28. *The Plastico-Elastic Deformation of a Semi-infinite Solid Body due to an Internal Force.*

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In this paper I am going to find mathematically the deformation of the ground in the neighbourhood of a volcano due to internal pressure at its nucleus, if exists, the material of the ground being assumed to be of plastic as well as of elastic nature.

1. We shall take first a two-dimensional problem, in which the axis of x resides on the horizontal surface of a simi-infinite ground and the axis of y is drawn vertically upwards. Let the centre of the pressure nucleus be at a depth ξ from the free surface and the radius of the nuclelus be a, the distribution of the pressure at the nucleus being assumed to be symmetrical with respect to its centre. The state of the

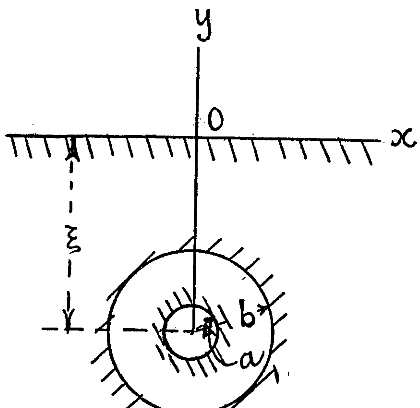

Fig. 1.

material in the very vicinity of the nucleus must be purely plastic, so that we have the equations of the equilibrium such that

$$\left.\begin{aligned} \frac{d}{dr}(r\sigma_r)-\sigma_t &= 0, \\ \sigma_r-\sigma_t &= 2k, \end{aligned}\right\} \quad [a<r<b] \quad \cdots\cdots(1)^{1)}$$

where $r^2 = x^2+(y+\xi)^2$ and k is the constant of the plastic state such that the maximum shearing stress takes this value in such a state. σ_r and σ_t are the radial and transverse components of the symmetrical stress in the region, $a < r < b$. In the region, $r > b$, the state of the material is assumed to be elastic owing to the fact that the maximum shearing stress does not reach the value k.

At the inner boundary, $r = a$, we may put

$$r = a; \qquad \sigma_r = -p. \quad \cdots\cdots(2)$$

From (1) and (2) we have

$$\left.\begin{aligned} \sigma_r &= -p+2k \log \frac{a}{r}, \\ \sigma_t &= -p-2k\left(1- \log \frac{a}{r}\right). \end{aligned}\right\} [a<r<b] \quad \cdots\cdots(3)$$

At the outer elastic region we have

$$\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right) = 0, \qquad [r>b] \cdots\cdots(4)$$

the solution of which is

$$U = \frac{B}{r}, \quad \cdots\cdots(5)$$

and therefore the radial and the transverse stresses in this region are

$$\sigma_r' = -\frac{2\mu B}{r^2}, \qquad \sigma_t' = \frac{2\mu B}{r^2}. \quad \cdots\cdots(6)$$

At the surface, $r = b$, we get

$$r = b; \qquad \sigma_r = \sigma_r', \qquad \sigma_r'-\sigma_t' = 2k. \quad \cdots\cdots(7)^{2)}$$

1) St. VENANT, *C. R.*, 70 (1870) and 73 (1871). These equations are not contradictory with laws in recent metallurgy and strength of materials; no other criterion than that involved in these equations can seem to explain the phenomena of slip bands and the stress-strain relation of the strained materials and also the behaviour of a material under hydrostatic external pressure.

2) In the case of special materials such as mild steel, in which the critical stress of the elastic region is different from the constant maximum shearing stress of the plastic region, we have to take, in place of (7), the forms:

$$r = b; \qquad \sigma_r = \sigma_r', \qquad \sigma_r'-\sigma_t' = 2\mu k,$$

where μ is the ratio of the critical stress of the shear of the elastic region to the maximum shearing stress of the plastic region.

From (3), (6), (7) we find

$$\left.\begin{aligned} B &= \frac{pb^2}{2\mu} + \frac{kb^2}{\mu} \log \frac{b}{a}, \\ \frac{b}{a} &= e^{\left(\frac{1}{4} + \frac{p}{2k}\right)}. \end{aligned}\right\} \qquad \text{(8)}$$

x- and y-components of the displacement U are expressed by

$$\left.\begin{aligned} u_0 &= U\frac{x}{r} = B\int_0^\infty e^{-k(y+\xi)} \sin kx\, dk, \\ v_0 &= U\frac{y+\xi}{r} = B\int_0^\infty e^{-k(y+\xi)} \cos kx\, dk. \end{aligned}\right\} [-\xi<y] \qquad \text{(9)}$$

We may now suppose at $y = \xi$ the image of the previous nucleus, all conditions being to be similar. Then the radial displacement U' and its x- and y-components (u', v') are expressed by

$$U' = \frac{B'}{r'}, \qquad [r'^2 = x^2 + (\xi - y)^2] \qquad \text{(10)}$$

$$\left.\begin{aligned} u' &= U'\frac{x}{r'} = B'\int_0^\infty e^{-k(\xi-y)} \sin kx\, dk, \\ v' &= -U'\frac{\xi-y}{r} = -B'\int_0^\infty e^{-k(\xi-y)} \cos kx\, dk. \end{aligned}\right\} [y<\xi] \qquad \text{(11)}$$

The effect of the image on the conditions in the neighbourhood of the nucleus is not so great on account of its large distance from the image.

Add (11) to (9), and consider the tangential and vertical stresses on $y = 0$. Then we have

$$\frac{\partial(v_0+v')}{\partial x} + \frac{\partial(u_0+u')}{\partial y} = 0, \qquad \text{(12)}$$

provided

$$B = B'; \qquad \text{(13)}$$

and hence we get

$$\lambda\Delta + 2\mu\frac{\partial(v_0+v')}{\partial y} = -4\mu B\int_0^\infty e^{-k\xi} \cos kx\, k\, dk. \qquad \text{(14)}$$

To annul the stress in (14), we shall introduce the following displacement which is mainly accumulated in the neighbourhood of the free surface.

$$\left.\begin{aligned} \Delta_1 &= Ce^{ky}\cos kx, \\ u_1 &= C\left(\frac{\lambda+\mu}{2\mu}y+\alpha\right)e^{ky}\sin kx, \\ v_1 &= -C\left(\frac{\lambda+\mu}{2\mu}y+\beta\right)e^{ky}\cos kx, \end{aligned}\right\} [y<0] \quad \cdots\cdots(15)$$

where

$$k(\alpha-\beta) = \frac{\lambda+3\mu}{2\mu}. \quad \cdots\cdots(16)$$

This displacement makes

$$\frac{\partial v_1}{dx}+\frac{\partial u_1}{\partial y} = 0. \quad \cdots\cdots(17)$$

When the vertical component of the stress formed by this displacement is added to the expression (14), and the sum:

$$\lambda\Delta_1+2\mu\frac{\partial v_1}{\partial y}+\lambda\Delta+2\mu\frac{\partial(v_0+v')}{dy} \quad \cdots\cdots(18)$$

is made zero, we find

$$\alpha = \frac{1}{2k}, \quad \beta = -\frac{\lambda+2\mu}{2\mu k}, \quad C = \frac{4\mu B}{(\lambda+\mu)}e^{-k\xi}k\,dk. \quad \cdots\cdots(19)$$

Substituting these values in (15) and evaluating the total displacement $(u_0+u'+u_1,\ v_0+v'+v_1)$, we obtain

$$\left.\begin{aligned} u = u_0+u'+u_1 &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)}\int_0^\infty e^{-k\xi}\sin kx\,dk \\ &= 2B\frac{(\lambda+2\mu)}{(\lambda+\mu)}\frac{x}{(\xi^2+x^2)}, \\ v = v_0+v'+v_1 &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)}\int_0^\infty e^{-k\xi}\cos kx\,dk \\ &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)}\frac{\xi}{(\xi^2+x^2)} \end{aligned}\right\} \quad \cdots\cdots(20)$$

on $y = 0$, where the constant B, and hence the boundary, $r = b$, can be found from the expression (8).

2. Secondly, we shall consider a three-dimensional problem, in which r is taken in lieu of x and z in lieu of y, both of the foregoing case. As we are to study the case of a symmetrical nucleus, we can easily construct the equations of equilibrium of the plastic region, such that

$$\left.\begin{aligned}\frac{\partial\sigma_R}{\partial R}+\frac{2(\sigma_R-\sigma_t)}{R}&=0,\\ \sigma_R-\sigma_t&=2k,\end{aligned}\right\}\quad [a<R<b] \qquad \cdots\cdots(21)$$

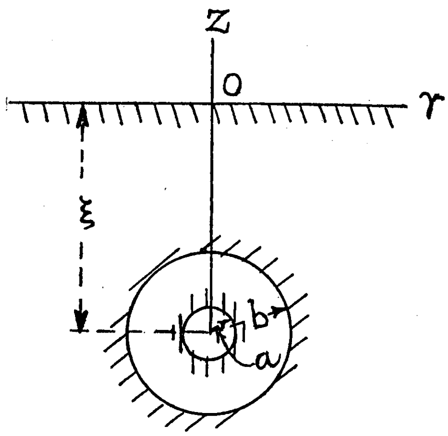

Fig. 2.

where $R^2=r^2+(z+\xi)^2$ and σ and σ_t are the radial and the transverse components of the stress. From the conditions that

$$R=a; \qquad \sigma_R=-p, \qquad \cdots\cdots(22)$$

we get

$$\left.\begin{aligned}\sigma_R&=-p+2k\log\frac{a}{R},\\ \sigma_t&=-p-2k\left(1-\log\frac{a}{R}\right).\end{aligned}\right\} \qquad \cdots\cdots(23)$$

At the elastic region we have

$$\frac{\partial}{\partial R}\left(\frac{\partial U}{\partial R}+\frac{2U}{R}\right)=0, \qquad [R>b] \qquad \cdots\cdots(24)$$

the solution of which is

$$U=\frac{B}{R^2}, \qquad \cdots\cdots(25)$$

so that we obtain

$$\sigma_R' = -\frac{4\mu B}{R^3}, \qquad \sigma_t' = \frac{2\mu B}{R^3} \quad \cdots\cdots(26)$$

At the surface $R = b$ we get

$$R = b\,; \quad \sigma_R = \sigma_R', \qquad \sigma_R' - \sigma_t' = 2k. \quad \cdots\cdots(27)^{3)}$$

From (23), (26), (27), we find

$$\left.\begin{aligned} B &= \frac{pb^3}{4\mu} + \frac{kb^3}{2\mu}\log\frac{b}{a}, \\ \frac{b}{a} &= e^{\left(\frac{1}{6}+\frac{p}{4k}\right)}. \end{aligned}\right\} \quad \cdots\cdots(28)$$

r- and z-components of the displacements U are expressed by

$$\left.\begin{aligned} u_0 &= U\frac{r}{R} = B\int_0^\infty e^{-k(z+\xi)}J_1(br)k\,dk, \\ v_0 &= U\frac{y+\xi}{R} = B\int_0^\infty e^{-k(z+\xi)}J_0(kr)k\,dk. \end{aligned}\right\} \quad [\xi<z] \quad \cdots\cdots(29)$$

Suppose the image at $z = \xi$, then the radial displacement U' and its r- and z-components (u', w') due to the image are written by

$$U' = \frac{B'}{R'^2}, \qquad [R'^2 = r^2+(\xi-z)^2] \quad \cdots\cdots(30)$$

$$\left.\begin{aligned} u' &= U'\frac{r}{R'} = B'\int_0^\infty e^{-k(\xi-z)}J_1(kr)k\,dk \\ v' &= -U'\frac{\xi-z}{R'} = -B'\int_0^\infty e^{-k(\xi-z)}J_0(kr)k\,dk. \end{aligned}\right\} \quad [z<\xi] \quad \cdots\cdots(31)$$

Add (31) to (29) and take the tangential and the vertical stresses on $z = 0$. Then, we get

$$\frac{\partial(w_0+w')}{\partial r} + \frac{\partial(u_0+u')}{dz} = 0, \quad \cdots\cdots(32)$$

3) For special materials as already cited concerning the equation (7), we have to put the following forms in place of (27),

$$R = b; \qquad \sigma_R = \sigma_R', \qquad \sigma'_R - \sigma_t' = 2\mu k,$$

where μ has the same meaning as that of the preceding case.

provided

$$B = B' \; ; \quad \cdots\cdots(33)$$

and hence we obtain

$$\lambda \Delta + 2\mu \frac{\partial (w_0 + w')}{\partial z} = -4\mu B \int_0^\infty e^{-k\xi} J_0(kr) k^2 dk. \quad \cdots\cdots(34)$$

We shall again consider the displacement which is accumulated in the neighbourhood of the surface, $z = 0$, such that

$$\left.\begin{aligned} \Delta_1 &= C e^{kz} J_0(kr) \\ u_1 &= C\left(\frac{\lambda+\mu}{2\mu} z + \alpha\right) e^{kz} J_1(kr), \\ w_1 &= -C\left(\frac{\lambda+\mu}{2\mu} z + \beta\right) e^{kz} J_0(kr), \end{aligned}\right\} [z<0] \quad \cdots\cdots(35)$$

where

$$k(\alpha - \beta) = \frac{\lambda + 3\mu}{2\mu}. \quad \cdots\cdots(36)$$

This displacement naturally satisfies the condition:

$$\frac{\partial w_1}{\partial r} + \frac{\partial u_1}{\partial z} = 0, \quad \cdots\cdots(37)$$

and, when the vertical component of the stress is added to (34), and the sum is made zero, such that

$$\lambda \Delta_1 + 2\mu \frac{\partial w_1}{\partial z} + \lambda \Delta + 2\mu \frac{\partial (w_0 + w_1)}{\partial z} = 0, \quad \cdots\cdots(38)$$

we get

$$\alpha = \frac{1}{2k}, \quad \beta = -\frac{(\lambda + 2\mu)}{2\mu k}, \quad C = \frac{4\mu B e^{-k\xi}}{(\lambda+\mu)} k^2 dk. \quad \cdots\cdots(39)$$

We find finally the components of the displacement on $z = 0$ in the forms:

$$\left.\begin{aligned} u = u_0 + u' + u_1 &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)} \int_0^\infty e^{-k\xi} J_1(kr) k\, dk \\ &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)} \frac{r}{(\xi^2 + r^2)^{\frac{3}{2}}}, \qquad (a) \\ w = w_0 + w' + w_1 &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)} \int_0^\infty e^{-k\xi} J_0(kr) k\, dk \\ &= \frac{2B(\lambda+2\mu)}{(\lambda+\mu)} \frac{\xi}{(\xi^2 + r^2)^{\frac{3}{2}}}, \qquad (b) \end{aligned}\right\} \quad \cdots\cdots(40)$$

where the constant B and hence the boundary $r = b$ can be determined from (28).

3. From the preceding calculation it may be easily concluded that

i) When the plastic region is mainly localised in the neighbourhood of the pressure nucleus, the general character of the distribution of the displacements on the free surface is not altered due to the size of the (spherical) nucleus or of the (spherical) plastic region; but the absolute value of the surface displacement changes considerably due to the size of the above elements and also due to the internal pressure at the nucleus.

ii) When the radius of the pressure nucleus is a, its pressure is p and the plastic constant is k, the boundary separating the solid into plastic and elastic regions is denoted by

$$b = a\, e^{\left(\frac{1}{4} + \frac{p}{2k}\right)}$$

in a two-dimensional case; and

$$b = a\, e^{\left(\frac{1}{6} + \frac{p}{4k}\right)}$$

in a three-dimensional case.

iii) In the two-dimensional case the vertical component of the displacement on a point on the free surface is proportional to the inverse square of the distance of that point from the centre of the nucleus, while in the three-dimensional case this component is inversely proportional to the cube of the distance.

To ascertain the validity of the present result for the practical purpose, I have introduced the result of the levelling survey which was carried out by Dr. C. TSUBOI[4] at the very vicinity of Volcano Komagatake at the epoch shortly after its eruption. The observed result concerning the movement being substituted in the formula (40b), I have found that the centre of the nucleus resides at 0～3 km vertically downwards from the summit of the volcano. Owing to the seismometrical observation made by Dr. F. KISHINOUYE[5] at Komagatake at the same epoch, the focal depths of the frequent seismic shocks seem to be very near to the depths of the nucleus which I have calculated here, so that, I think, my

4) C. TSUBOI, Part V of the paper " The Eruption of Komagatake, Hokkaidô, in 1929," *Bull. Earthq. Res. Inst.*, **8** (1930), 298–300.

5) F. KISHINOUYE, Part III of the same paper, 274–289.

mathematical result is not without value to the geophysical problem of the volcanoes.

It may be added that the equilibrium of the boundary separating the plastic and the elastic regions is in general of a somewhat complex nature, especially in some cases of metals; but in the case of the rocks the problem is very simple and the present analysis will be sufficient at least for the purpose of the geophysical investigation. For the discussion of the separation boundary of the general substances the result due to ROBERTSON and COOK[6] will be of some interest.

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28. 內部に力核を有する半無限プラスチコ彈性體の變形

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火山に內核(若しありとすれば)があつて、其爲に半無限固體である地殼が如何なる變形をなすべきかといふ事を計算して見たのである。固體はプラスチック性と彈性とを有するものと假定してあるのであり、從てこの樣な靜力學的の問題に可なりあてはまると思ふ。

內核の附近は應力が大である爲に最大剪斷應力がプラスチックの常數 k に達する。其等の所をプラスチックの領域と考へ、それから外の部分は彈性の領域としてよい譯である。此等の各々に平衡の微分方程式を應用し、又內核の壓力の加はる所の條件や、上述の兩領域の接ぎ目や半無限體表面の條件等を滿足する樣な解を求めたのである。

計算の結果の重要な事項を擧げて見ると、i) プラスチックの領域が餘り擴つて居らぬ時に、固體表面の變位の分布の割合は、內核の大さ、壓力及びプラスチック彈性兩領域境界面の大さなどに無關係となる。ii) 內核の半徑を a とし、其壓力を p、プラスチックの常數を k と書く時、プラスチック、彈性兩領域境界面の半徑 b は次式で表はされる。

$$b = a\,e^{\left(\frac{1}{4}+\frac{p}{2k}\right)} \qquad (\text{二次元})$$

$$b = a\,e^{\left(\frac{1}{6}+\frac{p}{4k}\right)} \qquad (\text{三次元})$$

iii) 固體表面上の點の上下の變位は二次元の時には其點から核心までの距離の二乘に反比例し、三次元の時にはその距離の三乘に反比例する。

この計算結果の適合性を確める爲に、駒ケ岳火山爆發後坪井忠二所員が行つた水準測量結果から火山の核心の深さを求めて見た。それが 0〜3 粁と出るのである。此を岸上所員が大體同じ頃に同火山の地震の源點を地震計的に觀測せる結果と比較して見ると可なり近い値を取る樣に思はれるのである。

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6) A. ROBERTSON and G. COOK, "The Transition from the Elastic to the Plastic State in Mild Steel," *Proc. Roy. Soc.*, London, 88 (1913), 462.