2}{dx_2}=\frac{du_1}{dx_1},\ -E_2 a_2 k_2^2 \frac{d^2u_2}{dx_2^2}=M_2. \qquad (35), (36), (37)$$

The condition of the fixing moment at this joint is

$$M_0 - M_2 = 0. \tag{38}$$

The boundary conditions at the other end are written by

$$x_2=l_2,\ x_1'=l_1\ ;\ u_2=0,\ \frac{du_2}{dx_2}=\frac{du_1'}{dx_1'},\ -E_2 a_2 k_2^2 \frac{d^2u_2}{dx_2^2}=M_1, \qquad (39), (40), (41)$$

where u_1' is the solution of the member, $0_3 0_4$, and has the same type as that of u_1 such that

$$u_1' = A_1'(\cos m_1 x_1' - \cosh m_1 x_1') + B_1'(\sin m_1 x_1' - \sinh m_1 x_1'). \tag{42}$$

The boundary conditions of the pillar, $0_3 0_4$, at the joint are

$$x_1'=l_1\ ;\ -E_1 a_1 k_1^2 \frac{d^2u_1}{dx_1'^2}=M_0',\ \ -E_1 a_1 k_1^2 \frac{d^3u_1'}{dx_1'^3}=\rho_2 a_2 l_2 p^2 u_1'. \qquad (43), (44)$$

The form of the solution of the beam, $0_4 0_6$, is written by

$$u_2' = A_2' \cos m_2 x_2' + B_2' \sin m_2 x_2' + C_2' \cosh m_2 x_2' + D_2' \sinh m_2 x_2'. \tag{45}$$

The boundary conditions of this bar at 0_4 are denoted by

$$x_2'=0,\ x_1=l_1\ ;\ u_2'=0,\ \frac{du_2'}{dx_2'}=\frac{du_1'}{dx_1'},\ -E_2 a_2 k_2^2 \frac{d^2u_2'}{dx_2'^2}=M_2'. \qquad (46), (47), (48)$$

The condition of the fixing moment at the point, 0_4, is

$$M_0' + M_1 - M_2' = 0. \tag{49}$$

The solution of the vibration of the pillar, $0_5 0_6$, is expressed by

$$u_1'' = A_1''(\cos m_1 x_1'' - \cosh m_1 x_1'') + B_1''(\sin m_1 x_1'' - \sinh m_1 x_1''). \tag{50}$$

The conditions of this pillar at the top end are given by

$$x_1''=l_1\ ;\ -E_1 a_1 k_1^2 \frac{d^2u_1''}{dx_1''^2}=M_0'',\ -E_1 a_1 k_1^2 \frac{d^3u_1''}{dx_1''^3}=\frac{\rho_2 a_2 l_2 p^2 u_1''}{2}. \qquad (51), (52)$$

The condition of the beam at the end, 0_6, are

$$x_2'=l_2,\ x_1''=l_1\ ;\ u_2'=0,\ \frac{du_2'}{dx_2'}=\frac{du_1''}{dx_1''},\ -E_2 a_2 k_2^2 \frac{p^2 u_2'^2}{dx_2^2}=M_1'. \ (53),(54),(55)$$

The condition of the fixing moment at the joint, 0_6, is

$$M_0'' + M_1' = 0. \tag{56}$$

Write

$$m_1 l_1 \equiv \alpha,\ m_2 l_2 \equiv \beta,\ \frac{\rho_2 a_2 l_2 p^2}{E_1 a_1 k_1^2 m_1^3} \equiv \gamma,\ \frac{m_2}{m_1} \equiv \xi,\ \frac{E_2 a_2 k_2^2 m_2^2}{E_1 a_1 k_1^2 m_1^2} \equiv \eta,$$

then, from (31) and (33) we find

$$A_1\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \\ -B_1\{2(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}=0. \quad (57)$$

Again, from (34) and (35)

$$C_2+A_2=0, \quad (58)$$

while (31), (34), (36) give us

$$\xi(B_2+D_2)+A_1(\sin\alpha+\sinh\alpha)-B_1(\cos\alpha-\cosh\alpha)=0. \quad (59)$$

From (39)

$$A_2(\cos\beta-\cosh\beta)+B_2\sin\beta+D_2\sinh\beta=0. \quad (60)$$

From (42), (34), (40) we find

$$\xi\{-A_2(\sin\beta+\sinh\beta)+B_2\cos\beta+D_2\cosh\beta\}+A_1'(\sin\alpha+\sinh\alpha) \\ -B_1'(\cos\alpha-\cosh a)=0. \quad (61)$$

By means of (32), (37), (38) we get

$$A_1(\cos\alpha+\cosh\alpha)+B_1(\sin\alpha+\sinh\alpha)-2\eta A_2=0. \quad (62)$$

(42), (44) give us

$$A_1'\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \\ -B_1'\{(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}=0. \quad (63)$$

(45), (46) give us

$$C_2'+A_2'=0, \quad (64)$$

while (42), (45), (47) give us

$$\xi(B_2'+D_2')+A_1'(\sin\alpha+\sinh\alpha)-B_1'(\cos\alpha-\cosh\alpha)=0. \quad (65)$$

From (53)

$$A_2'(\cos\beta-\cosh\beta)+B_2'\sin\beta+D_2'\sinh\beta=0. \quad (66)$$

By means of (45), (50), (54) we find

$$\xi\{-A_2'(\sin\beta+\sinh\beta)+B_2'\cos\beta+D_2'\cosh\beta\}+A_1''(\sin\alpha+\sinh\alpha) \\ -B_1''(\cos\alpha-\cosh\alpha)=0. \quad (67)$$

From (43), (41), (48), (49)

$$A_1'(\cos\alpha+\cosh\alpha)+B_1'(\sin\alpha+\sinh\alpha) \\ +\eta\{A_2(\cos\beta+\cosh\beta)+B_2\sin\beta-D_2\sinh\beta-2A_2'\}=0. \quad (68)$$

From (50), (52)

$$A_1''\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \\ -B_1''\{2(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}=0. \quad (69)$$

By means of (51), (55), (56) we obtain

$$A_1''(\cos\alpha+\cosh\alpha)+B_1''(\sin\alpha+\sinh\alpha)$$
$$+\eta\{A_2'(\cos\beta+\cosh\beta)+B_2'\sin\beta-D_2'\sinh\beta\}=0. \quad (70)$$

From (61), (65)

$$A_2(\sin\beta+\sinh\beta)-B_2\cos\beta-D_2\cosh\beta+B_2'+D_2'=0. \quad (71)$$

Eliminating A_1 between (57), (59), we find

$$(B_2+D_2)\xi\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2B_1\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}=0. \quad (72)$$

Eliminaing A_1 between (57) and (62)

$$B_1\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$-A_2\eta\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}=0. \quad (73)$$

Eliminating B_1 between (72), (73) we obtain

$$(B_2+D_2)\xi\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+2A_2\eta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}=0. \quad (74)$$

Eliminating A_1' between (63), (65) we get

$$(B_2'+D_2')\xi\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2B_1'\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}=0. \quad (75)$$

Eliminating A_1' between (63), (68)

$$2B_1'\{1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta\{A_2(\cos\beta+\cosh\beta)+B_2\sin\beta-D_2\sinh\beta-2A_2'\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}=0. \quad (76)$$

Eliminating A_1'' between (69), (70)

$$2B_1''\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta\}A_2'(\cos\beta+\cosh\beta)+B_2'\sin\beta-D_2'\sinh\beta\{$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}=0. \quad (77)$$

Eliminating B_2 between (60), (74) we obtain

$$D_2\xi(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+A_2[2\eta\sin\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$-\xi(\cos\beta-\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}]=0. \quad (78)$$

Eliminating B_2 between (60), (76)

$$B_1'\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta(A_2\cosh\beta-D_2\sinh\beta-A_2')\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}=0. \quad (79)$$

Eliminating B_2 between (60), (71)

$$A_2(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$+(B_2'+D_2')\sin\beta=0. \quad (80)$$

Eliminating B_2' between (66), (67)

$$A_2'\xi(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2'\xi(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$-A_1''\sin\beta(\sin\alpha+\sinh\alpha)+B_1''\sin\beta(\cos\alpha-\cosh\alpha)=0. \quad (81)$$

Eliminating B_2' between (66), (75)

$$\xi\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\{A_2'(\cos\beta-\cosh\beta)-D_2'(\sin\beta-\sinh\beta)\}$$
$$-2B_1'\sin\beta\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}=0. \quad (82)$$

Eliminating B_2' between (66), (77)

$$B_1''\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta(A_2'\cosh\beta-D_2'\sinh\beta)\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}=0. \quad (83)$$

Eliminating B_2' between (66), (80)

$$A_2(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$-A_2'(\cos\beta-\cosh\beta)+D_2'(\sin\beta-\sinh\beta)=0. \quad (84)$$

Eliminating A_1'' between (69), (81)

$$\xi\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{A_2'(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2'(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\}$$
$$-2B_1''\sin\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}=0. \quad (85)$$

Eliminating B_1'' between (83), (85)

$$-D_2'[2\eta\sin\beta\sinh\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$-\xi(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}]=0. \quad (86)$$

Eliminating D_2 between (78), (84)

$$\xi(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{-A_2'(\cos\beta-\cosh\beta)+D_2'(\sin\beta-\sinh\beta)\}+2A_2\sin\beta$$
$$\times[\xi(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$-\eta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}]=0. \quad (87)$$

Eliminating D_2 between (79), (87)

$$\eta\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\{A_2(\sinh\beta-\sin\beta)$$
$$+A_2'(-2\cos\beta\sinh\beta+\sinh\beta\cosh\beta+\sin\beta\cosh\beta)$$
$$+D_2'\sinh\beta(\sin\beta-\sinh\beta)\}+B_1'(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}=0. \quad (88)$$

Eliminating A_2 between (87), (88), we find finally

$$
\begin{aligned}
&4\eta^3 \sin\beta \sinh\beta (\cos\beta \sinh\beta - \sin\beta \cosh\beta) \\
&\quad \times \{2(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\}^2 \\
&\quad \times \{(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\} \\
&-\xi\eta^2 \{2(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\} \\
&\quad \times [(\cos\beta \sinh\beta - \sin\beta \cosh\beta)^2 \\
&\quad \times \{2(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha) \\
&\quad \times \{(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\}] \\
&+2\{2(\cos\beta \sinh\beta - \sin\beta \cosh\beta)^2 - (\sin\beta - \sinh\beta)^2\} \\
&\quad \times \{2(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\} \\
&\quad \times \{(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\}] \\
&+2\eta\xi^2(1 - \cos\beta \cosh\beta)(\cos\beta \sinh\beta - \sin\beta \cosh\beta) \\
&\quad \times \{2(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\} \\
&\quad \times [\{2(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\} \\
&\quad \times \{(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\} \\
&\quad + \{2(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\} \\
&\quad \times \{(\cos\alpha \sinh\alpha + \sin\alpha \cosh\alpha) - \gamma(1 - \cos\alpha \cosh\alpha)\}] \\
&-\xi^3(1 - \cos\beta \cosh\beta)^2 \{2(1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\}^2 \\
&\quad \times \{1 + \cos\alpha \cosh\alpha) + \gamma(\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha)\} = 0. \qquad (89)
\end{aligned}
$$

The natural period of the vibrations may be determined from this equation. We have calculated the three cases of $\frac{\rho_1}{\rho_2}$ as follows. ($l_1 = l_2$, $E_1 = E_2$, $a_1 = a_2$)

	$\frac{\rho_1}{\rho_2} = \frac{1}{4}$	$\frac{1}{2}$	1
$\alpha = \left(\frac{\rho_1 p^2}{E_1 k_1^2}\right)^{\frac{1}{4}} l_1$ for two spans	$=1{\cdot}117$	$1{\cdot}365$	$1{\cdot}73$
for inf. no. of spans	$=1{\cdot}224$	$1{\cdot}429$	$1{\cdot}645$

It may be seen that the periods of vibrations of the structure with two spans are approximately equal to those of the structure with an infinite number of spans. For smaller ratio of $\frac{\rho_1}{\rho_2}$ the value of α for two spans is somewhat less than that of α for an infinite number of spans, while for larger ratio of $\frac{\rho_1}{\rho_2}$ the inverse is the case.

IV. Forced Vibrations of Framed Structure of Two Spans.

We take the case where the bottom of each pillar is oscillating horizontally with amplitude b and with the period $\frac{2\pi}{p'}$. Then the boundary conditions at the bottom ends of the pillars are denoted by

$$\left.\begin{aligned} x_1=0\,; \quad u_1=b, \quad \frac{du_1}{dx_1}=0. \\ x_1'=0\,; \quad u_1'=b, \quad \frac{du_1'}{dx_1'}=0. \\ x_1''=0\,; \quad u_1''=b. \quad \frac{du_1''}{dx_1''}=0. \end{aligned}\right\} \tag{90}$$

and other conditions are the same as in the preceding case.

Write

$$m_1l_1\equiv\alpha,\ m_2l_2\equiv\beta,\ \frac{\rho_2a_2l_2p'^2}{E_1a_1k_1^2m_1^3}\equiv\gamma,\ \frac{m_2}{m_1}\equiv\xi,\ \frac{E_2a_2k_2^2m_2^2}{E_1a_1k_1^2m_1^2}\equiv\eta,$$

then, from (31), (33)

$$A_1\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}-B_1\{2(\cos\alpha+\cosh\alpha) \\ -\gamma(\sin\alpha-\sinh\alpha)\}=-b(2\sinh\alpha+\gamma\cosh\alpha) \tag{57'}$$

From (34), (35)

$$C_2+A_2=0. \tag{58'}$$

From (31), (34), (36), we have

$$\xi(B_2+D_2)+A_1(\sin\alpha+\sinh\alpha)-B_1(\cos\alpha-\cosh\alpha)=b\sinh\alpha \tag{59'}$$

From (39)

$$A_2(\cos\beta-\cosh\beta)+B_2\sin\beta+D_2\sinh\beta=0. \tag{60'}$$

From (42), (34), (40)

$$\xi\{-A_2'(\sin\beta+\sinh\beta)+B_2\cos\beta+D_2\cosh\beta\}+A_1'(\sin\alpha+\sinh\alpha) \\ -B_1'(\cos\alpha-\cosh\alpha)=b\sinh\alpha \tag{61'}$$

From (32), (37), (38)

$$A_1(\cos\alpha+\cosh\alpha)+B_1(\sin\alpha+\sinh\alpha)-2\eta A_2=b\cosh\alpha \tag{62'}$$

From (42), (44)

$$A_1'\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}-B_1'\{(\cos\alpha+\cosh\alpha) \\ -\gamma(\sin\alpha-\sinh\alpha)\}=-b(\sinh\alpha+\gamma\cosh\alpha) \tag{63'}$$

(45), (46) give us

$$C_2'+A_2'=0. \tag{64'}$$

From (42), (45), (47)

$$\xi(B_2'+D_2')+A_1'(\sin\alpha+\sinh\alpha)-B_1'(\cos\alpha-\cosh\alpha)=b\sinh\alpha \tag{65'}$$

From (53)

$$A_2'(\cos\beta-\cosh\beta)+B_2'\sin\beta+D_2'\sinh\beta=0. \tag{66'}$$

From (45), (50), (54)

$$\xi\{-A_2'(\sin\beta+\sinh\beta)+B_2'\cos\beta+D_2'\cosh\beta\}+A_1''(\sin\alpha+\sinh\alpha)$$
$$-B_1''(\cos\alpha-\cosh\alpha)=b\sinh\alpha \quad (67')$$

By means of (43), (41), (48), (49) we find

$$A_1'(\cos\alpha+\cosh\alpha)+B_1'(\sin\alpha+\sinh\alpha)$$
$$+\eta\{A_2(\cos\beta+\cosh\beta)+B_2\sin\beta-D_2\sinh\beta-2A_2'\}=b\cosh\alpha \quad (68')$$

From (50), (52)

$$A_1''\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-B_1''\{2(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}=-b(2\sinh\alpha+\gamma\cosh\alpha) \quad (69')$$

From (51), (55), (56)

$$A_1''(\cos\alpha+\cosh\alpha)+B_1''(\sin\alpha+\sinh\alpha)$$
$$+\eta\{A_2'(\cos\beta+\cosh\beta)+B_2'\sin\beta-D_2'\sinh\beta\}=b\cosh\alpha \quad (70')$$

From (61′), (65′)

$$A_2(\sin\beta+\sinh\beta)-B_2\cos\beta-D_2\cosh\beta+B_2'+D_2'=0. \quad (71')$$

Eliminating A_1 between (57′), (59′),

$$(B_2+D_2)\xi\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2B_1\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$=b\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\} \quad (72')$$

Eliminating A_1 between (57′), (62′),

$$B_1\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$-A_2\eta\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$=b\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\} \quad (73')$$

Eliminating B_1 between (72′), (73′),

$$(B_2'+D_2')\xi\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+2A_2\eta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$=-b\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \quad (74')$$

Eliminating A_1' between (63′), (65′), we get

$$(B_2'+D_2')\xi\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2B_1'\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$=b\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\} \quad (75')$$

Eliminating A_1' between (63′), (68′),

$$2B_1'\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta\{A_2(\cos\beta+\cosh\beta)+B_2\sin\beta-D_2\sinh\beta-2A_2'\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$=b\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\} \quad (76')$$

Eliminating A_1'' between (69′), (70′),

$$2B_1''\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta\{A_2'(\cos\beta+\cosh\beta)+B_2'\sin\beta-D_2'\sinh\beta\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$=2b\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\} \quad (77')$$

Eliminating B_2 between (60′), (74′),

$$D_2\xi(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+A_2[2\eta\sin\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$-\xi(\cos\beta-\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$=-b\sin\beta\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \quad (78')$$

Eliminating B_2 between (60′), (76′),

$$B_1'\{1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta(A_2\cosh\beta-D_2\sinh\beta-A_2')\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$=\frac{1}{2}b\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\} \quad (79')$$

Eliminating B_2 between (60′), (71′),

$$A_2(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$+(B_2'+D_2')\sin\beta=0. \quad (80')$$

Eliminating B_2' between (66′), (67′),

$$A_2'\xi(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2'\xi(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$-A_1''\sin\beta(\sin\alpha+\sinh\alpha)+B_1''\sin\beta(\cos\alpha-\cosh\alpha)$$
$$=-b\sin\beta\sinh\alpha \quad (81')$$

Eliminating B_2' between (66′), (75′),

$$\xi\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{A_2'(\cos\beta-\cosh\beta)-D_2'(\sin\beta-\sinh\beta)\}$$
$$-2B_1'\sin\beta\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$=-b\sin\beta\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\} \quad (82')$$

Eliminating B_2' between (66′), (77′),

$$B_1''\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$+\eta(A_2'\cosh\beta-D_2'\sinh\beta)\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$=b\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\} \quad (83')$$

Eliminating B_2' between (66′), (80′),

$$A_2(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$-A_2'(\cos\beta-\cosh\beta)+D_2'(\sin\beta-\sinh\beta)=0. \quad (84')$$

Eliminating A_1'' between (69′), (81′),

$$\xi\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{A_2'(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)+D_2'(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\}$$

$$-2B_1''\sin\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$=-b\sin\beta\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\} \quad (85')$$

Eliminating B_1'' between (83′), (85′),

$$\begin{aligned}
&A_2'[2\eta\sin\beta\cosh\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad+\xi(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)\\
&\quad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}]\\
&\quad-D_2'[2\eta\sin\beta\sinh\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad-\xi(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}]\\
&\qquad\qquad=b\sin\beta\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\} \quad (86')
\end{aligned}$$

Eliminating D_2 between (78′), (84′),

$$\begin{aligned}
&\xi(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{-A_2'(\cos\beta-\cosh\beta)+D_2'(\sin\beta-\sinh\beta)\}\\
&\quad+2A_2\sin\beta[\xi(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)\\
&\quad+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}-\eta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}]\\
&\quad=b\sin\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\{2(\sin\alpha-\sinh\alpha)\\
&\qquad\qquad+\gamma(\cos\alpha-\cosh\alpha)\} \quad (87')
\end{aligned}$$

Eliminating D_2 between (79′), (84′),

$$\begin{aligned}
&\eta\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\{-A_2(\sin\beta-\sinh\beta)\\
&\quad+A_2'(-2\cos\beta\sinh\beta+\sinh\beta\cosh\beta+\sin\beta\cosh\beta)\\
&\quad+D_2'\sinh\beta(\sin\beta-\sinh\beta)\}\\
&\quad+B_1'(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad=\frac{1}{2}b(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\qquad\qquad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\} \quad (88')
\end{aligned}$$

From (82′), (86′), (87′), (88′) we find the coefficients, B_1', A_2, A_2', D_2'. The other coefficients are derived from these constants thus determined. The result are shown below.

i) The denominator of each constant is given by twice the expression of (88)

ii) The numerator of $\frac{A_1}{b}$ is

$$\begin{aligned}
&4\eta^3\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{4\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}
\end{aligned}$$

$$
\begin{aligned}
&+\eta^2\xi\Big\{(\sin\beta-\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad\times\{2(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}\\
&\;+2(\sin\beta-\sinh\beta)^2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{4\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}\\
&\;-(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2\\
&\quad\times[4\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-\gamma\sin\alpha\cosh\alpha\}\\
&\;+2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{4\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}\\
&\;+\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{4\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}]\Big\}\\
&+\eta\xi^2(1-\cos\beta\cosh\beta)\Big\{(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times[4\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-\gamma\sin\alpha\cosh\alpha\}\\
&\;+2\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-\gamma\sin\alpha\cosh\alpha\}\\
&\;+\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{4\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}]\\
&\;-(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha\}\\
&\quad\times\{2(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}\Big\}\\
&-2\xi^3(1-\cos\beta\cosh\beta)^2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-\gamma\sin\alpha\cosh\alpha\}
\end{aligned}
$$

iii) The numerator of $\frac{B_1}{b}$ is

$$
\begin{aligned}
&4\eta^3\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}
\end{aligned}
$$

$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}$$
$$+\eta\xi^2\Big\{2(\sin\beta-\sinh\beta)^2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}$$
$$+(\sin\beta-\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2$$
$$\times[4\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\}$$
$$+2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}$$
$$+\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}]\Big\}$$
$$+\eta\xi^2(1-\cos\beta\cosh\beta)\Big\{(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times[4\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\}$$
$$+2\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\}$$
$$+\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{4\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}]$$
$$-(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\Big\}$$
$$-2\xi^3(1-\cos\beta\cosh\beta)^2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+\gamma\cos\alpha\cosh\alpha\}$$

iv) The numerator of $\dfrac{A_1'}{b}$ is

$$
\begin{aligned}
&4\eta^3\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}^2\\
&\quad\times\{2\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}\\
&+\eta^2\xi\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times[2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)(\sin\beta-\sinh\beta)\\
&\quad\times\{(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}\\
&\quad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad-(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-2\gamma\sin\alpha\cosh\alpha\}\\
&\quad-2\{2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2-(\sin\beta-\sinh\beta)^2\}\\
&\quad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\{2\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}]\\
&+2\eta\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\Big\{(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times[\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times\{2\cos\alpha\sinh\alpha-\gamma(1-\cos\alpha\cosh\alpha+\sin\alpha\sinh\alpha)\}\\
&\quad+\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-2\gamma\sin\alpha\cosh\alpha\}]\\
&\quad-(\sin\beta-\sinh\beta)\{(\cos\alpha+\cosh\alpha)-\gamma(\sin\alpha-\sinh\alpha)\}\\
&\quad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\Big\}\\
&-\xi^3(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}^2\\
&\quad\times\{(1+\cos\alpha\cosh\alpha-\sin\alpha\sinh\alpha)-\gamma\sin\alpha\cosh\alpha\}
\end{aligned}
$$

v) The numerator of $\frac{B_1'}{b}$ is

$$
\begin{aligned}
&4\eta^3\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}^2\\
&\quad\times\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}\\
&+\eta^2\xi\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times[2(\sin\beta-\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad-(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\}\\
&\quad-2\{2(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2-(\sin\beta-\sinh\beta)^2\}\\
&\quad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}
\end{aligned}
$$

$$\times\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)\}]$$
$$-2\eta\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\Big\{(\sin\beta-\sinh\beta)\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times[\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\}$$
$$+\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2\sin\alpha\sinh\alpha+\gamma(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha\ \}]\Big\}$$
$$-\xi^3(1-\cos\beta\cosh\beta)^2\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}^2$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)+2\gamma\cos\alpha\cosh\alpha\}]$$

vi) $A_1''=A_1$

vii) $B_1''=B_1$

viii) The numerator of $\dfrac{A_2}{b}$ is

$$-4\eta^2\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+\eta\xi[(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+4\sin\beta\sinh\beta(1-\cos\beta\cosh\beta)$$
$$\times 2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\sin\beta-\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$-\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times[(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\sin\beta-\sinh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)]$$

ix) The numerator of $\dfrac{B_2}{b}$ is

$$
\begin{aligned}
&2\eta^2 \sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\quad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times[\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+2\cos\beta\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]\\
&-\eta\xi[4\cos\beta\sinh\beta(1-\cos\beta\cosh\beta)\\
&\qquad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+\{2\sinh\beta(1-\cos\beta\cosh\beta)+(\cos\beta\sinh\beta-\sin\beta\cosh\beta)(\cos\beta-\cosh\beta)\}\\
&\qquad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+(1-\cos\beta\cosh\beta-\sin\beta\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\qquad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]\\
&+\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\quad\times[(\cos\beta-\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+(1-\cos\beta\cosh\beta-\sin\beta\sinh\beta)\\
&\qquad\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]
\end{aligned}
$$

x) The numerator of $\dfrac{D_2}{b}$ is

$$
\begin{aligned}
&-2\eta^2\sin\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)\\
&\qquad\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\quad\times[\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+2\cosh\beta\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]\\
&+\eta\xi[4\sin\beta\cosh\beta(1-\cos\beta\cosh\beta)\\
&\qquad\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}\\
&\qquad\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}\\
&\qquad\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}\\
&\quad+\{(\cos\beta-\cosh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)+2\sin\beta(1-\cos\beta\cosh\beta)\}
\end{aligned}
$$

$$\times\{2(1+\cos\alpha\cosh\alpha)+\beta(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$+\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times[(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-(\cos\beta-\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$

xi) The numerator of $\dfrac{A_2'}{b}$ is

$$-2\eta^2\sin\beta\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}^2$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+\eta\xi\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times[(\sin\beta-\sinh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+\{2\sin\beta\sinh\beta(1-\cos\beta\cosh\beta)+(\cos\beta\sinh\beta-\sin\beta\cosh\beta)^2\}$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$-\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times[(\sin\beta-\sinh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$

xii) The numerator of $\dfrac{B_2'}{b}$ is

$$2\eta^2\sinh\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times[\cos\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$

$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$-\eta\xi[4\sinh\beta(1-\cos\beta\cosh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+\{4\cos\beta\sinh\beta(1-\cos\beta\cosh\beta)+(\sin\beta-\sinh\beta)(\cos\beta-\cosh\beta)\}$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\cos\beta-\cosh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$+\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times[(1-\cos\beta\cosh\beta-\sin\beta\sinh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\cos\beta-\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$

xiii) The numerator of $\dfrac{D_2'}{b}$ is

$$-2\eta^2\sin\beta(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times[\cosh\beta\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+2\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$+\eta\xi[4\sin\beta(1-\cos\beta\cosh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+\{4\sin\beta\cosh\beta(1-\cos\beta\cosh\beta)+(\sin\beta-\sinh\beta)(\cos\beta-\cosh\beta)\}$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$+(\cos\beta-\cosh\beta)(\cos\beta\sinh\beta-\sin\beta\cosh\beta)$$
$$\times\{2(\cos\alpha\sinh\alpha+\sin\alpha\cosh\alpha)-\gamma(1-\cos\alpha\cosh\alpha)\}$$

$$\times\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$
$$+\xi^2(1-\cos\beta\cosh\beta)\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}.$$
$$\times[(1-\cos\beta\cosh\beta+\sin\beta\sinh\beta)$$
$$\times\{2(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}$$
$$-(\cos\beta-\cosh\beta)\{(1+\cos\alpha\cosh\alpha)+\gamma(\cos\alpha\sinh\alpha-\sin\alpha\cosh\alpha)\}$$
$$\times\{2(\sin\alpha-\sinh\alpha)+\gamma(\cos\alpha-\cosh\alpha)\}]$$

The general expressions of the displacements of all members are

$$y_1=\{A_1(\cos m_1x_1-\cosh m_1x_1)+B_1(\sin m_1x_1-\sinh m_1x_1)\}\cos p't,$$
$$y_1'=\{A_1'(\cos m_1x_1'-\cosh m_1x_1')+B_1'(\sin m_1x_1'-\sinh m_1x_1')\}\cos p't,$$
$$y_1''=\{A_1''(\cos m_1x_1''-\cosh m_1x_1'')+B_1''(\sin m_1x_1''-\sinh m_1x_1'')\}\cos p't,$$
$$y_2=\{A_2(\cos m_2x_2-\cosh m_2x_2)+B_2\sin m_2x_2+D_2\sinh m_2x_2\}\cos p't,$$
$$y_2'=\{A_2'(\cos m_2x_2'-\cosh m_2x_2')+B_2'\sin m_2x_2'+D_2'\sinh m_2x_2'\}\cos p't.$$

As an example of calculation we have taken up the case $\rho_1/\rho_2=1$, $\alpha=1$ and plotted the result in the following figure. From this result, it may be seen that the distribution of the displacement is the same as that in the case of the framed structure with an infinite number of spans. The transverse deflection of the beam is very small in comparison with that

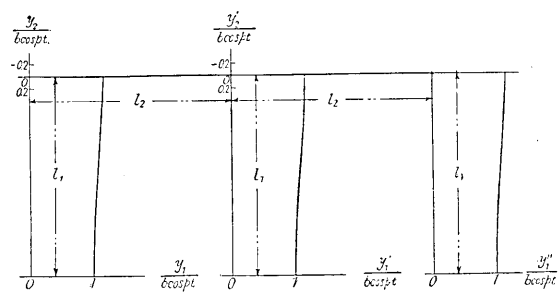

Fig. 36. $\dfrac{\rho_1}{\rho_2}=1$, $\alpha=1$.

of the pillar. This fact has been found also in the case of the structure of infinite number of spans. At any rate, it is not without importance that the natural periods as well as the deformation due to forced vibration of a framed structure with two spans are the same as those of the structure with an infinite number of spans. It is clear that the case of three or more spans far approximates to that of the infinite number of spans, so that it seems that the treatment of the case of an infinite number of spans covers practically all the problems of horizontal vibrations of framed structures with any number (larger than two) of spans in general.

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49. 單層架構構造の震動

地震研究所 { 妹澤克惟 / 金井清

架構構造の振動問題は内外の多くの學者によつて手をつけられて居るが、力學上や境界條件の立場から見て完全なものが殆ど見つからない。著者の一人が帶板を有するタービン翼の振動問題に試みた方法は建築構造が水平地震動を受ける場合の振動にも應用出來る事がわかつた。それで著者等は建築其他の構造物にあてはまる様な種々の場合を計算して見た所が其手續きは非常に面倒ではあつたけれども、しかし重要なる二三の結論に到達した。

計算を行つた場合といふのは張間が無限にある時の固有振動と其強制振動、張間が二つある時の同じく固有振動と強制振動とであるが、張間が二つある時のすべての問題は其が無限にある時のそれに近いといふ事がわかつた事は、張間が二つ以上幾個ある時でも無限にある時と同じに見て計算をしてよいといふ事を示すものであつて、現在の計算が何等の假定のなき根據を有するものだけに今後の斯る計算を正確に而も單純化せしめるものである。

又振動問題としてわかつた事は、共振の時に梁の所が最も大なる振幅をもつ事はいふ迄もなく、地震の週期的振動週期が固有振動週期よりも短い時には梁即ち屋根の所が餘り搖れなくて寧ろ柱の屈曲モーメントが非常に大きくなる事、地震の週期が固有振動週期よりも長い時には屋根が地動と同じく大きく搖れ、しかし屈曲モーメントは靜力學的に考へたのと同じ様になる事などもわかつた。其他質量や剛度の分布等についても種々の事柄がわかつたけれども説明を省略して置いた。終りに、地震動の週期と建物の週期との割合によつて、柱の屈曲モーメントが大になつた時には屋根の變位がそれ程大きくなく、屈曲モーメントの小なる時に反て屋根の變位が大きくなる事は地震の時に瓦の落ちた家とそうでなく構造が破壞した家との區別をつける事に好都合であらうといふ事を寺田先生から教へられた事をつけ加へて置く。

Corrigenda to the Bull. Earthq. Res. Inst.

Volume	Part	Page	Line	for	read	Discoverer
IX	4	400	3	$e^{\left(\frac{1}{4}+\frac{p}{2k}\right)}$	$e^{-\left(\frac{1}{2}+\frac{p}{2k}\right)}$	C. Tsuboi
〃	〃	405	14			
〃	〃	406	23			
〃	〃	403	7	$e^{\left(\frac{1}{6}+\frac{p}{2k}\right)}$	$e^{-\left(\frac{2}{3}+\frac{p}{2k}\right)}$	
〃	〃	405	16			
〃	〃	406	24			
X	2	305	4, 7, 9, 11, 13, 15	$(-1)^m \frac{d^m P_n(\cos\theta)}{d\theta^m}$	$P_n^m(\cos\theta)$	H. Kawasumi
〃	2	306	1, 3, 5			

These errors are not effective on remaining parts of the respective papers.