10. *On the Transmission of Seismic Waves on the Bottom Surface of an Ocean.*

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I. Introduction.

The object of this paper is to find the effect of an ocean water upon the two-dimensional transmission of Rayleigh-type waves on the bottom surface of that ocean, and especially to know the nature of the dispersion of an arbitrary wave progressing outwards from an epicentre. Even though Bromwich[1] and Stoneley[2] treated of similar problems, yet their cases are relatively particular. Bromwich studied so long ago as 1898 the influence of gravity of water on a one-dimensional propagation of elastic waves of harmonic type of an infinite extent, while Stoneley gave recently a like problem in which the compressibility of water is taken into account. As these authors appeared to aim at the formulation of the velocity equation of the waves, their treatments did not involve any result as to the behaviour of arbitrary waves, or as to the transmission of waves on a two-dimensional surface of the bottom of the ocean, or even as to the propagation of annular waves on the same surface; all of these have hitherto been left unknown. The investigation of these unknown cases will give us some clue to the dispersive phenomena of the seismic waves at the bottom surface of an ocean, rather than to enable us to understand the nature of the movement of the earth's solid due to a regular train of harmonic waves. There are moreover several data[3] of seismic observations concerning surface waves transmitted through a long distance along the bottom of the ocean; and, by comparing these data with results of the mathematical analysis, a good deal of obscurity which attaches to the distant earthquakes may perhaps be cleared.

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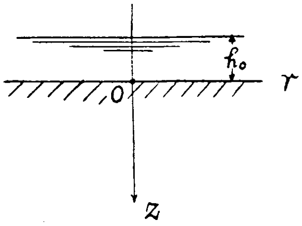
1) T. J. I'A. BROMWICH, *Proc. Math. Soc.*, London, 30 (1898), 98–120.
2) R. STONELEY, *M.N.R.A.S. Geophys. Suppl.*, 1 (1926), 349–356.
3) B. GUTENBERG, *Handb. d. Geophys.*, 4 (1930), 254–255.

II. Motion of a Semi-infinite Elastic Body.



Let the axis of z is directed downwards from the surface of the body, the senses of r, θ being horizontal. Let u, v, w be the components of the displacement in radial, azimuthal and vertical directions, and ρ, λ, μ, the density and Lamé's elastic constants. The equations of motion of the body in cylindrical co-ordinates are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \varpi_z}{\partial \theta} + 2\mu\frac{\partial \varpi_\theta}{\partial z}, \\ \rho\frac{\partial^2 v}{\partial t^2} &= (\lambda+2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial \theta} - 2\mu\frac{\partial \varpi_r}{\partial z} + 2\mu\frac{\partial \varpi_z}{\partial r}, \\ \rho\frac{\partial^2 w}{\partial t^2} &= (\lambda+2\mu)\frac{\partial \Delta}{\partial z} - 2\mu\frac{1}{r}\frac{\partial}{\partial r}(r\varpi_\theta) + 2\mu\frac{1}{r}\frac{\partial \varpi_r}{\partial \theta} + \rho g, \end{aligned}\right\} \qquad \cdots\cdots\cdots(1)$$

where

$$\left.\begin{aligned} \Delta &= \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}, & 2\varpi_r &= \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \\ 2\varpi_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, & 2\varpi_z &= \frac{1}{r}\left(\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right). \end{aligned}\right\} \qquad \cdots\cdots(2)$$

If we eliminate u, v, w in (1) by means of (2), we get

$$\left.\begin{aligned} \rho\frac{\partial^2 \Delta}{\partial t^2} &= (\lambda+2\mu)\left[\frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r}\frac{\partial \Delta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Delta}{\partial \theta^2} + \frac{\partial^2 \Delta}{\partial z^2}\right], \\ \rho\frac{\partial^2 \varpi_r}{\partial t^2} &= \mu\left[\frac{\partial^2 \varpi_r}{\partial r^2} + \frac{3}{r}\frac{\partial \varpi_r}{\partial r} + \frac{\varpi_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 \varpi_r}{\partial \theta^2} + \frac{\partial^2 \varpi_r}{\partial z^2} + \frac{2}{r}\frac{\partial \varpi_z}{\partial z}\right], \\ \rho\frac{\partial^2 \varpi_\theta}{\partial t^2} &= \mu\left[\frac{\partial^2 \varpi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi_\theta}{\partial r} - \frac{\varpi_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 \varpi_\theta}{\partial \theta^2} + \frac{\partial^2 \varpi_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial \varpi_r}{\partial \theta}\right], \\ \rho\frac{\partial^2 \varpi_z}{\partial t^2} &= \mu\left[\frac{\partial^2 \varpi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varpi_z}{\partial \theta^2} + \frac{\partial^2 \varpi_z}{\partial z^2}\right]. \end{aligned}\right\} \qquad (3)$$

As it is difficult to solve the original equations in (1), we shall investigate the equations in (3). From the first and the last equations in (3), we obtain the solutions of the forms:

$$\left.\begin{aligned} \varDelta &= A_m H_m^{(2)}(kr)\, e^{-\alpha z+int} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta, \\ 2\varpi_z &= B_m H_m^{(2)}(kr)\, e^{-\beta z+int} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\theta, \end{aligned}\right\} \qquad \cdots\cdots(4)$$

where

$$k^2 = \alpha^2+h^2 = \beta^2+j^2,\quad h^2 = \frac{\rho n^2}{\lambda+2\mu},\quad j^2 = \frac{\rho n^2}{\mu}.$$

Substituting the solution in the second of (4) in the second of the equations (3), and finding the particular and the complementary solutions, we get

$$2\varpi_r = \left\{C_m \frac{H_m^{(2)}(kr)}{r} - \frac{\beta}{k^2} B_m \frac{\partial H_m^{(2)}(kr)}{\partial r}\right\} e^{-\beta z+int} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\theta. \qquad \cdots\cdots(5)$$

Substituting this in the third of (3), and, proceeding as before, we find

$$2\varpi_\theta = \left\{\frac{C_m}{m} \frac{\partial H_m^{(2)}(kr)}{\partial r} - \frac{\beta_m}{k^2} B_m \frac{H_m^{(2)}(kr)}{r}\right\} e^{-\beta z+int} \begin{matrix}\sin\\ \cos\end{matrix}\Big\} m\theta. \qquad \cdots(6)$$

Displacement answering to $\varDelta$ in (4) and satisfying $\varpi_r = \varpi_\theta = \varpi_z = 0$ in (2), is expressed by

$$\left.\begin{aligned} u_1 &= -A_m \frac{1}{h^2} \frac{\partial H_m^{(2)}(kr)}{\partial r} e^{-\alpha z+int} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta, \\ v_1 &= A_m \frac{m}{h^2} \frac{H_m^{(2)}(kr)}{r} e^{-\alpha z+int} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\theta, \\ w_1 &= A_m \frac{\alpha}{h^2} H_m^{(2)}(kr)\, e^{-\alpha z+int} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta. \end{aligned}\right\} \qquad \cdots\cdots(7)$$

Displacement answering to ϖ_z in (4) besides the second terms of the right-hand sides of ϖ_r and ϖ_θ, both given in (5) and (6), under the condition that $\varDelta = 0$ in (2), is expressed by

$$\left.\begin{aligned} u_2 &= B_m \frac{m}{k^2} \frac{H_m^{(2)}(kr)}{r} e^{-\beta z+int} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta, \\ v_2 &= -B_m \frac{1}{k^2} \frac{\partial H_m^{(2)}(kr)}{\partial r} e^{-\beta z+int} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\theta, \\ w_2 &= 0. \end{aligned}\right\} \qquad \cdots\cdots(8)$$

Displacement derived from the values of the first terms of the right-hand sides of the expressions of ϖ_r and ϖ_θ, both given in (5) and (6), and fulfilling the conditions $\varDelta = \varpi_z = 0$ in (2), is written by

$$\left.\begin{aligned} u_3 &= C_m \frac{\beta}{mj^2} \frac{\partial H_m^{(2)}(kr)}{\partial r} e^{-\beta z+int} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta, \\ v_3 &= -C_m \frac{\beta}{j^2} \frac{H_m^{(2)}(kr)}{r} e^{-\beta z+int} \begin{Bmatrix}\sin\\ -\cos\end{Bmatrix} m\theta, \\ w_3 &= -C_m \frac{k^2}{mj^2} H_m^{(2)}(kr)\, e^{-\beta z+int} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta. \end{aligned}\right\} \qquad \cdots\cdots(9)$$

Of these three components of the displacement, namely (u_1, v_1, w_1), (u_2, v_2, w_2) and (u_3, v_3, w_3), the first and the third are available to obtain Rayleigh-type waves, and the second is useful to get Love-type waves. In the present case the former set of combination will be considered. The latter set will perhaps be employed in some future occasion.

III. Motion of the Superficial Water.

Let u', v', w' be the components of the displacement of the superficial water in r, θ, z directions, ρ_0' the density of water in an undisturbed state, p_1 the charge of the pressure in a disturbed state, and μ' the viscosity of the water. The equations of motion of the water in Eulerean system are then expressed by

$$\left.\begin{aligned} \rho_0' \frac{\partial^2 u'}{\partial t^2} &= -\frac{\partial p_1}{\partial r} + \mu' \frac{\partial}{\partial t}\left\{ \frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\frac{\partial u'}{\partial r} \right. \\ &\qquad \left. -\frac{u'}{r^2} + \frac{1}{r^2}\frac{\partial^2 u'}{\partial \theta^2} + \frac{\partial^2 u'}{\partial z^2} - \frac{2}{r^2}\frac{\partial v'}{\partial \theta} \right\}, \\ \rho_0' \frac{\partial^2 v'}{\partial t^2} &= -\frac{1}{r}\frac{\partial p_1}{\partial \theta} + \mu' \frac{\partial}{\partial t}\left\{ \frac{\partial^2 v'}{\partial r^2} + \frac{1}{r}\frac{\partial v'}{\partial r} \right. \\ &\qquad \left. -\frac{v'}{r^2} + \frac{1}{r^2}\frac{\partial^2 v'}{\partial \theta^2} + \frac{\partial^2 v'}{\partial z^2} + \frac{2}{r}\frac{\partial u'}{\partial \theta} \right\}, \\ \rho_0' \frac{\partial^2 w'}{\partial t^2} &= -\frac{\partial p_1}{\partial z} + \mu' \frac{\partial}{\partial t}\left\{ \frac{\partial^2 w'}{\partial r^2} + \frac{1}{r}\frac{\partial^2 w'}{\partial r} \right. \\ &\qquad \left. + \frac{1}{r^2}\frac{\partial^2 w'}{\partial \theta^2} + \frac{\partial^2 w'}{\partial z^2} \right\} + \rho' g. \end{aligned}\right\} \qquad \cdots(10)$$

The actual density ρ' and the similar pressure p are denoted by

$$\rho' = \rho_0' + \rho_1', \qquad p = p_0 + p_1, \qquad \cdots\cdots(11)$$

where ρ_1' is the change of the density due to the disturbance and p_0 the statical pressure. The equation of continuity of a compressible fluid is written by

$$\frac{D\rho'}{Dt}+\rho_0'\frac{\partial}{\partial t}\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{\partial w'}{\partial z}\right\}=0, \quad\cdots\cdots\cdots\cdots(12)$$

where D/Dt denotes a differentiation following the motion of the fluid. Again, the change of pressure and that of density are related in a form:

$$\frac{Dp}{Dt}=c^2\frac{D\rho'}{Dt}, \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)$$

in which $c^2=\frac{\varkappa}{\rho_0'}=\left(\frac{dp}{d\rho'}\right)_{\rho'=\rho_0'}=\frac{\gamma p_0}{\rho_0'}$ in adiabatic law. ($\varkappa=2{\cdot}045\times10^{10}$ dyne per sq. cm for water at 15°C.)

From (11), (12), (13), we get

$$\frac{\partial p'}{\partial t}+\rho_0'c^2\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{\partial w'}{\partial z}\right\}+\rho_0'g\frac{\partial w'}{\partial t}=0, \quad\cdots(14)$$

in which use is made of

$$\frac{Dp}{Dt}\doteqdot\frac{\partial p}{\partial t}+\frac{\partial p}{\partial(z+w')}\frac{\partial(z+w')}{\partial t}=\frac{\partial p_1}{\partial t}+\frac{\partial p_0}{\partial z}\frac{\partial w'}{\partial t}. \quad\cdots\cdots\cdots(15)$$

Eliminating p_1, ρ_0' between (10), (12), (14), we obtain

$$\left.\begin{aligned}
\rho_0'\frac{\partial^3u'}{\partial t^3}&=\frac{\partial}{\partial r}\left[\rho_0'c^2\frac{\partial}{\partial t}\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{\partial w'}{\partial z}\right\}+\rho_0'g\frac{\partial w'}{\partial t}\right]\\
&\quad+\mu'\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2u'}{\partial r^2}+\frac{1}{r}\frac{\partial u'}{\partial r}-\frac{u'}{r^2}+\frac{1}{r^2}\frac{\partial^2u'}{\partial\theta^2}+\frac{\partial^2u'}{\partial z^2}-\frac{2}{r^2}\frac{\partial v'}{\partial\theta}\right],\\
\rho_0'\frac{\partial^3v'}{\partial t^3}&=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\rho_0'c^2\frac{\partial}{\partial t}\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{\partial w'}{\partial z}\right\}+\rho_0'g\frac{\partial w'}{\partial t}\right]\\
&\quad+\mu'\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2v'}{\partial r^2}+\frac{1}{r}\frac{\partial v'}{\partial r}-\frac{v'}{r^2}+\frac{1}{r^2}\frac{\partial^2v'}{\partial\theta^2}+\frac{\partial^2v'}{\partial z^2}+\frac{2}{r^2}\frac{\partial u'}{\partial\theta}\right],\\
\rho_0'\frac{\partial^3w'}{\partial t^3}&=\frac{\partial}{\partial z}\left[\rho_0'c^2\frac{\partial}{\partial t}\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{\partial w'}{\partial z}\right\}+\rho_0'g\frac{\partial w'}{\partial t}\right]\\
&\quad+\mu'\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2w'}{\partial r^2}+\frac{1}{r}\frac{\partial w'}{\partial r}+\frac{1}{r^2}\frac{\partial^2w'}{\partial\theta^2}+\frac{\partial^2w'}{\partial z^2}\right].
\end{aligned}\right\}\quad(16)$$

These may be satisfied by a function ϕ which is connected with u', v', w' in the forms:

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi}{\partial r}, \quad \frac{\partial v'}{\partial t} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad \frac{\partial w'}{\partial z} = -\frac{\partial \phi}{\partial z}, \quad \cdots\cdots\cdots(17)$$

provided

$$\frac{\partial^2 \phi}{\partial t^2} = \left(c^2 + \nu\frac{\partial}{\partial t}\right)\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}\right\} + g\frac{\partial \phi}{\partial z}, \quad \cdots(18)$$

where $\nu = \mu'/\rho_0'$.

Now, putting

$$\phi = \Phi H_m^{(2)}(kr) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(19)$$

in (18), and solving the resulting equation, we get

$$\phi = H_m^{(2)}(kr) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\theta \; e^{-\frac{gz}{2c^2}} [D_m \cosh k_1 z + E_m \sinh k_1 z] \; e^{-\frac{\nu k^2 t}{2}} e^{in_1 t}, \quad \cdots(20)$$

where

$$k_1^2 = \frac{g^2}{4c^4} - k^2\left(\frac{c_1^2}{c^2} - 1\right), \quad n_1^2 = k^2 c_1^2 - \frac{(\nu k^2)^2}{4}, \quad \cdots\cdots\cdots\cdots(21)$$

in which $c_1^2 = n^2/k^2$.

The values of ν at 10°C is nearly 0·0131, while $k = 2{\cdot}09 \times 10^{-5}\,\text{cm}^{-1}$ for waves of 3 km length and $k = 2{\cdot}09 \times 10^{-6}\,\text{cm}^{-1}$ for waves of 30 km length, so that we have $e^{-\frac{\nu k^2 t}{2}} = e^{-2{\cdot}86\times 10^{-9}}$ and $e^{-\frac{\nu k^2 t}{2}} = e^{-2{\cdot}86\times 10^{-11}}$ for respective waves at the time $t = 100$ sec. Again, if we assume that the velocity of the waves is of the order 3 km per sec, then $k^2 c_1^2 = 39{\cdot}4\,\text{sec}^{-2}$ for 3 km wave length, and $k^2 c_1^2 = 0{\cdot}394\,\text{sec}^{-1}$ for 30 km wave length; and hence $n_1^2 = k^2 c_1^2 - \frac{(\nu k^2)^2}{4} = (39{\cdot}4 - 2{\cdot}86 \times 10^{-12})\,\text{sec}^{-2} \doteqdot n^2$ and $n_1^2 = (0{\cdot}394 - 2{\cdot}86 \times 10^{-14})\,\text{sec}^{-2} = n^2$ in respective cases. We may thus write $e^{-\frac{\nu k^2 t}{2}} = 1$ and $e^{in_1 t} = e^{int}$ without any sensible error in all practical problems.

Lastly, we may note that $e^{-\frac{gz}{2c^2}}$ may also be taken to be unity, but $\cosh k_1 z$ cannot be replaced by $\cosh kz$. Because, if we take $z = -1$ km and -10 km, we obtain $e^{-\frac{gz}{2c^2}} = e^{2{\cdot}40\times 10^{-3}}$ and $e^{-\frac{gz}{2c^2}} = e^{2{\cdot}40\times 10^{-2}}$ for respective cases, both of the exponentials being thus very near unity; while $\cosh k_1 z \doteqdot \cosh\sqrt{1-\frac{c_1^2}{c^2}}\,kz = \cos 1{\cdot}84\,kz$, which is obviously different from $\cosh kz$. This fact was also pointed out by Stoneley.[4)]

We may therefore write the expression of ϕ in the following form:

$$\phi = [D_m \cosh k_1 z + E_m \sinh k_1 z]\; H_m^{(2)}(kr) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta\, e^{int}, \quad \cdots\cdots(22)$$

where $k_1^2 = -k^2\left(\frac{c_1^2}{c^2}-1\right)$. The values of the corresponding components of the displacement is written by

$$\left.\begin{aligned} u_1' &= \frac{i}{n}[D_m \cosh k_1 z + E_m \sinh k_1 z]\frac{\partial H_m^{(2)}(kr)}{\partial r} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta\, e^{int}, \\ v' &= -\frac{im}{n}[D_m \cosh k_1 z + E_m \sinh k_1 z]\frac{H_m^{(2)}(kr)}{r} \left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\} m\theta\, e^{int}, \\ w' &= \frac{ik_1}{n}[D_m \sinh k_1 z + E_m \cosh k_1 z]\; H_m^{(2)}(kr) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta\, e^{int}. \end{aligned}\right\} \quad \cdots\cdots(23)$$

These expressions are obtained from the relation in (17) without difficulty.

IV. Boundary Conditions.

As there is no water pressure at the free surface of the ocean, we must have

$$z = -h_0; \qquad p = 0, \qquad (\text{or} = \text{constant},) \quad \cdots\cdots\cdots\cdots\cdots(24)$$

where h_0 is the depth of the ocean.

The theorem of Bernoulli concerning the pressure distribution of fluid in wave motion is such that

$$\frac{p}{\rho'} = \frac{\partial\phi}{\partial t} + gw'. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(25)$$

We know, however, that

$$\frac{\partial w'}{\partial t} = -\frac{\partial\phi}{\partial z}.$$

Substituting this in (25), the condition (24) may be replaced by

4) R. Stoneley, *loc. cit.*

$$z = -h_0\,; \qquad \frac{\partial^2\phi}{\partial t^2} = g\frac{\partial\phi}{\partial z}. \qquad \cdots\cdots(26)$$

Hence, from (22) and this equation, we get

$$(gk_1 \cosh k_1h_0 - n^2 \sinh k_1h_0)\, E_m = (gk_1 \sinh k_1h_0 - n^2 \cosh k_1h_0)D_m. \cdots(27)$$

When the depth of the water is relatively small, we have approximately

$$(gk_1 - n^2k_1h_0)E_m = (gk_1^2h_0 - n^2)D_m. \qquad \cdots\cdots(27')$$

The boundary conditions at the bottom of the ocean are such that

i. The displocement of the solid and that of the water are continuous in all components.

ii. The normal and shearing stresses are also continuous.

Since the viscosity of water, as we have seen in the last chapter, gives little effect on the motion of the seismic waves, it is evident that the tangential stresses of the water are practically zero at the bottom surface, and therefore the shearing stresses of the solid at that surface should be taken to be zero. The tangential displacements of the solid and of the water may accordingly be discontinuous at the same boundary. The vertical displacements and the similar stresses must be continuous on account of the reason that the active forces except the viscosity of the water act in this direction and also that the material points should be continuous in the normal sense of the contacting boundary.

From the condition that the component of the displacement in the normal direction is continuous, we have

$$z = 0\,; \qquad w = w', \qquad \cdots\cdots(28)$$

where $w = w_1 + w_3$. Substituting from (7), (9), (23), we get

$$k_1E_m = -in\left(\frac{\alpha}{h^2}A_m - \frac{k^2}{j^2}\frac{C_m}{m}\right). \qquad \cdots\cdots(29)$$

From the conditions of the tangential stresses, we have

$$z = 0; \quad \left.\begin{aligned} &\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = 0, \\ &\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} = 0, \end{aligned}\right\} \qquad \cdots\cdots(30)$$

in which $u = u_1 + u_3$, $v = v_1 + v_3$, $w = w_1 + w_3$. Substituting from (7), (9) in (30), we obtain

$$\frac{C_m}{m}\bigg/A_m=\frac{2\mathfrak{j}^2\sqrt{k^2-h^2}}{h^2(2k^2-\mathfrak{j}^2)}\qquad\cdots\cdots(31)$$

From the condition of the normal stress, we get

$$\lambda\Delta+2\mu\frac{\partial w}{\partial z}-g(\rho-\rho')w+\rho'\frac{\partial\phi}{\partial t}=0.\qquad\cdots\cdots(32)$$

Substituting from (4), (7), (9), (22), we obtain

$$\left(2-\frac{\mathfrak{j}^2}{k^2}\right)^2-\frac{4\alpha\beta}{k^2}-\frac{g\rho\mathfrak{j}^2\alpha}{\mu k^4}+\frac{\rho'\mathfrak{j}^2\alpha}{\mu k^4}\left\{g+n^2\frac{g-n^2h_0}{gk_1^2h_0-n^2}\right\}=0.\quad\cdots(33)$$

V. Velocity- and Displacement-Equations.

The Equation (33) may be written in the form:

$$\left(2-\frac{\mathfrak{j}^2}{k^2}\right)^2-\frac{4\alpha\beta}{k^2}-\frac{g\rho}{\mu k}\frac{\mathfrak{j}^2\alpha}{k^3}+\frac{\rho'\mathfrak{j}^2\alpha}{\mu k^4}\left\{\frac{\dfrac{g^2k_1^2h_0}{n^2}-n^2h_0}{\dfrac{gk_1^2h_0}{n^2}-1}\right\}=0.\quad\cdots\cdots(33')$$

When $\frac{gk_1^2h_0}{n^2}$, which is very small in the ordinary case of the ocean, is omitted, we have

$$\left(2-\frac{\mathfrak{j}^2}{k^2}\right)^2-\frac{4\alpha\beta}{k^2}-\frac{g\rho}{\mu k}\frac{\mathfrak{j}^2\alpha}{k^3}+\frac{\rho'}{\rho}(\alpha h_0)\frac{\mathfrak{j}^4}{k^4}=0.\qquad\cdots\cdots(34)$$

If $\frac{gk_1^2h_0}{n^2}$ is not omitted, we obtain, neglecting squares of small quantities,

$$\left(2-\frac{\mathfrak{j}^2}{k^2}\right)^2-\frac{4\alpha\beta}{k^2}-\frac{g\rho}{\mu k}\frac{\mathfrak{j}^2\alpha}{k^3}+\frac{\rho'}{\rho}(\alpha h_0)\frac{\mathfrak{j}^4}{k^4}\left\{1-\frac{g^2k^2\rho^2}{\mu^2\mathfrak{j}^4}+\frac{g^2\rho}{\mu\mathfrak{j}^2c^2}+\frac{h_0gk^2\rho}{\mu\mathfrak{j}^2}-\frac{h_0g}{c^2}\right\}=0.\quad\cdots\cdots(35)$$

If we write $\frac{\mathfrak{j}^2}{k^2}=j'^2$, $\frac{h^2}{k^2}=h'^2$, we have

$$(2-j'^2)^2-4\sqrt{(1-k'^2)(1-j'^2)}-\left(\frac{g\rho}{\mu k}\right)j'^2\sqrt{1-h'^2}+\left(\frac{\rho'h_0k}{\rho}\right)j'^4\sqrt{1-h'^2}$$
$$-\frac{\rho\rho'h_0g^2}{\mu^2k}\sqrt{1-h'^2}+\frac{\rho'h_0g^2j'^2}{\mu kc^2}\sqrt{1-h'^2}+\frac{\rho'h_0^2gk}{\mu}j'^2\sqrt{1-h'^2}$$
$$-\frac{\rho'h_0gk}{\rho c^2}j'^4\sqrt{1-h'^2}=0.\quad\cdots\cdots(35')$$

This is the equation to determine the velocity of waves propagated in two dimensions on the bottom surface of an ocean. Now it is exceedingly difficult to solve the equation (34) or (35), so that the following approximation has been made. As $j'^2 = \frac{\rho n^2}{\mu k^2} = \left(\frac{\rho}{\mu}\right) \times$ (velocity)^2, we may write

$$j'^2 = \xi_0 + \delta\xi, \quad \cdots\cdots\cdots\cdots(36)$$

under the assumption that the velocity of propagation of the waves of the present case is not much different from that of Rayleigh-waves on the free surface of a semi-infinite body. In (36), $\xi_0 \times (\mu/\rho)$ is the square of the velocity, V_0, of the ordinary Rayleigh-waves, and $\delta\xi \times (\mu/\rho)$ is the deviation of the square of the velocity, V, of the waves on the bottom surface of an ocean, from that of the ordinary Rayleigh-waves. In the case, in which the solid is incompressible ($h'=0$) and also $\frac{gk_1^2 h_0}{n^2}$ is very small, we obtain

$$2\delta\xi\left[\frac{1}{\sqrt{1-\xi_0}} - (2-\xi_0)\right] - \left(\frac{\rho g}{\mu k}\right)\xi_0 + \left(\frac{\rho'}{\rho} h_0 k\right)\xi_0^2 = 0. \quad \cdots\cdots(37)$$

The value of ξ_0 in the case of $h' = 0$ is 0·91262 and $V_0 = \sqrt{\frac{\mu\xi_0}{\rho}}$, $V = \sqrt{\frac{\mu(\xi_0 + \delta\xi)}{\rho}}$. Hence

$$\frac{V - V_0}{V_0} \doteqdot \frac{1}{2}\frac{\delta\xi}{\xi_0} \doteqdot 0{\cdot}109\frac{g\rho}{\mu k} - 0{\cdot}0994\frac{\rho'}{\rho}h_0 k. \cdots\cdots \cdots\cdots(38)$$

To get the order of the numerical value of this expression, we put $\mu/\rho = 3{\cdot}14^2 \text{km}^2/\text{sec}^2$, $k = 2{\cdot}09 \times 10^{-6}\,\text{cm}^{-1}$, (nearly 10 sec. period.), $g = 980$ dyne, $h_0 = 2$km, $\rho'/\rho = 0{\cdot}4$, $\rho' = 1$. Then we have

$$\frac{V - V_0}{V_0} = 0{\cdot}00129 - 0{\cdot}0166.$$

In the case of a solid whose Poisson's ratio is 1/4, we have

$h'^2 = \frac{1}{3}j'^2$ and the equation similar to (37) is expressed by

$$\delta\xi(2-\xi_0)\left[\frac{4}{3}\frac{1}{\sqrt{(1-\xi_0)\left(1-\frac{\xi_0}{3}\right)}}-2\right]-\left(\frac{g\rho}{\mu k}\right)\xi_0\sqrt{1-\frac{\xi_0}{3}}$$

$$+\left(\frac{\rho'}{\rho}h_0 k\right)\xi_0^2\sqrt{1-\frac{\xi_0}{3}} = 0. \quad\cdots\cdots\cdots\cdots(39)$$

The value of ξ_0 in the case $h'^2 = \frac{1}{3}j'^2$ is approximately 0·8453. Therefore

$$\frac{V-V_0}{V_0} = 0{\cdot}183\frac{g\rho}{\mu k} - 0{\cdot}155\frac{\rho'}{\rho}h_0 k. \quad\cdots\cdots\cdots\cdots(40)$$

Generally, we can write the velocity equation in the form:

$$V = V_0 + \frac{c_1}{k} - c_2 k, \quad\cdots\cdots\cdots\cdots(41)$$

where $c_1 = 0{\cdot}109\frac{g\rho}{\mu}V_0$ and $c_2 = 0{\cdot}0994\frac{\rho'}{\rho}h_0 V_0$ for an incompressible solid and $c_1 = 0{\cdot}183\frac{g\rho}{\mu}V_0$ and $c_2 = 0{\cdot}155\frac{\rho}{\rho'}h_0 V_0$ for a material whose Poisson's ratio is 1/4. For a material whose Poisson's ratio is between $\frac{1}{2}$ and $\frac{1}{4}$, the respective values of c_1 and c_2 are intemediate between their two particular values written above.

The expressions of the displacement of the bottom of the ocean is now written by

$$\left.\begin{aligned} u &= A_m\frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{\partial H_m^{(2)}(kr)}{\partial r}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\theta\, e^{i(V_0 k+c_1-c_2k^2)t},\\ v &= -A_m\frac{m(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{H_m^{(2)}(kr)}{r}\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix}m\theta\, e^{i(V_0 k+c_1-c_2k^2)t},\\ w &= A_m H_m^{(2)}(kr)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\theta\, e^{i(V_0 k+c_1-c_2k^2)t}, \end{aligned}\right\} \quad\cdots\cdots(42)$$

and the motion of the water particle is expressed by (23), where D_m and E_m are to be determined from (27), (29), (31).

From these expressions it may be seen that the azimuthal component of the displacement of the solid as well as that of the water become quiescent as the distance from the epicentre increases. The ratio of the vertical and the horizontal components of the displacement of the solid is not much different from that of the ordinary Rayleigh-waves. It is certain, however, that the ratio of the horizontal and the vertical components of the displacement of the bottom surface at a certain distance from the epicentre is more or less altered from that of Rayleigh-waves propagated on the free surface of a semi-infinite body. When the material is incompressible, the ratio of both components of the displacement of the ordinary Rayleigh-waves is

$$\frac{u_{\text{max.}}}{w_{\text{max.}}}=\frac{2-\xi_0-2\sqrt{1-\xi_0}}{\xi_0}\Big(=0{\cdot}543\Big)\,;\qquad\text{(43)}$$

while that of the waves propagated on the ocean bottom is

$$\frac{u_{\text{max.}}}{w_{\text{max}}}=\frac{2-\xi_0-2\sqrt{1-\xi_0}}{\xi_0}+\left(\frac{2-\xi_0}{\sqrt{1-\xi_0}}-2\right)\delta\xi$$

$$\left(=0{\cdot}543+0{\cdot}403\frac{g\rho}{\mu k}-0{\cdot}367\frac{\rho'}{\rho}h_0k\right).\qquad\text{(44)}$$

If we take $\mu/\rho=3{\cdot}14^2\,\text{km}^2/\text{sec}^2$, $k=2{\cdot}09\times10^{-6}\text{cm}^{-1}$, $g=980$ dyne, $h_0=2$ km, $\rho'/\rho=0.4$, $\rho'=1$; we get from (44)

$$\frac{u_{\text{max.}}}{w_{\text{max.}}}=0{\cdot}543+0{\cdot}00477-0{\cdot}0614,$$

so that we find that the change of the ratio of the amplitudes in some cases is not so small. From the equation (44), we may conclude that the ratio of the horizontal component of the displacement to the vertical component becomes large for long waves and the ratio becomes small for short waves.

Lastly, it may be seen from (23) and (27′) that the motion of the water particle on the free surface is mainly of a vertical type. This nature is quite different from that of the ordinary long water waves, and may perhaps be due to the fact that the displacement of the water caused by the movement of the ocean bed is chiefly in the sense of the vertical but not of the horizontal.

VI. Transmission of Arbitrary Waves.

We may now proceed to study the transmission of arbitrary waves on the two-dimensional surface of the bottom of an ocean. The equation of harmonic vibrations of the bottom surface of the ocean similar to (42) is written directly in the forms:

$$\left.\begin{aligned} u &= A_m \frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{\partial J_m(kr)}{\partial r}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(V_0k+c_1-c_2k^2)t, \\ v &= -A_m \frac{m(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{J_m(kr)}{r}\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix} m\theta \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(V_0k+c_1-c_2k^2)t, \\ w &= A_m J_m(kr)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(V_0k+c_1-c_2k^2)t. \end{aligned}\right\} \quad (45)$$

When the initial disturbance is of the forms:

$$t=0,\ z=0;\ w=f(r)\cos m\theta,\quad \frac{\partial w}{\partial t}=\varphi(r)\cos q(\theta+\varepsilon);\ \cdots\cdots(46)$$

we have the general expressions of the displacement as follows:

$$\left.\begin{aligned} u &= \cos m\theta\int_0^\infty F(k)\frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{\partial J_m(kr)}{\partial r}\cos(V_0k+c_1-c_2k^2)t\,kdk \\ &\quad+\cos q(\theta+\varepsilon)\int_0^\infty \Phi(k)\frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{\partial J_m(kr)}{\partial r}\frac{\sin(V_0k+c_1-c_2k^2)t}{V_0k+c_1-c_2k^2}kdk, \\ v &= -m\sin m\theta\int_0^\infty F(k)\frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{J_m(kr)}{r}\cos(V_0k+c_1-c_2k^2)t\,kdk \\ &\quad-q\sin q(\theta+\varepsilon)\int_0^\infty \Phi(k)\frac{(2k^2-j^2-2\alpha\beta)}{\alpha j^2}\frac{J_m(kr)}{r}\frac{\sin(V_0k+c_1-c_2k^2)t}{V_0k+c_1-c_2k^2}kdk, \\ w &= \cos m\theta\int_0^\infty F(k)J_m(kr)\cos(V_0k+c_1-c_2k^2)t\,kdk \\ &\quad+\cos q(\theta+\varepsilon)\int_0^\infty \Phi(k)J_m(kr)\frac{\sin(V_0t+c_1-c_2k^2)}{V_0k+c_1-c_2k^2}kdk; \end{aligned}\right\} \quad (47)$$

provided

$$F(k)=\int_0^\infty f(\sigma)J_m(k\sigma)\sigma d\sigma,\qquad \Phi(k)=\int_0^\infty \varphi(\sigma)J_m(k\sigma)\sigma d\sigma.\ \cdots\cdots\cdots\cdots(48)$$

If we put particularly

$$\left.\begin{aligned} f(r) &= A\frac{r}{a}e^{-\frac{r^2}{a^2}}, \quad \varphi(r)=0, \quad [m>1] \\ f(r) &= A\,e^{-\frac{r^2}{a^2}}, \quad \varphi(r)=0, \quad [m=0] \end{aligned}\right\} \cdots\cdots\cdots\cdots(49)$$

we obtain

$$F(k)=\frac{A}{a}\int_0^\infty e^{-\frac{\sigma^2}{a^2}}J_m(k\sigma)\sigma^2 d\sigma=\frac{A}{a}\sum_{q=0}^\infty \frac{(-1)^q\left(\frac{k}{2}\right)^{m+2q}}{q!\,\Gamma(m+q+1)}\frac{\Gamma\left(\frac{m}{2}+\frac{3}{2}+q\right)}{\left(\frac{2}{a}\right)^{m+2q+3}},$$

$$[m>2] \quad \cdots(50)^{5)}$$

$$F(k)=\frac{A}{a}\int_0^\infty e^{\frac{\sigma^2}{a^2}}J_1(k\sigma)\sigma^2 d\sigma = A\frac{a^3}{4}k\,e^{-\frac{a^2k^2}{4}}, \qquad [m=1] \quad \cdots(51)$$

$$F(k)=A\int_0^\infty e^{-\frac{\sigma^2}{a^2}}J_0(k\sigma)\sigma d\sigma = A\frac{a^2}{2}e^{-\frac{a^2k^2}{4}}; \qquad [m=0] \quad \cdots(52)$$

so that we find the expression of w as follows:

$$w=\frac{A\cos m\theta}{a}\sum_{q=0}^\infty\frac{(-1)^q\Gamma\left(\frac{m}{2}+q+\frac{3}{2}\right)}{q!\,2^{m+2q}\,\Gamma(m+q+1)}\left(\frac{a}{2}\right)^{m+2q+3}$$

$$\times\int_0^\infty J_m(kr)\cos(V_0k+c_1-c_2k^2)t\,k^{m+2q+1}dk, \qquad [m>2]\cdots(53)$$

$$w=\frac{Aa^3}{4}\cos\theta\int_0^\infty e^{-\frac{a^2k^2}{4}}J_1(kr)\cos(V_0k+c_1-c_2k^2)t\,k^2dk, \qquad [m=1]\cdots(54)$$

$$w=\frac{Aa^2}{2}\int_0^\infty e^{-\frac{a^2k^2}{2}}J_0(kr)\cos(V_0k+c_1-c_2k^2)t\,kdk; \qquad [m=0]\cdots(55)$$

the expressions of u and v can be similarly written.

In (53), (54), (55), the asymptotic expansions of $J_m(kr)$ may be taken for a relatively large value of kr. We shall employ such expansions hereafter, because of the reason that we are to investigate the behaviour of waves in transmission through a certain epicentral distance. For the

5) G. N. WATSON, *Theory of Bessel Functions* (Cambridge, 1922), 394.

evaluation of the integrals contained in (53), (54), (55), we have introduced the idea of stationary points of an oscillating function due to Debye[6] and Lord Kelvin[7]. The integrals in (53), (54), (55) may be of the forms:

$$\frac{1}{2}\sqrt{\frac{1}{2\pi r}}\int_0^\infty\Big[e^{i\left\{\left(c_1t-\frac{2m+1}{4}\pi\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t-\frac{2m+1}{4}\pi\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{i\left\{\left(c_1t+\frac{2m+1}{4}\pi\right)+(V_0t-r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t+\frac{2m+1}{4}\pi\right)+(V_0t-r)k-c_2tk^2\right\}}\Big]k^{m+2q+\frac{1}{2}}dk,\quad [m>2] \quad (56)$$

$$\frac{1}{2}\sqrt{\frac{1}{2\pi r}}\int_0^\infty e^{-\frac{a^2k^2}{4}}\Big[e^{i\left\{\left(c_1t-\frac{3}{4}\pi\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t-\frac{3}{4}\pi\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{i\left\{\left(c_1t+\frac{3}{4}\pi\right)+(V_0t-r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t+\frac{3}{4}\pi\right)+(V_0t-r)k-c_2tk^2\right\}}\Big]k^{\frac{3}{2}}dk,\qquad [m=1] \quad (57)$$

$$\frac{1}{2}\sqrt{\frac{1}{2\pi r}}\int_0^\infty e^{-\frac{a^2k^2}{4}}\Big[e^{i\left\{\left(c_1t-\frac{\pi}{4}\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t-\frac{\pi}{4}\right)+(V_0t+r)k-c_2tk^2\right\}}$$

$$+e^{i\left\{\left(c_1t+\frac{\pi}{4}\right)+(V_0t-r)k-c_2tk^2\right\}}$$

$$+e^{-i\left\{\left(c_1t+\frac{\pi}{4}\right)+(V_0t-r)k-c_2tk^2\right\}}\Big]k^{\frac{1}{2}}dk.\qquad [m=0] \quad (58)$$

Now, we have to equate the derivative of the factor $\{\quad\}$ in the index of each exponential of these integrals to zero as follows:

$$\frac{d}{dk}\Big\{\qquad\Big\}=0, \quad\cdots\cdots\cdots\cdots\cdots\cdots(59)$$

6) P. DEBYE, *Math. Ann.*, **67**, 535.
7) W. THOMSON, *Proc. Roy. Soc.*, **42** (1887), 80.

the solution of this equation being named k_1. We get, for the first and the second terms of each integrand of (56), (57), (58),

$$k_1 = \frac{V_0 t + r}{2c_2 t}; \qquad \cdots\cdots(60)$$

and, for the third and the fourth terms of the same integrand,

$$k_1 = \frac{V_0 t - r}{2c_2 t}. \qquad \cdots\cdots(60')$$

In (60) and (60') k_1 should be positive, because of the fact that k_1 must lie within the range of integration, namely between zero and infinity in the present case. Hence we must have

$$V_0 t > r. \qquad [t > 0,\ r > 0] \quad \cdots\cdots(61)$$

We have also to find the value of

$$\frac{d^2}{dk^2}\left\{ \qquad \right\}, \qquad \cdots\cdots(62)$$

its value being

$$\mp 2c_2 t. \qquad \cdots\cdots(63)$$

In this the upper sign is taken for the first and the third terms of each integrand, and the lower sign for the second and the fourth terms of the same integrand.

Thus, by the criterion of Kelvin and Debye, we obtain the integrals of (56), (57), (58) in the forms:

$$\frac{1}{\sqrt{2c_2 tr}}\left[\left\{\frac{V_0 t - r}{2c_2 t}\right\}^{m+2q+\frac{1}{2}} \cos\left\{c_1 t + \frac{2m+1}{4}\pi + \frac{(V_0 t - r)^2}{4c_2 t} - \frac{\pi}{4}\right\}\right.$$
$$\left. + \left\{\frac{V_0 t + r}{2c_2 t}\right\}^{m+2q+\frac{1}{2}} \cos\left\{c_1 t - \frac{2m+1}{4}\pi + \frac{(V_0 t + r)^2}{4c_2 t} - \frac{\pi}{4}\right\}\right],$$
$$[V_0 t > r; \quad m > 2] \quad \cdots\cdots(64)$$

$$\frac{1}{\sqrt{2c_2 tr}}\left[\left\{\frac{V_0 t - r}{2c_2 t}\right\}^{\frac{3}{2}} e^{-\frac{a^2}{4}\left(\frac{V_0 t - r}{2c_2 t}\right)^2} \cos\left\{c_1 t + \pi + \frac{(V_0 t - r)^2}{4c_2 t}\right\}\right.$$
$$\left. + \left\{\frac{V_0 t + r}{2c_2 t}\right\}^{\frac{3}{2}} e^{-\frac{a^2}{4}\left(\frac{V_0 t + r}{2c_2 t}\right)^2} \cos\left\{c_1 t - \pi + \frac{(V_0 t + r)^2}{4c_2 t}\right\}\right],$$
$$[V_0 t > r; \quad m = 1] \quad \cdots\cdots(65)$$

and similar expression when $m = 0$.

The expressions of w then become

$$w=\frac{A\cos m\theta}{\sqrt{c_2tr}}\sum_{q=0}^{\infty}\frac{(-1)^q a^{m+2q+2}}{2^{3m+6q+4}q!}\frac{\Gamma\left(\frac{m}{2}+q+\frac{3}{2}\right)}{\Gamma(m+q+1)}\left[\left(\frac{V_0t-r}{c_2t}\right)^{m+2q+\frac{1}{2}}\cos\left\{c_1t+\frac{m\pi}{2}+\frac{(V_0t-r)^2}{4c_2t}\right\}+\left(\frac{V_0t+r}{c_2t}\right)^{m+2q+\frac{1}{2}}\cos\left\{c_1t-\frac{m+1}{2}\pi+\frac{(V_0t+r)^2}{4c_2t}\right\}\right]$$

$$[V_0t>r;\quad m>2]\;\cdots\cdots\cdots(66)$$

$$w=\frac{Aa^3}{16}\frac{\cos\theta}{\sqrt{c_2tr}}\left[\left(\frac{V_0t-r}{c_2t}\right)^{\frac{3}{2}}e^{-\frac{a^2}{4}\left(\frac{V_0t-r}{2c_2t}\right)^2}\cos\left\{c_1t+\frac{(V_0t-r)^2}{4c_2t}+\pi\right\}+\left(\frac{V_0t+r}{c_2t}\right)^{\frac{3}{2}}e^{-\frac{a^2}{4}\left(\frac{V_0t+r}{2c_2t}\right)^2}\cos\left\{c_1t+\frac{(V_0t+r)^2}{4c_2t}-\pi\right\}\right],$$

$$[V_0t>r;\quad m=1]\;\cdots\cdots\cdots(67)$$

$$w=\frac{Aa^2}{4}\frac{1}{\sqrt{c_2tr}}\left[\sqrt{\frac{V_0t-r}{c_2t}}\,e^{-\frac{a^2}{4}\left(\frac{V_0t-r}{2c_2t}\right)^2}\cos\left\{c_1t+\frac{(V_0t-r)^2}{4c_2t}\right\}+\sqrt{\frac{V_0t+r}{c_2t}}\,e^{-\frac{a^2}{4}\left(\frac{V_0t+r}{2c_2t}\right)^2}\cos\left\{c_1t+\frac{(V_0t+r)^2}{4c_2t}-\frac{\pi}{2}\right\}\right].$$

$$[V_0t>r;\quad m=0]\;\cdots\cdots\cdots(68)$$

When $V_0t<r$, the integrals in (56), (57), (58) disappear owing to the cancelling nature of the fluctuating functions of the integrands. Hence we may write

$$w=0.\qquad[V_0t<r]\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(69)$$

Though the expressions of u and v may be obtained in similar manners as that employed in the evaluation of w, I have omitted the investigation of such components for the sake of simplicity.

A few examples of the space distribution of the displacements of the case in which $\frac{c_1a}{V_0}=\frac{0.001}{3}$ and $\frac{c_2}{V_0a}=0.06$ are shown in Figs. 1, 2, 3, 4. Fig. 1 gives the case, $m=0$, $\frac{V_0t}{a}=7.5$, while Fig. 2 the case, $m=0$, $\frac{V_0t}{a}=15$. Both of these cases belong to the waves due the initial disturbance of the type:

$$t=0, \quad z=0: \quad f(r)=A\,e^{-\frac{r^2}{a^2}}; \quad \cdots\cdots\cdots\cdots\cdots\cdots(70)$$

but correspond to different stages of the transmission of the disturbance. Figs. 3 and 4, both belonging to the kind, $m=1$, give the cases, $\frac{V_0 t}{a}=7{\cdot}5$ and $\frac{Vt}{a}=15$ respectively. The original disturbance is of the type:

$$t=0, \quad z=0: \quad f(r)\cos\theta = A\frac{r}{a}e^{-\frac{r^2}{a^2}}\cos\theta. \quad \cdots\cdots\cdots\cdots(71)$$

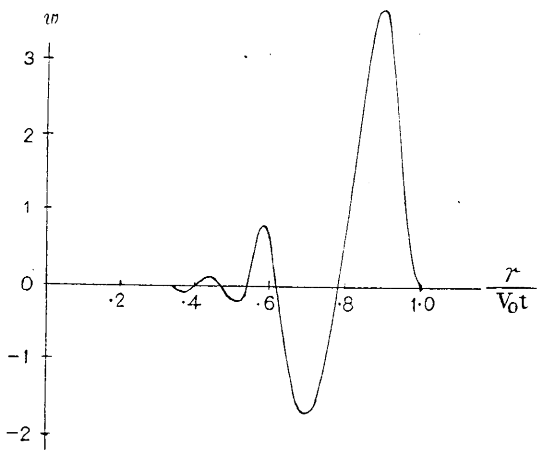


Fig. 1. $m=0, \quad \frac{V_0 t}{a}=7{\cdot}5.$

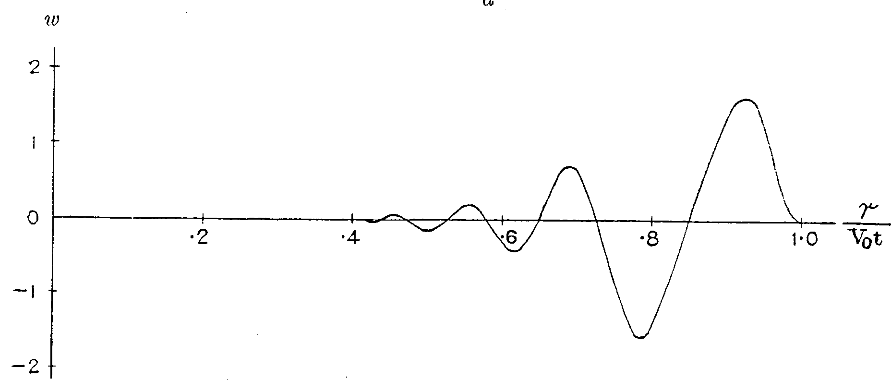


Fig. 2. $m=0, \quad \frac{V_0 t}{a}=15.$

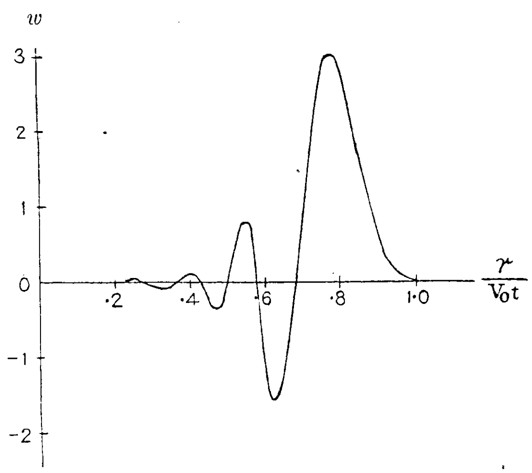


Fig. 3. $m = 1, \quad \frac{V_0 t}{a} = 7{\cdot}5.$

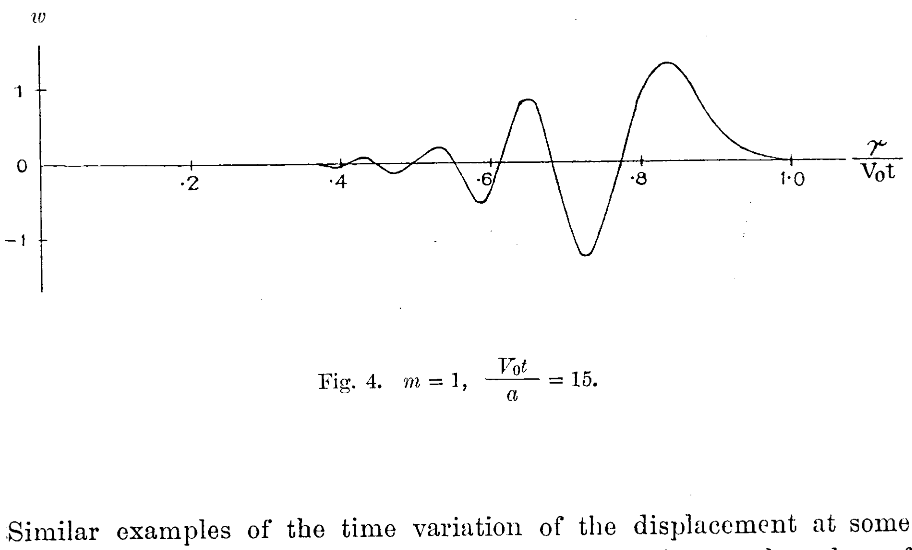


Fig. 4. $m = 1, \quad \frac{V_0 t}{a} = 15.$

Similar examples of the time variation of the displacement at some distances from the epicentre are shown in Figs. 5, 6, 7, 8; the values of $\frac{c_1 a}{V_0}$ and $\frac{c_2}{V_0 a}$ being taken to be the same as those of the last examples.

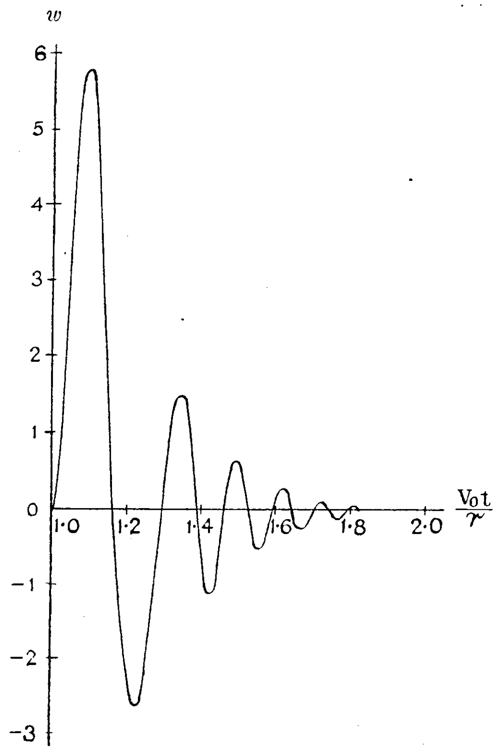


Fig. 5. $m = 0, \quad \frac{r}{a} = 15.$

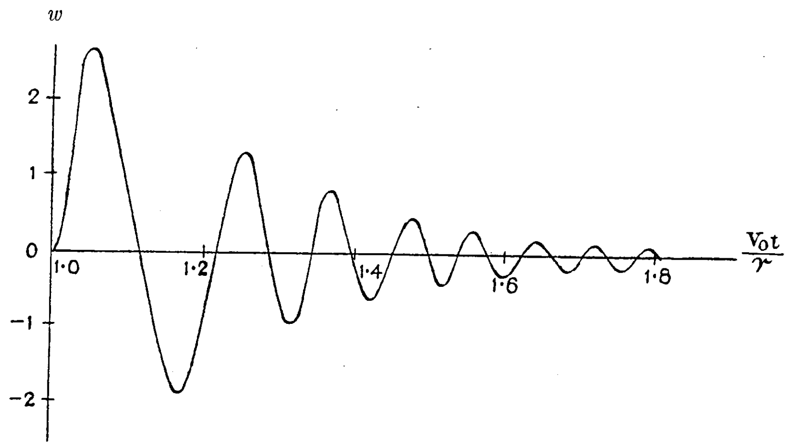


Fig. 6. $m = 0, \quad \frac{r}{a} = 30.$

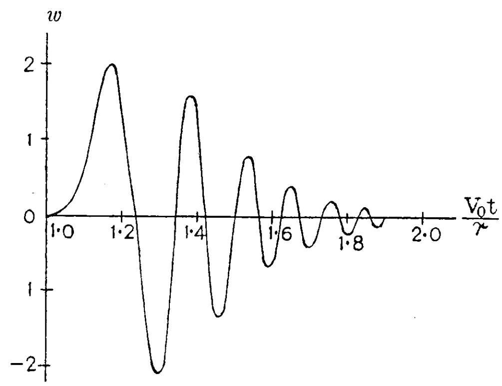


Fig. 7. $m = 1$, $\frac{r}{a} = 15$.

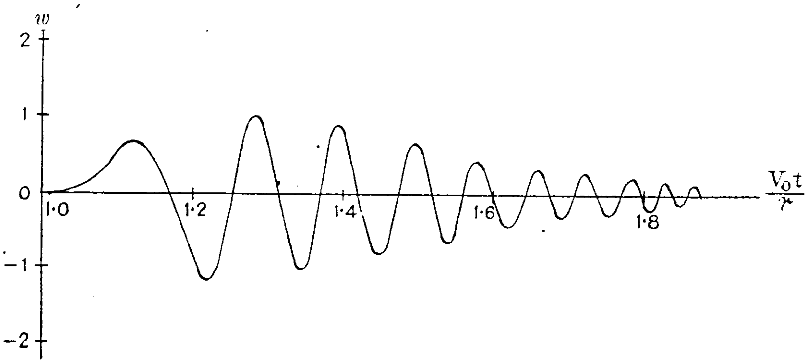


Fig. 8. $m = 1$, $\frac{r}{a} = 30$.

Among these figures, Figs. 5 and 6 correspond to the case $m = 0$ and Figs. 7 and 8 to the case $m = 1$, the types of the original disturbances being the same as (70), (71) in respective cases.

From these figures and the preceding equations, it may be seen that the transmitted waves are more and more dispersed as the distance from the epicentre increases, and the leading part of the dispersed waves is propagated with the definite velocity of the ordinary Rayleigh-waves. It appears, moreover, that in spite of the concentrated type of the original disturbance, the propagated waves are of harmonic types, with gradually decreasing periods and amplitudes. This fact is in good accordance with some of seimic records. The wave length of the oscillatory part becomes larger and the form of the displacement is more and more flattened as the waves proceed. We have not introduced any element like solid viscosity having the nature of decaying the waves; yet we find that the dispersive waves of the present case indicate the feature of diffusing ones.

It is also to be noticed that the phase of the leading part of the disturbed portion of the waves is not unchangeable, but varies during the progression of the waves. This tendency will be seen in Fig. 9. In this case, $m=0$, $\dfrac{c_1 a}{V_0}=\dfrac{0{\cdot}1}{3}$; $\dfrac{c_2}{aV_0}=0{\cdot}06$ are assumed and the original disturbance is taken to be the same as (70).

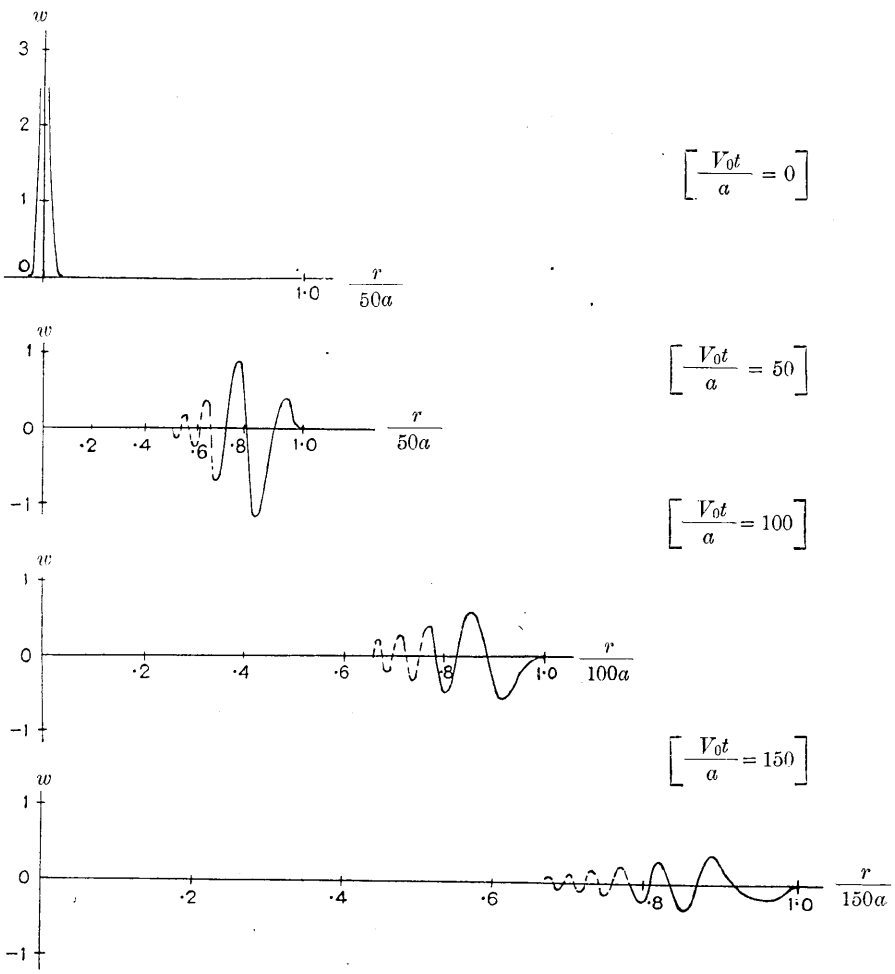


Fig. 9. $m=0$, $\dfrac{c_1 a}{V_0}=\dfrac{0{\cdot}1}{3}$, $\dfrac{c_2}{aV_0}=0{\cdot}06$,

The variation of the phase in the neighbourhood of the leading part of the waves is mainly due to the fact that the velocity of the phase of the waves is not equal to that of the leading part. Consider

$$\begin{matrix}\cos\\ \sin\end{matrix}\Big\} \left\{c_1 t + \frac{(V_0 t - r)^2}{4c_2 t}\right\}. \qquad (72)$$

Differentiating the argument of this harmonic function with respect to t, we find

$$\frac{\partial r}{\partial t} = V_0 + \frac{2c_1c_2 t}{(V_0 - r)} - \frac{V_0 t - r}{2t}. \qquad (73)$$

In this equation $\frac{\partial r}{\partial t}$ is the phase velocity, while V_0 that of the leading part. We may thus easily find that the phase velocity in many cases is greater than that of the leading part and directed in the sense of the transmission of the disturbance; and accordingly it may be concluded that the phase of the waves in this vicinity changes in the course of the transmission of the disturbance in such a manner that the wave forms within the disturbed portion proceed from the tail part towards the leading part where the wave forms suddenly disappear. This fact may also be seen from the reason that each of the nodal points, which are to be determined from the equation (72), disappears discontinuously one after another, when t is grudually increased.

VII. Group and Phase Velocities.

It is not without importance to remark here that both of the group and the phase velocities are variable with respect to the wave length and the time. We can write from (41) the expression of the group velocity U such that

$$\begin{aligned} U &= \frac{d(Vk)}{dk} \\ &= \frac{d(V_0 k + c_1 - c_2 k^2)}{dk} \\ &= V_0 - 2c_2 k. \qquad (74) \end{aligned}$$

The maximum group velocity is evidently V_0, which is the same as the velocity of propagation of Rayleigh-waves. This maximum occurs when the length of the elementary waves is infinitely long. The other group velocities are all less than V_0. Thus, we find why the wave length in the neighbourhood of this part is relatively long. The reason that the length of the disturbed portion is more and more prolonged as the waves are transmitted, is also understood. Nextly, the phase velocity, as it may be seen from the equation (73), is also variable with respect to r and t. In the immediate neighbourhood of the point, $r = V_0t$, it becomes exceedingly large. This, however, only occurs at the very point of $r = V_0t$ and at a point very near to this the velocity is not so high, so that there needs a certain interval of time before the phase of the leading part changes its sign.

When we examine the forms of the equations (66), (67), (68), we shall find that c_2 plays an important part in the action of flattening the wave forms and of the elongation of the wave length, while c_1 is most effective in changing the phases of the leading part of the disturbed portion of the waves. We know that c_1 and c_2 are connected with the gravity force of the solid and the superposition of the water respectivety, so that it may be concluded that the gravity mainly changes the phase of the waves and the superposed water the form of waves.

VIII. Undispersive Waves.

Lastly, we may add a few remarks on the propagation of Rayleigh-waves on the surface of a simple semi-infinite solid body. In this case the solution can be obtained by putting $c_1 = c_2 = 0$ in (53), (54), (55). If we take only the cases $m = 0$ and $m = 1$, we get, by means of the formula,

$$\left.\begin{aligned} \int_0^\infty J_0(kr)\sin kV_0t\,dk &= \frac{1}{\sqrt{(Vt)^2-r^2}}, \quad [Vt > r] \\ \text{,,} \qquad &= 0, \qquad\qquad [Vt < r] \end{aligned}\right\}^{8)} \cdots\cdots\cdots(75)$$

the solutions of the forms:

$$\left.\begin{aligned} w &= -\frac{Aa^2}{2}\left[\frac{(Vt)}{\{(Vt)^2-r^2\}^{\frac{3}{2}}}+\frac{a(Vt)\{6(Vt)^2+5r^2\}}{4\{(Vt)^2-r\}^{\frac{7}{2}}}+\cdots\cdots\right], \\ &\qquad\qquad\qquad\qquad [m=0;\quad Vt>r] \\ \text{,,} &= 0, \qquad\qquad\qquad [m=0;\quad Vt<r] \end{aligned}\right\}\cdots\cdots(76)$$

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8) H. LAMB, *Proc. Math. Soc.*, London [2], 7 (1907), 122; S. SANO, *Bull. Centr. Met. Obs.*, Japan, 2 (1913), 13.

and

$$\left.\begin{aligned} w &= \frac{Aa^3}{4}\cos\theta\left[\frac{3(Vt)r}{\{(Vt)^2-r^2\}^{\frac{5}{2}}}+\frac{a^2(Vt)r\{52(Vt)^2+25r^2\}}{4\{(Vt)^2-r^2\}^{\frac{9}{2}}}+\cdots\cdots\right\}, \\ &\qquad\qquad\qquad\qquad [m=1;\quad Vt>r] \\ ,, &= 0. \qquad\qquad\qquad\quad [m=1;\quad Vt<r] \end{aligned}\right\} \quad (77)$$

The results for which $m = 0$ and $m = 1$ are shown in Figs. 10, 11, 12, 13. It may be seen that no fluctuating phenemenon happens, but

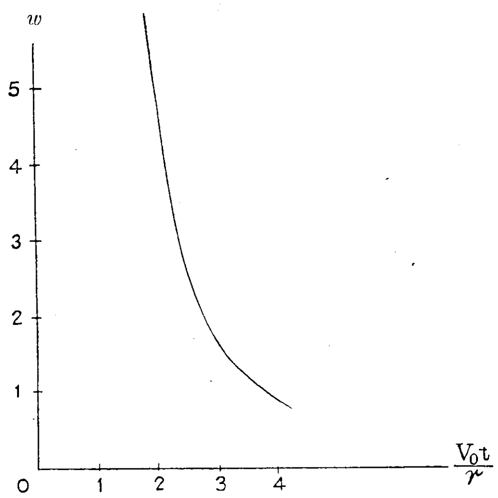

Fig. 10. $m = 0, \quad \frac{r}{a} = 2.$

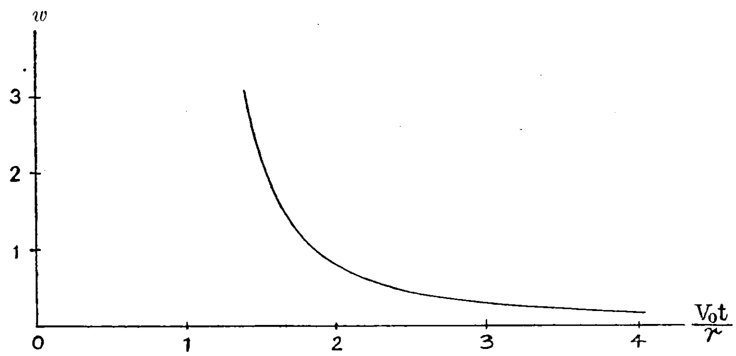

Fig. 11. $m = 0, \quad \frac{r}{a} = 5.$

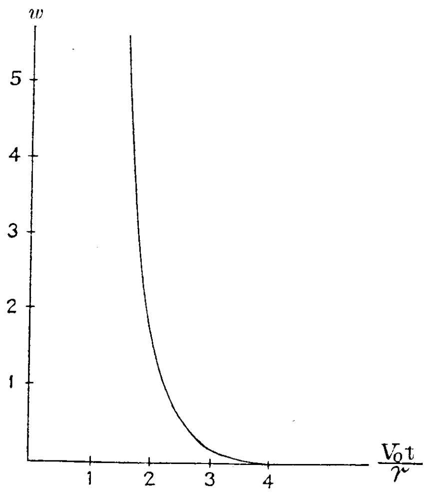

Fig. 12. $m = 1, \quad \frac{r}{a} = 2.$

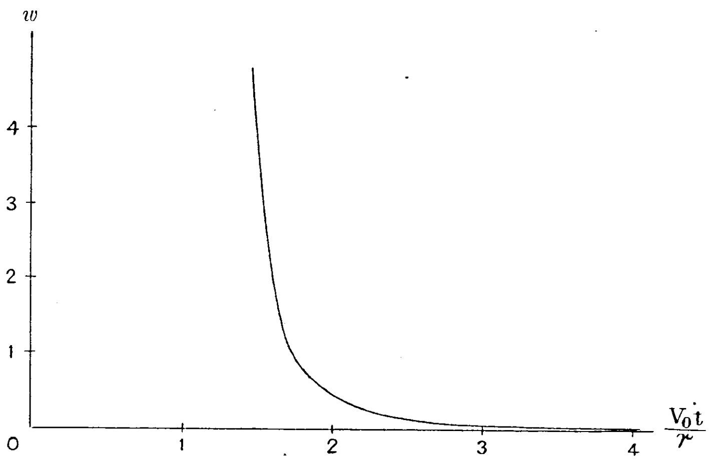

Fig. 13. $m = 1, \quad \frac{r}{a} = 5$

an infinitely prolonged tail follows the leading part. The movement is of a gradually subsiding type without oscillation. This is, as already noticed by Lamb[9] for the propagation of sound or tidal waves, a characteristic feature of the waves transmitted in two dimensions.

9) H. LAMB, *Proc. Math. Soc.*, London, 35 (1902), 141.

XI. Summary.

We may now complete this paper with a brief summary.

1. The viscosity of water gives no influence upon the transmission of Rayleigh-type waves on the bottom surface of an ocean.

2. The compressibility of water is effective only on the movement of the superposed water, but not on the motion of the ground.

3. The pressure of the water in the sense of forming surface water waves is of a great importance in connection with the transmission of the seismic waves.

4. The velocity of harmonic waves changes with the wave length, and this velocity is quick or slow according as the wave length is long or short. There is no definite critical velocity for harmonic waves of an infinite extent.

5. In a two-dimensional transmission of waves, the azimuthal component of the displacements of the solid as well as of the water becomes quiescent as the distance from the epicentre is increased.

6. For long waves, the ratio of the horizontal component of the displacement to the vertical component is larger than that of the ordinary Rayleigh-waves, while this ratio is smaller than that of the latter for short waves.

7. The motion of the water particle is mainly of a vertical type.

8. However concentrated the original disturbance may be, the transmitted waves are of a dispersed form. As the distance from the epicentre is increased, this tendency becomes more distinct.

9. The dispersed waves are apparently of an oscillatory type, having a tail with gradually decreasing periods and amplitudes.

10. The leading part of the dispersed waves is propagated with the velocity of the ordinary Rayleigh-waves.

11. The phase of the leading part changes as the waves proceed to a long distance.

12. The wave lengths of the oscillatory part become larger and larger and the forms of the waves are more and more flattened in the course of the progression of the disturbance.

13. The disturbed portion has various group velocities depending upon the harmonic elements involved in that portion. The maximum of the group velocities is the same as that of Rayleigh-waves, and this maximum velocity corresponds with the harmonic element of an infinitely long wave length.

14. The phase velocity is also variable for different parts of the disturbed waves.

15. When the effects of gravity and of water are omitted, the transmitted waves due to a concentrated disturbance is of a gradually subsiding type without oscillation.

In conclusion I must express my sincerest thank to Dr. G. Nishimura who has assisted me in preparing this paper. All of the numerical calculations, of which the result is shown in Figs. 1–13, have, indeed, been carried out by him in his accomplished mathematical ability.

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10. 洋底に傳はる地震波に就て

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此論文の目的は地震波が洋底に於て二次元的に波動する場合の諸現象を數學的に研究する事にある。調和波が一次元的に傳播する場合に其波動速度を求める事は既に一二の學者によつて手をつけられて居るけれども、それ以外の事は未だ誰も注目して居らぬ樣に見える。二次元的の場合、殊に勝手な震動が發生した場合に其が傳播して行く模樣を研究して見ると、種々の不思議な事柄が見出されるのである。其等を取纏めて箇條書きとし、且つ計算中に得た特別な結果を附加へて見ると大體次の如くなる。

1. 水の粘性は洋底を傳はるレーリー型地震波に對して殆ど影響を與へぬものである。

2. 水の壓縮性は地震と共に動く水の運動には非常に影響を與へるけれども、土地それ自身の震動にはあまり異常を起さぬものと考へられる。

3. 水の重力的壓力は地震の波動に非常に關係がある。

4. 洋底を傳播する地震波が調和型である場合に、共速度は波長によつて異なり、長波の場合には速に、短波の場合には遲くなる。而して其波動速度の局限などといふものは見當らぬ。

5. 二次元的傳播に於て波動方向に直角な向の振動は、土地の震動に就ても水の動きに就ても、振動部が震源から遠かるに從て非常に速に減少する事がわかる。

6. 波長が長い場合には波動方向に平行な向の振幅と其に直角な鉛直方向の向の振幅との比が普通のレーリー波の場合の其比よりも大なる事がわかり、又波長が短い時は其が逆になる事が知られる。

7. 水の振動は鉛直方向が一番大である。

8. 源點の震動が如何に集中性のものであつても、傳播する波は分散性を有し、震源から距離が遠くなるに從て波動部分が一層分散して行くものである。

9. 分散した波は割合に規則正しい繰替しの振動性を有し、振動形の後の部位程、振幅は小さく、振動週期は短くなるのである。

10. 振動の初動部は水のない場合の純粹のレーリー波の速度で傳播する。

11. 初動部の位相は地震波が遠方へ進むに從て漸次變つて行く事がわかる。

12. 分散波動の波長は地震波が遠方へ進むに從て次第に長くなり、振幅の形狀は次第に低く平たくなる。

13. 擾亂部分は種々の群波速度を有し、其最大極限の速度はレーリー波の速度に等しい。此等の群波速度は擾亂部分中にある調和的の波素に因るのであつて、其最大極限の速度は無限に長い波長の波素によるものと考へられる。

14. 位相波の速度は擾亂部分の種々の場所によつて夫々違つて居る樣に思はれる。

15. 重力と水との影響を除いて考へると、波動は初動部から振動する事なく漸次衰へる形狀をなす事がわかる。

此研究に當て種々の數學的考究事項に遭遇した。それを適當に解決したのではあるけれども、猶數多の計算上の困難があつた。しかしこの計算的困難は幸にも西村學士の助力によつて殆ど切抜ける事が出來た事を附記して置く。