23. *On the Expressions of the Deformation of a Semi-infinite Elastic Body due to the Temperature Variation.*

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1. Nearly two years ago, using the Fourier's double integral formula I[1] studied the deformation of a semi-infinite elastic body due to temperature distribution of different kinds. The temperature distribution in question corresponded to the state at an instant of heat conduction in the solid, but it did not contain the effect of time variation, and hence the problem of diffusion of heat in the interior of solid was not treated in that case. In the present paper I have studied the deformation of the solid due to the temperature distribution and variation satisfying all the conditions of the theory of heat conduction. Recently Dr. H. Arakawa[2] treated of a similar problem as my present one using rectangular coordinates with some critisism on my preceding paper; but, as he did not seem to give the consideration concerning the boundary conditions of stress at the free surface of elastic solid, his result is not theoretically complete.

Using cylindrical coordinates I have treated the following two cases, *i. e.* the case where the inertia of solid is neglected, and the other where the inertia effect is taken into account. Studying these two cases, we found some interesting facts which we would not be able to get in the case of the temperature distribution without time variation. I believe that the present study on these two cases is also interesting from theoretical point of view.

Part I. Problem in the Case of Neglected Inertia Term.

2. Cylindrical coordinates (r, θ, z) are used. The plane $z=0$ is

1) G. NISHIMURA, "The Effect of Temperature Distribution on the Deformation of a Semi-infinite Elastic Body," *Bull. Earthq. Res. Inst.*, 8 (1930), 91—142.

2) H. ARAKAWA, "The Effect of Temperature on the Deformation of Infinite or Semi-infinite Elastic Body," *Geophys. Mag.*, Tôkyô, 64 (1931).

the surface of elastic solid, and r is the distance from the origin o, θ the azimuthal angle around z-axis. The axis of z is taken vertically downwards. (Fig. 1.)

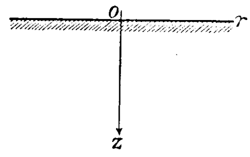

Fig. 1.

The equations of motion of elastic body when temperature variation is taken into consideration, and the inertia force is neglected, are expressed by the following three forms:

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi_z}{\partial\theta}+2\mu\frac{\partial\varpi_\theta}{\partial z}=\alpha\frac{\partial T}{\partial r},\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial\varpi_r}{\partial z}+2\mu\frac{\partial\varpi_z}{\partial r}=\alpha\frac{1}{r}\frac{\partial T}{\partial\theta},\\
&(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_\theta)+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}=\alpha\frac{\partial T}{\partial z},
\end{aligned}\right\}\quad\ldots\ldots\ldots(1)$$

in which λ, μ are Lamé's elastic constants of solid, T is the temperature variation of the same solid, and

$$\alpha=\left(\lambda+\frac{2}{3}\mu\right)c,\quad\ldots\ldots\ldots\ldots\ldots(2)^{3)}$$

where c is the cubical expansion coefficient of solid.

The dilatation Δ, and the components of rotation $2\varpi_r$, $2\varpi_\theta$, $2\varpi_z$ are connected with the displacement in the following forms:

$$\left.\begin{aligned}
&2\varpi_r=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z},\\
&2\varpi_\theta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial v},\\
&2\varpi_z=\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial\theta},
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(3)$$

where u, v, w are the radial, azimuthal, and axial components of displacement of solid.

For the purpose of finding u, v, w from the equations (1) and (3), we must first find the suitable form of T which satisfies the equation of conduction of heat in elastic solid such that

3) In two-dimensional problems, α should be taken as $\frac{2}{3}(\lambda+\mu)c$. In the second part of my preceding paper, *loc. cit.*, pp. 95—115, the meaning of notation α must be taken to be $\frac{2}{3}(\lambda+\mu)c$. My thanks are due to Dr. H. Arakawa for his kind notice, in his paper, *loc. cit.*, on my inattention on this point in my preceding paper.

$$\frac{\partial T}{\partial t}=\kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right), \qquad (4)$$

where t is time and κ the diffusivity of solid such that

$$\kappa=\frac{K}{\gamma\rho}. \qquad (5)$$

K is the thermal conductivity, γ the specific heat and ρ the density of solid.

There are many types of elementary solutions of equation (4), and we take the following form of T for our purpose:

$$T=e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} mz \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta, \qquad (6)$$

where m, n and k_s are any constants to be determined by the initial and boundary conditions of temperature, and $C_n(x)$ is the Cylindrical function of any form of order n.

Now we know the following identical relation between ϖ_r, ϖ_θ and ϖ_z:

$$\frac{1}{r}\frac{\partial(r\varpi_r)}{\partial r}+\frac{1}{r}\frac{\partial \varpi_\theta}{\partial \theta}+\frac{\partial \varpi_z}{\partial z}=0. \qquad (7)$$

Using this relation and the equations (1), we obtain the following differential equations:

$$\frac{\partial^2}{\partial r^2}[(\lambda+2\mu)\Delta-\alpha T]+\frac{1}{r}\frac{\partial}{\partial r}[(\lambda+2\mu)\Delta-\alpha T]$$

$$+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[(\lambda+2\mu)\Delta-\alpha T]+\frac{\partial^2}{\partial z^2}[(\lambda+2\mu)\Delta-\alpha T]=0, \qquad (8)$$

$$\frac{\partial^2 \varpi_z}{\partial r^2}+\frac{1}{r}\frac{\partial \varpi_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \varpi_z}{\partial \theta^2}+\frac{\partial^2 \varpi_z}{\partial z^2}=0. \qquad (9)$$

Substituting the expression (6) for T in the equation (8), we obtain the following differential equation:

$$\frac{\partial^2 \Delta}{\partial r^2}+\frac{1}{r}\frac{\partial \Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Delta}{\partial \theta^2}+\frac{\partial^2 \Delta}{\partial z^2}=-\frac{\alpha(k_s^2+m^2)}{(\lambda+2\mu)}e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} mz \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta. \qquad (10)$$

The particular solution of the equation (10) is written in the form:

$$\Delta_1=\frac{\alpha}{(\lambda+2\mu)}e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} mz \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta. \qquad (11)$$

The dilatation expressed by (11) is that due to the elementary temperature distribution expressed by (6).

The complementary solution of equation (10), which is useful for the present study, is

$$\Delta_2 = A_m e^{-\kappa(k_s^2+m^2)t} e^{-k_s z} C_n(k_s r) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} n\theta, \quad \ldots\ldots\ldots\ldots (12)$$

where A_m is an arbitrary constant to be determined by the boundary conditions of elastic solid.

The solution of (9), which is suitable for our problem, is of the form of

$$2\varpi_z = B_m e^{-\kappa(k_s^2+m^2)t} e^{-k_s z} C_n(k_s r) \begin{Bmatrix}-\cos\\ \sin\end{Bmatrix} n\theta, \quad \ldots\ldots\ldots (13)$$

where B_m is also an arbitrary constant to be determined by the elasticity conditions of solid.

Substituting the expressions (6), (11), (12), and (13) for T, Δ and ϖ_z in equations (1), we obtain the following solutions of $2\varpi_r$ and $2\varpi_\theta$:

$$2\varpi_r = \left[\frac{n(\lambda+2\mu)}{k_s\mu} A_m \frac{C_n(k_s r)}{r} - B_m \frac{1}{k_s}\frac{\partial C_n(k_s r)}{\partial r}\right] e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} \begin{Bmatrix}-\cos\\ \sin\end{Bmatrix} n\theta, \quad \ldots (14)$$

$$2\varpi_\theta = \left[\frac{(\lambda+2\mu)}{k_s\mu} A_m \frac{\partial C_n(k_s r)}{\partial r} - B_m \frac{n}{k_s}\frac{C_n(k_s r)}{r}\right] e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} n\theta. \quad \ldots (15)$$

Now we can find the following differential equations concerning ru, rv, w, Δ, $2\varpi_r$, $2\varpi_\theta$ and $2\varpi_z$ by means of the relations of (3):

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} = \frac{\partial \Delta}{\partial z} + \frac{2}{r}\frac{\partial \varpi_r}{\partial \theta} - \frac{2}{r}\frac{\partial}{\partial r}(r\varpi_\theta), \quad \ldots (16)$$

$$\frac{\partial^2 (rv)}{\partial r^2} + \frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 (rv)}{\partial \theta^2} + \frac{\partial^2 (rv)}{\partial z^2} = \frac{\partial \Delta}{\partial \theta} - 2r\frac{\partial \varpi_r}{\partial z} + \frac{2}{r}\frac{\partial}{\partial r}(r^2\varpi_z), \quad (17)$$

$$\frac{\partial^2 (ru)}{\partial r^2} - \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 (ru)}{\partial \theta^2} + \frac{\partial^2 (ru)}{\partial z^2} - \frac{2}{r^2}\frac{\partial (rv)}{\partial \theta}$$

$$= r\frac{\partial \Delta}{\partial r} + 2r\frac{\partial \varpi_\theta}{\partial z} - 2\frac{\partial \varpi_z}{\partial \theta}. \quad \ldots\ldots (18)$$

Substituting (11), (12), (13), (14) and (15) for Δ, $2\varpi_r$, $2\varpi_\theta$ and $2\varpi_z$ in the two differential equations (16), (17), we obtain the particular solutions w, rv in the following forms:

$$w = -\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} C_n(k_s r) \begin{Bmatrix}\cos\\ -\sin\end{Bmatrix} mz \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} n\theta$$

$$-\frac{A_m}{2k_s\mu}\left[(\lambda+3\mu)+(\lambda+\mu)k_s z\right] e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} C_n(k_s r) \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} n\theta, \quad \ldots (19)$$

and

$$rv = \frac{n\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} C_n(k_s r) \begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta$$
$$-\frac{n(\lambda+\mu)}{2\mu k_s} A_m z e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} \frac{1}{r} C_n(k_s r) \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta$$
$$-B_m \frac{e^{-k_s z}}{k_s^2} e^{-\kappa(k_s^2+m^2)t} \frac{\partial C_n(k_s r)}{\partial r} \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta. \quad \ldots\ldots\ldots\ldots(20)$$

The complementary solution of (16), (17) are written easily in the following forms:

$$w = -k_s E_m e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} C_n(k_s r) \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \quad \ldots\ldots\ldots(21)$$

$$rv = -n E_m e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} C_n(k_s r) \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta, \quad \ldots\ldots(22)$$

where E_m is also an arbitrary constant to be determined by the boundary conditions.

Now substituting (11), (12), (13), (15), (20) and (22) for the expressions of Δ, $2\varpi_\theta$, $2\varpi_z$ and rv in the equation (18), we can find the particular solution of ru, which are favourable to the present study, in the following forms:

$$ru = -\frac{\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} r \frac{\partial C_n(k_s r)}{\partial r} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$
$$+\frac{(\lambda+\mu)}{2k_s\mu} A_m z e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} r \frac{\partial C_n(k_s r)}{\partial r} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$
$$+\frac{n}{k_s} B_m e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} C_n(k_s r) \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$
$$+E_m e^{-k_s z} e^{-\kappa(k_s^2+m^2)t} r \frac{\partial C_n(k_s r)}{\partial r} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \ldots\ldots\ldots\ldots\ldots(23)$$

There is also the complementary solution ru of the equation (18), but it does not satisfy the equation of motion (1).

As the components of stress $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{r\theta}$, $\widehat{rz}$ and $\widehat{\theta z}$ are expressed by

$$\left.\begin{aligned}
&\widehat{rr} = \lambda\Delta + 2\mu\frac{\partial u}{\partial r} - \alpha T, \quad \widehat{\theta\theta} = \lambda\Delta + 2\mu\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right) - \alpha T, \\
&\widehat{zz} = \lambda\Delta + 2\mu\frac{\partial w}{\partial z} - \alpha T, \quad \widehat{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right), \\
&\widehat{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right), \quad \widehat{\theta z} = \mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right),
\end{aligned}\right\} \quad (24)$$

we can find the components of stress which are necessary for the present study, by using (6), (11), (12), (19), (20), (21), (22) and (23).

Now the surface ($z=0$) of elastic solid is a free surface, so that we have the following conditions on this surface:

$$z=0;\quad \widehat{zz}=0,\ \widehat{rz}=0,\ \widehat{\theta z}=0. \qquad \ldots\ldots\ldots\ldots(25)$$

From these relations we can find the following forms of constants A_m, B_m and E_m:

$$\left.\begin{aligned} A_m &= \frac{2\mu\alpha m k_s}{(\lambda+\mu)(\lambda+2\mu)(k_s^2+m^2)}, \quad B_m=0, \\ E_m &= -\frac{m\alpha}{(\lambda+\mu)(k_s^2+m^2)k_s}. \end{aligned}\right\} \qquad \ldots\ldots(26)^{4)}$$

Substituting these relations (26) for A_m, B_m and E_m in the components of displacement expressed by (19), (20), (21), (22) and (23), we obtain the components of displacement u, v, w in the following forms:

$$\begin{aligned} u = &-\frac{\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} \frac{\partial C_n(k_s r)}{\partial r} \sin mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta \\ &+\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} z e^{-k_s z} \frac{\partial C_n(kr)}{\partial r} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta \\ &-\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+\mu)(k_s^2+m^2)k_s} e^{-k_s z} \frac{\partial C_n(k_s r)}{\partial r} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \end{aligned} \qquad \ldots\ldots\ldots(27)$$

$$\begin{aligned} v = &\frac{n\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} \frac{C_n(k_s r)}{r} \sin mz \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta \\ &-\frac{nm\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} z e^{-k_s z} \frac{C_n(k_s r)}{r} \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta \\ &+\frac{nm\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+\mu)(k_s^2+m^2)} e^{-k_s z} \frac{C_n(k_s r)}{r} \begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta, \end{aligned} \qquad \ldots\ldots\ldots(28)$$

$$\begin{aligned} w = &-\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} C_n(k_s r)\cos mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta \\ &-\frac{m k_s \alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+2\mu)(k_s^2+m^2)} z e^{-k_s z} C_n(k_s r) \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta \\ &-\frac{m\mu\alpha e^{-\kappa(k_s^2+m^2)t}}{(\lambda+\mu)(\lambda+2\mu)(k_s^2+m^2)} e^{-k_s z} C_n(k_s r) \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \end{aligned} \qquad \ldots(29)$$

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4) To obtain these results, we used the upper part of $\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz$ of T expressed by (6).

The displacement thus obtained is of elementary type which corresponds to the temperature variation expressed by (6) and satisfies the boundary conditions at the free surface $z=0$ expressed by (25).

3. When the temperature varies as any function of time t, coordinates r and θ on the free surface ($z=0$) of elastic solid and the initial temperature is zero at every point in the solid, the problem is more complicated; and I shall show the method of treatment as follows.

Now the conditions with regards to temperature T are as follows (in mathematical notations):

$$t=0; \quad T=0, \qquad \ldots\ldots (30)_1$$

$$z=0; \quad T=f(r,\theta,t), \qquad \ldots\ldots (30)_2$$

$$z=\infty; \quad T=0, \qquad \ldots\ldots (30)_3$$

$$r=\infty; \quad T=0. \qquad \ldots\ldots (30)_4$$

Substituting the Bessel function of first kind $J_n(k_s r)$ for $C_n(k_s r)$, and taking $\sin mz$ and $\cos n\theta$ in the elementary equation (6), we obtain the following expression of T which satisfies the equation of heat conduction (4) and the initial and boundary conditions expressed by $(30)_1$, $(30)_2$, $(30)_3$ and $(30)_4$:

$$T=\frac{\kappa}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\cos n(\theta-\varphi).J_n(\sigma r)$$
$$J_n(\sigma\zeta)\sin\xi z\sigma\xi\zeta\, d\eta d\varphi d\zeta d\sigma d\xi, \qquad \ldots\ldots (31)$$

where $\delta_0=1$, $\delta_1=\delta_2=\delta_3=\ldots\ldots=\delta_n=2$.

Now we have to obtain the displacement corresponding to the temperature variation expressed by (31) and satisfying the boundary conditions (25) in Section 2. Using the elementary solutions expressed by (27), (28) and (29), we obtain the final solutions, which satisfy all conditions, in the following forms:

$$u=-\frac{\kappa\alpha}{\pi^2(\lambda+2\mu)}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\int_0^{2\pi}e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\frac{\left[\sin\xi z+\left(\xi z-\frac{\xi}{\sigma}\right)e^{-\sigma z}\right]}{(\sigma^2+\xi^2)}$$
$$\cos n(\theta-\varphi)f(\zeta,\varphi,\eta)\frac{\partial J_n(\sigma r)}{\partial r}J_n(\sigma\zeta)\sigma\xi\zeta\, d\eta d\varphi d\zeta d\sigma d\xi, \qquad \ldots (32)$$

$$v=\frac{\kappa\alpha}{\pi^2(\lambda+2\mu)}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\int_0^{2\pi}e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}$$
$$\frac{n\left[\sin\xi z-\xi z e^{-\sigma z}+\frac{(\lambda+2\mu)}{(\lambda+\mu)}\frac{\xi}{\sigma}e^{-\sigma z}\right]}{(\sigma^2+\xi^2)}$$

$$\sin n(\theta-\varphi)f(\zeta,\varphi,\eta)\frac{J_n(\sigma r)}{r}J_n(\sigma\zeta)\sigma\xi\zeta d\eta d\varphi d\zeta d\sigma d\xi, \quad \ldots .(33)$$

$$w=-\frac{\kappa\alpha}{\pi^2(\lambda+2\mu)}\sum_{n=3}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\frac{\left[\cos\xi z+\frac{u}{\lambda+\mu}e^{-\sigma z}+\sigma z e^{-\sigma z}\right]}{(\sigma^2+\xi^2)}$$
$$\cos n(\theta-\varphi)f(\zeta,\varphi,\eta)J_n(\sigma r)J_n(\sigma\zeta)\sigma\xi^2\zeta d\eta d\varphi d\zeta d\sigma d\xi. \quad \ldots .(34)$$

We can easily formulate the components of thermal stress $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{r\theta}$, $\widehat{rz}$, $\widehat{\theta z}$ at any point in the solid due to the temperature variation expressed by (31), but in the present study we omit this procedure.

In this section, we have obtained the general expressions of displacement u, v, w due to temperature T in the semi-infinite elastic body satisfying the conditions expressed by (30).

4. When the distribution of temperature has no azimuthal variation, and is given by the form:

$$z=0; \quad T=\phi(r,t), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

we have the following form of temperature distribution in solid:

$$T=\frac{\kappa}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)J_0(\sigma r)J_0(\sigma\zeta)\sin\xi z\sigma\xi\zeta d\eta d\zeta d\sigma d\xi. \quad \ldots\ldots\ldots.(36)$$

The equation (36), of cource, satisfies the conditions $(30)_1$, $(30)_3$, $(30)_4$.

The components of displacement (u, w) corresponding to the temperature variation (36) are easily formulated as in the follow:

$$u=-\frac{\kappa\alpha}{\pi(\lambda+2\mu)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)\frac{\left[\sin\xi z+(z+1)e^{-\sigma z}\xi\right]}{(\sigma^2+\xi^2)}\frac{\partial J_0(\sigma r)}{\partial r}$$
$$J_0(\sigma\zeta)\sigma\xi\zeta d\zeta d\eta d\sigma d\xi, \quad \ldots .(37)$$

$$w=-\frac{\kappa\alpha}{\pi(\lambda+\mu)(\lambda+2\mu)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)$$
$$\frac{\left[(\lambda+\mu)\cos\xi z+\zeta e^{-\sigma z}+(\lambda+\mu)\sigma z e^{-\sigma z}\right]}{(\sigma^2+\xi^2)}$$
$$J_0(\sigma r)J_0(\sigma\zeta)\sigma\xi^2\zeta d\zeta d\eta d\sigma d\xi. \quad \ldots\ldots .(38)$$

The surface deformation due to the temperature variation may be studied by the following two components of displacement:

$$u_{z=0}=-\frac{\kappa\alpha}{\pi(\lambda+2\mu)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)\frac{J_0(\sigma\zeta)}{(\sigma^2+\xi^2)}\frac{\partial J_0(\sigma r)}{\partial r}$$
$$\xi^2\zeta d\zeta d\eta d\sigma d\xi, \quad \ldots\ldots .(39)$$

$$w_{z=0} = -\frac{\kappa\alpha}{\pi(\lambda+\mu)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)\frac{J_0(\sigma r)}{(\sigma^2+\xi^2)} J_0(\sigma\zeta)\sigma\xi^2\zeta\, d\eta\, d\zeta\, d\sigma\, d\xi \, . . (40)$$

Part II. Problems When the Inertia Effect is Considered.

5. Using the same coordinates as that in Part I, the equations of motion of an elastic body of temperature variation T are expressed by the following forms:

$$\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+2\mu)\frac{\partial \varDelta}{\partial r} - \frac{2\mu}{r}\frac{\partial \varpi_z}{\partial \theta} + 2\mu\frac{\partial \varpi_\theta}{\partial z} - \alpha\frac{\partial T}{\partial r}, \\
\rho\frac{\partial^2 v}{\partial t^2} &= (\lambda+2\mu)\frac{1}{r}\frac{\partial \varDelta}{\partial \theta} - 2\mu\frac{\partial \varpi_r}{\partial z} + 2\mu\frac{\partial \varpi_z}{\partial r} - \alpha\frac{1}{r}\frac{\partial T}{\partial \theta}, \\
\rho\frac{\partial^2 w}{\partial t^2} &= (\lambda+2\mu)\frac{\partial \varDelta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_\theta) + \frac{2\mu}{r}\frac{\partial \varpi_r}{\partial \theta} - \alpha\frac{\partial T}{\partial z}.
\end{aligned}\right\} \, . . (41)$$

For solving the equations (41), we must first find the suitable form of the expression of temperature T. We take the following form of T as that in the preceding part:

$$T = e^{-\kappa(k_s^2+m^2)t} C_n(k_s r)\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \quad \ldots\ldots\ldots\ldots (6')$$

Now, using the identical relations expressed by (7) and the relations of (3) in Part I, we find the following equations:[5)]

$$\rho\frac{\partial^2 \varDelta}{\partial t^2} = \frac{\partial^2}{\partial r^2}[(\lambda+2\mu)\varDelta - \alpha T] + \frac{1}{r}\frac{\partial}{\partial r}[(\lambda+2\mu)\varDelta - \alpha T] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[(\lambda+2\mu)\varDelta - \alpha T] + \frac{\partial^2}{\partial z^2}[(\lambda+2\mu)\varDelta - \alpha T], \ldots (42)$$

$$\rho\frac{\partial^2 \varpi_z}{\partial t^2} = \mu\left[\frac{\partial^2 \varpi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varpi_z}{\partial \theta^2} + \frac{\partial^2 \varpi_z}{\partial z^2}\right], \quad \ldots\ldots\ldots\ldots (43)$$

$$\rho\frac{\partial^2 \varpi_r}{\partial t^2} = \mu\left[\frac{\partial^2 \varpi_r}{\partial r^2} + \frac{3}{r}\frac{\partial \varpi_r}{\partial r} + \frac{\varpi_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 \varpi_r}{\partial \theta^2} + \frac{\partial^2 \varpi_r}{\partial z^2} + \frac{2}{r}\frac{\partial \varpi_z}{\partial z}\right], \ldots (44)$$

$$\rho\frac{\partial^2 \varpi_\theta}{\partial t^2} = \mu\left[\frac{\partial^2 \varpi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi_\theta}{\partial r} - \frac{\varpi_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 \varpi_\theta}{\partial \theta^2} + \frac{\partial^2 \varpi_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial^2 \varpi_r}{\partial \theta}\right]. \ldots (45)$$

Substituting (6′) for T in equation (42), we obtain the differential equation of the form:

5) The reductions of these equations when $T=0$ have been made by Prof. K. Sezawa. K. SEZAWA, *Bull. Earthq. Res. Inst.*, 6 (1929).

$$\rho\frac{\partial^2 \Delta}{\partial t^2}-(\lambda+2\mu)\left[\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{\partial^2\Delta}{\partial z^2}\right]$$
$$=\alpha(k_s^2+m^2)e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \quad ..(46)$$

We have the particular solution of equation (46) of the form:

$$\Delta_1=\frac{\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}}C_n(k_s r)\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \quad ..(47)$$

The dilatation expressed by (47) is due to the elementary temperature variation expressed by (6′).

The complementary solution of equation (46), which is necessary for the present study, is

$$\Delta_2=A_m e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \quad ....(48)$$

where A_m is an arbitrary constant to be determined by the boundary conditions of elastic solid.

The solution of (43), which is favourable for the present study, is of the form:

$$2\varpi_z=B_m e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\} n\theta, \quad ..(49)$$

B_m being also an arbitrary constant to be determined by the boundary conditions.

Substituting (49) for $2\varpi_z$ in equation (44), we obtain the following solution of (44):

$$2\varpi_r=C_m\frac{C_n(k_s r)}{r}e^{-\kappa(k_s^2+m^2)t}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta$$
$$-B_m\frac{\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}}{k_s^2}\frac{\partial C_n(k_s r)}{\partial r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}-\cos\\ \sin\end{matrix}\Big\} n\theta. \quad (50)$$

Using the expression (50), we can find the solution of $2\varpi_\theta$ shown below:

$$2\varpi_\theta=C_m\frac{1}{n}\frac{\partial C_n(k_s r)}{\partial r}e^{-\kappa(k_s^2+m^2)t}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$
$$+B_m\frac{n\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}}{k_s^2}\frac{C_n(kr)}{r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \quad (51)$$

In the expressions (50), (51), C_m is also an arbitrary constant.

Substituting (47), (48), (49), (50) and (51) for Δ, $2\varpi_r$, $2\varpi_\theta$ and $2\varpi_z$ in the equations (16), (17) and (18), we obtain the following solutions of the components of displacement (u, v, w) after some reductions:

$$u=-\frac{\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}}\frac{\partial C_n(k_s r)}{\partial r}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$

$$+\frac{(\lambda+2\mu)A_m e^{-\kappa(k_s^2+m^2)t}}{\rho\kappa^2(k_s^2+m^2)^2}\frac{\partial C_n(k_s r)}{\partial r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$

$$-\frac{\mu\left\{\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}}{n\rho\kappa^2(k_s^2+m^2)^2}C_m e^{-\kappa(k_s^2+m^2)t}\frac{\partial C_n(k_s r)}{\partial r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$

$$-\frac{n}{k_s^2}B_m e^{-\kappa(k_s^2+m^2)t}\frac{C_n(k_s r)}{r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \qquad \ldots\ldots\ldots (52)$$

$$v=-\frac{n\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}}\frac{C_n(k_s r)}{r}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} mz \begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta$$

$$+\frac{n(\lambda+2\mu)}{\rho\kappa^2(k_s^2+m^2)^2}A_m e^{-\kappa(k_s^2+m^2)t}\frac{C_n(k_s r)}{r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta$$

$$-\frac{\mu\left\{\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}}{\rho\kappa^2(k_s^2+m^2)^2}C_m e^{-\kappa(k_s^2+m^2)t}\frac{C_n(k_s r)}{r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta$$

$$-\frac{1}{k_s^2}B_m e^{-\kappa(k_s^2+m^2)t}\frac{\partial C_n(k_s r)}{\partial r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta, \qquad \ldots . (53)$$

$$w=-\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}}C_n(k_s r)\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} mz \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$

$$-\frac{(\lambda+2\mu)}{\rho\kappa^2(k_s^2+m^2)^2}\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}A_m C_n(k_s r)e^{-\kappa(k_s^2+m^2)t}$$

$$\times e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta$$

$$+\frac{k_s^2\mu}{n\rho\kappa^2(k_s^2+m^2)^2}C_m e^{-\kappa(k_s^2+m^2)t}C_n(k_s r)e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta. \quad (54)$$

In each expression of the component of displacement, u, v, and w, the first term having the trigonometrical functions of mz corresponds to the temperature variation (6′) only, and the other terms having e-functions of z correspond to the displacement which may be used for the purpose of fulfilment of the boundary conditions of elastic solid.

Using these components of displacement, (52), (53), (54) and dilatations (47), (48), and temperature variation (6′), we can obtain the components of traction $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{r\theta}$, $\widehat{\theta z}$, $\widehat{rz}$.

Now the surface ($z=0$) of elastic solid is free from traction as in the case of Part I. Therefore we have

$$z=0\,;\quad \widehat{zz}=0,\ \widehat{rz}=0,\ \widehat{\theta z}=0. \qquad \ldots\ldots\ldots\ldots (55)$$

These conditions give us the values of constants A_m, B_m and C_m in the following forms:

$$B_m=0,$$

$$A_m=\frac{-4\mu^2 k_s^2 m\alpha\rho\kappa^2(k_s^2+m^2)\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}}{(\lambda+2\mu)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\left[4\mu^2k_s^2\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right\}^{1/2}\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}-\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}^2\right]},$$

$$C_m=-\frac{2mn\alpha\rho\kappa^2(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\left[4\mu^2k_s^2\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right\}^{1/2}\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}-\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}^2\right]}.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

Then, substituting these expressions (56) for the constants A_m, B_m and C_m in the components of displacement expressed by (52), (53) and (54), we have the components of displacement u, v, w of the forms:

$$u=-\frac{\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}}\frac{\partial C_n(k_s r)}{\partial r}\begin{matrix}\sin\\ \cos\end{matrix}\Big\}mz\begin{matrix}\sin\\ \cos\end{matrix}\Big\}n\theta$$

$$-\frac{4\mu^2k_s^2m\alpha\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}F(k_s,m)}\frac{\partial C_n(k_s r)}{\partial r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}}\begin{matrix}\sin\\ \cos\end{matrix}\Big\}n\theta$$

$$+\frac{2m\alpha\mu\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}}{(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}F(k_s,m)}\frac{\partial C_n(k_s r)}{\partial r}e^{-\kappa(k_s^2+m^2)t}$$

$$\times e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{matrix}\sin\\ \cos\end{matrix}\Big\}n\theta, \qquad \ldots (57)$$

$$v=-\frac{n\alpha e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}}\frac{C_n(k_sr)}{r}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}mz\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}n\theta$$

$$-\frac{4\mu^2k_s^2mn\alpha\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}e^{-\kappa(k_s^2+m^2)t}}{\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}\{k_s^2+m^2\}F(k_s,m)}\frac{C_n(k_sr)}{r}e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}n\theta$$

$$+\frac{2mn\alpha\mu\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}}{(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}F(k_s,m)}\frac{C_n(k_sr)}{r}$$

$$\times e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}n\theta, \ldots. (58)$$

$$w=-\frac{m\alpha e^{-\kappa(k_s^2+m^2)t}}{(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}}C_n(k_sr)\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}mz\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\theta$$

$$+\frac{4\mu^2k_s^2m\alpha\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right\}^{1/2}\left\{\dfrac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}e^{-\kappa(k_s^2+m^2)t}}{(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}F(k_s,m)}$$

$$\times C_n(k_sr)e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right]^{1/2}z}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\theta$$

$$-\frac{2k_s^2m\mu\alpha\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}e^{-\kappa(k_s^2+m^2)t}}{(k_s^2+m^2)\{\rho\kappa^2(k_s^2+m^2)+(\lambda+2\mu)\}F(k_s,m)}$$

$$\times C_n(k_sr)e^{-\left[\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right]^{1/2}z}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}n\theta. \quad (59)$$

In the above expressions (57), (58) and (59), $F(k_s, m)$ represents the relation:

$$F(k_s,m)=\left[4\mu^2k_s^2\left\{\frac{\rho\kappa^2(k_s^2+m^2)^2}{(\lambda+2\mu)}+k_s^2\right\}^{1/2}\left\{\frac{\rho\kappa^2(k_s^2+m^2)^2}{\mu}+k_s^2\right\}^{1/2}\right.$$

$$\left.-\{\rho\kappa^2(k_s^2+m^2)^2+2\mu k_s^2\}^2\right]. \quad (60)$$

The displacement thus obtained is the elementary solution which corresponds to the temperature variation of the type (6′) and satisfies the boundary conditions at the free surface ($z=0$).

6. In this section we shall study the case where the temperature of the semi-infinite solid is subjected to the following conditions:

$$t=0\,; \qquad T=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(61)$$

$$z=0\,; \qquad T=f(r,\theta,t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots(62)$$

$$r=\infty\,; \qquad T=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(63)$$

$$z=\infty\,; \qquad T=0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(64)$$

These conditions are the same one as that in Section 3, and therefore the form of T is

$$T=\frac{\kappa}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\cos n(\theta-\varphi)J_n(\sigma r)$$

$$J_n(\sigma\zeta)\sin\xi z\sigma\xi\zeta\,d\eta\,d\varphi\,d\zeta\,d\sigma\,d\xi\ldots.(65)$$

Using the elementary solutions (57), (58), and (59), we obtain the following form of displacement which corresponds to this temperature variation (65) and satisfies the boundary conditions (55):

$$u=-\frac{\kappa\alpha}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t \frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\cos n(\theta-\varphi)}{\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}\{\sigma^2+\xi^2\}}\frac{\partial J_n(\sigma r)}{\partial r}$$

$$J_n(\sigma\zeta)\sin\xi z\,\sigma\xi\zeta\,d\eta\,d\varphi\,d\zeta\,d\sigma\,d\xi$$

$$-\frac{2\mu\alpha\kappa}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t\frac{\xi\left\{\dfrac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}f(\zeta,\varphi,\eta)e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}}{(\sigma^2+\xi^2)\{\rho\kappa^2(\sigma^2+\xi^2)^2+(\lambda+2\mu)\}F(\sigma,\xi)}$$

$$\frac{\partial J_n(\sigma r)}{\partial r}\left[2\mu\sigma^2e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right]^{1/2}z}+\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right]^{1/2}z}\right]$$

$$\cos n(\theta-\varphi)J_n(\sigma\zeta)\sigma\xi\zeta\,d\eta\,d\varphi\,d\zeta\,d\sigma\,d\xi,\qquad(66)$$

$$v=\frac{\kappa\alpha}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}nf(\zeta,\varphi,\eta)\sin n(\theta-\varphi)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}}\frac{J_n(\sigma r)}{r}$$

$$J_n(\sigma\zeta)\sin\xi z\sigma\xi\zeta\,d\eta\,d\varphi\,d\zeta\,d\sigma\,d\xi$$

$$+\frac{2\mu\alpha\kappa}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^t\frac{ne^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\left\{\dfrac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}\sin n(\theta-\varphi)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}F(\sigma,\xi)}$$

$$\left[2\mu\sigma^2e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right]^{1/2}z}+\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right]^{1/2}z}\right]$$

$$J_n(\sigma r)J_n(\sigma\zeta)\sigma\xi^2\zeta\,d\eta\,d\varphi\,d\zeta\,d\sigma\,d\xi,\qquad(67)$$

$$w=-\frac{\kappa\alpha}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\int_0^{t}\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\cos n(\theta-\varphi)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}}J_n(\sigma r)$$

$$J_n(\sigma\zeta)\cos\xi z\sigma\xi^2\zeta\, d\eta\, d\varphi\, d\zeta\, d\sigma\, d\xi$$

$$+\frac{2\mu\kappa\alpha}{\pi^2}\sum_{n=0}^{\infty}\delta_n\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\int_0^{t}\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}f(\zeta,\varphi,\eta)\xi\sigma^2\cos n(\theta-\varphi)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}\,F(\sigma,\xi)}$$

$$J_n(\sigma r)\left[2\mu\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right\}^{1/2}\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right]^{1/2}z}\right.$$

$$\left.-\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right]^{1/2}z}\right]J_n(\sigma\zeta)\sigma\xi\zeta\, d\eta\, d\varphi\, d\zeta\, d\sigma\, d\xi. \quad (68)$$

In the above expressions (66), (67) and (68), $F(\sigma,\xi)$ is written in place of

$$F(\sigma,\xi)=\left[4\mu^2\sigma^2\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right\}^{1/2}\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}\right.$$

$$\left.-\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}^2\right]. \quad \ldots.(69)$$

7. When the surface distribution of temperature has no azimuthal variation and is given by the form:

$$z=0;\quad T=\phi(r,t), \quad \ldots\ldots\ldots\ldots\ldots\ldots(70)$$

the expression of T satisfying (70), (61), (63) and (64) can be written as follows:

$$T=\frac{\kappa}{\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{t}e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)J_0(\sigma r)J_0(\sigma\zeta)\sin\xi z\sigma\xi\zeta\, d\eta\, d\zeta\, d\sigma\, d\xi, \quad (71)$$

and the expressions of the components of displacement in the solid are written in the forms:

$$u=-\frac{\kappa\alpha}{\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{t}\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)\sin\xi z}{\{\rho\kappa^2(\sigma^2+\xi^2)^2+(\lambda+2\mu)\}(\sigma^2+\xi^2)}\frac{\partial J_0(\sigma\zeta)}{\partial r}$$

$$J_0(\sigma\zeta)\sigma\xi\zeta\, d\eta\, d\zeta\, d\sigma\, d\xi$$

$$-\frac{2\mu\alpha\kappa}{\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{t}\frac{\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}\phi(\zeta,\eta)J_0(\sigma\zeta)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)^2+(\lambda+2\mu)\}F(\sigma,\xi)}\frac{\partial J_0(\sigma r)}{\partial r}$$

$$e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\left[2\mu\sigma^2 e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right]^{1/2}z}\right.$$

$$+\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right]^{1/2}z}\Big]\sigma\xi^2\zeta d\eta d\zeta d\sigma d\xi, \quad \ldots\ldots(72)$$

$$w=-\frac{\kappa\alpha}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)J_0(\sigma r)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}}$$

$$J_0(\sigma\zeta)\cos\zeta z\sigma\xi^2\zeta d\eta d\zeta d\sigma d\xi$$

$$+\frac{2\mu\kappa\alpha}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}\phi(\zeta,\eta)\xi\sigma^2J_0(\sigma r)J_0(\sigma\zeta)}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}F(\sigma,\xi)}$$

$$\left[2\mu\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right\}^{1/2}\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{(\lambda+2\mu)}+\sigma^2\right]^{1/2}z}\right.$$

$$-\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}e^{-\left[\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right]^{1/2}z}\Big]\sigma\xi\zeta d\eta d\zeta d\sigma d\xi, \quad \ldots\ldots(73)$$

where $F(\sigma,\xi)$ has the same meaning as that of (69) in Section 6.

The surface deformation due to the temperature distribution (71) may be written by the following equations:

$$u_{z=0}=-\frac{2\mu\alpha\kappa}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\frac{\left\{\frac{\rho\kappa^2(\sigma^2+\xi^2)^2}{\mu}+\sigma^2\right\}^{1/2}\phi(\zeta,\eta)e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}}{\{\sigma^2+\xi^2\}\{\rho\kappa^2(\sigma^2+\xi^2)^2+(\lambda+2\mu)\}F(\sigma,\xi)}$$

$$[\rho\kappa^2(\sigma^2+\xi^2)^2+4\mu\sigma^2]\frac{\partial J_0(\sigma r)}{\partial r}J_0(\sigma\zeta)\sigma\xi^2\zeta d\eta d\zeta d\sigma d\xi, \quad \ldots\ldots(74)$$

$$w_{z=0}=\frac{2\kappa^2\alpha\rho}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^t\frac{e^{-\kappa(\sigma^2+\xi^2)(t-\eta)}(\sigma^2+\xi^2)\{\rho\kappa^2(\sigma^2+\xi^2)^2+2\mu\sigma^2\}}{\{\rho\kappa^2(\sigma^2+\xi^2)+(\lambda+2\mu)\}F(\sigma,\xi)}\phi(\zeta,\eta)$$

$$J_0(\sigma r)J_0(\sigma\zeta)\sigma\xi^2\zeta d\zeta d\eta d\sigma d\xi. \quad \ldots\ldots(75)$$

8. Using the cylindrical coordinates, we have obtained the expressions of the displacement of a semi-infinite elastic solid due to the temperature variation which satisfies completely the equations of heat conduction and the initial and boundary conditions of the solid. We have studied the problem separating into two cases: one of them is the case where the inertia of solid is neglected, and the other the case where the inertia effect is taken into account. In both cases the resulting equations contain the term of density in spite of the fact that the inertia term is neglected in one case and not in the other.

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23. 溫度變化による半無限彈性體の變形式

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曩に「溫度分布が半無限彈性體の變形に及ぼす影響」と題して著者は彈性體の溫度分布による變形問題をフリーエーの二重積分公式を利用して色々な溫度分布に就て取扱つてみた事がある。本研究に於ては熱傳導論に基いて、半無限彈性體の變形問題を理論的に取扱つたのである。即ち彈性體中に起る溫度變化を全く傳導論的に取扱ひ、それによつて生ずる變形問題を彈性體の慣性を考へに入れない場合と、彈性體の慣性を入れた場合の二つに就き、變形式を求めてみた。式は可なり復雜であるが兩者を比較研究すると共に、全く時間的の變化を考へに入れない場合の彈性體の變形式との間に、色々面白い事項が含まれてゐる事がわかるのである。