21. *The Displacement independent of the Dilatation and the Rotation in a Solid Body.*

By **Katsutada SEZAWA** and **Genrokuro NISHIMURA,**

Earthquake Research Institute.

(Received Sept. 20, 1929.)

1. Very recently Professor Terada and Dr. Miyabe[1] pointed out that the deformation of the earth crust can be composed of the dilatation, the rotation and the shear. As they dealt with the actual problem of the earth crust and they seem to have given no definite specification on the nature of the solid body, whether the body is elastic or plastic, we cannot give any authentic discussion on the above problem. Even though the body may be assumed to be plastic in the sense of St. Venant[2] and Th. v. Kármán,[3] yet the natures of the dilatation, the rotation and others, we think, are not essentially different from those in the case of the elastic deformation. Leaving the problem of the difference between the elasticity and the plasticity, besides the discussion on the plastic deformation, for the present, we shall first consider the deformation of the elastic body.

It is of recent time[4] that the dilatation, rotation, and other components in elastic solid bodies are theoretically analysed with the view of arriving at the complete solutions of the mathematical equations of the equilibrium of the bodies. This problem has attracted very little attention of the investigators probably because of the existence of the other methods of solving the equations without such an analytical consideration.

According to our analytical investigation, the idea of expressing the deformation of a solid body by means of the dilatation, the rotation and the shear is not completely theoretical. The idea of expressing the displacement by means of the components of the dilatation, the rotation and the shear is

1) T. TERADA & N. MIYABE, 39th Colloq. Meeting of the Institute (June 18, 1929). The paper will perhaps appear in *Bull. Earthq. Res. Inst.*, **7** (1929) Part 2, which is now going to be published.

2) ST. VENANT, C. R. (1868–1872).

3) HAAR U. TH. V. KÁRMÁN, *Enc. d. Math. Wiss.*, **4** (1914).

4) K. SEZAWA & B. MIYAZAKI. *Journ. Soc. Mech. Eng., Tokyo*, **31** (1928), 625–634. K. SEZAWA & G. NISHIMURA, *Bull. Earthq. Res. Inst.*, **6** (1929), 47–62. G. NISHIMURA, *Journ. Soc. Ord.-Expl.*, **23** (1929).

due to Stokes and others. This idea seems to represent the method of expressing the infinitesimal displacement of a material point, but not involve any view on the essential nature of the deformable body, such as the consideration of the total dilatation. Thus the thought of taking the dilatation, the rotation and the shear in the analysis of a body seems to be violent. Our investigation shews that the deformation of a solid body should be composed of the dilatation, the rotation and another component which is neither dilatational nor rotational. The last of these is not such a meaningless component as to give the bodily movement to the solid; but it plays its important *role* in giving the strain or the stress to the solid both in normal and tangential directions. It has also been cleared that the dilatation and the rotation are by no means of different types. The component of the displacement which gives the dilatation is of the same form as that which gives the rotation; the ratio of the magnitudes of both components is invariable when the elasticities of the material and the type of the solution are specified. These will be seen presently at th part of th mathmatical investigation. The proposed fact[5] that the deformation of the land surface involves the rotational component is also self-evident.

I. The Problems of a Spherical Heterogeneity in a Stressed Body.[6]

2. The equations of motion of elastic bodies in spherical coordinates, where the azimuthal component of the motion is omitted, are expressed by

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi}{\partial\theta}-\frac{2\mu}{r}\varpi\cot\theta=0, \qquad (1)$$

$$\frac{\lambda+2\mu}{r}\frac{\partial\Delta}{\partial\theta}+2\mu\frac{\partial\varpi}{\partial r}+2\mu\frac{\varpi}{r}=0, \qquad (2)$$

where u, v are radial and colatitudinal components of displacement, and

$$\Delta=\frac{\partial u}{\partial r}+\frac{2u}{r}-\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{v}{r}\cot\theta, \qquad (3)$$

$$2\varpi=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}. \qquad (4)$$

Eliminating u and v in (1), (2) by means of (3) and (4), we get

$$\frac{\partial^2\Delta}{\partial r^2}+\frac{2}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\Delta}{\partial\theta}\cot\theta=0, \qquad (5)$$

5) S. FUJIWHARA & T. TAKAYAMA, *Bull. Earthq. Res. Inst.*, **6** (1929), 174.
6) K. SEZAWA & MIYAZAKI, *loc. cit.*, 389.

$$\frac{\partial^2 \varpi}{\partial r^2}+\frac{2}{r}\frac{\partial \varpi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \varpi}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial \varpi}{\partial \theta}\cot\theta-\frac{\varpi}{r^2}(1+\cot^2\theta)=0. \quad \ldots\ldots(6)$$

Solving (5) and (6), we obtain

$$\left.\begin{aligned} \Delta &= \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta), \quad \ldots\ldots\ldots(7) \\ 2\varpi &= \left(A_n' r^n + \frac{B_n'}{r^{n+1}}\right) \frac{dP_n(\cos\theta)}{d\theta} \quad \ldots\ldots(8) \end{aligned}\right\}[n=0,\ 1,\ 2,\ldots\ldots]$$

Displacement $(u_1,\ v_1)$ answering to Δ in (7) and satisfying $\varpi=0$ in (8) is denoted by

$$\left.\begin{aligned} u_1 &= \left[\frac{A_n(n+2)}{2(2n+3)} r^{n+1} + \frac{B_n(n-1)}{2(2n-1)r^n}\right] P_n(\cos\theta), \\ v_1 &= \left[\frac{A_n}{2(2n+3)} r^{n+1} - \frac{B_n}{2(2n-1)r^n}\right] \frac{dP_n(\cos\theta)}{d\theta}, \end{aligned}\right\} \ldots\ldots\ldots\ldots(9)$$

in which $n=0, 1, 2,\ldots.$ for A_n and $n=1, 2, 3,\ldots.$ for B_n.

Displacement $(u_2,\ v_2)$ derived from the value of ϖ in (8) under the condition, $\Delta=0$, is expressed by

$$\left.\begin{aligned} u_2 &= \left[\frac{A_n' n(n+1)}{2(2n+3)} r^{n+1} - B_n' \frac{n(n+1)}{2(2n-1)r^n}\right] P_n(\cos\theta), \\ v_2 &= \left[\frac{A_n'(n+2)}{2(2n+3)} r^{n+1} + \frac{B_n'(n-2)}{2(2n-1)r^n}\right] \frac{dP_n(\cos\theta)}{d\theta}. \end{aligned}\right\}[n=1,\ 2,\ldots.]\ldots(10)$$

Displacement $(u_3,\ v_3)$ which satisfies $\Delta=0$, $\varpi=0$ is expressed by

$$\left.\begin{aligned} u_3 &= \left[A_n'' n r^{n-1} - \frac{B_n''(n+1)}{r^{n+2}}\right] P_n(\cos\theta), \\ v_3 &= \left[A_n'' r^{n-1} + \frac{B_n''}{r^{n+2}}\right] \frac{dP_n(\cos\theta)}{d\theta}. \end{aligned}\right\}[n=0,\ 1,\ 2,\ldots.]\ldots..(11)$$

Now A_n and A_n' or B_n and B_n' are not independent. When we refer to the equation (1) or (2), we find that there is a fixed relation between A_n and A_n' or between B_n and B_n', the relation being expressed by

$$A_n' = -\frac{\lambda+2\mu}{\mu(n+1)} A_n; \qquad B_n' = \frac{\lambda+2\mu}{\mu n} B_n. \quad \ldots\ldots\ldots\ldots(12)$$

From the above treatment, it will be seen that there are two kinds of constants, one of which corresponds to $(A_n,\ A_n')$ or $(B_n,\ B_n')$ and the other to A_n'' or B_n''. The practical examples can be solved definitely by these two kinds of the arbitrary constants. It will also be remarked that the form of the displacement $(u_1,\ v_1)$ and that of $(u_2,\ v_2)$ are completely of the same types, while that of (u_3, v_3) is of a different type. We may thus conclude that the first of two kinds of displacements gives the dilatation as well as the rotation, the proportion of the dilatational displacement to the rotational one being determined by the elastic constants and the order of the harmonics of the straining; while the second kind gives only the displacement which is neither dilatational nor rotational. Thus the decomposition of the actual deformation into the dilatation and the rotation is only significant in the sense that we may determine what order of the harmonics of the straining is prevalent in this deformation. The straining, however, corresponding to (u_3, v_3) is to be let behind in this case. It will not be useless to tabulate the principal natures thus obtained in the forms:

Kind of displacement	Types of solutions	Physical meaning	Arbitrary constants	
1st kind	$r^{n+1}\times\ldots,\ \frac{1}{r^n}\times\ldots,$	dilatation	$A_n,\quad B_n$	$A_n'=\frac{-(\lambda+2\mu)}{\mu(n+1)}A_n,$
		rotation	$A_n',\quad B_n'$	$B_n'=\frac{(\lambda+2\mu)}{\mu n}B_n$
2nd kind	$r^{n-1}\times\ldots,\ \frac{1}{r^{n+2}}\times\ldots,$	indil. and irrot.	$A_n'',\ B_n''$	

3. We will now shew an example applied to a practical problem of the equilibrium of a solid body. Let us suppose that an elastic solid body has a spherical cavity of the radius a and the body is subjected to an uniform tension T or compression T, then the boundary conditions are expressed by

$$\widehat{rr}=0,\qquad \widehat{r\theta}=0\qquad \text{on } r=a,\ldots\ldots\ldots\ldots(13)$$

$$\widehat{rr}=T\cos^2\theta,\qquad \widehat{\theta\theta}=T\sin^2\theta\qquad \text{at } r=\infty\ldots\ldots\ldots(14)$$

By applying the general solutions in (9), (10), (11), we get the distribution of the stress components in all the point of the body as in the following forms:

$$\left.\begin{aligned}\widehat{rr}&=T\left[\frac{1}{3}\left(1-\frac{a^3}{r^3}\right)+\left\{\frac{2}{3}-\frac{10(9\lambda+10\mu)}{3(9\lambda+14\mu)}\frac{a^3}{r^3}+\frac{24(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}P_2(\cos\theta)\right],\\ \widehat{\theta\theta}&=T\left[\left\{\frac{2}{3}+\frac{(9\lambda+34\mu)}{6(9\lambda+14\mu)}\frac{a^3}{r^3}+\frac{2(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}\right.\\ &\qquad\left.-\left\{\frac{2}{3}-\frac{10\mu}{3(9\lambda+14\mu)}\frac{a^3}{r^3}+\frac{14(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}P_2(\cos\theta)\right],\end{aligned}\right\}(15)$$

$$\left.\begin{aligned}\widehat{\phi\phi}&=T\Bigg[\left\{\frac{3\lambda-2\mu}{2\,(9\lambda+14\mu)}\frac{a^3}{r^3}-\frac{2\,(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}\\&\qquad+\left\{\frac{10\mu}{(9\lambda+14\mu)}\frac{a^3}{r^3}-\frac{10\,(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}P_2(\cos\theta)\Bigg],\\\widehat{r\theta}&=T\left\{\frac{1}{3}+\frac{(15\lambda+10\mu)}{3\,(9\lambda+14\mu)}\frac{a^3}{r^3}-\frac{8\,(\lambda+\mu)}{(9\lambda+14\mu)}\frac{a^5}{r^5}\right\}\frac{dP_2(\cos\theta)}{d\theta}.\end{aligned}\right\}$$

At the point $r=a$, $\theta=\pi/2$, we have

$$\widehat{\theta\theta}=\frac{39\lambda+54\mu}{2\,(9\lambda+14\mu)}T, \qquad\qquad (16)$$

and, if $\lambda=\mu$, this becomes

$$\widehat{\theta\theta}=2.02T; \qquad\qquad (17)$$

while, if $\lambda=\infty$, $\widehat{\theta\theta}$ takes the value

$$\widehat{\theta\theta}=2.17T. \qquad\qquad (18)$$

This shews that, when there is a spherical cavity in a solid body, the stress $\widehat{\theta\theta}$ at the boundary of the cavity is about twice the magnitude of the applied traction. As to the maximum shear stress, the induction of such high stress at some portion of the solid due to the cavity is also easily calculated. Though this solutions have been obtained by one of us, the similar results obtained by other methods are frequently used in the discussion of the failure of bodies due to the statical loads and alternating stresses.

4. We may take another example for the confirmation of the applicability of the present problem. Suppose that the elastic body, which is stressed in one direction, has an imbedded spherical grain in itself. Then, taking the external and internal solutions, we get finally, when $r<a$,

$$\left.\begin{aligned}\widehat{rr'}&=T\left[\frac{(\lambda+2\mu)}{(3\lambda+2\mu)}\frac{(3\lambda'+2\mu')}{(3\lambda'+2\mu'+4\mu)}+4\mu'\frac{D_1}{D}P_2(\cos\theta)\right],\\\widehat{\theta\theta'}&=T\left[\frac{(\lambda+2\mu)}{(3\lambda+2\mu)}\frac{(3\lambda'+2\mu')}{(3\lambda'+2\mu'+4\mu)}+2\mu\frac{D_1}{D}-4\mu'\frac{D_1}{D}P_2(\cos\theta)\right],\\\widehat{\phi\phi'}&=T\left[\frac{(\lambda+2\mu)}{(3\lambda+2\mu)}\frac{(3\lambda'+2\mu')}{(3\lambda'+2\mu'+4\mu)}+2\mu'\frac{D_1}{D}\right],\\\widehat{r\theta'}&=T\left[2\mu'\frac{D_1}{D}\frac{dP_2(\cos\theta)}{d\theta}\right].\end{aligned}\right\} \qquad (19)$$

and when $r>a$,

$$\left.\begin{aligned}
\widehat{rr}&=T\left[\frac{1}{3}+\frac{4\mu}{3\lambda+2\mu}\,\frac{3(\lambda'-\lambda)+2(\mu'-\mu)}{9\lambda'+6\mu'+12\mu}\,\frac{a^3}{r^3}\right.\\
&\qquad\left.+\left(\frac{2}{3}+\frac{9\lambda+10\mu}{3}\,\frac{D_2}{D}\,\frac{a^3}{r^3}+24\mu\frac{D_3}{D}\,\frac{a^5}{r^5}\right)P_2(\cos\theta)\right],\\
\widehat{\theta\theta}&=T\left[\frac{2}{3}-\left(\frac{2\mu}{3\lambda+2\mu}\,\frac{3(\lambda'-\lambda)+2(\mu'-\mu)}{9\lambda'+6\mu'+12\mu}+\frac{\mu}{3}\,\frac{D_2}{D}\right)\frac{a^3}{r^3}\right.\\
&\qquad\left.+2\mu\frac{D_3}{D}\,\frac{a^5}{r^5}-\left(\frac{2}{3}+\frac{\mu}{3}\,\frac{D_2}{D}\,\frac{a^3}{r^3}+14\mu\frac{D_3}{D}\,\frac{a^5}{r^5}\right)P_2(\cos\theta)\right],\\
\widehat{\phi\phi}&=T\left[\left(\frac{\mu}{3}\,\frac{D_2}{D}-\frac{2\mu}{3\lambda+2\mu}\,\frac{3(\lambda'-\lambda)+2(\mu'-\mu)}{9\lambda'+6\mu'+12\mu}\right)\frac{a^3}{r^3}\right.\\
&\qquad\left.-2\mu\frac{D_3}{D}\,\frac{a^5}{r^5}-\left(\mu\frac{D_2}{D}\,\frac{a^3}{r^3}+10\mu\frac{D_3}{D}\,\frac{a^5}{r^5}\right)P_2(\cos\theta)\right],\\
\widehat{r\theta}&=T\left[\left(\frac{1}{3}-\frac{3\lambda+2\mu}{6}\,\frac{D_2}{D}\,\frac{a^3}{r^3}-8\mu\frac{D_3}{D}\,\frac{a^5}{r^5}\right)\frac{dP_2(\cos\theta)}{d\theta}\right].
\end{aligned}\right\}\quad(20)$$

where λ, μ are elastic constants of the external medium and λ', μ' are those of the internal grain, while

$$D=\begin{vmatrix}
\frac{\lambda'}{7}, & 4\mu', & -\frac{9\lambda+10\mu}{3}, & -24\mu\\
-\frac{8\lambda'+7\mu'}{21}, & 2\mu', & \frac{3\lambda+2\mu}{6}, & 8\mu\\
-\frac{\lambda'}{7\mu'}, & 2, & \frac{3\lambda+5\mu}{6\mu}, & 3\\
-\frac{5\lambda'+7\mu'}{42\mu'}, & 1, & \frac{1}{6}, & -1
\end{vmatrix},\qquad
D_2=\begin{vmatrix}
\frac{\lambda'}{7}, & 4\mu', & \frac{2}{3}, & -24\mu\\
-\frac{8\lambda'+7\mu'}{21}, & 2\mu', & \frac{1}{3}, & 8\mu\\
-\frac{\lambda'}{7\mu'}, & 2, & \frac{1}{3\mu'}, & 3\\
-\frac{5\lambda'+7\mu'}{42\mu'}, & 1, & \frac{1}{6\mu'}, & -1
\end{vmatrix},$$

$$D_1=\begin{vmatrix}
\frac{\lambda'}{7}, & \frac{2}{3}, & -\frac{9\lambda+10\mu}{3}, & -24\mu\\
-\frac{8\lambda'+7\mu'}{21}, & \frac{1}{3}, & \frac{3\lambda+2\mu}{6}, & 8\mu\\
-\frac{\lambda'}{7\mu'}, & \frac{1}{3\mu'}, & \frac{3\lambda+5\mu}{6\mu}, & 3\\
-\frac{5\lambda'+7\mu'}{42\mu'}, & \frac{1}{6\mu'}, & \frac{1}{6}, & -1
\end{vmatrix},\qquad
D_3=\begin{vmatrix}
\frac{\lambda'}{7}, & 4\mu', & -\frac{9\lambda+10\mu}{3}, & \frac{2}{3}\\
-\frac{8\lambda'+7\mu'}{21}, & 2\mu', & \frac{3\lambda+2\mu}{6}, & \frac{1}{3}\\
-\frac{\lambda'}{7\mu'}, & 2, & \frac{3\lambda+5\mu}{6\mu}, & \frac{1}{3\mu}\\
-\frac{5\lambda'+7\mu'}{42\mu}, & 1, & \frac{1}{6}, & \frac{1}{6\mu}
\end{vmatrix}.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\quad(21)$$

The distribution of stresses, $\widehat{\theta\theta}'$ and $\widehat{\theta\theta}$, in the section $\theta=\frac{\pi}{2}$ for the two cases; (1) $\lambda=\mu$, $\lambda'=\mu'$, $\frac{\mu'}{\mu}=\frac{1}{2}$ and (2) $\lambda=\mu$, $\lambda'=\mu'$, $\frac{\mu'}{\mu}=2$ are illustrated in Fig. 1 and Fig. 2.

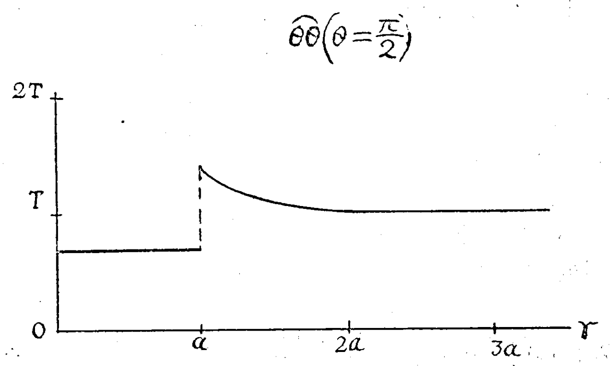


Fig. 1. $\left(\lambda=\mu,\quad \lambda'=\mu',\quad \frac{\mu'}{\mu}=\frac{1}{2}.\right)$

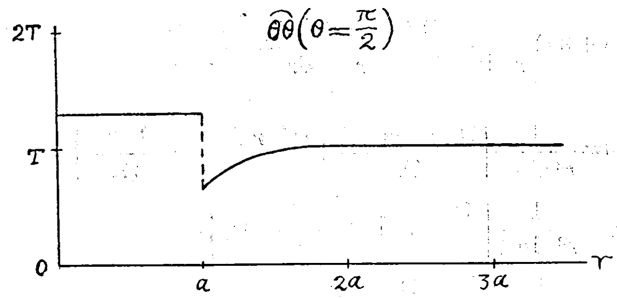


Fig. 2. $\left(\lambda=\mu,\quad \lambda'=\mu',\quad \frac{\mu'}{\mu}=2.\right)$

It is to be noticed that the stresses in the imbedded sphere are independent of the radial distance from the centre of th sphere. It is equally important fact that, when the internal grain is soft compared with the external medium, the stresses are accumulated at the neighbourhood of the external solid and the stresses in the grain is diminished more than the uniform stress, while in the case of a rigid grain imbedded in a soft medium, the stresses are accumulated in the imbedded grain and the stresses at the boundary region of the external solid is the more lessened than the uniform mean stress. The mean of the stresses in the plane surface $\theta=\pi/2$ is, of course, equal to the uniform traction.

The recent problem of block movements in the earth crust of our country will probably be nothing but the effect of such heterogeneity of physical constants on the stresses or the displacements.

II. The Equilibrium of a Spherical Body under Normal Pressures.[7]

5. The problem of the solid earth which is subjected to the normal boundary pressure of any distribution, when the curvature of the surface is taken into consideration, can be solved in the following manner.

Let r, θ, ϕ be spherical polar coordinates, and let us denote u, v, w as the components of the displacement in the direction of the radius, colatitude and azimuth, then the equations of equilibrium of the body can be expressed by

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial \Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\frac{\partial(\varpi_\phi\sin\theta)}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial\varpi_\theta}{\partial\phi}=0,\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial\theta}-\frac{2\mu}{r\sin\theta}\frac{\partial\varpi_r}{\partial\phi}+\frac{2\mu}{r}\frac{\partial(\varpi_\phi r)}{\partial r}=0,\\
&(\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial \Delta}{\partial\phi}-\frac{2\mu}{r}\frac{\partial(\varpi_\theta r)}{\partial r}+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}=0,
\end{aligned}\right\}\quad\cdots\cdots\cdots(22)$$

where

$$\left.\begin{aligned}
&\Delta=\frac{1}{r^2\sin\theta}\left[\frac{\partial(ur^2\sin\theta)}{\partial r}+\frac{\partial(vr\sin\theta)}{\partial\theta}+\frac{\partial(wr)}{\partial\phi}\right],\\
&2\varpi_r=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(wr\sin\theta)-\frac{\partial}{\partial\phi}(vr)\right],\\
&2\varpi_\theta=\frac{1}{r\sin\theta}\left[\frac{\partial u}{\partial\phi}-\frac{\partial(wr\sin\theta)}{\partial r}\right],\\
&2\varpi_\phi=\frac{1}{r}\left[\frac{\partial(vr)}{\partial r}-\frac{\partial u}{\partial\theta}\right].
\end{aligned}\right\}\quad\cdots\cdots\cdots(23)$$

The elimination of u, v, w among (22) and (23) gives us that

$$\left.\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Delta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Delta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Delta}{\partial\phi^2}=0,\\
&\frac{\partial^2\varpi_r}{\partial r^2}+\frac{4}{r}\frac{\partial\varpi_r}{\partial r}+\frac{2}{r^2}\varpi_r+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varpi_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_r}{\partial\phi^2}=0,\\
&\frac{1}{r}\frac{\partial^2(\varpi_\theta r)}{\partial r^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_\theta}{\partial\phi^2}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\varpi_\phi\sin\theta)}{\partial\phi\,\partial\theta}-\frac{1}{r}\frac{\partial^2\varpi_r}{\partial r\partial\theta}=0,\\
&\frac{1}{r}\frac{\partial^2(\varpi_\phi r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial(\varpi_\phi\sin\theta)}{\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\varpi_\theta}{\partial\phi}\\
&\qquad\qquad-\frac{1}{r\sin\theta}\frac{\partial^2\varpi_r}{\partial r\partial\phi}=0.
\end{aligned}\right\}\quad\cdots(24)$$

7) K. SEZAWA & G. NISHIMURA, *loc. cit.*, 389.

These gives us

$$\left.\begin{aligned}
\Delta&=\left(A_{mn}r^{n}+\frac{A'_{mn}}{r^{n+1}}\right)P_n^m(\cos\theta)\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,\\
2\varpi_r&=\left(B_{mn}r^{n-1}+\frac{B'_{mn}}{r^{n+2}}\right)P_n^m(\cos\theta)\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi,\\
2\varpi_\theta&=\left[\left(D_{mn}r^{n}+\frac{D'_{mn}m}{r^{n+1}}\right)\frac{P_n^m(\cos\theta)}{\sin\theta}\right.\\
&\qquad\left.+\left(\frac{B_{mn}}{n}r^{n-1}-\frac{B'_{mn}}{(n+1)\,r^{n+2}}\right)\frac{dP_n^m(\cos\theta)}{d\theta}\right]\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi,\\
2\varpi_\phi&=\left[\left(D_{mn}r^{n}+\frac{D'_{mn}}{r^{n+1}}\right)\frac{dP_n^m(\cos\theta)}{d\theta}\right.\\
&\qquad\left.+\left(\frac{B_{mn}}{n}r^{n-1}-\frac{B'_{mn}}{(n+1)\,r^{n+2}}\right)\frac{P_n^m(\cos\theta)}{\sin\theta}\right]\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi.
\end{aligned}\right\}\quad ..(25)$$

in which A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn}, C'_{mn} are arbitrary constants.

Displacement (u_1, v_1, w_1) answering to Δ in (25) and satisfying $\varpi_r=\varpi_\theta=\varpi_\phi=0$ is expressed by

$$\left.\begin{aligned}
u_1&=\left[\frac{A_{mn}(n+2)}{2(2n+3)}r^{n+1}+\frac{A'_{mn}(n-1)}{2(2n-1)\,r^{n}}\right]P_n^m(\cos\theta)\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,\\
v_1&=\left[\frac{A_{mn}}{2(2n+3)}r^{n+1}-\frac{A'_{mn}}{2(2n-1)\,r^{n}}\right]\frac{dP_n^m(\cos\theta)}{d\theta}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,\\
w_1&=-\left[\frac{A_{mn}m}{2(2n+3)}r^{n+1}-\frac{A'_{mn}}{2(2n-1)\,r^{n}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi.
\end{aligned}\right\}\quad ....(26)$$

Displacement (u_2, v_2, w_2) answering to ϖ_r together with the second terms in the expression of ϖ_θ, ϖ_ϕ, all given in (25) under the condition that $\Delta=0$, is expressed by

$$\left.\begin{aligned}
u_2&=0,\\
v_2&=\left[\frac{mB_{mn}}{n(n+1)}r^{n}+\frac{mB_{mn}}{n(n+1)\,r^{n+1}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,\\
w_2&=-\left[\frac{B_{mn}}{n(n+1)}r^{n}+\frac{B'_{mn}}{n(n+1)\,r^{n+1}}\right]\frac{dP_n^m(\cos\theta)}{d\theta}\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi.
\end{aligned}\right\}\quad ....(27)$$

Displacement (u_4, v_4, w_4) derived from the values of the first terms of ϖ_θ, ϖ_ϕ, in (25) fulfilling the conditions, $\Delta=\varpi_r=0$ is written by

$$\left.\begin{aligned} u_4&=\left[\frac{D_{mn}\,n(n+1)}{2(2n+3)}r^{n+1}+\frac{D'_{mn}\,n(n+1)}{2(2n-1)\,r^n}\right]P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ v_4&=\left[\frac{D_{mn}(n+3)}{2(2n+3)}r^{n+1}+\frac{D'_{mn}(n-2)}{2(2n-1)\,r^n}\right]\frac{1}{m}\,\frac{dP_n^m(\cos\theta)}{d\theta}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ w_4&=-\left[\frac{D_{mn}\,m(n+3)}{2(2n+3)}r^{n+1}+\frac{D'_{mn}\,m(n-2)}{2(2n-1)\,r^n}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\}m\phi.\end{aligned}\right\} \qquad (28)$$

Displacement (u_3, v_3, w_3) which satisfies $\Delta=\varpi_r=\varpi_\theta=\varpi_\phi=0$, is expressed by

$$\left.\begin{aligned} u_3&=\left[C_{mn}\,n r^{n-1}-\frac{C'_{mn}(n+1)}{r^{n+2}}\right]P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}\Big]m\phi,\\ v_3&=\left[C_{mn}r^{n-1}+\frac{C'_{mn}}{r^{n+2}}\right]\frac{dP_n^m(\cos\theta)}{d\theta}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ w_3&=-m\left[C_{mn}r^{n-1}+\frac{C'_{mn}}{r^{n+2}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\}m\phi.\end{aligned}\right\} \qquad \ldots\ldots\ldots(29)$$

In the above solutions, the displacement (u_2, v_2, w_2) in (27) is without significance for the problem of normal pressure, so that this displacement may be excluded from the present study. It is the only proper term when the sphere is subjected to the distortional stress at the surface.

Now A_{mn} and D_{mn} or A'_{mn} and D'_{mn} are not independent of each other. When we refer to the equation (22), we find that there is a fixed relation between A_{mn} and D_{mn} or A'_{mn} and D'_{mn}, the relations being expressed by

$$\frac{D_{mn}}{A_{mn}}=-\frac{\lambda+2\mu}{\mu(n+1)}\,;\qquad \frac{D'_{mn}}{A'_{mn}}=\frac{\lambda+2\mu}{\mu n}\ \ \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

Proceeding in the same manner as in the preceding part we may write down the principal items in the following table:

Kind of displacement	Types of solutions	Physical meaning	Arbitray constants	
1st kind	$r^{n+1}\times\ldots,\ \frac{1}{r^n}\times\ldots,$	dilatation	$A_{mn},\ A'_{mn}$	$D_{mn}=-\frac{\lambda+2\mu}{\mu(n+1)}A_{mn},$
		rotation	$D_{mn},\ D'_{mn}$	$D'_{mn}=\frac{(\lambda+2\mu)}{\mu n}A'_{mn}$
2nd kind	$r^{n-1}\times\ldots,\ \frac{1}{r^{n+2}}\times\ldots,$	indil. and irrot.	$C_{mn},\ C'_{mn}$	

Examples of application have been omitted in this paper, as they have already appeared in the Bulletin VI.

III. Spheroidal Problems of Elasticity solved in Curvilinear Coordinates.

6. The equations of motion of elastic bodies in spheroidal coordinates, when the azimuthal component of the motion is omitted, are expressed by

$$\left.\begin{aligned}(\lambda+2\mu)\,h_1\frac{\partial\Delta}{\partial\xi}-2\mu\,h_2\,h_3\frac{\partial}{\partial\eta}\left(\frac{\varpi}{h_3}\right)=0,\\(\lambda+2\mu)\,h_2\frac{\partial\Delta}{\partial\eta}+2\mu\,h_1\,h_3\frac{\partial}{\partial\xi}\left(\frac{\varpi}{h_3}\right)=0,\end{aligned}\right\}\quad\text{.............}(31)$$

in which

$$\left.\begin{aligned}\Delta=h_1h_2h_3\left[\frac{\partial}{\partial\xi}\left(\frac{u}{h_2\,h_3}\right)+\frac{\partial}{\partial\eta}\left(\frac{v}{h_1\,h_2}\right)\right],\\2\varpi=h_1h_2\left[\frac{\partial}{\partial\xi}\left(\frac{v}{h_2}\right)-\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right],\end{aligned}\right\}\quad\text{............}(32)$$

and

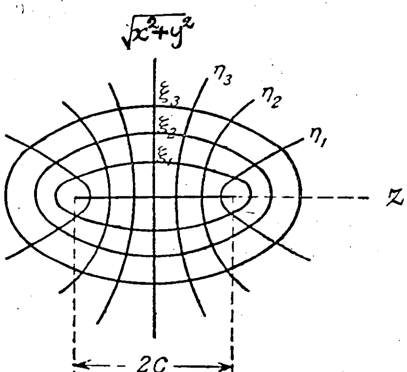
$$\frac{1}{h_1^2}=\frac{1}{h_2^2}=c^2(\mathrm{ch}^2\xi-\cos^2\eta),\qquad\frac{1}{h_3^2}=c^2\,\mathrm{sh}^2\,\xi\,\sin^2\eta,\ \text{.......}(33)$$

in the case of a prolate-spheroidal coordinates (ξ, η) connected with Cartesian coordinates in the form:

$$z+i\sqrt{x^2+y^2}=c\cosh(\xi+i\eta),\ \text{...................}(34)$$

or in real forms,

$$\frac{x^2+y^2}{c^2\,\mathrm{sh}^2\,\xi}+\frac{z^2}{c^2\,\mathrm{ch}^2\,\xi}=1,\qquad\frac{z^2}{c^2\cos^2\eta}-\frac{x^2+y^2}{c^2\sin^2\eta}=1.\ \text{..........}(35)$$



In equations (32) u, v are the displacement in the direction of ξ and η and $2c$ is the length of the major axis of the spheroid.

Eliminating u, v between (31) and (32), we obtain

$$\left.\begin{aligned}&\frac{\partial^2 \Delta}{\partial \xi^2}+\operatorname{cth}\xi\frac{\partial \Delta}{\partial \xi}+\frac{\partial^2 \Delta}{\partial \eta^2}+\cot\eta\frac{\partial \Delta}{\partial \eta}=0,\\&\frac{\partial^2 \varpi}{\partial \xi^2}+\operatorname{cth}\xi\frac{\partial \varpi}{\partial \xi}-\frac{\varpi}{\operatorname{sh}^2\xi}+\frac{\partial^2 \varpi}{\partial \eta^2}+\cot\eta\frac{\partial \varpi}{\partial \eta}-\frac{\varpi}{\sin^2\eta}=0.\end{aligned}\right\}\quad\ldots\ldots\ldots(36)$$

Solving (36), we find

$$\Delta=\{A_nP_n(\operatorname{ch}\xi)+B_nQ_n(\operatorname{ch}\xi)\}P_n(\cos\eta),\qquad [n=0,1,2..]\quad\ldots.(37)$$

$$2\varpi=\left\{A_n'\frac{dP_n(\operatorname{ch}\xi)}{d\xi}+B_n'\frac{dQ_n(\operatorname{ch}\xi)}{d\xi}\right\}\frac{dP_n(\cos\eta)}{d\eta}.\qquad [n=1,2,3..]\quad\ldots.(38)$$

Displacement (u_1, v_1) answering to Δ in (37) and fulfilling the condition that $\varpi=0$ is expressed by

$$\begin{aligned}u_1=\frac{c}{\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}}\Bigg[&\frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2}\left(A_n\frac{dP_{n+2}(\operatorname{ch}\xi)}{d\xi}+B_n\frac{dQ_{n+2}(\operatorname{ch}\xi)}{d\xi}\right)P_n(\cos\eta)\\&+\frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2}\left(A_n\frac{dP_n(\operatorname{ch}\xi)}{d\xi}+B_n\frac{dQ_n(\operatorname{ch}\xi)}{d\xi}\right)P_{n+2}(\cos\eta)\qquad [n=0,1,2,..]\\&-\frac{n(n-1)}{2(2n-1)^2(2n+1)}\left(A_n\frac{dP_{n-2}(\operatorname{ch}\xi)}{d\xi}+B_n\frac{dQ_{n-2}(\operatorname{ch}\xi)}{d\xi}\right)P_n(\cos\eta)\\&-\frac{n(n-1)}{2(2n-1)^2(2n+1)}\left(A_n\frac{dP_n(\operatorname{ch}\xi)}{d\xi}+B_n\frac{dQ_n(\operatorname{ch}\xi)}{d\xi}\right)P_{n-2}(\cos\eta)\Bigg],\qquad [n=2,3,..]\end{aligned}$$

$$\begin{aligned}v_1=\frac{c}{\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}}\Bigg[&\frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2}\Big(A_nP_{n+2}(\operatorname{ch}\xi)+B_nQ_{n+2}(\operatorname{ch}\xi)\Big)\frac{dP_n(\cos\eta)}{d\eta}\\&+\frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2}\Big(A_nP_n(\operatorname{ch}\xi)+B_nQ_n(\operatorname{ch}\xi)\Big)\frac{dP_{n+2}(\cos\eta)}{d\eta}\qquad [n=0,1,2,..]\end{aligned}$$

$$-\frac{n(n-1)}{2(2n-1)^2(2n+1)}\left(A_n P_{n-2}(\operatorname{ch}\xi) + B_n Q_{n-2}(\operatorname{ch}\xi)\right)\frac{dP_n(\cos\eta)}{d\eta} - \frac{n(n-1)}{2(2n-1)^2(2n+1)}\left(A_n P_n(\operatorname{ch}\xi) + B_n Q_n(\operatorname{ch}\xi)\right)\frac{dP_{n-2}(\cos\eta)}{d\eta}\Bigg]. \quad [n=2, 3, ..]$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (39)$$

Displacement (u_2, v_2) corresponding to ϖ in (38) under the condition that $\Delta=0$ is expressed by

$$u_2 = \frac{cn(n+1)}{\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}}\Bigg[\frac{n(n+1)}{2(2n+1)(2n+3)^2}\left(A_n' \frac{dP_{n+2}(\operatorname{ch}\xi)}{d\xi} + B_n' \frac{dQ_{n+2}(\operatorname{ch}\xi)}{d\xi}\right)P_n(\cos\eta) + \frac{(n+2)(n+3)}{2(2n+1)(2n+3)^2}\left(A_n' \frac{dP_n(\operatorname{ch}\xi)}{d\zeta} + B_n' \frac{dQ_n(\operatorname{ch}\xi)}{d\xi}\right)P_{n+2}(\cos\eta) \quad [n=0, 1, 2, ..]$$

$$-\frac{n(n+1)}{2(2n-1)^2(2n+1)}\left(A_n' \frac{dP_{n-2}(\operatorname{ch}\xi)}{d\xi} + B_n' \frac{dQ_{n-2}(\operatorname{ch}\xi)}{d\xi}\right)P_n(\cos\eta) - \frac{(n-1)(n-2)}{2(2n-1)^2(2n+1)}\left(A_n' \frac{dP_n(\operatorname{ch}\xi)}{d\xi} + B_n' \frac{Q_n(\operatorname{ch}\xi)}{d\xi}\right)P_{n-2}(\cos\eta)\Bigg], \quad [n=2, 3, ..]$$

$$v_2 = \frac{cn(n+1)}{\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}}\Bigg[\frac{(n+2)(n+3)}{2(2n+1)(2n+3)^2}\left(A_n' P_{n+2}(\operatorname{ch}\xi) + B_n' Q_{n+2}(\operatorname{ch}\xi)\right)\frac{dP_n(\cos\eta)}{d\eta} + \frac{n(n+1)}{2(2n+1)(2n+3)^2}A_n' P_n(\operatorname{ch}\xi) + B_n' Q_n(\operatorname{ch}\xi)\Big)\frac{dP_{n+2}(\cos\eta)}{d\zeta} \quad [n=0, 1, 2, ..]$$

$$
\left.\begin{aligned}
&-\frac{(n-1)(n-2)}{2(2n-1)^2(2n+1)}\Big(A_n' P_{n-2}(\mathrm{ch}\,\xi) \\
&\qquad\qquad + B^{n\prime} Q^{n-2}(\mathrm{ch}\,\xi)\Big)\frac{dP_n(\cos\eta)}{d\eta} \\
&-\frac{n(n+1)}{2(2n-1)^2(2n+1)}\Big(A_n' P_n(\mathrm{ch}\,\xi) \\
&\qquad\qquad + B_n' Q_n(\mathrm{ch}\,\xi)\Big)\frac{dP_{n-2}(\cos\eta)}{d\eta}\Bigg].
\end{aligned}\right\} [n=2,3,..] \qquad (40)
$$

Displacement (u_3, v_3) which satisfies $\Delta=0$, $\varpi=0$ can be written in the forms:

$$
\left.\begin{aligned}
u_3 &= \frac{c}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\left[A_n''\frac{dP_n(\mathrm{ch}\,\xi)}{d\xi} + B_n''\frac{dQ_n(\mathrm{ch}\,\xi)}{d\xi}\right]P_n(\cos\eta), \\
v_3 &= \frac{c}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\left[A_n'' P_n(\mathrm{ch}\,\xi) + B_n'' Q_n(\mathrm{ch}\,\xi)\right]\frac{dP_n(\cos\eta)}{d\eta}.
\end{aligned}\right\} [n=0,1,2,..] \qquad (41)
$$

Now A_n and A_n' or B_n and B_n' are not independent and there is a fixed relation between A_n and A_n' or B_n and B_n', the relations, which are obtained from (31), being expressed by

$$
A_n' = -\frac{\lambda+2\mu}{\mu n(n+1)}A_n; \quad B_n' = -\frac{\lambda+2\mu}{\mu n(n+1)}B_n. \qquad (42)
$$

The tabulated elements of the nature of the solutions are as follows:

Kind of displacement	Types of solutions	Physical meaning	Arbitrary constants	
1st kind	P_{n+2}, P_n, P_{n-2}....	dilatation	A_n, B_n	$\left.\begin{matrix}A_n'\\B_n'\end{matrix}\right\} = -\frac{\lambda+2\mu}{\mu n(n+1)}\left\{\begin{matrix}A_n\\B_n.\end{matrix}\right.$
	Q_{n+2}, Q_n, Q_{n-2}....	rotation	A_n', B_n'	
2nd kind	P_n, Q_n	indil. and irrot.	A_n'', B_n''	

7. In the case of the oblate spheroid, we should write

$$\frac{1}{h_1^2}=\frac{1}{h_2^2}=c^2(\mathrm{ch}^2\xi-\sin^2\eta);\qquad \frac{1}{h_3^2}=c^2\,\mathrm{ch}^2\xi\,\sin^2\eta,\quad\text{........ (43)}$$

where $2c$ is the magnitude of the major axis.

From (31) and (43), we obtain

$$\left.\begin{aligned}&\frac{\partial^2\Delta}{\partial\xi^2}+\mathrm{th}\,\xi\frac{\partial\Delta}{\partial\xi}+\frac{\partial^2\Delta}{\partial\eta^2}+\cot\eta\frac{\partial\Delta}{\partial\eta}=0,\\&\frac{\partial^2\varpi}{\partial\xi^2}+\mathrm{th}\,\xi\frac{\partial\varpi}{\partial\xi}+\frac{\varpi}{\mathrm{ch}^2\xi}+\frac{\partial^2\varpi}{\partial\eta^2}+\cot\eta\frac{\partial\varpi}{\partial\eta}-\frac{\varpi}{\sin^2\eta}=0,\end{aligned}\right\}\quad\text{....... (44)}$$

the solutions of which are

$$\Delta=\{A_nP_n(i\,\mathrm{sh}\,\xi)+B_nQ_n(i\,\mathrm{sh}\,\xi)\}P_n(\cos\eta),\quad\text{................ (45)}$$

$$2\varpi=\left\{A_n'\frac{dP_n(i\,\mathrm{sh}\,\xi)}{d\xi}+B_n'\frac{dQ_n(i\,\mathrm{sh}\,\xi)}{d\xi}\right\}\frac{dP_n(\cos\eta)}{d\eta}.\quad\text{........... (46)}$$

The corresponding displacements are of the similar forms except the negative signs of (u_1, v_1), (u_2, v_2) and also the argument of the function of the type $P_n(i\,\mathrm{sh}\,\xi)$. The other explanations have been omitted for the sake of simplicity.

IV. The Problem of an Elastic Cylinder.[8]

8. The equations of motion of an elastic body in cylindrical coordinates are expressed by

$$\left.\begin{aligned}&(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi_z}{\partial\theta}+2\mu\frac{\partial\varpi_\theta}{\partial z}=0,\\&(\lambda+2\mu)\frac{\partial\Delta}{r\partial\theta}-2\mu\frac{\partial\varpi_r}{\partial z}+2\mu\frac{\partial\varpi_z}{\partial r}=0,\\&(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_\theta)+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}=0,\end{aligned}\right\}\quad\text{............ (47)}$$

where

$$\left.\begin{aligned}&\Delta=\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z},\\&2\varpi_r=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z},\\&2\varpi_\theta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r},\\&2\varpi_z=\frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial\theta},\end{aligned}\right\}\quad\text{................. (48)}$$

8) G. Nishimura, *Journ. Soc. Ord. and Expl.*, **23** (1929).

in which u, v, w are the components of displacement in r-, θ-, z-directions.

Eliminating u, v, w in (47) and (48), we get

$$\left.\begin{aligned}&\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{\partial^2\Delta}{\partial z^2}=0,\\&\frac{\partial^2\varpi_r}{\partial r^2}+\frac{3}{r}\frac{\partial\varpi_r}{\partial r}+\frac{\varpi_r}{r^2}+\frac{1}{r^2}\frac{\partial^2\varpi_r}{\partial\theta^2}+\frac{\partial^2\varpi_r}{\partial z^2}+\frac{2}{r}\frac{\partial\varpi_\theta}{\partial z}=0,\\&\frac{\partial^2\varpi_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi_\theta}{\partial r}-\frac{\varpi_\theta}{r}+\frac{1}{r^2}\frac{\partial^2\varpi_\theta}{\partial\theta^2}+\frac{\partial^2\varpi_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial\varpi_r}{\partial\theta}=0,\\&\frac{\partial^2\varpi_z}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varpi_z}{\partial\theta^2}+\frac{\partial^2\varpi_z}{\partial z^2}=0.\end{aligned}\right\}\quad\cdots\cdots\cdots(49)$$

The solutions of these can be expressed by

$$\Delta=\left\{A_{mk}\,I_m(kr)+A'_{mk}\,K_m(kr)\right\}\begin{Bmatrix}\sin\\\cos\end{Bmatrix}kz\begin{Bmatrix}\sin\\\cos\end{Bmatrix}m\theta,$$

$$2\varpi_r=\left\{B_{mk}\frac{1}{k}\frac{\partial I_m(kr)}{r}+C_{mk}\frac{I_m(kr)}{r}\right.$$

$$\left.+B'_{mk}\frac{1}{k}\frac{\partial K_m(kr)}{\partial r}+C'_{mk}\frac{K_m(kr)}{\partial r}\right\}\begin{Bmatrix}-\cos\\\cos\end{Bmatrix}kz\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}m\theta,$$

$$2\varpi_\theta=\left\{B_{mk}\frac{m}{k}\frac{I_m(kr)}{r}+C_{mk}\frac{1}{m}\frac{\partial I_m(kr)}{\partial r}\right.$$

$$\left.+B'_{mk}\frac{m}{k}\frac{K_m(kr)}{r}+C'_{mk}\frac{1}{m}\frac{\partial K_m(kr)}{\partial r}\right\}\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}kz\begin{Bmatrix}\sin\\\cos\end{Bmatrix}m\theta,$$

$$2\varpi_z=\left\{B_{mn}\,I_m(kr)+B'_{mk}\,K_m(kr)\right\}\begin{Bmatrix}\sin\\\cos\end{Bmatrix}kz\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}m\theta.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(50)$$

Displacement (u_1, v_1, w_1) which gives Δ in (50) and satisfies $\varpi_r=\varpi_\theta=\varpi_z=0$ are expressed by

$$\left.\begin{aligned}u_1&=\frac{1}{2k^2}\left\{A_{mk}\frac{\partial}{\partial r}\left(r\frac{\partial I_m(kr)}{\partial r}\right)\right.\\&\qquad\left.+A'_{mk}\frac{\partial}{\partial r}\left(r\frac{\partial K_m(kr)}{\partial r}\right)\right\}\begin{Bmatrix}\sin\\\cos\end{Bmatrix}kz\begin{Bmatrix}\sin\\\cos\end{Bmatrix}m\theta,\\v_1&=-\frac{m}{2k^2}\left\{A_{mk}\frac{\partial I_m(kr)}{\partial r}+A'_{mk}\frac{\partial K_m(kr)}{\partial r}\right\}\begin{Bmatrix}\sin\\\cos\end{Bmatrix}kz\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}m\theta,\\w_1&=-\frac{1}{2k}\left\{A_{mk}\,r\frac{\partial I_m(kr)}{\partial r}+A'_{mk}\,r\frac{\partial K_m(kr)}{\partial r}\right\}\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}kz\begin{Bmatrix}\sin\\\cos\end{Bmatrix}m\theta.\end{aligned}\right\}\cdot\cdot(51)$$

Displacement which satisfies ϖ_z and 1st and 3rd terms of ϖ_r and ϖ_θ

in (50), with the condition that $\varDelta=0$ is written by

$$\left.\begin{aligned} u_2&=-\frac{m}{k^2}\left\{B_{mk}\frac{I_m(kr)}{r}+B'_{mk}\frac{K_m(kr)}{r}\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}\sin\\\cos\end{matrix}\Bigg\} m\theta,\\ v_2&=\frac{1}{k}\left\{B_{mk}\frac{\partial I_m(kr)}{\partial r}+B'_{mk}\frac{\partial K_m(kr)}{\partial r}\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}-\cos\\\sin\end{matrix}\Bigg\} m\theta,\\ w_2&=0.\end{aligned}\right\}\quad\cdots\cdots(52)$$

Displacement (u_3, v_3, w_3) satisfying the 2nd and 4th terms of ϖ_r, ϖ_θ in (50) and fulfilling the condition $\varDelta=\varpi_z=0$ is expressed by

$$\begin{aligned} u_3&=-\frac{1}{mk}\left\{C_{mk}\frac{\partial}{\partial r}\left(r\frac{\partial I_m(kr)}{\partial r}\right)+C'_{mk}\frac{\partial}{\partial r}\left(r\frac{\partial K_m(kr)}{\partial r}\right)\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}\sin\\\cos\end{matrix}\Bigg\} m\theta,\\ v_3&=\frac{1}{k}\left\{C_{mk}\frac{\partial I_m(kr)}{\partial r}+C'_{mk}\frac{\partial K_m(kr)}{\partial r}\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}-\cos\\\sin\end{matrix}\Bigg\} m\theta,\\ w_3&=-\frac{1}{2m}\left\{C_{mk}\,r\frac{\partial I_m(kr)}{\partial r}+C'_{mk}\,r\frac{\partial K_m(kr)}{\partial r}\right\}\begin{matrix}-\cos\\\sin\end{matrix}\Bigg\} kz\ \begin{matrix}\sin\\\cos\end{matrix}\Bigg\} m\theta.\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(53)$$

Displacement (u_4, v_4, w_4) which answers to $\varDelta=\varpi_r=\varpi_\theta=\varpi_z=0$ is expressed by

$$\left.\begin{aligned} u_4&=-\frac{1}{k}\left\{E_{mk}\frac{\partial I_m(kr)}{\partial r}+E'_{mk}\frac{\partial K_m(kr)}{\partial r}\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}\sin\\\cos\end{matrix}\Bigg\} m\theta,\\ v_4&=\frac{m}{k}\left\{E_{mk}\frac{I_m(kr)}{r}+E'_{mk}\frac{K_m(kr)}{r}\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\} kz\ \begin{matrix}-\cos\\\sin\end{matrix}\Bigg\} m\theta,\\ w_4&=\left\{E_{mk}I_m(kr)+E'_{mk}K_m(kr)\right\}\begin{matrix}-\cos\\\sin\end{matrix}\Bigg\} kz\ \begin{matrix}\sin\\\cos\end{matrix}\Bigg\} m\theta.\end{aligned}\right\}\cdots\cdots(54)$$

In the above equations, the displacement (u_2, v_2, w_2) in (52) has no significance for the extensional and flexural problems, but they are useful for the torsion problem. Now, taking the case of the extensional or flexural problem, it will be easily seen that A_{mk} and C_{mk} or A'_{mk} and C'_{mk} are not independent. There are the following connections:

$$\frac{C_{mk}}{A_{mk}}=-\frac{(\lambda+2\mu)\,m}{\mu k},\quad \frac{C'_{mk}}{A'_{mk}}=-\frac{(\lambda+2\mu)\,m}{\mu k}\cdots\cdots\cdots\cdots(55)$$

Thus, in this case, we have only two types of solutions, one of which corresponds to (u_1, v_1, w_1) and (u_3, v_3, w_3), while the other to (u_4, v_4, w_4). The tabulated summary is as follows:

Kind of displacement	Types of solutions	Physical meaning	Arbitrary constants
1st kind	$r\frac{\partial I_m(kr)}{\partial r}, \ldots$	dilatation rotation	$\frac{C_{mk}}{A_{mk}} = -\frac{(\lambda+2\mu)m}{\mu k}, \quad \frac{C'_{mk}}{A'_{mk}} = -\frac{(\lambda+2\mu)m}{\mu k}.$
2nd kind	$I_m(kr), \ldots$	indilatational irrotational	$E_{mk}, \quad E'_{mk}$

The similar process can be applied to Boussinesq's problem on the equilibrium of a semi-infinite solid body under the normal traction or shearing stress on the free surface and we may find that the displacements consist of two kinds as already cited in the forgoing parts. The problem itself is too well-known to study here in any analytical manner.

V. Résumé on the Elastic Deformation.

9. We have now ascertained that the deformation of a solid body should be composed of the dilatation, the rotation and another component which is neither dilatational nor distortional and also that the dilatational component and the distortional component are not independent but they are connected both in types and in magnitudes. Thus, to get an analytical result from the data of the land survey, the mere consideration of the dilatation, the rotation and the shear is not sufficient and also the classification of the deformations into the dilatation and the rotation has not so much meaning. Such classification would be equivalent to seek the order of the harmonics of the deformation of the first kind. It is also theoretically useless to take up only the resulting rotation (or only the resulting dilatation) of the movement of the earthcrust, excepting the case where the rotation or the dilatation is thought to be a type of the nuclei of the strain. In general, it would be reasonable to take the stress components or the ordinary strain components in the analysis of the observed data of the deformation of the land surface. It would be rather better at least to consider the displacements only relative to some assigned axes.

VI. The Components of Displacement in Inelastic Deformation.

10. The most theoretical definition of the plastic deformation is such that, i) the stresses of all kinds acting at any element of the material should be in equilibrium, ii) the stresses are connected with the displacement of the material in certain laws of stress-strain relations and of compatibility, iii) the maximum shear stress at any point should not exceed a given limit,[9] iv) the time variation fo the deformation due to the stresses, the duration of the acting force and etc. obey definite rules, v) the material in a plastic state sometimes flows as if it were an imcompressible matter.

Among these, the criterions i), iii) and v) are generally taken up by the recent students in applied mechanics. We think that the criterion v) is not so important as to consider it to be a necessary condition. Such a criterion is probably based upon the fact that the behaviour of the plastic material resembles that of the incompressible fluid. We believe that the plastic material may be compressible or incompressible. Even the elastic solid can be treated as compressible or incompressible. Thus, in this part, the material is considered to be compressible in general and to be incompressible in some particular cases.

Again, the condition iv) would be the most important nature in the plastic deformation. In spite of its importance in geophysical problems, this nature is in many cases diregarded by the students due to no importance in the problems of the engineering and also due to the complicated behaviour which is yet unknown. A special case of this, however, attracted the keen attention of the theoreticians[10] of some classical manner, though they do not seem to care for the other conditions. In the present case we will not take this criterion iv) under the assumption that the behaviour of the plastic materials during relatively short intervals of time is only considered.

We have taken into account of the condition ii) which seems to be often disregarded by the recent investigators.

11. Now, we take a particular case of a plane polar coordinates, which will shew the nature of the displacement in a somewhat simple manner.

For the sake of simplicity, take the equation of the equilibrium of an inelastic body in polar coordinates in the forms:

i)
$$\frac{\partial \widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}=0, \qquad (56)$$

$$\frac{\partial \widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta}+\frac{2\widehat{r\theta}}{r}=0, \qquad (57)$$

where $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$ are normal and tangential stresses in r-, θ- and $r\theta$-directions.

9) O. Mohr, *V. D. I.* (1900), (1901).
Th. v. Kármán, *V. D. I.*, (1911), 1749.
10) L. Bolzmann, *Ann. d. Phys.*, Erg.-Bd., 7 (1876), 625.
H. Shôji, *Sci. Rep. Tohoku Imp. Univ.*, 18 (1929), No. 1, 1–10.

The condition of the maximum shear stress is expressed in the manner of St. Venant[12] as in the forms:

$$\text{iii)} \qquad \widehat{r\theta}^2+\left(\frac{\widehat{rr}-\widehat{\theta\theta}}{2}\right)^2\leq K^2, \qquad\qquad (58)$$

in which K is a constant.

The stress-displacement relation is of the form:

$$\text{ii)} \qquad \frac{\widehat{r\theta}}{\dfrac{\partial v}{\partial r}-\dfrac{v}{r}+\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}}=\frac{\widehat{rr}-\widehat{\theta\theta}}{2\left(\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}-\dfrac{u}{r}\right)}, \qquad\qquad (59)$$

where u, v[12] are the displacement of a material point in r- and θ-directions, though it may be better to take u, v as velocity components in the sense of Tresca.[13]

The condition v) may temporarily be expressed by

$$\text{v)} \qquad \frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{1}{r}\frac{\partial v}{\partial \theta}=0. \qquad\qquad (60)$$

From five equations (56), (57), (58), (59), (60), we can determine the values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$, u and v. The equations actually used by the plasticians are only three of them, i.e. (56), (57) and (58).

As the equation (60) is not important and the equation (59) is ambiguous, we can write in place of (59) and (60) the equivalent equations of the following forms, which involve more or less the idea of v. Mises[14] and which are more improved than those of v. Mises.

$$\text{v)} \qquad \frac{\lambda_1}{2(\lambda_1+\mu_1)}=\sigma_1 \qquad \text{or equivalent formulae.}^{15)} \qquad (61)$$

$$\text{ii)}' \qquad \delta\widehat{rr}=\lambda_1\,\delta\Delta+2\mu_1\,\delta\frac{\partial u}{\partial r}, \qquad\qquad (62)$$

$$\text{ii)}'' \qquad \delta\widehat{\theta\theta}=\lambda_1\,\delta\Delta+2\mu_1\,\delta\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right), \qquad\qquad (63)$$

---

11) ST. VENANT, *loc. cit.*, 389.

12) (u, v) may be taken to be the displacement in the present case, but not to be the velocity of the point. Though the students in Europe seem to take u, v as the velocity components, yet we think that they should be taken as the displacements, as far as the statical case is taken up and Boltmann's idea is neglected.

13) TRESCA, *C. R.* (1867).

14) R. v. MISES, *Nachr. Ges. Wiss., Göttingen* (1913).

15) The meaning of the equivalent formulae is that λ_1 and μ_1 are known functions of the strain.

ii)'''
$$\delta \widehat{r\theta} = \mu_1 \, \delta \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right), \quad \ldots\ldots (64)$$

where λ_1 (or μ_1) is a given function of a certain typical strain; and

$$\varDelta = \frac{1}{r} \frac{\partial}{\partial r}(ur) + \frac{1}{r} \frac{\partial v}{\partial \theta}. \quad \ldots\ldots (65)$$

Here we shall understand that, in place of (59) and (60), four equations (61), (62), (63), (64) having two quantities λ_1 and μ_1 are introduced. λ_1 and μ_1 should not be invariables like Lame's constants; they may take various values for different magnitudes of the strains or the stresses. This nature will perhaps be suited to the deformation in which the stresses and the strains are not related linearly. The equation (61) expresses the degree of the compressibility and it will not be much erroneous that we take σ_1 is constant for the ordinary plastic state of a given material. It is not, however, constant in a large range[16] of plasticity where the hydrostatis pressure changes to many thousand atmosphere. If the material is practically incompressible at the plastic region, we may take $\lambda_1 = \infty$ or $\sigma_1 = \frac{1}{2}$. If the material fulfills Poisson's condition at the plastic state, we have $\lambda_1 = \mu_1$ or $\sigma_1 = \frac{1}{4}$. Thus the condition of the incompressibility is satisfied if we take $\sigma_1 = \frac{1}{2}$ in (61). This nature can be easily comprehended if we know the fact that $\lambda_1 \varDelta$ has a finite value even in an incompressible material.

Again, the introduction of (62), (63), (64) will enable us to consider the equation of compatibility even for a complex case of stress-strain relation.

Now we have found seven equations (56), (57), (58), (61), (62), (63), (64) to determine seven unknown values $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$, u, v, limiting value of λ_1 and that of μ_1.[17] The problem is, thus, reduced to be determinate and soluble in an easy manner.

To solve the equations of the example, we put first (62), (63), (64) in (56), (57). Then we obtain

$$(\lambda_1 + 2\mu_1) \frac{\partial \varDelta}{\partial r} - \frac{2\mu_1}{r} \frac{\partial \varpi}{\partial \theta} = 0, \quad \ldots\ldots (66)$$

$$(\lambda_1 + 2\mu_1) \frac{1}{r} \frac{\partial \varDelta}{\partial \theta} + 2\mu_1 \frac{\partial \varpi}{\partial r} = 0, \quad \ldots\ldots (67)$$

---

16) P. W. BRIDGMAN, *Proc. Amer. Acad.*, 58 (1923), 166.

17) The meaning of the limiting value of λ_1 or μ_1 is such that at such value of λ_1 or μ_1 (both are functions of the strain) the plastic deformation takes place owing to Mohr's condition of the maximum shearing stress.

where

$$\Delta=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}, \qquad 2\varpi=\frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\alpha\theta} \quad\text{..........(68)}$$

In these equations, λ_1 and μ_1 are constants to express the tangent of the neighbourhood of a certain point of a properly described stress-strain curve.

From (66), (67) we obtain

$$\left.\begin{aligned}\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}=0,\\ \frac{\partial^2\varpi}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varpi}{\partial\theta^2}=0.\end{aligned}\right\} \quad\text{..................(69)}$$

The solutions of these can be expressed by

$$\left.\begin{aligned}\Delta=\left(A_n r^n+B_n\frac{1}{r^n}\right)\begin{matrix}\sin\\ \cos\end{matrix}\Big\}n\theta,\\ 2\varpi=\left(A_n' r^n+B_n'\frac{1}{r^n}\right)\begin{matrix}\cos\\ -\sin\end{matrix}\Big\}n\theta,\end{aligned}\right\}[n>1] \quad\text{...........(70)}$$

and

$$\left.\begin{aligned}\Delta=A_0\log r,\\ 2\varpi=A_0'\log r.\end{aligned}\right\}[n=0] \quad\text{....................(70')}$$

The displacements satisfying Δ in (70) and fulfilling the condition $\varpi=0$ are expressed by

$$\left.\begin{aligned}u_1=\left[\frac{(n+2)}{4(n+1)}A_n r^{n+1}+\frac{(n-2)}{4(n-1)}\frac{B_n}{r^n}\right]\begin{matrix}\sin\\ \cos\end{matrix}\Big\}n\theta,\\ v_1=\left[\frac{n}{4(n+1)}A_n r^{n+1}-\frac{n}{4(n-1)}\frac{B_n}{r^{n-1}}\right]\begin{matrix}\cos\\ -\sin\end{matrix}\Big\}n\theta.\end{aligned}\right\}\begin{bmatrix}n>1 \text{ for } A_n\\ n>2 \text{ for } B_n\end{bmatrix} \quad\text{..(71)}$$

$$\left.\begin{aligned}u_1=\frac{(\log r+1)}{2}B_1\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\theta,\\ v_1=\frac{\log r}{2}B_1\begin{matrix}\cos\\ -\sin\end{matrix}\Big\}\theta,\end{aligned}\right\}[n=1 \text{ for } B_1] \quad\text{............(71')}$$

$$\left.\begin{aligned}u_0=A_0\frac{r}{2}\left(\log r-\frac{1}{2}\right),\\ v_0=0.\end{aligned}\right\}[n=0] \quad\text{..................(71'')}$$

The displacements corresponding to ϖ in (70) with the condition, $\Delta=0$, are writted by

$$\left.\begin{aligned} u_2&=\left[\frac{n}{4(n+1)}A_n' r^{n+1}-\frac{n}{4(n+1)}\frac{B_n'}{r^{n-1}}\right]\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \\ v_2&=\left[\frac{(n+2)}{4(n+1)}A_n' r^{n+1}+\frac{(n-2)}{4(n-1)}\frac{B_n'}{r^{n-1}}\right]\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta, \end{aligned}\right\}\begin{bmatrix} n>1 \text{ for } A_n' \\ n>2 \text{ for } B_n' \end{bmatrix}\dots\dots(72)$$

$$\left.\begin{aligned} u_1'&=\frac{\log r}{2}B_1' \begin{matrix}\sin\\ \cos\end{matrix}\Big\}\theta, \\ v_1'&=\frac{(\log r+1)}{2}B_1' \begin{matrix}\cos\\ -\sin\end{matrix}\Big\}\theta. \end{aligned}\right\}[n=1 \text{ for } B_1']\dots\dots\dots\dots(72')$$

$$\left.\begin{aligned} u_0'&=0, \\ v_0'&=B_0'\frac{r}{2}\left(\log r-\frac{1}{2}\right). \end{aligned}\right\}[n=0]\dots\dots\dots\dots\dots\dots(72'')$$

The displacements satisfying $\Delta=\varpi=0$ are denoted by

$$\left.\begin{aligned} u_3&=\left(A_n'' r^{n-1}+\frac{B_n''}{r^{n+1}}\right)\begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta, \\ v_3&=\left(A_n'' r^{n-1}-\frac{B_n''}{r^{n+1}}\right)\begin{matrix}\cos\\ -\sin\end{matrix}\Big\} n\theta. \end{aligned}\right\}[n>0]\dots\dots\dots\dots(73)$$

It is worth noticing that the displacements belonging to the system of (71) and those attached to the system of (72) are not independent; they are connected by the formulae of the types:

$$A_n'=-\frac{\lambda_1+2\mu_1}{\mu_1}A_n, \qquad B_n'=\frac{\lambda_1+2\mu_1}{\mu_1}B_n, \dots\dots\dots(74)$$

while the solutions in (73) give quite independent displacements which are free from the dilatation and the rotation.

Now suppose that λ_1 is the function of Δ such that

$$\lambda_1=f(\Delta), \dots\dots\dots\dots\dots\dots(75)$$

then, by means of (61), we find

$$\mu_1=\frac{1-2\sigma_1}{2\sigma_1}f(\Delta), \dots\dots\dots\dots\dots\dots(76)$$

so that, from (62), (63), (64), we get

$$\left.\begin{aligned} \widehat{rr}&=\int_0^{\Delta_1} f(\Delta)\left\{1+\frac{1-2\sigma_1}{\sigma_1}\frac{\partial\dfrac{\partial u}{\partial r}}{\partial\Delta}\right\}d\Delta, \\ \widehat{\theta\theta}&=\int_0^{\Delta_1} f(\Delta)\left\{1+\frac{1-2\sigma_1}{\sigma_1}\frac{\partial\left(\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}+\dfrac{u}{r}\right)}{\partial\Delta}\right\}d\Delta, \end{aligned}\right\}\dots\dots\dots(77)$$

$$\widehat{r\theta}=\int_0^{\Delta_1}\frac{1-2\sigma_1}{2\sigma_1}f(\Delta)\frac{\partial\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)}{\partial\Delta}d\Delta,$$

where $\frac{\partial u}{\partial r}$, $\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}$, $\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}$ are the known functions of Δ having the paramater $f(\Delta)$. When we apply the boundary conditions of stresses to represent A_n'' or B_n'' in terms of A_n or B_n, the parameter $f(\Delta)$ will necessarily be induced.

Now, the limit of the integration Δ_1 at various positions can be determined for the plastic state by means of the equation (58). i.e.

$$\widehat{r\theta}^2+\left(\frac{\widehat{rr}-\widehat{\theta\theta}}{2}\right)^2=K^2. \qquad\qquad (58')$$

When the left-hand side of this equation becomes independent of r and θ, the derived relation has its applicability to all the positions; but, if it were not so, the problem becomes somewhat complicated and we must need some special consideration from the start.

Again, we have seen in this section that the displacements in an inelastic deformation are composed of two kinds, one of which is dilatational–distortional and the other is free from the dilatation and the rotation. This nature is quite similar to the case of an elastic deformation.

The application of the present theory to actual examples has been left to the future occasion, as our present intention is not to find the solutions of a plastic problem, but is to find a component of the displacement which is neither dilatational nor distortional.

12. Although we have treated in the preceding section some special case of the equations of the inelastic deformation, we can generalise the problem to more extended cases without any difficulty. Thus, in the case of the plane Cartesian coordinates, we get

$$\left.\begin{aligned}\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0,\\ \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0,\end{aligned}\right\} \qquad\qquad (83)$$

$$\left(\frac{X_x-Y_y}{2}\right)^2+X_y^2\leq K^2, \text{[18)]} \qquad\qquad (84)$$

18) The extended expressions of this equation of the limitation may be obtained in the manner of HENCKY, *Zeits. f. A. M. M.* **4** (1924), 323 & *V. D. I.* **69** (1925) 695.

$$\frac{\lambda_1}{2(\lambda_1+\mu_1)}=\sigma_1 \quad \text{.......................... (85)}$$

$$\left.\begin{aligned}\delta X_x&=\lambda_1\,\delta\Delta+2\mu_1\,\delta\frac{\partial u}{\partial x},\\ \delta Y_y&=\lambda_1\,\delta\Delta+2\mu_1\,\delta\frac{\partial v}{\partial y},\\ \delta X_y&=\mu_1\,\delta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),\end{aligned}\right\} \quad \text{...................... (86)}$$

where $\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$, and u, v are the components of displacement in x- and y-directions.

In a similar manner we can arrive at the equations in the cases of spherical, cylindrical and other coordinates.

VII. Some Notes on the Prevailing Theory of the Plasticity, with a Reference to the Study on the Distribution of the Cracks in a Body.

13. The present paper has aimed at the discovery of a component of the displacement which is neither dilatational nor distortional. The problem, however, is concerned with the deformation of the earthcrust, so that it will not be without significance to give some discussions on the theory of plasticity.

The theory of plasticity now prevailing among the students in applied mechanics should be discussed. In the first place, the equations[19] actually used in the calculations are limited only to those of the stress-equilibrium and that of the limiting value of the maximum shear stress. A little consideration will enable us to know that the conditions of the compatibility through the displacements of the material points are most important[20] excepting very simple cases as the deformation of simple bars. In our present idea these conditions have been taken into consideration. In the second place, the theory of plasticity developed by the european students in the engineering side does not involve the nature of the time effect on the behaviour of the plastic deformation, which shall not be left in the geophysical problems. As this effect will perhaps act during such a long

19) C. Carathéodory & E. Schmidt, *Zeits. f. A. M. and M.*, 3 (1923), 468.
L. Prandtl, *Proc. 1. Int. Nat. Congr. Appl. Mech.* (1924), 43.
Th. v. Kármán, *Verh. 2. Int. Kong. f. Tech.-Mech.* (1927), 23.

20) W. Jennne, *Z. A. M. M.*, 8 (1928), 18–44.

time as many thousand years, we have neglected such behaviour in the present paper.

14. Next, we have to draw the attention of those who apply the plastic theory on the problem of the earth pressure or the movement of the earthcrust. The students seem to explain frequently the figures and the positions of the cracks only by the theory of the plasticity and they always assert that the theory of the elasticity can never explain the figures and the positions of the cracks or slip surface. We can not think that their idea would be true. Though the equation of the limiting value of the maximum shear stress gives us the extent of the proportionality of stress and the region of the plastic part, we can not yet determine the figure and the position of the sharp cracks as observed in a finite scale. Such indeterminateness can be clearly known from the mathematical equations of the plasticity. The observed figures and the position of the cracks are often produced by the boundary conditions either in plastic or in elastic state of the materials.

We think that these cracks are not formed by the general movements of the plastic body, but these are produced in an early stage of the deformation, which is principally of the elastic nature, at some heavy stressed portion[21] of the material. Thus the creation of the cracks or the slip surfaces may rather take place in an elastic body. Indeed, the theoretical plasticians in England seem to consider the deformed plastic body as an elactic body having an accumulation of cracks as well as the dead spaces of elastic regions.

The problem of so-called block movements, which are advocated by Japanese seismologists of the recent time in the manner of the physiographists, involves yet many doubtful points from the true sense of the plastic theory as well as from some other theoretical considerations.

VIII. Concluding Remarks.

15. We have now found some facts which may be important on the geophysical problem as well as on the applied mechanics. Although the general theory was considered by means of somewhat restricted examples, yet we may say that the theory would never lose its applicability to all cases of the problem. This can be clearly known in the light of other

21) K. SUYEHIRO, *Engineering,* (Sept. 1, 1911).
C. E. INGLIS, *Trans. Inst. Nav. Arch., London,* **60** (1913), 219.
A. A. GRIFFITH, *Phil. Trans. Ray, Soc.,* **221** (1921), 163–198.
R. V. SOUTHWELL & H. J. *Gough, Phil. Mag.,* [7] **1** (1926), 71–96.
I. NAKAYAMA, *Journ. Soc. Mech. Eng., Tokyo,* **29** (1926).

problems of mechanics in which the similar examples contribute very often to the determination of the general nature. To our dissatisfaction the various cases of the inelastic deformation could not be solved owing to the lack of time. These will be shortly published in conclusive forms.

The principal results of the present investigation are as follows:

1. The deformation of an elastic solid body in equilibrium is composed of the dilatation, the rotation and another component which is neither dilatational nor rotational and which is an important component in affording the stresses or the strains.

2. The dilatational and the rotational components of displacement are not independent; the ratio of the constant factors belonging to these is determined only by the elastic constants of the material and the order of the harmonics of the straining. The component which is not dilatational nor rotational has an independent arbitrary constant factor.

3. The types of the solutions of the dilatational and the rotational components resemble one another, while that of the third component takes a different form.

4. By applying the present method of calculation to various cases of examples, we find easily the solutions of the various problems. Especially the solution of the heterogeneous distributions of the material in the displacements and stresses gives us the fact that the so-called block-movement may be nothing else the effect of such a heterogeneity of physical constants on the movements of the material points.

5. In an inelestic solid body, the similar nature as in the elastic body manifests itself. The method of the addition of all components to obtain a complete solution in this case is somewhat complicated.

6. A new idea of investigating the problem of the plasticity is obtained. The defect of the prevailing theory of plasticity, in which the condition of the compatibility is neglected, can be replaced by some improved theory.

7. The unnecessary condition of the incompressibility of the plastic material may be replaced by another controllable equation indicating the state of the material.

8. It is pointed out that the position and the number of the cracks in a strained material cannot be explained by the prevailing plastic theory to the same degree as the ordinary application of the elastic theory and that the above question can be solved even in the theory of elasticity by considering the effect of all the boundary conditions as in the manner of the students in England.

9. It is also added that the so-called block-movement advocated by the

seismologists cannot be explained by the plastic theory; rather it involves a great deal of doubtful points when it is considered from the true theory of inelasticity as well as from some other theoretical points.

---

21. 固體のダイレテーションとロテーションとに無關係な變位に就いて

地震研究所 { 妹澤克惟
西村源六郎

全く普遍的ではないが、成るべく理想的な數種の計算によつて、變形固體にダイレテーション及びロテーションに無關係な變位が存在する事を示し、併せて不完全な彈性變形の事にも言ひ及ぼした。不完全な彈性變形の計算は種々面白い場合があるが、時日の都合で餘り研究が進まなかつた。全體として主要な結果を摘錄すれば、

1. 固體が靜的變形を行ふ時、ダイレテーションにもロテーションにも無關係な變位が存在する。而もこの變位が固體の應力や歪みを與へる事には重要な役目をするものである。
2. ダイレテーションの變位とロテーションの變位とは互に無關係でなく、各の係數の間には固體の彈性恒數や變形次數によつて定まる關係が存在する。然るにダイレテーションとロテーションに無關係な變位の係數は全く獨立のものである。
3. ダイレテーションに相當する變位とロテーションに相當する變位とは其解が類似型をして居るが、ダイレテーションとロテーションとに無關係な變位は別の型を持つ。
4. この論文の計算法を多くの例題にあてはめて、種々の場合の解が容易に得られた。中でも材料の異質混合の場合は、所謂地塊運動の如きものが、材料の斯樣な分布によつて說明してもよい事を我々に敎へて吳れる。
5. 不完全な彈性體の場合にも彈性體の場合と殆んど同樣な數學的性質がある事がわかる。但しこの場合は充分な解を得る爲め各分變位を加へ合せる事が大分複雜となる。
6. この論文ではプラスチツク體の新しい研究方法を示してあるが、殊に現在流行して居る解法には缺けて居る所の變位の融通を考慮した方法を與へて置いた。
7. 以前のプラスチツク力學に存在する材料の非壓縮性といふ條件は餘り意味がないから、之等の性質に相當の自由を許す別の條件で上述の條件に代つて置き換へた。
8. 歪み材料の切れ目の位置や數は現在流體するプラスチツク論では、彈性力學を普通の方法で均一材料に應用するのと同樣の程度に役に立たぬ事を示し、又之等は英國の學者がやつて居る樣に種々の境界條件を充分に考慮して所々高歪になつた場所を求める方法によれば彈性力學の方法でも可なりの程度まで解決出來るものである事を指摘した。
9. 地震學者が近來唱へる地塊運動なるものは純粹のプラスチツク論に照す時は說明出來るどころか反つ都合が惡く、尙又別の見地からしても可なり怪しいものである事を附加へて置いた。