

7. *The Effect of Temperature Distribution on the Deformation of a Semi-infinite Elastic Body.*

By Genrokuro NISHIMURA,

Earthquake Research Institute.

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1. The problem of the deformation of a semi-infinite elastic solid due to the surface loading was solved by many mathematicians: H. Hertz,¹⁾ H. Nagaoka,²⁾ K. Terazawa,³⁾ H. Lamb⁴⁾ and others. And recently K. Sezawa⁵⁾ studied the case of the deformation of a semi-infinite elastic body due to the internal nuclei of strain. Presumably the results of the investigations of these authorities may give some clue to the questions of the surface deformation and the internal state of the earth.

The recent geophysical researches tell us the incessant variation of the surface form of the earth crust, and although the phenomenon may take place by the action of the crustal force or by the physico-chemical changes in the interior of the earth, we cannot yet neglect the effect of the temperature distribution and variation in the neighbourhood of the surface. Professor K. Sezawa seems to have stuck a long time on his view that the deformation of the crust observed at a local portion is chiefly due to such an effect of the temperature, and he urged the author to solve the problem of this effect. The temperature distribution and variation, of course, depend upon the heat radiated from the sun and also upon other meteorological and topographical circumstances. Even if the above effect should necessarily be diminished towards the interior of the earth crust, we do not yet know in any exact manner how much effects the temperature distribution on the surface of the earth crust as well as in the interior of the same crust may have upon the deformation of the earth.

1) H. HERTZ, *Ges. Werke*, 1 (1895).

2) H. NAGAOKA, *Tokyo Math.-Phys. Soc. Proc.*, 6 (1912), 208.

3) K. TERAZAWA, *Phil. Trans. Roy. Soc.*, (A), 217 (1916), 35.

4) H. LAMB, *Proc. Roy. Soc.*, London, (A), 93 (1917), 293.

5) K. SEZAWA, *Bull. Earthq. Res. Inst.*, Tokyo, 7 (1929), 1.

Thus, it will not be useless to study the present problem in order to know the true nature of the deformation of the surface of the earth and to give some step on the advancement of the seismological science.

Recently Dr. R. Takahasi,⁶⁾ the assistant professor of the Earthquake Research Institute, Tokyo, using the Michelson tube and the Ishimoto tiltmeter, observed the tilt of the house of the Institute, and found the result that the chief deformation of the house is of the daily and annular variation of the tilt due to the heat of the sun on the surface of the house. The similar nature may be found, when fully examined, in the deformation of the local portion of the actual earth crust.

Although the bending of a relatively long house due to the temperature change is very simply analysed in the manner of the engineering mechanics, yet the theoretical researches concerning to the problem of a semi-infinite body have not been much developed, and the present paper, the author thinks, may throw some lights on the study of this problem.

It is important to bring the problem of the diffusion of heat distributed on the surface of the earth in the interior of the crust in order to deal with the effect of the temperature distribution on the deformation of a semi-infinite elastic solid. But this problem of conduction is not so important on the deformation of the crust. Theoretically Dr. Y. Kodaira⁷⁾ at the Central Meteorological Observatory, Tokyo, studied the conduction of heat evolved from magma in the volcano districts, but it is not concerned with the present problem.

The present author has obtained the equations of equilibrium of the solid, in which the case of steady state of heat distribution of some kinds or of a certain instant of non-steady state is taken. The assumption on the temperature distribution is very reasonable, not incoherent with but satisfying the state at an instant of conduction of heat in a solid. From the calculations which will appear presently, the author has obtained the important facts upon the geophysical researches as given in Section 16.

The mathematical parts below are composed of three parts: in the first part the author has obtained the general equations of equilibrium of the elastic solid when the variations of temperature occur, and the second part gives us the two-dimensional problem and the third the three-dimensional problem, and in these last two parts some examples have been solved by using Fourier's double integral formula.

6) R. TAKAHASI, 45th Colloq. Meeting of the Institute (Jan. 21, 1930).

7) Y. KODAIRA, *Geophysical Magazine*, Tokyo, 1 (1926).

Part I. General Theory.

2. The mathematical theory of elasticity shews that when an elastic homogeneous isotropic body is exposed to pressure in the condition of standard temperature, the stress and strain relations are given as follows :

$$\left. \begin{aligned} X_x &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial u}{\partial x}, \\ Y_y &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial v}{\partial y}, \\ Z_z &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial w}{\partial z}, \\ X_y &= \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad X_z = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \& \quad Y_z = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right). \end{aligned} \right\} \dots(1)$$

The meaning of the notations in the above equations are given in Love's text book⁸⁾ of elasticity.

When the temperature of the body is raised by T , and no external forces are applied, the strains are given as follows :

$$\text{and} \quad \left. \begin{aligned} \frac{\partial u}{\partial x} &= \frac{cT}{3}, \quad \frac{\partial v}{\partial y} = \frac{cT}{3}, \quad \frac{\partial w}{\partial z} = \frac{cT}{3}, \\ \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} = \dots\dots = 0, \end{aligned} \right\} \dots\dots\dots(2)$$

where c is the cubical expansion coefficient of the body. Now these strains in (2) are connected with the stresses, when the body is in the standard temperature, in the forms :

$$\text{and} \quad \left. \begin{aligned} X_x &= \left(\lambda + \frac{2}{3}\mu \right) cT, \quad Y_y = \left(\lambda + \frac{2}{3}\mu \right) cT, \quad Z_z = \left(\lambda + \frac{2}{3}\mu \right) cT, \\ X_y &= X_z = Y_z = 0. \end{aligned} \right\} \dots(3)$$

Therefore the general expressions of the stress and strain relations are written in the following forms :

8) LOVE, *The Mathematical Theory of Elasticity*, 4th ed., 102.

$$\left. \begin{aligned}
 X_x &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial u}{\partial x} - \left(\lambda + \frac{2}{3}\mu \right) cT, \\
 Y_y &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial v}{\partial y} - \left(\lambda + \frac{2}{3}\mu \right) cT, \\
 Z_z &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial w}{\partial z} - \left(\lambda + \frac{2}{3}\mu \right) cT,
 \end{aligned} \right\} \dots (4)$$

and

$$X_y = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad X_z = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad Y_z = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).$$

These relations have been given by Professor Love.⁹⁾

Now, the stress-equations of equilibrium of no body forces are

$$\left. \begin{aligned}
 \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial Z_x}{\partial z} &= 0, \\
 \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} &= 0, \\
 \frac{\partial Z_x}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} &= 0.
 \end{aligned} \right\} \dots (5)^{10)$$

If we put the relations (4) in the stress-equations (5), we get the following equations of equilibrium of an elastic body of the temperature distribution T , after some reduction :

$$\left. \begin{aligned}
 (\lambda + 2\mu) \frac{\partial \Delta}{\partial x} - 2\mu \frac{\partial \varpi_x}{\partial y} + 2\mu \frac{\partial \varpi_y}{\partial z} &= \alpha \frac{\partial T}{\partial x}, \\
 (\lambda + 2\mu) \frac{\partial \Delta}{\partial y} - 2\mu \frac{\partial \varpi_x}{\partial z} + 2\mu \frac{\partial \varpi_z}{\partial x} &= \alpha \frac{\partial T}{\partial y}, \\
 (\lambda + 2\mu) \frac{\partial \Delta}{\partial z} - 2\mu \frac{\partial \varpi_y}{\partial x} + 2\mu \frac{\partial \varpi_z}{\partial y} &= \alpha \frac{\partial T}{\partial z},
 \end{aligned} \right\} \dots (6)$$

where

$$\left. \begin{aligned}
 \Delta &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \\
 2\varpi_x &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \\
 2\varpi_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \\
 2\varpi_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},
 \end{aligned} \right\} \dots (7)$$

9) LOVE, *loc. cit.*, 108.

10) LOVE, *loc. cit.*, 85.

and

$$\alpha = \left(\lambda + \frac{2}{3} \mu \right) c. \quad \dots\dots\dots(8)$$

In order to determine the form of T in the equations (6), we must first solve the equation of conduction of heat, such that

$$\frac{\partial T}{\partial t} = z \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \quad \dots\dots\dots(9)$$

But, if we select a certain suitable form of T satisfying the condition of conduction at every moment and the conditions of the problem to be solved, we can easily solve the equations (6), and can finally complete our study.

Part II. Two-dimensional Problem.

3. In two-dimensional problem, we use the Cartesian coordinates (x, y) . Let the plane $y = 0$ be the surface of the semi-infinite elastic solid, and the axis of y be taken vertically downwards. (Fig. 1.)

The equations of equilibrium, when the temperature distribution T takes place, are given in the forms :

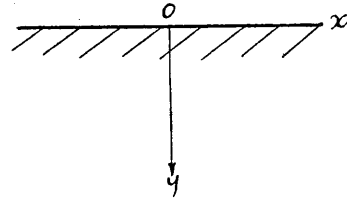


Fig. 1.

$$\left. \begin{aligned} (\lambda + 2\mu) \frac{\partial \Delta}{\partial x} - 2\mu \frac{\partial \varpi}{\partial y} &= \alpha \frac{\partial T}{\partial x}, \\ (\lambda + 2\mu) \frac{\partial \Delta}{\partial y} + 2\mu \frac{\partial \varpi}{\partial x} &= \alpha \frac{\partial T}{\partial y}, \end{aligned} \right\} \quad \dots\dots\dots(10)$$

where

$$\left. \begin{aligned} \Delta &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \\ 2\varpi &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \end{aligned} \right\} \quad \dots\dots\dots(11)$$

We can, now, construct the following equations from (10),

$$\frac{\partial^2}{\partial x^2} [(\lambda + 2\mu) \Delta - \alpha T] + \frac{\partial^2}{\partial y^2} [(\lambda + 2\mu) \Delta - \alpha T] = 0, \quad \dots\dots\dots(12)$$

$$\frac{\partial^2 \varpi}{\partial x^2} + \frac{\partial^2 \varpi}{\partial y^2} = 0. \quad \dots\dots\dots(13)$$

Now, in the problems at which we are aiming, the temperature should be gradually small according as the depth of the solid from the free surface $y = 0$ is increased, and at $y = \infty$ it must be zero. Therefore, we take the following standard form for the temperature distribution :

$$T = \mathfrak{U} e^{-\gamma y} \left\{ \frac{\cos}{\sin} \right\} \beta x, \dots\dots\dots (14)$$

where \mathfrak{U} is an arbitrary constant which is the measurement of the temperature magnitude.

As we have already discussed in Section 2, the equation (14), just we have assumed, should give the temperature distribution at a certain moment in the conduction of heat.

From (12) and (14), we have the following equation,

$$\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} = \frac{\alpha \mathfrak{U}}{\lambda + 2\mu} (\gamma^2 - \beta^2) e^{\gamma y} \left\{ \frac{\cos}{\sin} \right\} \beta x. \dots\dots\dots (15)$$

Now, we assume that the dilatation Δ is of the following form :

$$\Delta = F(y) \left\{ \frac{\cos}{\sin} \right\} \beta x. \dots\dots\dots (16)$$

Then we have the following particular solution Δ_1 and complementary solution Δ_2 of (15),

$$\Delta_1 = \frac{\alpha \mathfrak{U}}{\lambda + 2\mu} e^{-\gamma y} \left\{ \frac{\cos}{\sin} \right\} \beta x, \dots\dots\dots (17)$$

$$\Delta_2 = B e^{-\beta y} \left\{ \frac{\cos}{\sin} \right\} \beta x. \dots\dots\dots (18)$$

As the complementary solution, we have another form of $e^{\beta y} \left\{ \frac{\cos}{\sin} \right\} \beta x$, but it is omitted in our case. When $\gamma = \beta$, $\Delta_1 = 0$.

The solution of (13) which is suitable to the present purpose is

$$2 \varpi = D e^{-\beta y} \left\{ \frac{\sin}{\cos} \right\} \beta x. \dots\dots\dots (19)$$

If we put these dilatations and rotation in the equation (10), we have the following relation between the constants B and D ;

$$\left. \begin{aligned} (\lambda + 2\mu) B + \mu D &= 0, & [\gamma \neq \beta] \\ (\lambda + 2\mu) B + \mu D &= \alpha \mathfrak{U}, & [\gamma = \beta] \end{aligned} \right\} \dots\dots\dots (20)$$

Now we must have the displacement (u, v) from the equations (17), (18) and (19). Following the analytical method¹¹⁾ of Professor K. Sezawa, we obtain the following system of displacements.

If we assume that the displacement (u_1, v_1) satisfies Δ_1 of the equation (17) and $2\varpi = 0$, then

$$\left. \begin{aligned} u_1 &= \frac{\alpha \mathfrak{A}}{\lambda + 2\mu} \frac{\beta}{\beta^2 - \gamma^2} e^{-\gamma y} \left. \begin{aligned} &\sin \\ &-\cos \end{aligned} \right\} \beta x, \\ v_1 &= \frac{\alpha \mathfrak{A}}{\lambda + 2\mu} \frac{\gamma}{\beta^2 - \gamma^2} e^{-\gamma y} \left. \begin{aligned} &\cos \\ &\sin \end{aligned} \right\} \beta x. \end{aligned} \right\} \dots\dots\dots (21)$$

When $\gamma = \beta$, $u_1 = v_1 = 0$.

The displacement (u_2, v_2) satisfying Δ_2 expressed by (18), and $2\varpi = 0$, is written by

$$\left. \begin{aligned} u_2 &= \frac{1}{2} B y e^{-\beta y} \left. \begin{aligned} &\sin \\ &-\cos \end{aligned} \right\} \beta x, \\ v_2 &= -\frac{B}{2\beta} (1 - \beta y) e^{-\beta y} \left. \begin{aligned} &\cos \\ &\sin \end{aligned} \right\} \beta x. \end{aligned} \right\} \dots\dots\dots (22)$$

If (u_3, v_3) is the displacement corresponding to the rotation of (19) and satisfying $\Delta_1 + \Delta_2 = 0$,

then

$$\left. \begin{aligned} u_3 &= -\frac{D}{2\beta} (1 - \beta y) e^{-\beta y} \left. \begin{aligned} &\sin \\ &-\cos \end{aligned} \right\} \beta x, \\ v_3 &= \frac{D}{2} y e^{-\beta y} \left. \begin{aligned} &\cos \\ &\sin \end{aligned} \right\} \beta x, \end{aligned} \right\} \dots\dots\dots (23)$$

and if we assume that the displacement (u_4, v_4) is to satisfy the condition $\Delta_1 + \Delta_2 = 2\varpi = 0$, we can write

$$\left. \begin{aligned} u_4 &= F e^{-\beta y} \left. \begin{aligned} &\sin \\ &-\cos \end{aligned} \right\} \beta x, \\ v_4 &= F e^{-\beta y} \left. \begin{aligned} &\cos \\ &\sin \end{aligned} \right\} \beta x. \end{aligned} \right\} \dots\dots\dots (24)$$

Therefore the general expressions of the displacement (u, v) are of the following forms:

11) K. SEZAWA. *Bull. Earthq. Res. Inst.*, Tokyo, 2 (1927), 3 (1927), 4 (1928), 5 (1928), 6 (1929), 7 (1929).

Now, from the beginning of this section we have not fully examined the relation of β and γ , but in the following sections we shall consider the relation of β and γ . If not, we cannot proceed our course of study. In the 4th section we assume that β and γ are independent each other, and in the 5th section β and γ are connected by the following relation :

$$\gamma = \gamma'^n \beta, \quad \dots\dots\dots(34)$$

where γ' is a mere number which is arbitrary, and n is the number of 1, 2, 3.....

The above mentioned assumptions on the relation between β and γ do not lose the generality of the problem as will be explained in the following sections.

4. When the temperature distribution of the surface of a semi-infinite elastic body is of the form of $f(x)$, and the surface is free from tractions, we can construct the appropriate forms of the temperature distribution in the body, and the displacement at any point of the body by using the equations of (14) and (31) in the following system.

The temperature distribution in the solid, when the surface distribution is $f(x)$, is expressed in the form :

$$T = \frac{1}{\pi} \int_0^\infty d\beta \int_{-\infty}^\infty e^{-\gamma y} f(x) \cos \beta (x-z) dx. \quad \dots\dots\dots(35)$$

The components of the displacement (u, v) are as follows :

$$\begin{aligned} u = & -\frac{\alpha}{\pi(\lambda+2\mu)} \left[\frac{(\lambda+2\mu)\gamma}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \sin \beta (x-z) dx \right. \\ & - \frac{\mu}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \sin \beta (x-z) \beta dx \\ & + y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta+\gamma} \sin \beta (x-z) \beta dx \\ & \left. - e^{-\gamma y} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{\sin \beta (x-z) \beta}{\beta^2-\gamma^2} dx \right], \quad \dots\dots\dots(36) \end{aligned}$$

$$\begin{aligned}
v = & -\frac{\alpha}{\pi(\lambda+2\mu)} \left[\frac{\lambda+2\mu}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \cos \beta(x-z) \beta dz \right. \\
& - \frac{\mu\gamma}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \cos \beta(x-z) dz \\
& + y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta+\gamma} \cos \beta(x-z) \beta dz \\
& \left. - \gamma e^{-\gamma y} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{\cos \beta(x-z)}{\beta^2-\gamma^2} dz \right]. \dots\dots\dots(37)
\end{aligned}$$

The inclination $\frac{\partial v}{\partial x}$ can easily be written such that

$$\begin{aligned}
\frac{\partial v}{\partial x} = & \frac{\alpha}{\pi(\lambda+2\mu)} \left[\frac{\lambda+2\mu}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \sin \beta(x-z) \beta^2 dz \right. \\
& - \frac{\mu\gamma}{\lambda+\mu} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta^2-\gamma^2} \sin \beta(x-z) \beta dz \\
& + y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta+\gamma} \sin \beta(x-z) \beta^2 dz \\
& \left. - \gamma e^{-\gamma y} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{\sin \beta(x-z)}{\beta^2-\gamma^2} \beta dz \right]. \dots\dots\dots(38)
\end{aligned}$$

The stresses X_x , Y_y and X_y can easily be determined similarly, but as these are not needed in our study, these formulae are omitted in this paper.

5. In this section we take the example of the 4th section, that the temperature distribution of the surface of the body is limited by the following form :

$$f(x) = B, \quad a > x > -a. \dots\dots\dots(39)$$

Then the temperature distribution in the body is formed by using the formula (35):

$$t = \frac{B}{\pi} e^{-\gamma y} \int_0^{\infty} \frac{1}{\beta} \left\{ \sin \beta(x+a) - \sin \beta(x-a) \right\} d\beta \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \dots\dots\dots(40)$$

$$= B e^{-\gamma y}, \text{ for } a > x > -a,$$

$$= 0, \quad x > a \quad \text{and} \quad x < -a.$$

From the formulae (37) and (38), we can find the following forms of displacement v and inclination $\frac{\partial v}{\partial x}$:

$$v = -\frac{\alpha B}{\pi(\lambda+2\mu)} \left[\frac{\lambda+2\mu}{\lambda+\mu} \int_0^{\infty} \frac{e^{-\beta y}}{\beta^2-\lambda^2} \left\{ \sin \beta(x+a) - \sin \beta(x-a) \right\} d\beta \right.$$

$$\left. - \frac{\mu\gamma}{\lambda+\mu} \int_0^{\infty} \frac{e^{-\beta y}}{\beta^2-\gamma^2} \left\{ \sin \beta(x+a) - \sin \beta(x-a) \right\} \frac{d\beta}{\beta} \right.$$

$$+ y \int_0^{\infty} \frac{e^{-\beta y}}{\beta+\gamma} \left\{ \sin \beta(x+a) - \sin \beta(x-a) \right\} d\beta$$

$$\left. - \gamma e^{-\gamma y} \int_0^{\infty} \frac{\sin \beta(x+a) - \sin \beta(x-a)}{\beta^2-\gamma^2} \frac{d\beta}{\beta} \right], \dots\dots(41)$$

$$\frac{\partial v}{\partial x} = \frac{\alpha B}{\pi(\lambda+2\mu)} \left[\frac{\lambda+2\mu}{\lambda+\mu} \int_0^{\infty} \frac{e^{-\beta y}}{\beta^2-\lambda^2} \left\{ \cos \beta(x-a) - \cos \beta(x+a) \right\} \beta d\beta \right.$$

$$\left. - \frac{\mu\gamma}{\lambda+\mu} \int_0^{\infty} \frac{e^{-\beta y}}{\beta^2-\gamma^2} \left\{ \cos \beta(x-a) - \cos \beta(x+a) \right\} d\beta \right.$$

$$+ y \int_0^{\infty} \frac{e^{-\beta y}}{\beta+\gamma} \left\{ \cos \beta(x-a) - \cos \beta(x+a) \right\} \beta d\beta$$

$$\left. - \gamma e^{-\gamma y} \int_0^{\infty} \frac{\cos \beta(x-a) - \cos \beta(x+a)}{\beta^2-\gamma^2} d\beta \right]. \dots(42)$$

Now, to find the surface displacement and the surface inclination, we must first construct the following integral formulae¹²⁾:

12) These formulae are easily obtained from the integral formulae obtained by Prof. K. Sezawa.

K. SEZAWA, "Some Problems of Shocks transmitted in Bars and in Plate," *Report of Aeron. Res. Inst.*, Tokyo, 4, No. 4 (1928).

K. SEZAWA, "On the Propagation of the Leading & Trailing Parts of a Train of Elastic Waves," *Bull. Earthq. Res. Inst.*, Tokyo, 4 (1928).

for $a > 0$,

$$\left. \begin{aligned} \int_0^\infty \frac{\cos ax}{x-k} dx &= -\frac{\pi}{2} \sin ka - Cika \cos ka - Sika \sin ka, \\ \int_0^\infty \frac{\sin ax}{x-k} dx &= \frac{\pi}{2} \cos ka - Cika \sin ka + Sika \cos ka, \\ \int_0^\infty \frac{\cos ax}{x+k} dx &= -Cika \cos ka + \left(\frac{\pi}{2} - Sika\right) \sin ka, \\ \int_0^\infty \frac{\sin ax}{x+k} dx &= \left(\frac{\pi}{2} - Sika\right) \cos ka + Cika \sin ka, \end{aligned} \right\} \dots\dots(43)$$

and for $a < 0$,

$$\left. \begin{aligned} \int_0^\infty \frac{\cos ax}{x-k} dx &= \frac{\pi}{2} \sin ka - Cika \cos ka - Sika \sin ka, \\ \int_0^\infty \frac{\sin ax}{x-k} dx &= -\frac{\pi}{2} \cos ka - Cika \sin ka + Sika \cos ka, \\ \int_0^\infty \frac{\cos ax}{x+k} dx &= -Cika \cos ka - \left(\frac{\pi}{2} + Sika\right) \sin ka, \\ \int_0^\infty \frac{\sin ax}{x+k} dx &= -\left(\frac{\pi}{2} + Sika\right) \cos ka + Cika \sin ka, \end{aligned} \right\} \dots\dots(44)$$

where Cix and Six are the integralcosine and integralsine in the form of

$$\int_0^\infty \frac{\cos u}{u} du, \quad \int_x^\infty \frac{\sin u}{u} du.$$

By using the equations (43) and (44), we can evaluate the integrals in (41) and (42) when $y = 0$ as follows:

$x < -a$:—

$$\begin{aligned} v = & -\frac{\alpha B}{r\pi(\lambda + \mu)} \left[\frac{\pi}{2} \left\{ \cos r(x+a) - \cos r(x-a) \right\} \right. \\ & + Si\,r(x+a) \cos r(x+a) + Ci\,r(x-a) \sin r(x-a) \\ & \left. - Ci\,r(x+a) \sin r(x+a) - Si\,r(x-a) \cos r(x-a) \right], \quad \dots\dots\dots(45) \end{aligned}$$

— $a < x < a$:—

$$v = -\frac{\alpha B}{\gamma \pi (\lambda + \mu)} \left[\pi - \left\{ \frac{\pi}{2} - Si \gamma (x+a) \right\} \cos \gamma (x+a) \right. \\ \left. - Ci \gamma (x+a) \sin \gamma (x+a) - \left\{ \frac{\pi}{2} + Si \gamma (x-a) \right\} \cos \gamma (x-a) \right. \\ \left. + Ci \gamma (x-a) \sin \gamma (x-a) \right], \dots\dots(46)$$

and $x > a$:—

$$v = -\frac{\alpha B}{\gamma \pi (\lambda + \mu)} \left[\frac{\pi}{2} \left\{ \cos \gamma (x-a) - \cos \gamma (x+a) \right\} \right. \\ \left. + Si \gamma (x+a) \cos \gamma (x+a) + Ci \gamma (x-a) \sin \gamma (x-a) \right. \\ \left. - Ci \gamma (x+a) \sin \gamma (x+a) - Si \gamma (x-a) \cos \gamma (x-a) \right]. \dots\dots(47)$$

Similarly we can construct the displacement u and the inclination $\frac{\partial v}{\partial x}$. But we omit these operations in this section. From the results expressed by (45), (46) and (47), it may be noticed that the displacement of the surface of the body depends extraordinarily upon the tendency of the temperature distribution, and the surface form of the deformation is somewhat similar to the form of temperature distribution of the surface. The magnitude of the displacement at any point of the body is proportional to the temperature at that point, and also the distribution of the displacement is much affected by the temperature distribution in the vertical direction of the solid. When the temperature distribution of the surface of the body is constant but that of the interior of the body is variable according to the law of $e^{-\gamma y}$ in which γ is variable, the surface displacement is much affected by the temperature distribution in the interior of the body. When γ is very small, namely the temperature effect in the interior of the solid is not so small, the surface displacement is large. But, if γ is large, the surface displacement is less. The magnitude of γ changes the temperature distribution in the interior of the body, i.e, for larger γ the temperature distribution is more concentrated in the neighbourhood of the surface of the body. We know that large γ corresponds to the daily variation, and small γ the annular variation of the temperature distribution in the interior of the earth.

When we compile the result of (40), we get the curves in Figs. 2a & 2b. We shall perhaps understand from the figure what we have discussed in the above paragraph.

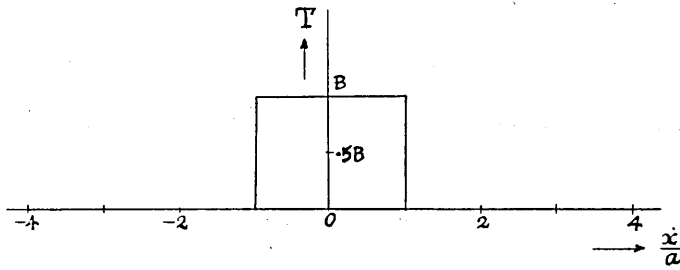


Fig. 2a. Temperature distribution of the surface of the solid.

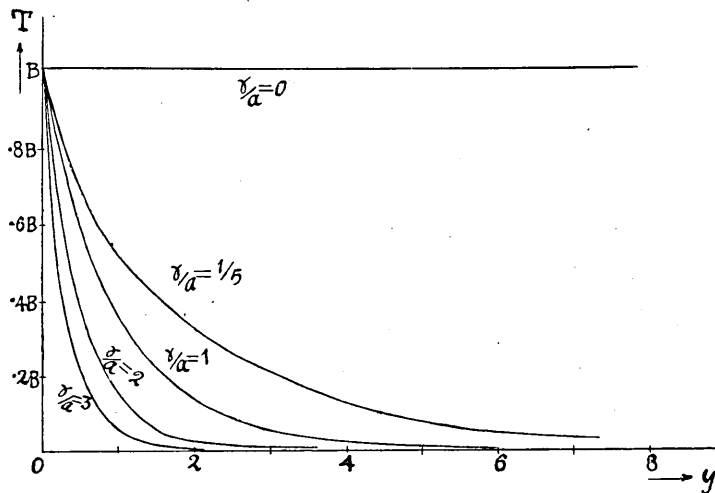
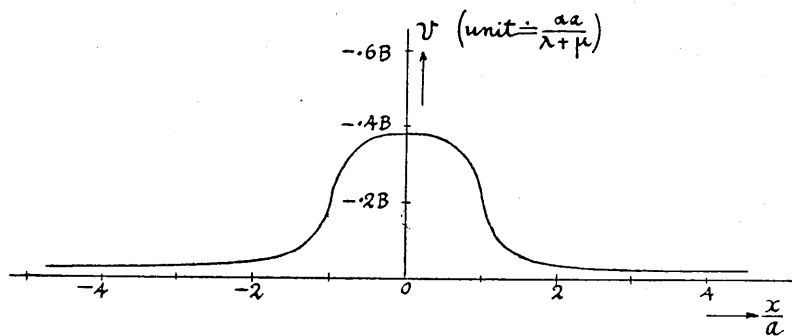


Fig. 2b. $\left[\frac{x}{a} = 0. \right]$

The surface displacement is drawn in Figure 3 which is obtained from (45), (46) and (47), when $\gamma = \frac{2}{a}$.

As actual examples, we take the following two cases in which the temperature of 100°C is distributed over the surface of the body about 20 meters long, and γ/a is taken $1/5$ and 2 ; namely the effects of the surface temperature in the earth at $x = 0$ are given by the following table. (next page)

Fig. 3. [$\gamma = 2/a$.]

distance from the surface (y)	$\gamma/a = 1/5$ (1st example)	$\gamma/a = 2$ (2nd example)
0 meter	100°C	100°C
1 meter	50°C	14°C
2 meters	32°C	2°C
4 meters	14°C	0°C

If we take about $.24 \times 10^{-4}$ as the cubical expansion coefficient and the conditions of the elasticity to be $\lambda = \infty$ & $\lambda = \mu$, we can calculate the surface displacements from the above formulae which are shown in the next table :

λ	$\gamma/a = 1/5$	$\gamma/a = 2$
∞	-.0332 cm	-.00896 cm
μ	-.0279 cm	-.00748 cm

6. In the same manner as we mentioned in Section 3, we shall proceed to the case that γ is a function of β , namely $\gamma = \gamma' \beta^n$, [$n=1, 2, 3 \dots$]. For the sake of simplicity, we take $n=1$, i.e. $\gamma = \gamma' \beta$, where γ' is a mere number.

When $\gamma = \gamma' \beta$, we can easily get the temperature distribution, displacement and inclination by the same method described in Section 4.

The temperature distribution, being expressed by

$$T = \frac{1}{\pi} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\gamma' \beta y} \cos \beta (x-z) dz, \dots\dots\dots (48)$$

the components of displacement (u , v) and the inclination $\frac{\partial v}{\partial x}$ are of the following expressions :

$$\begin{aligned}
 u = & -\frac{\alpha}{\pi(\lambda+2\mu)(1+\gamma')} \left[y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\beta y} \sin \beta(x-z) dx \right. \\
 & - \frac{\mu-(\lambda+2\mu)\gamma'}{(\lambda+\mu)(1-\gamma')} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta} \sin \beta(x-z) dx \\
 & \left. - \frac{1}{1-\gamma'} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\gamma'\beta y}}{\beta} \sin \beta(x-z) dx \right], \quad [\gamma' \neq 1] \\
 & \dots\dots\dots(49)
 \end{aligned}$$

$$\begin{aligned}
 v = & -\frac{\alpha}{\pi(\lambda+2\mu)(1+\gamma')} \left[y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\beta y} \cos \beta(x-z) dx \right. \\
 & - \frac{\mu\gamma'-(\lambda+2\mu)}{(\lambda+\mu)(1-\gamma')} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta} \cos \beta(x-z) dx \\
 & \left. - \frac{1}{1-\gamma'} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\gamma'\beta y}}{\beta} \cos \beta(x-z) dx \right], \quad [\gamma' \neq 1] \\
 & \dots\dots\dots(50)
 \end{aligned}$$

$$\left. \begin{aligned}
 u &= \frac{\alpha}{2\pi(\lambda+\mu)} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta} \sin \beta(x-z) dx, \\
 v &= -\frac{\alpha}{2\pi(\lambda+\mu)} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) \frac{e^{-\beta y}}{\beta} \cos \beta(x-z) dx,
 \end{aligned} \right\} \quad [\gamma' = 1]$$

\dots\dots\dots(51)

and

$$\begin{aligned}
 \frac{\partial v}{\partial x} = & \frac{\alpha}{\pi(\lambda+2\mu)(1+\gamma')} \left[y \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\beta y} \sin \beta(x-z) \beta dx \right. \\
 & - \frac{\mu\gamma'-(\lambda+2\mu)}{(\lambda+\mu)(1-\gamma')} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\beta y} \sin \beta(x-z) dx \\
 & \left. - \frac{1}{1-\gamma'} \int_0^\infty d\beta \int_{-\infty}^\infty f(x) e^{-\gamma'\beta y} \sin \beta(x-z) dx \right], \quad [\gamma' \neq 1] \\
 & \dots\dots\dots(52)
 \end{aligned}$$

$$\frac{\partial v}{\partial x} = \frac{\alpha}{2\pi(\lambda + \mu)} \int_0^\infty d\beta \int_{-\infty}^\infty f(z) e^{-\beta y} \sin \beta(x-z) dz. \quad [\gamma' = 1] \quad \dots\dots(53)$$

7. As an example of the section 8 just we have studied, we shall take the temperature distribution of the following form on the surface of the body :

$$f(x) = \frac{Ba^2}{b^2 + x^2}, \quad \dots\dots\dots(54)$$

where a and b are the constants which can be adjusted to obey to the distribution of the surface temperature to a wide range, and B is the measurement of temperature, having no dimension.

Then from (48), we can find the following form of temperature distribution :

$$T = \frac{Ba^2}{b} \int_0^\infty e^{-(\gamma'y+b)\beta} \cos \beta x d\beta, \quad \dots\dots\dots(55)$$

by the aid of

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-\alpha}. \quad \dots\dots\dots(56)$$

Now we can easily evaluate the integral of (55), using the integral formula

$$\int_0^\infty e^{-bf} \cos fx df = \frac{b}{b^2 + x^2}, \quad \dots\dots\dots(57)$$

in the form of

$$T = \frac{Ba^2}{b} \times \frac{b + \gamma'y}{x^2 + (b + \gamma'y)^2} \quad \dots\dots\dots(58)$$

Now, the inclination $\frac{\partial v}{\partial x}$ due to the temperature distribution expressed by the equation (58) is written from (52) in the following form :

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\alpha B a^2}{b(\lambda + 2\mu)(1 + \gamma')} \left[y \frac{\partial}{\partial x} \int_0^\infty e^{-(b+y)\beta} \cos \beta x d\beta \right. \\
&\quad - \frac{\mu\gamma' - (\lambda + 2\mu)}{(\lambda + \mu)(1 - \gamma')} \int_0^\infty e^{-(b+y)\beta} \sin \beta x d\beta \\
&\quad \left. - \frac{\gamma'}{1 - \gamma'} \int_0^\infty e^{-(b+\gamma'y)\beta} \sin \beta x d\beta \right] \\
&= \frac{\alpha B a^2}{b(\lambda + 2\mu)(1 + \gamma')} \left[\frac{\lambda + (2 - \gamma')\mu}{(\lambda + \mu)(1 - \gamma')} \frac{x}{x^2 + (b + y)^2} \right. \\
&\quad \left. - \frac{2xy(b + y)}{\{x^2 + (b + y)^2\}^2} - \frac{\gamma'}{1 - \gamma'} \frac{x}{x^2 + (b + \gamma'y)^2} \right], \\
&\quad [\gamma' \neq 1] \quad \dots\dots(59)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\alpha B a^2}{2(\lambda + \mu)b} \int_0^\infty e^{-(b+y)\beta} \sin \beta x d\beta \\
&= \frac{\alpha B a^2}{2b(\lambda + \mu)} \frac{x}{x^2 + (b + y)^2}. \quad [\gamma' = 1] \quad \dots\dots\dots(60)
\end{aligned}$$

The surface tilt due to temperature distribution of (58) is expressed by

$$\frac{\partial v}{\partial x} = \frac{\alpha B a^2}{b(\lambda + \mu)(1 + \gamma')} \frac{x}{(b^2 + x^2)}. \quad \dots\dots\dots(61)$$

We can see from the above formula that the inclination of the surface is extraordinarily affected by the number of γ' , namely the inclination $\frac{\partial v}{\partial x}_{y=0}$ is inversely proportional to $(1 + \gamma')$. The value of γ' , as we have discussed in the above section, changes the temperature distribution in the interior of the solid, and the larger the value of γ' is, the more intensively diminishes the effect of the surface distribution of the temperature in the interior of the solid. Hence we may suppose that the large value of γ' corresponds to the daily variation of the temperature, and the small value of γ' to the annular variation of the

temperature in the interior of the earth. These relations are given in Figures 4a, 4b, which are compiled from the equation of (58).

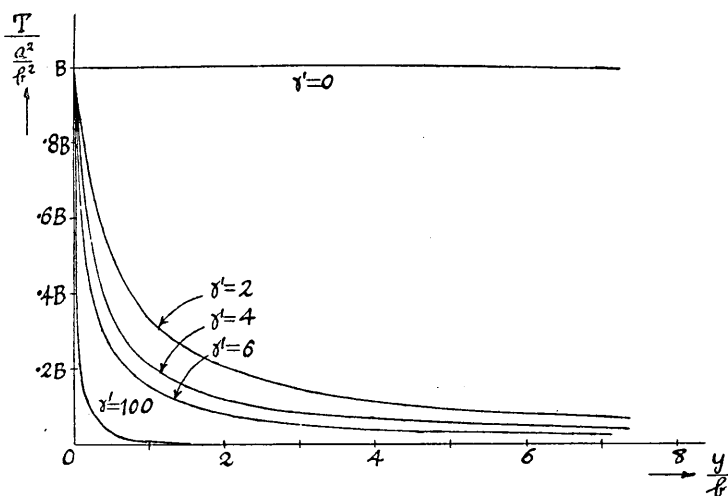


Fig. 4a. $\left[\frac{x}{b} = 0.\right]$

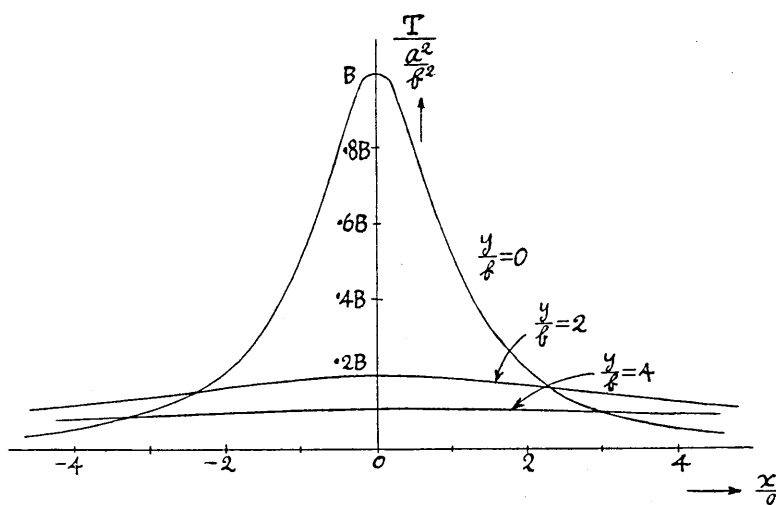


Fig. 4b. $[\gamma' = 2.]$

From the above figures, we can easily understand the above mentioned facts. In Fig. 4a the temperature distributions in the interior of the solid are compiled, when $x/a=0$ and $\gamma'=0, 2, 4, 6$ & 100 , and Fig. 4b shows the temperature distribution when $y/b=0, 2$ & 4 , and we can see that the effect of the surface distribution in the interior of the solid is extraordinarily small when the value of γ' is large.

Now, from the equations (59), (61) we can easily see that the surface tilt is limited by the value of γ' , namely the maximum inclination occurs when γ' is zero, that is the case of the constant temperature distribution at all depth in the interior of the solid. These relations will be seen in Figure 5.

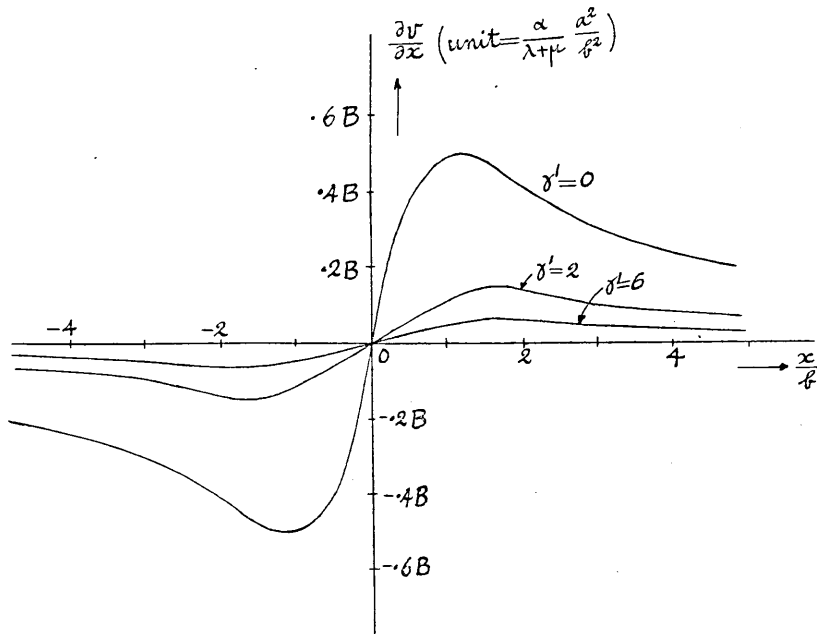


Fig. 5.

In the following two examples, we assume that γ' shall be taken 2 and 6, and the surface temperature distribution is given in Table I, and the effects of these surface temperature distribution in the interior of the solid at $x=0$ are written in Table II.

Table I.

distance from the origin (x)	surface temperature
0 meter	100°C
1 meter	50°C
2 meters	20°C
4 meters	5°C
6 meters	3°C

Table II.

distance from the surface (y)	$\gamma' = 2$ (1st example)	$\gamma' = 6$ (2nd example)
0 meter	100°C	100°C
1 meter	33°C	15°C
2 meters	22°C	7°C
4 meters	11°C	4°C
6 meters	8°C	3°C
8 meters	6°C	2°C

When we take these two cases, we have the following surface tilt at the point 2 meters far from the center, as given in the following table.

$\gamma' = 2$ (case I)	$\gamma' = 6$ (case II)
2.7×10^{-4}	1.3×10^{-4}

[$\lambda = \mu$ & $c = .24 \times 10^{-4}$.]

8. As a second example of the section 6, we shall take the case of the following surface temperature distribution of the body:

$$f(x) = B, \quad -a < x < a. \quad \dots\dots\dots(62)$$

This case corresponds to the example of Section 5, in which case γ is independent to β , but in this example $\gamma = \gamma'\beta$. The physical nature of temperature distribution is not different in two cases.

In this case we have from (48) the following form of temperature distribution by using the integral formula

$$\int_0^\infty e^{-ax} \frac{\sin px}{x} dx = \tan^{-1} \frac{p}{a}, \quad \dots\dots\dots(63)$$

$$T = \frac{2B}{\pi} \int_0^\infty \frac{e^{-\gamma'\beta y}}{\beta} \cos \beta x \sin \beta a d\beta$$

$$= \frac{B}{\pi} \left[\tan^{-1} \frac{x+a}{\gamma'y} + \tan^{-1} \frac{a-x}{\gamma'y} \right] \quad -a < x < a$$

$$= 0. \quad a < x, \quad -a > x \quad \dots\dots\dots(64)$$

In Section 5 we have studied the surface deformation, but in this section we shall discuss the inclinations at all depth of the body due to the surface temperature distribution represented by (62).

Then from the formula expressed by (52), we have the following formula of inclination at all depth of the body :

$$\begin{aligned} \frac{\partial v}{\partial x} = & \frac{2B\alpha}{\pi(\lambda+2\mu)(1+\gamma')} \left[y \int_0^\infty e^{-\beta y} \sin \beta x \sin \beta a \, d\beta \right. \\ & - \frac{\mu\gamma' - (\lambda+2\mu)}{(\lambda+\mu)(1-\gamma')} \int_0^\infty \frac{e^{-\beta y}}{\beta} \sin \beta x \sin \beta a \, d\beta \\ & \left. - \frac{\gamma'}{1-\gamma'} \int_0^\infty \frac{e^{-\gamma'\beta y}}{\beta} \sin \beta x \sin \beta a \, d\beta \right], \quad [\gamma' \neq 1] \dots (65) \end{aligned}$$

$$\frac{\partial v}{\partial x} = \frac{B\alpha}{\pi(\lambda+\mu)} \int_0^\infty \frac{e^{-\beta y}}{\beta} \sin \beta x \sin \beta a \, d\beta, \quad [\gamma' = 1] \dots (66)$$

Now using the following formulae,

$$\left. \begin{aligned} \int_0^\infty e^{-\beta x} \frac{\cos \alpha x - \cos \xi x}{x} \, dx &= \frac{1}{2} \log \frac{\beta^2 + \xi^2}{\beta^2 + \alpha^2}, \\ \int_0^\infty e^{-ax} \cos cx \, dx &= \frac{a}{a^2 + c^2}, \end{aligned} \right\} \dots (67)$$

we can easily evaluate the integrals of (65) & (66) in the following forms :

$$\begin{aligned} \frac{\partial v}{\partial x} = & \frac{B\alpha}{\pi(\lambda+2\mu)(1+\gamma')} \left[\frac{\lambda+(2-\gamma')\mu}{2(\lambda+\mu)(1-\gamma')} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right. \\ & - \frac{\gamma'}{2(1-\gamma')} \log \frac{(x+a)^2 + \gamma'^2 y^2}{(x-a)^2 + \gamma'^2 y^2} \\ & \left. + \frac{4axy^2}{\{(x+a)^2 + y^2\} \{(x-a)^2 + y^2\}} \right], \\ & [\gamma' \neq 1] \dots (68) \end{aligned}$$

$$\frac{\partial v}{\partial x} = \frac{B\alpha}{4\pi(\lambda+\mu)} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}, \quad [\gamma' = 1] \dots (69)$$

and the surface tilt is of the form of

$$\frac{\partial v}{\partial x_{y=0}} = \frac{B\alpha}{2\pi(\lambda + \mu)(1 + \gamma')} \log \frac{(x+a)^2}{(x-a)^2} \dots\dots\dots(68)'$$

From the equation (64), we can see that the temperature effect in the interior of the body is more decreased as larger the value of γ' becomes. This nature is shown in Figure 6b which is compiled from (64) at $x=0$.

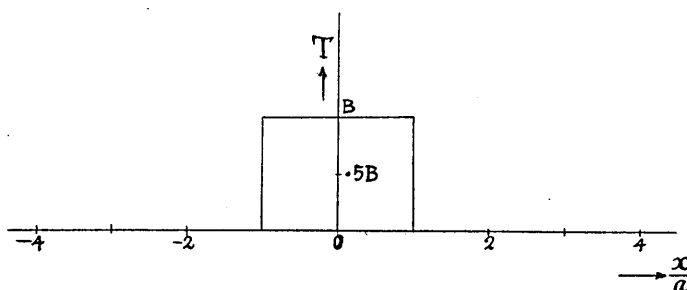


Fig. 6a. [$y = 0$.]

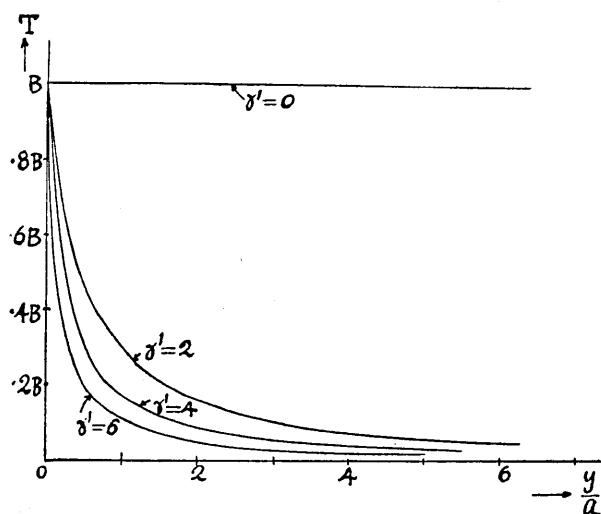


Fig. 6b. [$x = 0$.]

The equations (68), (69) teach us that for any temperature of the surface of the solid the inclination at a large depth is very small,

and the inclination of the surface is extraordinarily affected by the number of γ' , namely it is inversely proportional to $(1+\gamma')$. These natures have been shown in Section 5.

If we compile the inclination at the surface when $\gamma'=0, 1, 2$ & 4, we have Figure 7 from the formulae (68) & (69).

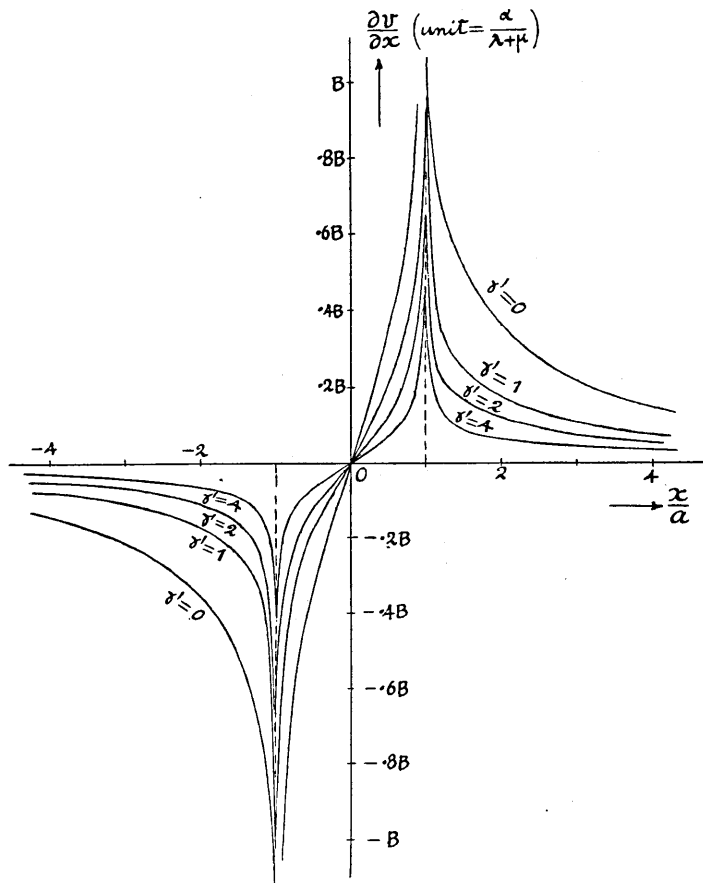


Fig. 7. [$y = 0$.]

At the point of $x = \pm a$, the inclination of the surface is infinite. This nature is not violent when we know the temperature distribution of the surface of the solid.

Part III. Three-dimensional Problem.

9. In the above sections we have studied the two-dimensional problem, and we have many valuable natures at the study of the effects of temperature distribution of the surface on the deformation of the solid. In the present and succeeding sections we shall take the three-dimensional problems using the cylindrical coordinates, which may give us some other natures of the temperature effects on the deformation of the solid.

The coordinates (r, θ, z) are taken, and the plane $z=0$ is the surface of the semi-infinite elastic solid, the axis of z is taken vertically downwards and r is the distance from the origin O , θ the azimuthal angle round the z -axis of the solid. (Fig. 8)

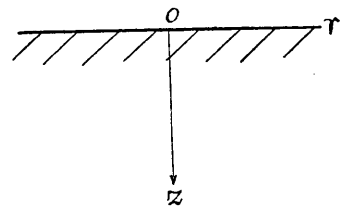


Fig. 8.

In Section 2 we have obtained the expressions (6) which are the general equations of equilibrium of an elastic solid when variations of temperature are supposed. If we take the cylindrical coordinates, we have the following forms of equations:

$$\left. \begin{aligned} (\lambda + 2\mu) \frac{\partial A}{\partial r} - \frac{2\mu}{r} \frac{\partial \varpi_z}{\partial \theta} + 2\mu \frac{\partial \varpi_\theta}{\partial z} &= \alpha \frac{\partial T}{\partial r}, \\ (\lambda + 2\mu) \frac{1}{r} \frac{\partial A}{\partial \theta} - 2\mu \frac{\partial \varpi_r}{\partial z} + 2\mu \frac{\partial \varpi_z}{\partial r} &= \alpha \frac{\partial T}{r \partial \theta}, \\ (\lambda + 2\mu) \frac{\partial A}{\partial r} - \frac{2\mu}{r} \frac{\partial}{\partial r} (r \varpi_\theta) + \frac{2\mu}{r} \frac{\partial \varpi_r}{\partial \theta} &= \alpha \frac{\partial T}{\partial z}, \end{aligned} \right\} \dots\dots\dots (70)$$

where

$$\left. \begin{aligned} A &= \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}, \\ 2\varpi_r &= \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \\ 2\varpi_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \\ 2\varpi_z &= \frac{1}{r} \frac{\partial (rv)}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \theta}, \end{aligned} \right\} \dots\dots\dots (71)$$

and T is the difference of the temperature, and

$$\alpha = \left(\lambda + \frac{2}{3} \mu \right) c. \dots\dots\dots (72)$$

Now we know that there is the following identical relation between ϖ_r , ϖ_θ and ϖ_z :

$$\frac{1}{r} \frac{\partial (r \varpi_r)}{\partial r} + \frac{1}{r} \frac{\partial \varpi_\theta}{\partial \theta} + \frac{\partial \varpi_z}{\partial z} = 0. \dots\dots\dots (73)$$

Using this relation, we can have the following expressions¹³⁾ from (70):

$$\begin{aligned} & \frac{\partial^2}{\partial r^2} [(\lambda + 2\mu) \Delta - \alpha T] + \frac{1}{r} \frac{\partial}{\partial r} [(\lambda + 2\mu) \Delta - \alpha T] \\ & + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} [(\lambda + 2\mu) \Delta - \alpha T] + \frac{\partial^2}{\partial z^2} [(\lambda + 2\mu) \Delta - \alpha T] = 0, \dots\dots\dots (74) \end{aligned}$$

$$\frac{\partial^2 \varpi_r}{\partial r^2} + \frac{3}{r} \frac{\partial \varpi_r}{\partial r} + \frac{\varpi_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 \varpi_r}{\partial \theta^2} + \frac{\partial^2 \varpi_r}{\partial z^2} + \frac{2}{r} \frac{\partial \varpi_z}{\partial z} = 0, \dots\dots\dots (75)$$

$$\frac{\partial^2 \varpi_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \varpi_\theta}{\partial r} - \frac{\varpi_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 \varpi_\theta}{\partial \theta^2} + \frac{\partial^2 \varpi_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial \varpi_r}{\partial \theta} = 0, \dots\dots\dots (76)$$

$$\frac{\partial^2 \varpi_z}{\partial r^2} + \frac{1}{r} \frac{\partial \varpi_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varpi_z}{\partial \theta^2} + \frac{\partial^2 \varpi_z}{\partial z^2} = 0. \dots\dots\dots (77)$$

Now we assume that the temperature distribution of the body is expressed by the following form:

$$T = \mathfrak{U} e^{-\beta z} J_m(kr) \frac{\sin}{\cos} m\theta. \dots\dots\dots (78)$$

This assumption is not made at random, but this equation gives the temperature distribution at a moment in the conduction of heat in the body, and the zero temperature at $z = \infty$. Therefore this equation is very favourable for our study. In (78) \mathfrak{U} is any constant, and a measurement of the magnitude of temperature, and m , k and β are any constants not defined in this section.

13) The reductions of these equations when $T = 0$ have been made by Prof. K. Sezawa.

K. SEZAWA, *Bull. Earthq. Res. Inst.*, 6 (1929), 1.

Then, from the equation (74), we have

$$\frac{\partial^2 \mathcal{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{A}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \mathcal{A}}{\partial \theta^2} + \frac{\partial^2 \mathcal{A}}{\partial z^2} = \frac{\alpha \mathfrak{A}}{\lambda + 2\mu} e^{-\beta z} J_m(kr) \left\{ \frac{\sin}{\cos} \right\} m\theta. \dots (79)$$

The particular solution of this equation is of the following form :

$$\mathcal{A}_1 = \alpha \mathfrak{A} \frac{e^{-\beta z}}{\lambda + 2\mu} J_m(kr) \left\{ \frac{\sin}{\cos} \right\} m\theta, \dots (80)$$

and the complementary solution, which is useful for our study, is

$$\mathcal{A}_2 = A_m e^{-kz} J_m(kr) \left\{ \frac{\sin}{\cos} \right\} m\theta. \dots (81)$$

The solution of (77), which is suitable to this case, is of the form of

$$2\varpi_z = B_m e^{-kz} J_m(kr) \left\{ \frac{-\cos}{\sin} \right\} m\theta. \dots (82)$$

We have thus the following expressions of ϖ_r and ϖ_θ from the equations (75) and (76):

$$2\varpi_r = \left[C_m \frac{J_m(kr)}{r} - B_m \frac{1}{k} \frac{\partial J_m(kr)}{\partial r} \right] e^{-kz} \left\{ \frac{-\cos}{\sin} \right\} m\theta, \dots (83)$$

$$2\varpi_\theta = \left[C_m \frac{1}{m} \frac{\partial J_m(kr)}{\partial r} - B_m \frac{m}{k} \frac{J_m(kr)}{r} \right] e^{-kz} \left\{ \frac{\sin}{\cos} \right\} m\theta. \dots (84)$$

In the above expressions of \mathcal{A} , ϖ_r , ϖ_θ and ϖ_z , A_m , B_m , C_m are arbitrary constants. From the equilibrium equations expressed by (70), we can find the relation between A_m and C_m :

$$A_m/C_m = \frac{k}{m} \frac{\mu}{\lambda + 2\mu}, \dots (85)$$

when $\beta = k$,

$$(\lambda + 2\mu) A_m - \frac{\mu k}{m} C_m = \alpha \mathfrak{A}. \dots (86)$$

As we know the relations among the displacement (u , v , w), the dilatations (\mathcal{A}_1 , \mathcal{A}_2) and the rotations (ϖ_r , ϖ_θ , ϖ_z) as seen in (71), we can find the displacements from \mathcal{A}_1 , \mathcal{A}_2 , ϖ_r , ϖ_θ , ϖ_z expressed by (80), (81), (82), (83) and (84) as the following system.¹⁴⁾

14) Prof. K. Sezawa devised this analytical method of obtaining the displacements except (u_1 , v_1 , w_1) *Bull. Earthq. Res. Inst.*, 6 (1929), 1. Even though the expressions of the displacement can also be obtained in the manner as Prof. K. Terazawa employed in his paper on the problem of tidal loading, *Jour. Coll. Soc.*, Tokyo, 37 (1916), his method of deduction is too complex to introduce in this analytical calculation.

Now the displacement (u_1, v_1, w_1) satisfying Δ_1 expressed by (80) and $\varpi_r = \varpi_\theta = \varpi_z = 0$, is expressed by

$$\left. \begin{aligned} u_1 &= \frac{\alpha \mathfrak{A}}{\lambda + 2\mu} \frac{e^{-\beta z}}{\beta^2 - k^2} \frac{\partial J_m(kr)}{\partial r} \frac{\sin}{\cos} \} m\theta, \\ v_1 &= -\frac{\alpha \mathfrak{A}}{\lambda + 2\mu} \frac{m e^{-\beta z}}{\beta^2 - k^2} \frac{J_m(kr)}{r} \frac{-\cos}{\sin} \} m\theta, \\ w_1 &= -\frac{\alpha \mathfrak{A}}{\lambda + 2\mu} \frac{\beta e^{-\beta z}}{\beta^2 - k^2} J_m(kr) \frac{\sin}{\cos} \} m\theta. \end{aligned} \right\} \dots\dots\dots(86)$$

The displacement (u_2, v_2, w_2) satisfying Δ_2 expressed by (81), and $\varpi_z = \varpi_\theta = \varpi_r = 0$ is written by

$$\left. \begin{aligned} u_2 &= -\frac{A_m}{2k} z e^{-kz} \frac{\partial J_m(kr)}{\partial r} \frac{\sin}{\cos} \} m\theta, \\ v_2 &= \frac{m A_m}{2k} z e^{-kz} \frac{J_m(kr)}{r} \frac{-\cos}{\sin} \} m\theta, \\ w_2 &= -\frac{A_m}{2k} (1 - kz) e^{-kz} J_m(kr) \frac{\sin}{\cos} \} m\theta. \end{aligned} \right\} \dots\dots\dots(87)$$

The displacement (u_3, v_3, w_3) answering to $\Delta = \Delta_1 + \Delta_2 = 0$ and satisfying $2\varpi_z$ expressed by (82), and the second term of ϖ_r and ϖ_θ expressed by (83) and (84) is of the form of

$$\left. \begin{aligned} u_3 &= B_m \frac{m}{k} e^{-kz} \frac{J_m(kr)}{r} \frac{\sin}{\cos} \} m\theta, \\ v_3 &= -B_m \frac{1}{k^2} e^{-kz} \frac{\partial J_m(kr)}{\partial r} \frac{-\cos}{\sin} \} m\theta, \\ w_3 &= 0. \end{aligned} \right\} \dots\dots\dots(88)$$

The displacement (u_4, v_4, w_4) answering to $\Delta = 2\varpi_z = 0$, and fulfilling the relations of the first of $2\varpi_r$ and $2\varpi_\theta$ expressed by (83) and (84) is expressed by

$$\left. \begin{aligned} u_4 &= \frac{1}{2m} C_m z e^{-kz} \frac{\partial J_m(kr)}{\partial r} \frac{\sin}{\cos} \} m\theta, \\ v_4 &= -\frac{C_m}{2} z e^{-kz} \frac{J_m(kr)}{r} \frac{-\cos}{\sin} \} m\theta, \\ w_4 &= -\frac{C_m}{2m} (1 + kz) e^{-kz} J_m(kr) \frac{\sin}{\cos} \} m\theta. \end{aligned} \right\} \dots\dots\dots(89)$$

Now the displacement to fulfill the condition that $\Delta = \varpi_r = \varpi_\theta = \varpi_z = 0$ is of the following form :

$$\left. \begin{aligned} u_5 &= E_m e^{-kz} \frac{\partial J_m(kr)}{\partial r} \frac{\sin}{\cos} \left\{ m\theta, \right. \\ v_5 &= -m E_m e^{-kz} \frac{J_m(kr)}{r} \frac{-\cos}{\sin} \left\{ m\theta, \right. \\ w_5 &= -k E_m e^{-kz} J_m(kr) \frac{\sin}{\cos} \left\{ m\theta. \right. \end{aligned} \right\} \dots\dots\dots(90)$$

The general expressions of the components of displacement favourable in our study are of the form of

$$\left. \begin{aligned} u &= u_1 + u_2 + u_3 + u_4 + u_5, \\ v &= v_1 + v_2 + v_3 + v_4 + v_5, \\ w &= w_1 + w_2 + w_3 + w_4 + w_5, \end{aligned} \right\} \dots\dots\dots(91)$$

where $u_1, v_1, w_1, u_2, \dots$ are expressed by the equations (86), (87), (88), (89) and (90). When $\beta=k$, $u_1=v_1=w_1=0$. We must, however, remember that there is a relation between A_m and C_m expressed by (85) or (86).

As the stresses are given by

$$\left. \begin{aligned} \widehat{rr} &= \lambda \Delta + 2\mu \frac{\partial u}{\partial r} - \alpha T, & \widehat{\theta\theta} &= \lambda \Delta + 2\mu \left(\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) - \alpha T, \\ \widehat{zz} &= \lambda \Delta + 2\mu \frac{\partial w}{\partial z} - \alpha T, & \widehat{r\theta} &= \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right), \\ \widehat{\theta z} &= \mu \left(\frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right), & \widehat{rz} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right), \end{aligned} \right\} \dots\dots\dots(92)$$

we can find the stresses which are necessary for our study by using the equations (78), (80), (81) and (91).

Now the surface ($z=0$) of the solid is assumed to be free from tractions, so that we have the following conditions on the surface $z=0$:

$$\widehat{zz} = \widehat{\theta z} = \widehat{rz} = 0. \dots\dots\dots(93)$$

From these relations we can find the following forms of constants A_m, C_m, E_m and B_m :

$$\left. \begin{aligned} A_m &= \frac{2\alpha\mu\mathfrak{A}}{(\lambda+\mu)(\lambda+2\mu)} \frac{k}{\beta+k}, \\ C_m &= \frac{2m\alpha\mathfrak{A}}{\lambda+\mu} \frac{1}{\beta+k}, \\ E_m &= -\frac{\alpha\mathfrak{A}}{(\lambda+\mu)(\lambda+2\mu)} \frac{(\lambda+2\mu)\beta-\mu k}{k(\beta^2-k^2)}, \\ B_m &= 0, \end{aligned} \right\} [\beta \neq k] \dots\dots\dots(94)$$

$$\left. \begin{aligned} A_m &= \frac{\alpha\mathfrak{A}}{\lambda+\mu}, & B_m &= 0, \\ C_m &= \frac{m}{k} \frac{\alpha\mathfrak{A}}{\lambda+\mu}, & E_m &= -\frac{\alpha\mathfrak{A}}{2(\lambda+\mu)k^2}. \end{aligned} \right\} [\beta = k] \dots\dots\dots(95)$$

Then, from (91), (92), (94) and (95), we get the following forms of u , v and w :

$$\left. \begin{aligned} u &= -\frac{\alpha\mathfrak{A}}{\lambda+2\mu} \left[\frac{(\lambda+2\mu)\beta-\mu k}{(\lambda+\mu)k(\beta^2-k^2)} e^{-kz} - \frac{ze^{-kz}}{\beta+k} - \frac{e^{-\beta z}}{\beta^2-k^2} \right] \frac{\partial J_m(kr)}{\partial r} \frac{\sin}{\cos} \} m\theta, \\ v &= \frac{\alpha\mathfrak{A}m}{\lambda+2\mu} \left[\frac{(\lambda+2\mu)\beta-\mu k}{(\lambda+\mu)k(\beta^2-k^2)} e^{-kz} - \frac{ze^{-kz}}{\beta+k} - \frac{e^{-\beta z}}{\beta^2-k^2} \right] \frac{J_m(kr)}{r} \frac{-\cos}{\sin} \} m\theta, \\ w &= -\frac{\alpha\mathfrak{A}}{\lambda+2\mu} \left[\frac{\mu\beta-(\lambda+2\mu)k}{(\lambda+\mu)(\beta^2-k^2)} e^{-kz} + \frac{kze^{-kz}}{\beta+k} + \frac{\beta e^{-\beta z}}{\beta^2-k^2} \right] J_m(kr) \frac{\sin}{\cos} \} m\theta, \end{aligned} \right\} [\beta \neq k] \dots\dots\dots(96)$$

$$\left. \begin{aligned} u &= -\frac{\alpha\mathfrak{A}}{2(\lambda+\mu)} e^{-kz} \frac{\partial J_m(kr)}{k^2 \partial r} \frac{\sin}{\cos} \} m\theta, \\ v &= \frac{\alpha\mathfrak{A}m}{2(\lambda+\mu)} e^{-kz} \frac{J_m(kr)}{k^2 r} \frac{-\cos}{\sin} \} m\theta, \\ w &= -\frac{\alpha\mathfrak{A}}{2(\lambda+\mu)} e^{-kz} \frac{J_m(kr)}{k} \frac{\sin}{\cos} \} m\theta. \end{aligned} \right\} [\beta = k] \dots\dots\dots(97)$$

The displacements thus we obtained are merely the elementary solutions satisfying the condition that the temperature distribution of the body is assumed to be of the type expressed by (78).

When the temperature distribution of the surface of the solid is expressed by $f(r, \theta)$, we can construct the favourable form of displacement by summing up the above elementary solutions by the aid of Fourier's series and the Fourier's Bessel double integral formula.

10. The temperature distribution at any point in the body, when the surface distribution is given by $f(r, \theta)$, is easily formulated by using (78) in the following form :

$$T = \sum_m \left\{ \int_0^\infty J_m(kr) e^{-\beta z} k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right\} \frac{\sin}{\cos} m\theta, \dots\dots\dots(98)$$

in which we assume that $f(r, \theta)$ can be expanded into a trigonometrical series of the form of

$$f(r, \theta) = \sum_m f_m(r) \frac{\sin}{\cos} m\theta, \dots\dots\dots(99)$$

where $f_m(r)$ is supposed to satisfy the theorem expressed by the following form :

$$f_m(r) = \int_0^\infty J_m(kr) k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi. \dots\dots\dots(100)$$

And the components of displacement (u, v, w) due to the temperature distribution above mentioned are easily written by

$$\begin{aligned} u = & -\frac{\alpha}{(\lambda + 2\mu)} \sum_m \left[\frac{1}{\lambda + \mu} \int_0^\infty \frac{(\lambda + 2\mu)\beta - \mu k}{\beta^2 - k^2} e^{-kz} \frac{\partial J_m(kr)}{\partial r} dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\ & \left. - z \int_0^\infty \frac{e^{-kz}}{\beta + k} \frac{\partial J_m(kr)}{\partial r} k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\ & \left. - \int_0^\infty \frac{e^{-\beta z}}{\beta^2 - k^2} \frac{\partial J_m(kr)}{\partial r} k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{\sin}{\cos} m\theta, \\ & [\beta \neq k] \dots\dots\dots(101) \end{aligned}$$

$$\begin{aligned}
 v = \frac{\alpha}{(\lambda + 2\mu)} \sum_m m \left[\frac{1}{\lambda + \mu} \int_0^\infty \frac{(\lambda + 2\mu) \beta - \mu k}{\beta^2 - k^2} e^{-kz} \frac{J_m(kr)}{r} dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\
 \left. - z \int_0^\infty \frac{e^{-kz}}{\beta + k} \frac{J_m(kr)}{r} k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\
 \left. - \int_0^\infty \frac{e^{-\beta z}}{\beta^2 - k^2} \frac{J_m(kr)}{r} k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{-\cos}{\sin} m\theta, \\
 [\beta \neq k] \quad \dots\dots(102)
 \end{aligned}$$

$$\begin{aligned}
 w = -\frac{\alpha}{(\lambda + 2\mu)} \sum_m \left[\frac{1}{\lambda + \mu} \int_0^\infty \frac{\mu\beta - (\lambda + 2\mu)k}{\beta^2 - k^2} e^{-kz} J_m(kr) k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\
 \left. + z \int_0^\infty \frac{e^{-kz}}{\beta + k} J_m(kr) k^2 dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right. \\
 \left. + \int_0^\infty \frac{\beta e^{-\beta z}}{\beta^2 - k^2} J_m(kr) k dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{\sin}{\cos} m\theta, \\
 [\beta \neq k] \quad \dots\dots(103)
 \end{aligned}$$

and when $\beta = k$,

$$\begin{aligned}
 u = -\frac{\alpha}{2(\lambda + \mu)} \sum_m \left[\int_0^\infty e^{-kz} \frac{\partial J_m(kr)}{k \partial r} dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{\sin}{\cos} m\theta, \quad [\beta = k] \\
 \dots\dots\dots(104)
 \end{aligned}$$

$$\begin{aligned}
 v = \frac{\alpha}{2(\lambda + \mu)} \sum_m m \left[\int_0^\infty e^{-kz} \frac{J_m(kr)}{kr} dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{-\cos}{\sin} m\theta, \quad [\beta = k] \\
 \dots\dots\dots(105)
 \end{aligned}$$

$$\begin{aligned}
 w = -\frac{\alpha}{2(\lambda + \mu)} \sum_m \left[\int_0^\infty e^{-kz} J_m(kr) dk \int_0^\infty f_m(\xi) J_m(k\xi) \xi d\xi \right] \frac{\sin}{\cos} m\theta. \quad [\beta = k] \\
 \dots\dots\dots(106)
 \end{aligned}$$

In the above equations (101), (102), (103), (104), (105) & (106), we assume that the temperature distribution $f(r, \theta)$ of the surface of the solid is of the following form :

$$f(r, \theta) = \sum_m f_m(r) \left\{ \frac{\sin}{\cos} \right\} m\theta,$$

and $f_m(r)$ is supposed to satisfy the theorem of (100).

We can easily formulate the thermal stresses \widehat{rr} , $\widehat{\theta\theta}$, \widehat{zz} , due to the heat distribution expressed by (98) by the aid of above mentioned method, but in our study we have omitted this procedure.

In this section, we have obtained the general expressions of temperature distribution in the body and the displacement (u, v, w) due to the surface temperature of which distribution is of the type of $f(r, \theta)$. In the actual cases, too, the temperature distribution having the azimuthal variation is most probable, but the equations thus we obtained are very complicated and it is sufficient to study the symmetrical cases to know the effect of the temperature distribution in the three-dimensional problem. Moreover this case is more simple than the case having the azimuthal variation.

11. When the temperature distribution in the body has no azimuthal variation, and the temperature distribution of the surface is given by the form of

$$T = f(r), \quad z = 0, \quad \dots\dots\dots(107)$$

we have the following temperature distribution instead of (98):

$$T = \int_0^\infty J_0(kr) e^{-\beta z} k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi, \quad \dots\dots\dots(108)$$

and the components of the displacement (u, v, w) have the following forms instead of (101), (102), (103), (104), (105), (106):

$$\begin{aligned} u = & -\frac{\alpha}{\lambda+2\mu} \left[\frac{1}{\lambda+\mu} \int_0^\infty \frac{(\lambda+2\mu)\beta-\mu k}{\beta^2-k^2} e^{-kz} \frac{\partial J_0(kr)}{\partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\ & - z \int_0^\infty \frac{e^{-kz}}{\beta+k} \frac{\partial J_0(kr)}{\partial r} k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\ & \left. - \int_0^\infty \frac{e^{-\beta z}}{\beta^2-k^2} \frac{\partial J_0(kr)}{\partial r} k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\ & [\beta \neq k] \quad \dots\dots\dots(109) \end{aligned}$$

$$\begin{aligned}
 w = & -\frac{\alpha}{(\lambda + 2\mu)} \left[\frac{1}{\lambda + \mu} \int_0^\infty \frac{\mu\beta - (\lambda + 2\mu)k}{\beta^2 - k^2} e^{-kz} J_0(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\
 & + z \int_0^\infty \frac{e^{-kz}}{\beta + k} J_0(kr) k^2 dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\
 & \left. + \int_0^\infty \frac{\beta e^{-kz}}{\beta^2 - k^2} J_0(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\
 & [\beta \neq k] \quad \dots\dots(110)
 \end{aligned}$$

$$\begin{aligned}
 u = & -\frac{\alpha}{2(\lambda + \mu)} \int_0^\infty e^{-kz} \frac{\partial J_0(kr)}{k \partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi, \\
 & [\beta = k] \quad \dots\dots(111)
 \end{aligned}$$

$$\begin{aligned}
 w = & -\frac{\alpha}{2(\lambda + \mu)} \int_0^\infty e^{-kz} J_0(kr) dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi. \\
 & [\beta = k] \quad \dots\dots(112)
 \end{aligned}$$

We can easily find the tilt $\frac{\partial w}{\partial r}$ in the following form :

$$\begin{aligned}
 \frac{\partial w}{\partial r} = & \frac{\alpha}{\lambda + 2\mu} \left[\frac{1}{\lambda + \mu} \int_0^\infty \frac{\mu\beta - (\lambda + 2\mu)k}{\beta^2 - k^2} e^{-kz} J_1(kr) k^2 dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\
 & + z \int_0^\infty \frac{e^{-kz}}{\beta + k} J_1(kr) k^3 dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\
 & \left. + \int_0^\infty \frac{\beta e^{-kz}}{\beta^2 - k^2} J_1(kr) k^2 dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\
 & [\beta \neq k] \quad \dots\dots(113)
 \end{aligned}$$

$$\frac{\partial w}{\partial r} = \frac{\alpha}{2(\lambda + \mu)} \int_0^\infty e^{-kz} J_0(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi. \quad [\beta = k] \quad \dots\dots(114)$$

12. In the last section we have not fully examined the relation of β and k , and when β is a arbitrary constant, not a function of k , it is not easy to evaluate the integrals; thus we assume that β is a function of k such that

$$\beta = \beta' k^n, \quad \dots\dots\dots(115)$$

where β' and n are mere numbers. This assumption does not lose the generality of our study as we have seen in the two-dimensional problems.

Now in our study of the following sections we shall consider only the case of $n=1$, namely $\beta=\beta'k$, for the sake of simplicity. Then we have the following form of temperature distribution when the surface distribution is of the form of $f(r)$ from (108):

$$T = \int_0^\infty e^{-\beta'kz} J_0(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi, \quad \dots\dots\dots(116)$$

and from (109) and (110), we have

$$\begin{aligned} u = & -\frac{\alpha}{(\lambda+2\mu)(\beta'^2-1)} \left[\frac{(\lambda+2\mu)\beta'-\mu}{\lambda+\mu} \int_0^\infty e^{-kz} \frac{\partial J_0(kr)}{k\partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\ & - z(\beta'-1) \int_0^\infty e^{-kz} \frac{\partial J_0(kr)}{\partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\ & \left. - \int_0^\infty e^{-\beta'kz} \frac{\partial J_0(kr)}{k\partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\ & [\beta' \neq 1] \quad \dots\dots(117) \end{aligned}$$

$$\begin{aligned} w = & -\frac{\alpha}{(\lambda+2\mu)(\beta'^2-1)} \left[\frac{\mu\beta'-(\lambda+2\mu)}{\lambda+\mu} \int_0^\infty e^{-kz} J_0(kr) dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\ & + z(\beta'-1) \int_0^\infty e^{-kz} J_0(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\ & \left. + \beta' \int_0^\infty e^{-\beta'kz} J_0(kr) dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\ & [\beta' \neq 1] \quad \dots\dots(118) \end{aligned}$$

and from (111) and (112),

$$u = -\frac{\alpha}{(\lambda + \mu)} \int_0^\infty e^{-kz} \frac{\partial J_0(kr)}{k \partial r} dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi, \quad [\beta' = 1] \quad \dots\dots(119)$$

$$v = -\frac{\alpha}{2(\lambda + \mu)} \int_0^\infty e^{-kz} J_0(kr) dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi, \quad [\beta' = 1] \quad \dots\dots(120)$$

and the tilt $\frac{\partial w}{\partial r}$ is easily formulated, namely from the equation (113), (114), we have

$$\begin{aligned} \frac{\partial w}{\partial r} = \frac{\alpha}{(\lambda + 2\mu)(\beta'^2 - 1)} & \left[\frac{\mu\beta' - (\lambda + 2\mu)}{\lambda + \mu} \int_0^\infty e^{-kz} J_1(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right. \\ & + z(\beta' - 1) \int_0^\infty e^{-kz} J_1(kr) k^2 dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \\ & \left. + \beta' \int_0^\infty e^{-\beta' kz} J_1(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi \right], \\ & [\beta' \neq 1] \quad \dots\dots(121) \end{aligned}$$

$$\frac{\partial w}{\partial r} = \frac{\alpha}{2(\lambda + \mu)} \int_0^\infty e^{-kz} J_1(kr) k dk \int_0^\infty f(\xi) J_0(k\xi) \xi d\xi. \quad [\beta' = 1] \quad \dots\dots(122)$$

13. For the example of Section 14, we shall take the temperature distribution of the following form on the surface of the solid:

$$f(r) = \frac{Ab}{\sqrt{b^2 + r^2}}, \quad b > 0, \quad \dots\dots\dots(123)$$

where b is the constant which can be adjusted to obey the distribution and A is the constant to express the measurement of temperature.

Then, by the aid of

$$\int_0^\infty \frac{J_0(kx)}{\sqrt{b^2 + x^2}} dx = \frac{e^{-bk}}{k}, \quad b > 0 \dots\dots\dots(124)^{15}$$

15) This formula is obtained by Prof. K. Terazawa and others.
K. TERAZAWA, "On Deep Sea Water Waves caused by a Local Disturbance on or beneath the Surface," *Proc. Roy. Soc., London*, A **82** (1916).

we have the following temperature distribution from (116):

$$T = Ab \int_0^{\infty} e^{-(\beta'z+b)k} J_0(kr) dk. \dots\dots\dots(125)$$

The evaluation of the integral of (125) is very easy, and we have the following form of the temperature distribution of the solid:

$$T = A \frac{b}{\sqrt{r^2 + (\beta'z + b)^2}}. \dots\dots\dots(126)$$

The Figures 9 and 10 are compiled from the expression of (126).

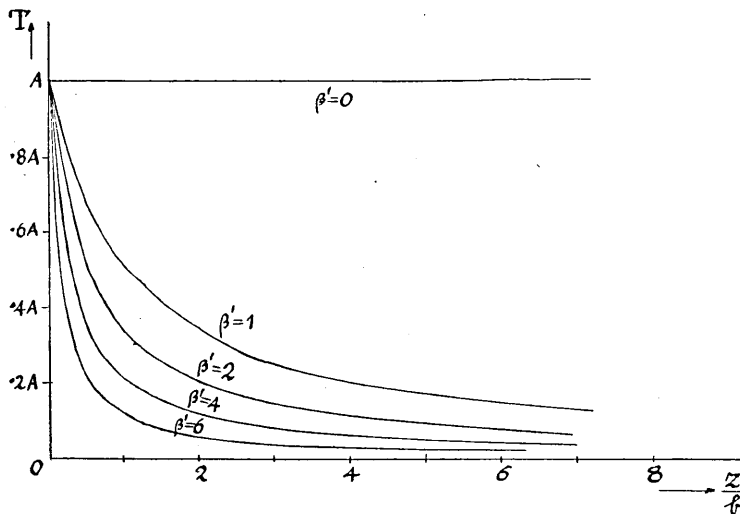
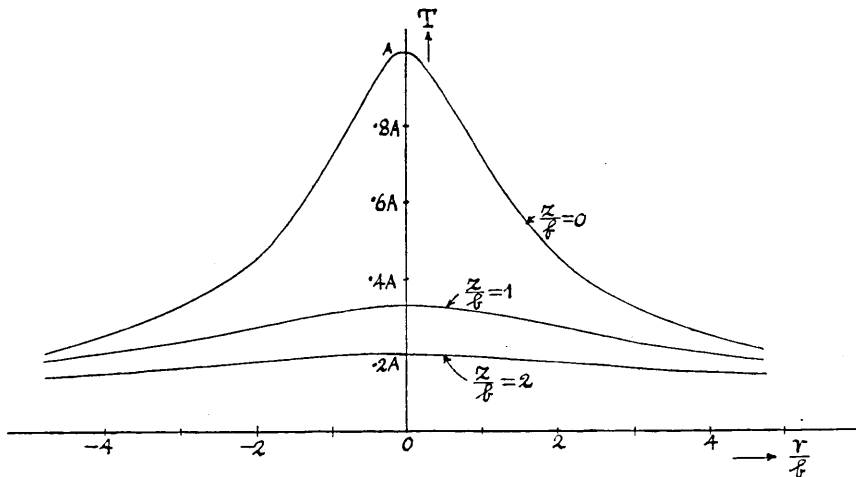


Fig. 9. $\left[\frac{r}{b} = 0. \right]$

Fig. 9 indicates the temperature distribution in the solid on the plane $r/b=0$ when $\beta'=0, 1, 2, 4$ & 6 . When $\beta'=2$, the temperature distributions on the planes $z/b=0, 1, 2$ are compiled in Fig. 10. From these two figures we can easily understand that the greater the number of β' is, the more intensively decreases the effect of temperature in the solid, namely the effect is localised only on the surface.

Fig. 10. $[\beta' = 2.]$

We have the following expression of the inclination of the solid :

$$\begin{aligned} \frac{\partial w}{\partial r} = \frac{\alpha A b}{(\lambda + 2\mu)(\beta'^2 - 1)} & \left[\frac{\mu\beta' - (\lambda + 2\mu)}{\lambda + \mu} \int_0^\infty e^{-(z+b)k} J_1(kr) dk \right. \\ & + (\beta' - 1) z \int_0^\infty e^{-(z+b)k} J_1(kr) k dk \\ & \left. + \beta' \int_0^\infty e^{-(\beta'z+b)k} J_1(kr) dk \right], \quad [\beta' \neq 1] \end{aligned}$$

.....(127)

$$\frac{\partial w}{\partial r} = \frac{\alpha A b}{2(\lambda + \mu)} \int_0^\infty e^{-(b+z)k} J_1(kr) dk. \quad [\beta' = 1] \quad \text{.....(128)}$$

by the above formula of (124).

Now we know the following relations :

$$\left. \begin{aligned} \int_0^\infty J_1(tx) e^{-ty} t dt &= \frac{x}{(x^2 + y^2)^{3/2}}, \\ \int_0^\infty J_1(tx) e^{-ty} dt &= \frac{\sqrt{x^2 + y^2} - y}{x\sqrt{x^2 + y^2}}. \end{aligned} \right\} \quad \text{.....(129)}$$

By the aid of this equation, we can reduce (127), (128) in the following forms:

$$\frac{\partial w}{\partial r} = \frac{\alpha Ab}{(\lambda + 2\mu)(\beta' - 1)} \left[\frac{\mu\beta' - (\lambda + 2\mu)}{\lambda + \mu} \frac{\sqrt{r^2 + (b+z)^2} - (b+z)}{r\sqrt{r^2 + (b+z)^2}} \right. \\ \left. + (\beta' - 1) \frac{rz}{\{r^2 + (b+z)^2\}^{3/2}} + \beta' \frac{\{\sqrt{r^2 + (b+\beta'z)^2} - (b+\beta'z)\}}{r\sqrt{r^2 + (b+\beta'z)^2}} \right], \\ [\beta' \neq 1] \quad \dots\dots(130)$$

$$\frac{\partial w}{\partial r} = \frac{\alpha Ab}{2(\lambda + \mu)} \frac{\sqrt{r^2 + (b+z)^2} - (b+z)}{r\sqrt{r^2 + (b+z)^2}}, \quad [\beta' = 1] \quad \dots\dots(131)$$

and the surface tilt is of the form of

$$\frac{\partial w}{\partial r} = \frac{\alpha Ab}{(\lambda + \mu)(1 + \beta')} \frac{\sqrt{r^2 + b^2} - b}{r\sqrt{r^2 + b^2}}. \quad \dots\dots\dots(130)$$

From the annexed Figure 11, in which we have compiled $\frac{\partial w}{\partial r}$ on the surface $z=0$ when $\beta'=2$ and 4, the surface inclination of the solid is much affected by the value of β' , and inversely proportional to $(1 + \beta')$, i.e. the more the effect of the surface temperature distribution in the solid diminishes, the smaller the amount of the surface inclination becomes, in spite of a constant surface temperature distribution.

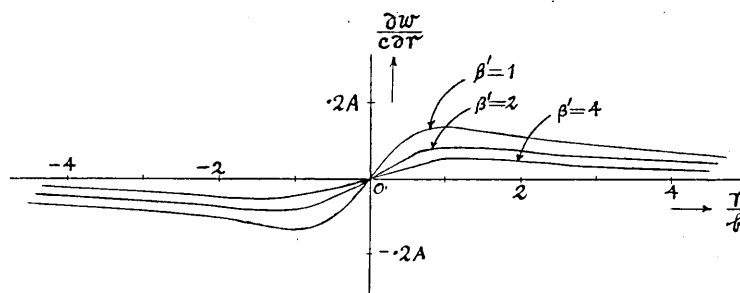


Fig. 11. $\left[\frac{z}{b} = 0 \quad \lambda = u_* \right]$

To understand how much the displacement w in the interior of the earth may be affected by β' , we calculate $\frac{\partial w}{\partial z}$ in the following form :

$$\begin{aligned} \frac{\partial w}{\partial z} = & - \frac{\alpha Ab}{(\lambda + 2\mu)(\beta'^2 - 1)} \left[\frac{\lambda\beta' + \mu}{\lambda + \mu} \int_0^\infty e^{-(z+b)k} J_0(kr) dk \right. \\ & \left. - (\beta' - 1) z \int_0^\infty e^{-(z+b)k} J_0(kr) k dk - \beta'^2 \int_0^\infty e^{-(\beta'z+b)k} J_0(kr) dk \right], \\ & [\beta' \neq 1] \quad \dots\dots(132) \end{aligned}$$

$$\frac{\partial w}{\partial z} = \frac{\alpha Ab}{2(\lambda + \mu)} \int_0^\infty e^{-(z+b)k} J_0(kr) dk. \quad [\beta' = 1] \quad \dots\dots(133)$$

By means of the formulae,

$$\left. \begin{aligned} \int_0^\infty J_0(tx) e^{-ty} t dt &= \frac{y}{(x^2 + y^2)^{3/2}}, \\ \int_0^\infty J_0(tx) e^{-ty} dt &= \frac{1}{\sqrt{x^2 + y^2}}, \end{aligned} \right\} \dots\dots\dots(134)$$

we get finally

$$\begin{aligned} \frac{\partial w}{\partial z} = & - \frac{\alpha Ab}{(\lambda + 2\mu)(\beta'^2 - 1)} \left[\frac{\lambda\beta' + \mu}{\lambda + \mu} \frac{1}{\sqrt{r^2 + (b+z)^2}} \right. \\ & \left. - (\beta' - 1) \frac{z(z+b)}{\{r^2 + (b+z)^2\}^{3/2}} - \frac{\beta'^2}{\sqrt{r^2 + (\beta'z+b)^2}} \right], \\ & [\beta' \neq 1] \quad \dots\dots(135) \end{aligned}$$

$$\frac{\partial w}{\partial z} = \frac{\alpha Ab}{2(\lambda + \mu)} \frac{1}{\sqrt{r^2 + (b+z)^2}}. \quad [\beta' = 1] \quad \dots\dots(136)$$

When $r=0$,

$$\frac{\partial w}{\partial z} = -\frac{\alpha Ab}{(\lambda + 2\mu)(\beta'^2 - 1)} \left[\frac{\lambda\beta' + \mu}{\lambda + \mu} \frac{1}{(b+z)} - \frac{(\beta' - 1)z}{(b+z)^2} - \frac{\beta'^2}{(b+\beta'z)} \right],$$

[$\beta' \neq 1$](137)

$$\frac{\partial w}{\partial z} = \frac{\alpha Ab}{2(\lambda + \mu)} \frac{1}{b+z}. \quad [\beta' = 1] \quad \text{.....(138)}$$

The compiled results of the normal strain $\frac{\partial w}{\partial z}$ on the plane $\frac{r}{b}=0$ of the solid is shown in Fig. 12.

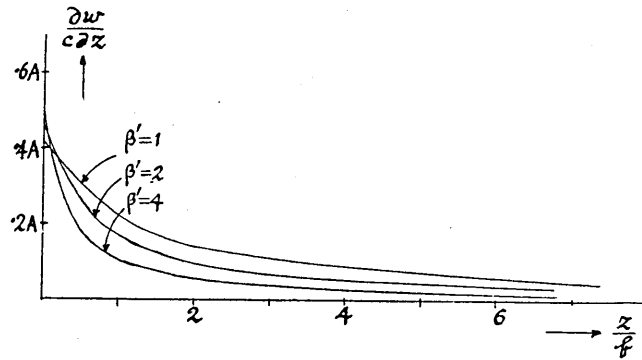


Fig. 12. $\left[\frac{r}{b} = 0, \lambda = \mu \right]$

This figure shews that the magnitude of the surface normal strain $\frac{\partial w}{\partial z}$ is the greatest of all the normal strains in the solid, and that the more the effect of the temperature distribution is concentrated in the neighbourhood of the surface of the solid, the larger the magnitude of the surface normal strain $\frac{\partial w}{\partial z}$ becomes, and the smaller becomes the normal strain $\frac{\partial w}{\partial z}$ in the interior of the solid.

14. If the surface temperature distribution is given by the form of

$$f(x) = \frac{A\alpha^2}{b^2 + r^2}, \quad \text{.....(139)}$$

where the physical meanings of constants a , b and A are same in the above sections.

Then we have the following forms of temperature and displacement:

$$T = \frac{Aa^2\pi i}{2} \int_0^\infty e^{-\beta'kz} H_{1,0}(kbi) J_0(kr) k dk, \quad \dots\dots\dots(140)$$

$$\begin{aligned} u = \frac{Aa^2\alpha\pi i}{2(\lambda+2\mu)(\beta'^2-1)} & \left[\frac{(\lambda+2\mu)\beta'-\mu}{\lambda+\mu} \int_0^\infty H_{1,0}(kbi) J_1(kr) e^{-kz} dk \right. \\ & - (\beta'-1)z \int_0^\infty H_{1,0}(kbi) J_1(kr) e^{-kz} k dk \\ & \left. - \int_0^\infty H_{1,0}(kbi) J_1(kr) e^{-\beta'kz} dk \right], \\ & [\beta' \neq 1] \quad \dots\dots(141) \end{aligned}$$

$$\begin{aligned} w = -\frac{Aa^2\alpha\pi i}{2(\lambda+2\mu)(\beta'^2-1)} & \left[\frac{\mu\beta'-(\lambda+2\mu)}{\lambda+\mu} \int_0^\infty H_{1,0}(kbi) J_0(kr) e^{-kz} dk \right. \\ & + (\beta'-1)z \int_0^\infty H_{1,0}(kbi) J_0(kr) e^{-kz} k dk \\ & \left. + \beta' \int_0^\infty H_{1,0}(kbi) J_0(kr) e^{-\beta'kz} dk \right] \\ & [\beta' \neq 1] \quad \dots\dots(142) \end{aligned}$$

$$u = \frac{Aa^2\alpha\pi i}{4(\lambda+\mu)} \int_0^\infty H_{1,0}(kbi) J_1(kr) e^{-kz} dk, \quad [\beta' = 1] \quad \dots\dots(143)$$

$$w = -\frac{Aa^2\alpha\pi i}{4(\lambda+\mu)} \int_0^\infty H_{1,0}(kbi) J_0(kr) e^{-kz} dk, \quad [\beta' = 1] \quad \dots\dots(144)$$

by means of

$$\int_0^\infty \frac{J_0(k\tilde{\xi})}{b^2+\tilde{\xi}^2} \tilde{\xi} d\tilde{\xi} = \frac{i\pi}{2} H_{1,0}(kbi). \quad \dots\dots\dots(145)$$

Using the following formula

$$\int_0^\infty H_{1,\nu}(ty) J_\rho(tx) t^\sigma dt = e^{\frac{\rho+\sigma-\nu}{2}} \frac{\pi i 2^\sigma x^\rho}{\pi y^{\rho+\sigma+1}} \frac{\Gamma\left(\frac{\rho+\sigma+\nu+1}{2}\right) \Gamma\left(\frac{\rho+\sigma-\nu+1}{2}\right)}{\Gamma(\rho+1)} \times$$

$$F\left(\frac{\rho+\sigma+\nu+1}{2}, \frac{\rho+\sigma-\nu+1}{2}, \rho+1, -\frac{x^2}{y^2}\right), \dots\dots\dots(146)$$

we can easily find the following forms of surface displacement by (143), (144):

$$u_{z=0} = \frac{Aa^2\alpha i}{2(\lambda+\mu)(\beta'+1)} e^{\frac{\pi i}{2}} \frac{r}{b^2 i^2} F\left(1, 1, 2, -\frac{r^2}{b^2}\right), \dots\dots\dots(147)$$

$$w_{z=0} = -\frac{Aa^2\alpha\pi}{2b(\lambda+\mu)(\beta'+1)} F\left(\frac{1}{2}, \frac{1}{2}, 1, -\frac{r^2}{b^2}\right). \dots\dots\dots(148)$$

When $r=0$, we have the following form of displacement:

$$\left. \begin{aligned} u_{z=0} &= 0, \\ w_{z=0} &= -\frac{Aa^2\alpha\pi}{2(\lambda+\mu)(\beta'+1)b}. \end{aligned} \right\} \dots\dots\dots(149)$$

15. When the surface temperature distribution is given by the form of

$$f(r) = \frac{Aa^3}{(a^2+r^2)^{3/2}}, \dots\dots\dots(150)$$

where the constants A, a are determined by experiments as in the above examples. Then we have the following forms of temperature distribution in the body and the displacement and its derivatives using the formulae (116), (117), (118), (119) and (120).

Applying the relation employed by Prof. K. Terazawa,¹⁶⁾ namely

16) K. TERAZAWA, *Phil. Trans. Roy. Soc., London*, (A) 217 (1916) and *J. Coll. Sci., Tokyo*, 37 (1916).

$$\int_0^{\infty} \frac{a J_0(f\lambda) \lambda d\lambda}{(a^2 + \lambda^2)^{3/2}} = e^{-fa}, \dots\dots\dots(151)$$

we find

$$T = a^2 A \int_0^{\infty} e^{-(a+\beta'z)k} J_0(kr) k dk, \dots\dots\dots(152)$$

$$u = \frac{Aa^2\alpha}{(\lambda+2\mu)(\beta'^2-1)} \left[\frac{(\lambda+2\mu)\beta' - \mu}{\lambda+\mu} \int_0^{\infty} e^{-(a+z)k} J_1(kr) dk \right. \\ \left. - (\beta'-1) z \int_0^{\infty} e^{-(a+z)k} J_1(kr) k dk - \int_0^{\infty} e^{-(a+\beta'z)k} J_1(kr) dk \right], \\ [\beta' \neq 1] \dots\dots(153)$$

$$w = -\frac{Aa^2\alpha}{(\lambda+2\mu)(\beta'^2-1)} \left[\frac{\mu\beta' - (\lambda+2\mu)}{\lambda+\mu} \int_0^{\infty} e^{-(a+z)k} J_0(kr) dk \right. \\ \left. + (\beta'-1) z \int_0^{\infty} e^{-(a+z)k} J_0(kr) k dk + \beta' \int_0^{\infty} e^{-(a+\beta'z)k} J_0(kr) dk \right], \\ [\beta' \neq 1] \dots\dots(154)$$

$$u = \frac{\alpha a^2 A}{2(\lambda+\mu)} \int_0^{\infty} e^{-(a+z)k} J_1(kr) dk, \quad [\beta' = 1] \dots\dots(155)$$

$$w = -\frac{\alpha a^2 A}{2(\lambda+\mu)} \int_0^{\infty} e^{-(a+z)k} J_0(kr) dk. \quad [\beta' = 1] \dots\dots(156)$$

Now we can easily evaluate the integrals in the above mentioned formulae by using

$$\left. \begin{aligned} \int_0^\infty J_1(tx) e^{-ty} dt &= \frac{\sqrt{x^2+y^2}-y}{x\sqrt{x^2+y^2}}, \\ \int_0^\infty J_\nu(tx) e^{-ty} t^\nu dt &= \frac{\Gamma(\nu+1/2)}{\sqrt{\pi}} \frac{(2x)^\nu}{(x^2+y^2)^{\nu+1/2}}, \\ \int_0^\infty J_\nu(tx) e^{-ty} t^{\nu+1} dt &= \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}} \frac{y(2x)^\nu}{(x^2+y^2)^{\nu+3/2}}, \end{aligned} \right\} \dots\dots\dots(157)$$

in the following forms :

$$T = Aa^2 \frac{(a+\beta'z)}{\{r^2+(a+\beta'z)^2\}^{3/2}}, \quad \dots\dots\dots(158)$$

$$\begin{aligned} u = \frac{Aa^2\alpha}{(\lambda+2\mu)(\beta'^2-1)} &\left[\frac{(\lambda+2\mu)\beta'-\mu}{\lambda+\mu} \frac{\sqrt{r^2+(a+z)^2}-(a+z)}{r\sqrt{r^2+(a+z)^2}} \right. \\ &\left. -(\beta'-1) \frac{rz}{\{r^2+(a+z)^2\}^{3/2}} - \frac{\sqrt{r^2+(a+\beta'z)^2}-(a+\beta'z)}{r\sqrt{r^2+(a+\beta'z)^2}} \right], \\ &[\beta' \neq 1] \quad \dots\dots(159) \end{aligned}$$

$$\begin{aligned} w = -\frac{Aa^2\alpha}{(\lambda+2\mu)(\beta'^2-1)} &\left[\frac{\mu\beta'-(\lambda+2\mu)}{\lambda+\mu} \frac{1}{\sqrt{r^2+(a+z)^2}} \right. \\ &\left. +(\beta'-1) \frac{z(a+z)}{\{r^2+(a+z)^2\}^{3/2}} + \frac{\beta'}{\sqrt{r^2+(a+\beta'z)^2}} \right], \\ &[\beta' \neq 1] \quad \dots\dots(160) \end{aligned}$$

$$u = \frac{Aa^2\alpha}{2(\lambda+\mu)} \frac{\sqrt{r^2+(a+z)^2}-(a+z)}{r\sqrt{r^2+(a+z)^2}}, \quad [\beta' = 1] \quad \dots\dots(161)$$

$$w = -\frac{Aa^2\alpha}{2(\lambda+\mu)} \frac{1}{\sqrt{r^2+(a+z)^2}}, \quad [\beta' = 1] \quad \dots\dots(162)$$

The temperature distribution in the solid when $\frac{r}{a} = 0$ is shown in Figure 13 where $\beta' = 2, 4$ & 6. Figure 14 gives the temperature distribution on the planes $\frac{z}{b} = 0, 1$ & 2 when $\beta' = 2$.

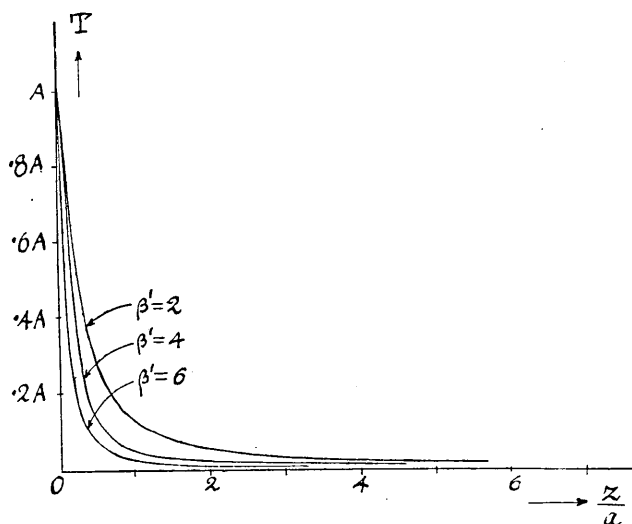


Fig. 13. $\left[\frac{r}{a} = 0. \right]$

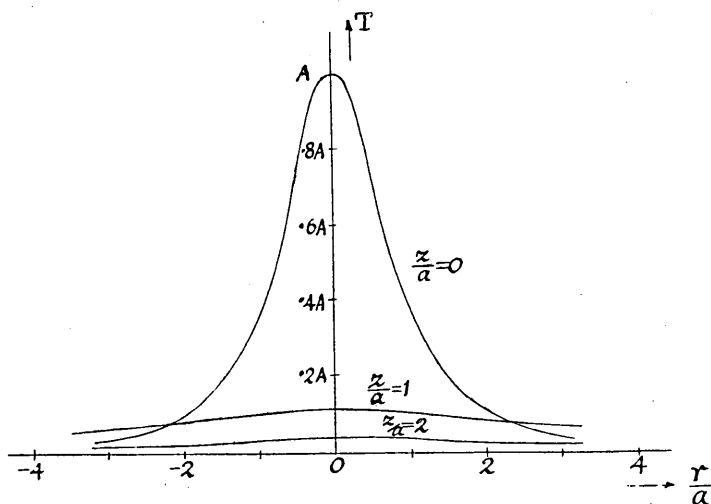


Fig. 14. $[\beta' = 2.]$

It may be seen from the above figures that the value of β' specifies the temperature distribution in the interior of the solid as we have already discussed in Section 13.

We will now study more fully how much effects the variation of the temperature distribution in the interior of the solid may have upon the deformation of the solid under the condition that the temperature distribution on the surface is of a certain sharp nature.

The equations (159), (160) shew us that the surface displacement (u, w) is much affected by the number of β' , namely it is inversely proportional to the value of $(1 + \beta')$.

The compiled result of the component w of displacement on the surface $\frac{r}{a} = 0$, when $\lambda = \mu$, is shown in Figure 15.

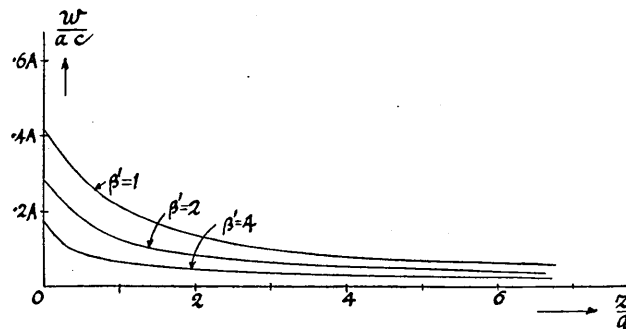
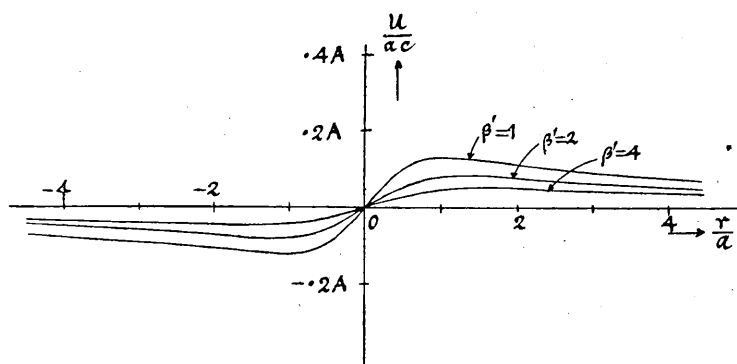
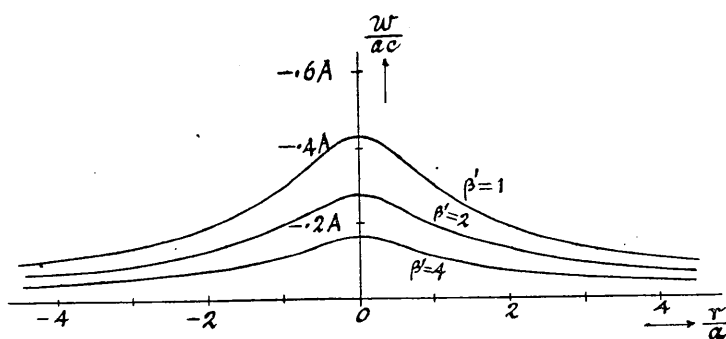
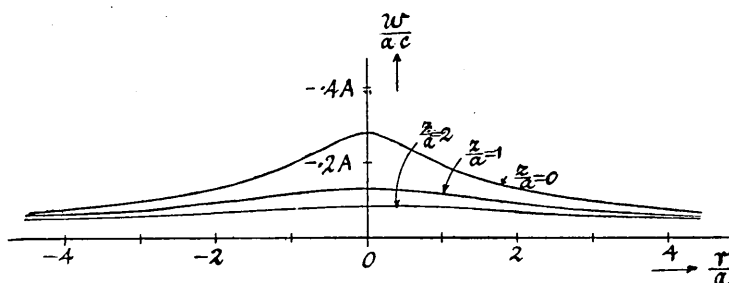


Fig. 15. $\left[\frac{r}{a} = 0, \lambda = \mu. \right]$

From this figure we can easily understand that if the temperature distribution of the surface of the solid is assigned to be a certain sharp nature, the change of the surface displacement due to the small variation of the temperature distribution in the interior of the solid is very sensible.

The calculated results of the displacement (u, w) on the surface $\frac{z}{a} = 0$, when $\beta' = 1, 2, 4$, are shown in the following Figures 16 and 17 and Figure 18 shews the displacement on the planes of $\frac{z}{a} = 0, 1$ and 2 when $\beta' = 2$.

Fig. 16. $\left[\frac{z}{a} = 0, \lambda = \mu.\right]$ Fig. 17. $\left[\frac{z}{a} = 0, \lambda = \mu.\right]$ Fig. 18. $[\beta' = 2, \lambda = \mu.]$

These three figures shew that the magnitude of the surface displacement becomes smaller according as the temperature is concentrated in the neighbourhood of the surface as we have already discussed in the two-dimensional problem, and that even if the temperature distribution of the surface of the solid is of a sharp nature, the surface as well as the inner deformation of the solid is moderate.

Again, using the results of (159), (160), (161), (162), we have compiled the normal strain $\frac{\partial w}{\partial z}$ of the plane of $\frac{r}{a} = 0$ and the surface inclination $\frac{\partial w}{\partial r}$ when $\beta' = 1, 2 \text{ \& } 4$ in Figures 19 and 20.

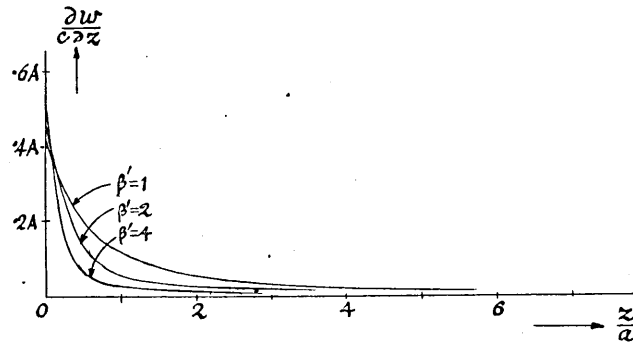


Fig. 19. $\left[\frac{r}{a} = 0, \lambda = \mu. \right]$

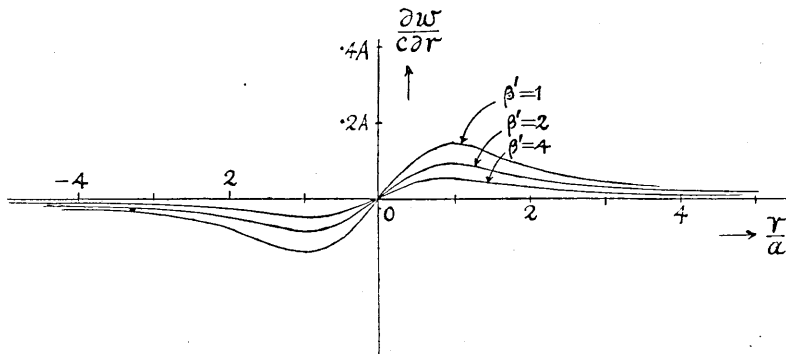


Fig. 20. $\left[\frac{z}{a} = 0, \lambda = \mu. \right]$

Fig. 19 indicates the reliableness of the result¹⁷⁾ which we have obtained in the section 13 concerning the relation between the normal strain $\frac{\partial w}{\partial z}$ and the variation of the temperature distribution in the interior of the solid. Also we confirm, from Fig. 11¹⁸⁾ and Fig. 20, that when the temperature distribution of the surface is of a certain sharp nature, the surface tilt is sharper than that when the temperature distribution of the surface is of a moderate nature.

Résumé.

16. We have now obtained some results which may be interesting on the geophysical problems, and of which the principal facts are enumerated as below :

1. The magnitude of the displacement at a given point in the solid is proportional to the temperature at that point.

2. The surface distribution of displacements is, of course, affected by the temperature distribution of the surface, and especially by the distribution in the interior of the body, i.e. the displacement or the tilt of the surface of the solid is inversely proportional to $(1 + \beta')$ where β' is a constant which specifies the temperature distribution in the interior of the solid.

3. The surface displacement becomes maximum when the temperature distribution is constant through all depth in the interior of the solid.

4. Even when the surface temperature is discontinuous, the surface as well as internal deformation of the body is continuous, and in this case the portion of no temperature change is also deformed.

5. Even if the temperature distribution of the surface of the solid is of a sharp nature, the surface deformation of the solid has a moderate nature.

6. When the temperature distribution of the surface of the solid is assigned to be of a certain sharp nature, the change of the surface displacement due to the small variation of the temperature distribution in the interior of the solid is very sensible.

In conclusion the author must express his cordial thanks to Professor K. Sezawa for his kind advices and supervision.

17) *loc. cit.*, 132.

18) *loc. cit.*, 130.

7. 温度分布が半無限弾性體の變形に及ぼす影響

地震研究所 西村源六郎

地球表面の絶えざる變形を起す原因は色々あるであらうが我々は決して温度分布による影響も見逃す理にはゆかない。地球表面に壓力の加はる場合や、地球内部に歪核のある場合に生ずる地表の變形は寺澤先生や妹澤先生によつて研究されてあるが、表面の温度分布による影響はまだ解いてない様である。この問題は種々地形變化觀測の方面に關係がある様に思はれるが、これ等の事に就て多少でも參考になる事があれば幸と思つてゐる。

研究の方法は、温度の影響を含んだ一般弾性體平衡方程式の解から出發して、二次元の問題と三次元の問題を二三の例に就て解いてみただけである。たゞ計算中、表面から地球内部へかけて温度が種々變つた場合に地殼の變形がどの様に變化されるかをみる爲め、この方面に主として力を入れた。

計算中所々から得た結果を擧げてみると、

1. 地殼中のある點の變位量はその點の温度に比例する。
 2. 地表の變形は勿論地表の温度分布の狀態の變化によつて支配されるが特に温度の地球内部に變る割合によつて非常に影響を受ける。即ち地表の變形或は傾斜は $(1+\beta')$ の價に逆比例する事がわかる。但し β' は温度の地球内部に變る割合を支配する或る常數である。
 3. 地表面上の變位は地中深く温度が變る時に、特にその量が多い。
 4. 表面の温度分布に不連續點があつても地表面及び内部に於ける變形には不連續點はないばかりか、温度變化のない所でも或る程度まで變形をうける。
 5. 地表の温度分布が集中性の時でも地表變形の分布は緩漫性を帯びる。
 6. 地表の温度分布が集中性を帯びてゐる時には地球内部での温度分布が少し變つても地表面の變位の量は非常に變化される。
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