21. *A Kind of Waves transmitted over a Semi-infinite Solid Body of Varying Elasticity.*

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In this paper I am going to study the propagation of surface waves over a semi-infinite solid body whose elastic constants are functions of the position of a material point of the body below its free surface. There are, of course, several kinds of waves according with the forms of the functions of varying elasticities and also with the types of the propagated waves; yet we now confine ourselves to the case such that

$$\Delta = 0, \qquad w = 0; \quad \cdots\cdots\cdots\cdots (1)$$

$$\mu = c(d+z_1), \quad \cdots\cdots\cdots\cdots (2)$$

where Δ is the dilation, w the vertical component of the displacement, μ the modulus of rigidity, and z_1 the depth of a material point below the free surface, c and d being certain constants. Although a similar kind of waves has already been dealt with by MEISSNER[1] and AICHI,[2] yet their cases belong to a one-dimensional problem and the methods of integration of differential equation are quite different from that of the present paper.

The differential equation of motion of a semi-infinite solid body, when the conditions in (1) are to be satisfied, are expressed by

$$\rho\frac{\partial^2 u}{\partial t^2} = -\frac{2\mu}{r}\frac{\partial \varpi_z}{\partial \theta} + 2\frac{\partial(\mu\varpi_\theta)}{\partial z}, \quad \cdots\cdots\cdots\cdots (3)$$

$$\rho\frac{\partial^2 v}{\partial t^2} = -2\frac{\partial(\mu\varpi_r)}{\partial z} + 2\mu\frac{\partial \varpi_z}{\partial r}, \quad \cdots\cdots\cdots\cdots (3)$$

1) E. MEISSNER, "Elastische Oberflächen-Querwellen," *Verh. 2. int. Kongr. f. tech. Mech.*, (Zürich, 1926), 3–11; and also *Vierteljahr. Nat. Forsch. Ges.*, Zürich, **67** (1921), 181.

2) K. AICHI, "On the Transversal Seismic Waves travelling upon the Surface of Heterogeneous Material," *Proc. Phys.-Math. Soc.*, Japan, [3], **4** (1922), 137–142.

provided

$$2\varpi_r = -\frac{\partial v}{\partial z},\ 2\varpi_\theta = \frac{\partial u}{\partial z},\ 2\varpi_z = \frac{1}{r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right), \quad (4)$$

in which ρ is the density of the solid and u, v are the radial and azimuthal components of the displacement. From the condition of the problem we have the identical relation such that

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0. \quad \cdots\cdots(5)$$

Eliminating u, v between (3) and (4), we obtain

$$\rho\frac{\partial^2 \varpi_z}{\partial t^2} = \mu\left(\frac{\partial^2 \varpi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varpi_z}{\partial \theta^2}\right) + \frac{\partial}{\partial z}\mu\frac{\partial \varpi_z}{\partial z}, \quad \cdots(7)$$

$$\rho\frac{\partial^2 \varpi_r}{\partial t^2} = \frac{\partial^2}{\partial z^2}\left(\mu\, \varpi_r\right) - \frac{\partial}{\partial z}\left(\mu\frac{\partial \varpi_z}{\partial r}\right), \quad \cdots\cdots(8)$$

$$\rho\frac{\partial^2 \varpi_\theta}{\partial t^2} = \frac{\partial^2}{\partial z^2}\left(\mu\, \varpi_\theta\right) - \frac{1}{r}\frac{\partial}{\partial z}\left(\mu\frac{\partial \varpi_z}{\partial \theta}\right). \quad \cdots\cdots(9)$$

These differential equations are satisfied by

$$\varpi_z = B_m \Phi(z) H_m^{(2)}(kr) \left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\} m\theta e^{ipt}, \quad \cdots\cdots(10)$$

$$\varpi_r = \frac{B_m}{k^2}\frac{\partial \Phi(z)}{\partial z}\frac{\partial H_m^{(2)}(kr)}{\partial r} \left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\} m\theta e^{ipt}, \quad \cdots\cdots(11)$$

$$\varpi_\theta = \frac{B_m m}{k^2}\frac{\partial \Phi(z)}{\partial z}\frac{H_m^{(2)}(kr)}{r} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta e^{ipt}, \quad \cdots\cdots(12)$$

where p is frequency of waves, $2\pi/k$ wave length, B_m a constant and $\Phi(z)$ is a certain function which satisfies the differential equation

$$\frac{d^2\Phi(z)}{dz^2} + \frac{1}{\mu}\frac{d\mu}{dz}\frac{d\Phi(z)}{dz} + \left(\frac{\rho p^2}{\mu} - k^2\right)\Phi(z) = 0. \quad \cdots\cdots(13)$$

The components of the displacement $(u,\ v)$ are found from (4) in the following forms:

$$u = 2B_m \frac{m}{k^2} \Phi(z) \frac{H_m^{(2)}(kr)}{r} \left.\begin{matrix} \cos \\ \sin \end{matrix}\right\} m\theta e^{ipt}, \cdots\cdots\cdots\cdots (14)$$

$$v = -2B_m \frac{1}{k^2} \Phi(z) \frac{\partial H_m^{(2)}(kr)}{\partial r} \left.\begin{matrix} \sin \\ -\cos \end{matrix}\right\} m\theta e^{ipt}, \cdots\cdots\cdots (15)$$

When z is taken vertically downwards and $z = d$ is assumed to be the free surface, the boundary conditions at that surface are written by

$$z = d: \begin{cases} \lambda\Delta + 2\mu\dfrac{\partial w}{\partial z} = 0, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (16) \\ \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} + \dfrac{\partial v}{\partial z} = 0, \cdots\cdots\cdots\cdots\cdots\cdots (17) \\ \dfrac{\partial w}{\partial r} + \dfrac{\partial u}{\partial z} = 0. \cdots\cdots\cdots\cdots\cdots\cdots\cdots (18) \end{cases}$$

From the nature of the problem, (16) is identically satisfied, and (17) and (18) are simply written by

$$z = d: \quad \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} = 0; \cdots\cdots\cdots\cdots (19)$$

and therefore

$$\left[\frac{d\Phi(z)}{dz}\right]_{z=d} = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)$$

is the only condition of the boundary.

When the distribution of the elastic constants is defined by (2), i.e.

$$\mu = cz \qquad (z = d + z_1), \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2')$$

the equation (13) can be transformed to

$$z\frac{d^2\Phi(z)}{dz^2} + \frac{d\Phi(z)}{dz} + \left(\frac{\rho p^2}{c} - k^2 z\right)\Phi(z) = 0. \cdots\cdots\cdots\cdots (21)$$

Putting

$$\Phi(z) = e^{kz}\,\Phi_1(z), \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (22)$$

we get

$$z\frac{\Phi_1^2(z)}{dz^2} + (1+2kz)\frac{d\Phi_1(z)}{dz} + k\left(1 + \frac{\rho p^2}{kc}\right)\Phi_1(z) = 0. \cdots\cdots (23)$$

Writing

$$1+\frac{\rho p^2}{kc} = 2\alpha, \quad \cdots\cdots(24)$$

the appropriate solution of (23) is expressed by

$$\Phi_1(z) = \left\{1-\frac{\alpha}{1.1}(2kz)+\frac{\alpha(\alpha+1)}{1.2.1.2}(2kz)^2-\cdots\cdots\right\}$$
$$= {}_1F_1(\alpha\,;\ 1\,;-2kz). \quad \cdots\cdots(25)^{3)}$$

Substituting (22) and (25) in (20), we have

$$\frac{d}{dz}\Big[e^{kz}\,{}_1F_1(\alpha\,;\ 1\,;-2kz)\Big]_{z=d} = 0, \quad \cdots\cdots(26)$$

which is to be written by

$$\frac{d\,{}_1F_1(\alpha\,;\ 1\,;-2kz)}{dz}+k\,{}_1F_1(\alpha\,;\ 1\,;-2kz) = 0 \quad \cdots\cdots(27)$$

at $z = d$.

When kd is very small, the solution of (27) is approximately expressed by

$$2\alpha = 1. \quad \cdots\cdots(28)$$

The second approximation gives us

$$2\alpha = 1+\frac{1}{2}(kd). \quad \cdots\cdots(29)$$

The third approximation shows that

$$2\alpha = 1+\frac{1}{2}(kd)+0. \quad \cdots\cdots(30)$$

The fourth approximation gives us

$$2\alpha = 1+\frac{1}{2}(kd)-\frac{1}{48}(kd).^3 \quad \cdots\cdots(31)$$

3) E. W. BARNES, *Camb. Trans.*, 20, 253: H. LAMB, *Proc. Roy. Soc.*, London, 86 (1911), 551.

From (24) and (31), we obtain

$$\left(\frac{p}{k}\right)^2 = (2\alpha-1)\frac{1}{k}\frac{c}{\rho}$$
$$= \frac{1}{2}\left\{1-\frac{1}{24}(kd)^2\right\}\frac{cd}{\rho}. \quad\cdots\cdots\cdots\cdots\cdots(32)$$

If we write the modulus of rigidity at the free surface by $\mu_0\,(=cd)$, we have

$$\text{Velocity of propagation} = \sqrt{\frac{1}{2}\left\{1-\frac{1}{24}(kd)^2\right\}}\sqrt{\frac{\mu_0}{\rho}}, \quad\cdots\cdots\cdots(33)$$

the calculated results being shown in the table below:

kd	0	0·1	0·5	1·0	1·5
Velocity $\Big/\sqrt{\frac{\mu_0}{\rho}}$	0·713	0·713	0·708	0·696	0·679

The components of the displacement are written by

$$\left.\begin{aligned} u &= 2B_m\frac{m}{k^2}e^{kz}{}_1F_1(\alpha\,;\;1\,;-2kz)\frac{H_m^{(2)}(kr)}{r}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\theta e^{ipt}, \\ v &= -2B_m\frac{1}{k^2}e^{kz}{}_1F_1(\alpha\,:\;1\,;-2kz)\frac{\partial H_m^{(2)}(kr)}{\partial r}\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix}m\theta e^{ipt}. \end{aligned}\right\} \cdots(34)$$

In some cases, we have

$$e^{kz}{}_1F_1(\alpha\,;\;1\,;-2kz)\rightarrow 0 \qquad (35)$$

as $z\rightarrow\infty$, on account of the reason that, in such cases, the value of ${}_1F_1(\alpha;\,1;-2kz)$ decays more quickly than that of e^{-kz} as z is increased continuously. Thus the waves written by (34) may have the characteristic of surface waves.

From the expression of (35) it may be noticed that the radial component of the displacement of the waves of the present type quickly disappears as the waves proceed outwards from the epicentre, while the transverse component is transmitted in the manner of the ordinary Love-waves, the azimuthal distribution of the azimuthal displacement being maintained for all radius. It may no doubt be significant on seimology that this kind of waves, in which the vertical displacement and the dilatation do not exist, and the velocity of propagation is different for different wave length, is possible to be transmitted over the free surface whose elastic constant increases continuously along the depth of the body.

21. 彈性が深さに從て變る半無限固體上に傳はる一地震波

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半無限固體の彈性が其表面からの深さに從て變化する場合に其ダイレテーションと上下の變位とが零になる様な場合に一種の地震波の存在し得る事を確めた。多少似た場合の研究はマイスナーや愛知博士などによつても行はれて居るけれども、其等は一次元的の問題であり、且つ一般の微分方程式の積分の方法などに於ても可なり違つて居る。さて地震波の性質をよくしらべて見ると波動の進行方向の變位は震央からの距動が遠くなるに從つて急に少くなるけれども、其方向に直角な向の水平變位はそれ程減少せず、普通の二次元傳播の法則に從つて居る事がわかる。又其様な横變位は震央からの距離に無關係に其割合が保持される事が示されて居る。波動の速度は波長が長くなる程多少は速くなるとはいへ、それ程大きくはならぬ樣である。一體ラブ波の如く違つた物質から成る表面層を特別に置かなくても物質が連續的に變化しさへすればラブ波に似た波の存在し得る事は驗震學上無意味の事とは思はれぬ次第である。