22. *Generation of Rayleigh-waves from a Sheet of Internal Sources.*

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1. In this paper I am going to show how Rayleigh-waves are generated from a splitting or a sliding surface in the interior of a semi-infinite solid body. The theory is based upon the conception that the splitting or the sliding surface is equivalent to a sheet of internal nuclei of disturbances. Although the ordinary treatment of the elastic waves under the assumption that they are transmitted from a point source can never violate the general mechanism of the generation of the waves and rather such a treatment gives us the rigorous and incomplex confirmation of the problem, yet this established manner of the treatment acquires frequently various objections from the side of the practical seismologists. To answer to the consideration of these practicians I have introduced the thought of a sheet of internal sources, which may, from the practical point of view, resemble the splitting or the sliding surface in the solid body.

An assigned distribution of energy at the surface of the crack may be obtained by applying the various strengths of the intensity to each elementary source which forms the sheet.

It may be questioned whether the waves in the neighbourhood of each source are transmitted with the velocity which is proper to the generation of the waves from a point source. A little consideration will enable us to know that the velocity at every point about the crack is not to be constant. It depends on the conditions at the boundary of the crack. A special case of this will be seen in one[1] of my papers where the propagation of elastic waves from a small elliptic or a spheroidal source of some regular form is investigated.

In the present case the crack is supposed to be so long compared with the wave length that the general form of the crack has no much effect on the motion of the solid body in the neighbourhood of the aperture. The

1) K. SEZAWA, "Propagation of Elastic Waves from an Elliptic or a Spheroidal Origin," *Bull. Earthq. Res. Inst.*, 2 (1927), 29–48.

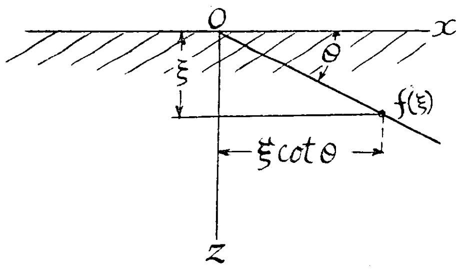
criterion of the sheet of the sources seems to be based upon a most suitable reasoning.

I. Two-dimensional Propagation.

2. We shall take the free surface of a semi-infinite solid to be $z=0$ and also that the positive sense of the axis of z is directed downwards. Let the sheet of the sources lie in a straight line making an angle θ with the horizontal line and suppose that the dilatational disturbance at the depth ξ is to be

$$-ie^{i\sigma t}H_0^{(2)}\{h\sqrt{(x-\xi\cot\theta)^2+(z-\xi)^2}\}f(\xi), \qquad (1)$$

where $h^2=\dfrac{\rho\sigma^2}{\lambda+2\mu}$, ρ is the density of the solid and λ, μ are Lame's elastic constants.



The expression (1) gives us an element of the sources in a splitting line denoted by

$$z=x\tan\theta. \qquad (2)$$

The generated dilatational waves are then given by

$$\varDelta_0'=\frac{e^{i\sigma t}}{\pi}\int_0^\infty f(\xi)\,d\xi\operatorname{cosec}\theta\int_{-\infty}^\infty\frac{e^{\alpha(z-\xi)+ik(x-\xi\cot\theta)}}{\alpha}\,dk. \qquad (3)$$

In this expression use has been made of

$$H_0^{(2)}(h\sqrt{x^2+z^2})=\frac{i}{\pi}\int_{-\infty}^\infty\frac{e^{-\alpha z+ikx}}{\alpha}\,dk,^{2)} \qquad (4)$$

where $\alpha^2=k^2-h^2$.

Superposing the effect $\varDelta_0''$ of the reflection at the free surface, we obtain

2) H. LAMB, *Phil. Trans. Roy, Soc.*, **203** (1904).

$$\Delta_0=\Delta_0'+\Delta_0''=\frac{2e^{i\sigma t}}{\pi}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty \frac{\text{ch}\,\alpha z\, e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{\alpha}dk.\ldots(5)$$

The corresponding components of displacement in x- and z-directions are written in the forms:

$$\left.\begin{aligned} u_0&=-\frac{1}{h^2}\frac{\partial\Delta_0}{\partial x}\\ &=-2\frac{ie^{i\sigma t}}{\pi h^2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty\frac{\text{ch}\,\alpha z\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{\alpha}k\,dk,\\ w_0&=-\frac{1}{h^2}\frac{\partial\Delta_0}{\partial z}\\ &=-2\frac{e^{i\sigma t}}{\pi h^2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty \text{sh}\,\alpha z\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}\,dk. \end{aligned}\right\}\ldots(6)$$

These waves give no tangential stress on $z=0$, but they give us a certain normal stress. We may superpose on these some free waves (without tangential stress) accumulated in the neighbourhood of the surface to annul the normal stress on that surface. The free waves are expressed by

$$\left.\begin{aligned} u_1&=\frac{2i\,e^{i\sigma t}}{\pi h_2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty\frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{\alpha F(k)}\\ &\qquad\qquad\times e^{-\alpha\xi+ik(x-\xi\cos\theta)}\,k\,dk,\\ w_1&=\frac{2e^{i\sigma t}}{\pi h^2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty\frac{j^2(2k^2-j^2)}{F(k)}e^{-\alpha\xi+ik(x-\xi\cot\theta)}\,dk, \end{aligned}\right\}\ldots..(7)$$

provided

$$F(k)=(2k^2-j^2)^2-4k^2\alpha\beta,\ldots\ldots\ldots\ldots(8)$$

$$\alpha^2=k^2-h^2,\quad \beta^2=k^2-j^2,\quad j^2=\frac{\rho\sigma^2}{\mu}.\ldots\ldots\ldots(9)$$

The resultant displacement on $z=0$ is thus written by

$$\left.\begin{aligned} u&=u_0+u_1=\frac{4ie^{i\sigma t}}{\pi h^2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty\frac{\beta kj^2\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{F(k)}dk,\\ w&=w_0+w_1=\frac{2e^{i\sigma t}}{\pi h^2}\int_0^\infty f(\xi)\,d\xi\,\text{cosec}\,\theta\int_{-\infty}^\infty\frac{(2k^2-j^2)\,j^2\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{F(k)}dk. \end{aligned}\right\}..(10)$$

Now we know that

$$\left.\begin{aligned} &\int_{-\infty}^\infty\frac{\beta j^2k\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{F(k)}dk\approx\frac{4\pi i\,\beta_1 j^2\kappa}{F'(\kappa)}e^{-\alpha_1\xi}\cos\kappa\,(x-\xi\cot\theta),\\ &\int_{-\infty}^\infty\frac{j^2(2k^2-j^2)\,e^{-\alpha\xi+ik(x-\xi\cot\theta)}}{F(k)}dk\approx-\frac{4\pi\,j^2(2\kappa^2-j^2)}{F'(\kappa)}e^{-\alpha_1\xi}\sin\kappa\,(x-\xi\cot\theta), \end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(11)$$

where κ is the real, positive root of $F(k)=0$ and α_1, β_1 are the corresponding values of α, β. These lead us to the results that

$$\left.\begin{aligned} u &\approx -\frac{16e^{i\sigma t}\beta_1 j^2\kappa}{h^2 F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\alpha_1\xi}\cos\kappa\,(x-\xi\cot\theta)\,d\xi,\\ w &\approx -\frac{8e^{i\sigma t} j^2(2\kappa^2-j^2)}{h^2 F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\alpha_1\xi}\sin\kappa\,(x-\xi\cot\theta)\,d\xi.\end{aligned}\right\}\quad\ldots(12)$$

These expressions are suitable to the transmission of waves of a finite extent of a harmonic train. To get the progressive waves of an infinite train, we must remember that the above expressions give a combination of two systems of progressive waves. The analysed waves are thus given by

$$\left.\begin{aligned} u &\approx -\frac{8\beta_1 j^2\kappa}{h^2F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\alpha_1\xi}\cos\{\kappa\,(x-\xi\cot\theta)\mp\sigma t\}\,d\xi,\\ v &\approx -\frac{4j^2(2\kappa^2-j^2)}{h^2F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\alpha_1\xi}\sin\{\kappa\,(x-\xi\cot\theta)\mp\sigma t\}\,d\xi.\end{aligned}\right\}\quad\ldots(13)$$

In these the negative and positive signs should be applied to each system of waves progressing towards the positive and negative directions of x-axis.

3. Let us suppose that

$$f(\xi)=\text{constant}=c,\quad\ldots\ldots\ldots\ldots(14)$$

then the surface displacement at any time, say $t=0$, is expressed by

$$\left.\begin{aligned} u &= -\frac{8\beta_1 j^2 c}{h^2F'(\kappa)}\left\{\operatorname{cosec}\theta\,\frac{\dfrac{\alpha_1}{\kappa}\cos\kappa x+\cot\theta\sin\kappa x}{\left(\dfrac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}\right\},\\ w &= -\frac{4j^2(2\kappa^2-j^2)\,c}{h^2\kappa F'(\kappa)}\left\{\operatorname{cosec}\theta\,\frac{\dfrac{\alpha_1}{\kappa}\sin\kappa x-\cot\theta\cos\kappa x}{\left(\dfrac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}\right\}.\end{aligned}\right\}\quad\ldots\ldots(15)$$

The absolute values of { } for different values of θ in the above equations are given by the lines named $n=1$ in Fig. 1 and Fig. 2 and the lines named $a=0$ in Fig. 3 and Fg. 4.

The fact that the amplitudes of the generated surface waves in an incompressible material are independent of the inclination of the splitting surface is apparently curious.

4. Next, consider that

$$f(\xi)=c^n\xi^{n-1},\quad\ldots\ldots\ldots\ldots(16)$$

then the surface displacement at $t=0$ is written by

$$\left.\begin{aligned} u &= -\frac{8\beta_1 j^2}{h^2 F'(\kappa)}\left(\frac{c}{\kappa}\right)^n \left\{\operatorname{cosec}\theta \sin^n\omega\,(\cos\kappa x\cos n\omega+\sin\kappa x\sin n\omega)\right\}, \\ w &= -\frac{4j^2(2\kappa^2-j^2)}{h^2\kappa F'(\kappa)}\left(\frac{c}{\kappa}\right)^n \left\{\operatorname{cosec}\theta \sin^n\omega\,(\sin\kappa x\cos n\omega-\cos\kappa x\sin n\omega)\right\}, \end{aligned}\right\} \quad \text{(17)}$$

where $\tan\omega=\cot\theta\Big/\frac{\alpha_1}{\kappa}$.

The absolute values of { } of the above expressions in the cases $n=1$, 2, 3, 4 are plotted in Fig. 1 and Fig. 2. Fig. 1 is the propagation of the waves in a medium whose Poisson's ratio is $\frac{1}{4}$, while Fig. 2 is that in an incompressible solid body. It will be seen that the amplitude of Rayleigh-waves has the maximum value at $\theta=90°$ and it decreases as the angle θ

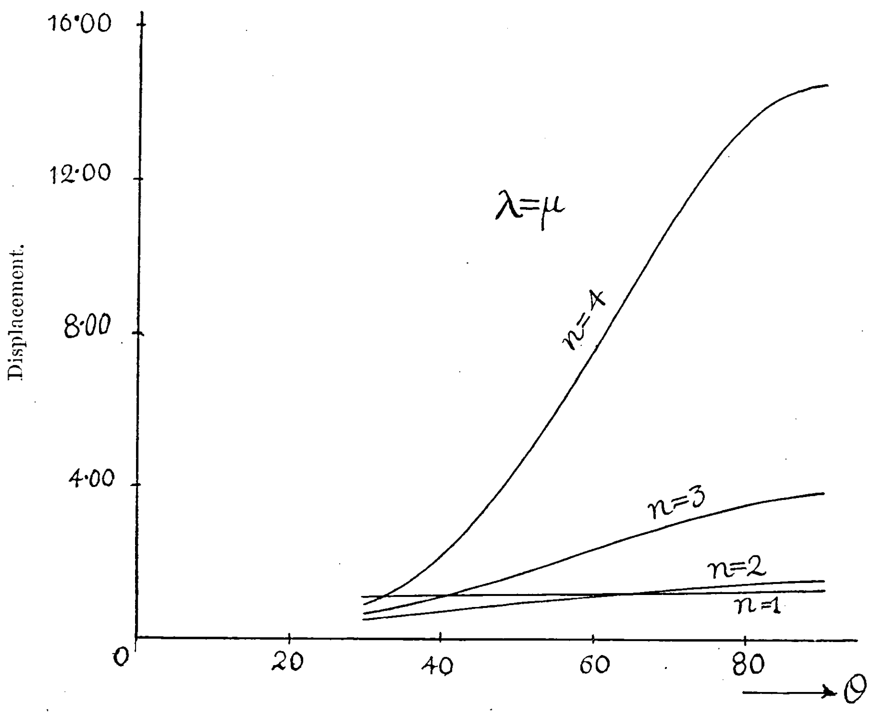


Fig. 1.

is diminished. It will also be noted that the decrease of the amplitude with the diminution of the angle θ is conspicuous for the greater values of n.

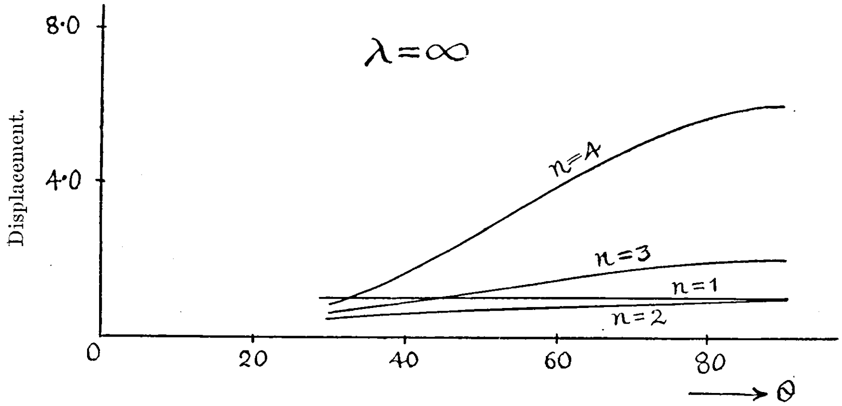


Fig. 2.

5. Now we suppose that

$$f(\xi)=ce^{-a\xi}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

then we find

$$\left.\begin{aligned} u&=-\frac{8\beta_1 j^2 c}{h^2F'(\kappa)}\left\{\operatorname{cosec}\theta\,\frac{\dfrac{\alpha_1+a}{\kappa}\cos\kappa x+\cot\theta\sin\kappa x}{\left(\dfrac{\alpha_1+a}{\kappa}\right)^2+\cot^2\theta}\right\},\\ v&=-\frac{4j^2(2\kappa^2-j^2)c}{h^2\kappa F'(\kappa)}\left\{\operatorname{cosec}\theta\,\frac{\dfrac{\alpha_1+a}{\kappa}\sin\kappa x-\cot\theta\cos\kappa x}{\left(\dfrac{\alpha_1+a}{\kappa}\right)^2+\cot^2\theta}\right\}.\end{aligned}\right\} \ldots\ldots (19)$$

The absolute values of { } of the above expressions in the cases $a/\kappa=0$,

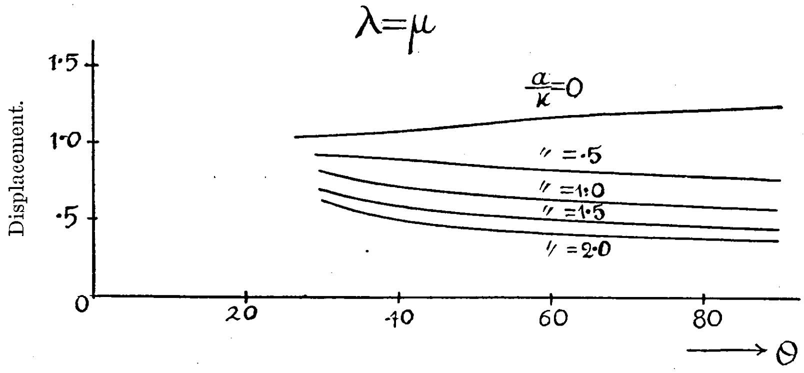


Fig. 3.

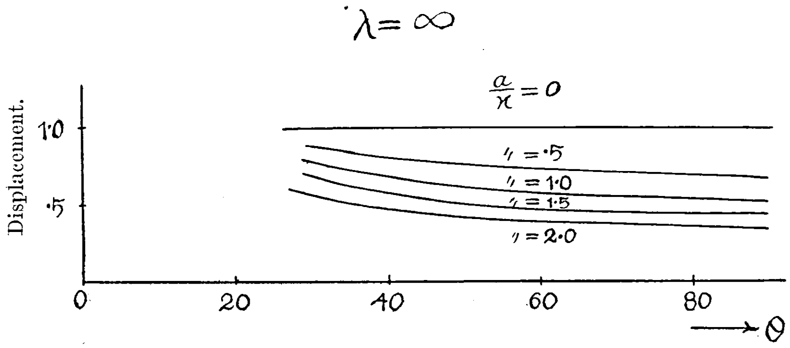


Fig. 4.

0.5, 1.0, 1.5, 2.0 are plotted in Fig. 3 and Fig. 4. Fig. 3 gives us the case $\lambda=\mu$ and Fig. 4 gives us that of $\lambda=\infty$. It will be noticed that the greater the value of a/κ becomes, the more diminishes the amplitude of the resulting surface waves. This can be easily anticipated from the assumed functions of the origin. The more remarkable fact is that, for a large value of a/κ the displacement is diminished as θ is increased, while, for a small value of a/κ, say $a/\kappa=0$, the displacement is enlarged with the increase of θ. This conforms with the result of the preceding section, in which the displacement increases with θ for a greater value of n.

6. If we put

$$\left.\begin{array}{ll} f(\xi)=c, & \xi_1<\xi<\xi_2 \\ f(\xi)=0, & 0<\xi<\xi_1 \text{ and } \xi_2<\xi<\infty \end{array}\right\} \quad\cdots\cdots\cdots\cdots(20)$$

we get without any difficulty the following result:

$$u=-\frac{8\beta_1 j^2 c}{h^2F'(\kappa)}\frac{\operatorname{cosec}\theta}{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}\Bigg[\cos\kappa x\Bigg\{e^{-\alpha_1\xi_2}\left(\cot\theta\sin\overline{\kappa\cot\theta\xi_2}-\frac{\alpha_1}{\kappa}\cos\overline{\kappa\cot\theta\xi^2}\right)$$
$$-e^{-\alpha_1\xi_1}\left(\cot\theta\sin\overline{\kappa\cot\theta\xi_1}-\frac{\alpha_1}{\kappa}\cos\overline{\kappa\cot\theta\xi_1}\right)\Bigg\}$$
$$-\sin\kappa x\Bigg\{e^{-\alpha_1\xi_2}\left(\cot\theta\cos\overline{\kappa\cot\theta\xi_2}+\frac{\alpha_1}{\kappa}\sin\overline{\kappa\cot\theta\xi_2}\right)$$
$$-e^{-\alpha_1\xi_1}\left(\cot\theta\cos\overline{\kappa\cot\theta\xi_1}+\frac{\alpha_1}{\kappa}\sin\overline{\kappa\cot\theta\xi_1}\right)\Bigg\}\Bigg],$$

$$w=-\frac{4j^2(2\kappa^2-j^2)\,c}{h^2\kappa F'(\kappa)}\,\frac{\operatorname{cosec}\theta}{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}\left[\sin\kappa x\begin{Bmatrix}\text{The same expression}\\ \text{as that in the first}\\ \text{bracket of the above}\end{Bmatrix}+\cos\kappa x\begin{Bmatrix}\text{The same expression}\\ \text{as that in the second}\\ \text{bracket of the above}\end{Bmatrix}\right] \qquad (21)$$

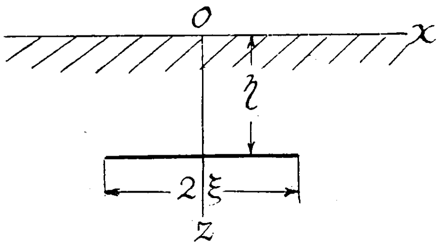
The numerical computation of the above equations has been omitted in the present paper.

7. If the source is at the depth η from the surface and if

$$f(\xi)=c, \qquad (-\xi_1<\xi<\xi_1, \quad \theta=0). \qquad (22)$$

then we obtain

$$\left.\begin{aligned} u_0&=\frac{16\beta_1 j^2 c}{h^2 F'(\kappa)}\,e^{-\alpha_1\eta}\cos\kappa x\sin\kappa\xi,\\ v_0&=\frac{8j^2(2\kappa^2-j^2)\,c}{h^2\kappa F'(\kappa)}\,e^{-\alpha_1\eta}\sin\kappa x\sin\kappa\xi.\end{aligned}\right\} \qquad (23)$$



8. When the disturbance is of a sliding type, the method of the analysis is somewhat modified. In this case we should consider the distortional disturbance as an element of the sources. Let the intensity of an element be

$$-ie^{i\sigma t}\,H_0^{(2)}\{j\sqrt{(x-\xi\cot\theta)^2+(z-\xi)^2}\}\,f(\xi), \qquad (24)$$

where $j^2=\frac{\rho\sigma^2}{\mu}$, then the general distortional waves from the sliding surface

$$z=x\tan\theta, \qquad (2')$$

are expressed in the forms:

$$2\varpi_0'=\frac{e^{i\sigma t}}{\pi}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\frac{e^{\beta(z-\xi)+ik(x-\xi\cot\theta)}}{\beta}\,dk, \qquad (25)$$

where $\beta^2=k^2-j^2$.

Superposing the effect of the images, we obtain finally

$$\left.\begin{aligned} u_0 &= \frac{2e^{i\sigma t}}{\pi j^2}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\operatorname{sh}\beta z\; e^{-\beta\xi+ik(x-\xi\cot\theta)}\,dk, \\ w_0 &= -\frac{2ie^{i\sigma t}}{\pi j^2}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\frac{\operatorname{ch}\beta z}{\beta}\,e^{-\beta\xi+ik(x-\xi\cot\theta)}\,k\,dk. \end{aligned}\right\}\quad\ldots\ldots(26)$$

The normal stress at the surface due to this displacement is nil, but the tangential stress is in existence. To cancel the tangential stress, the following surface waves should be superposed:

$$\left.\begin{aligned} u_1 &= \frac{2e^{i\sigma t}}{\pi j^2}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\frac{[(2k^2-j^2)\,e^{-\beta z}-2k^2\,e^{-\alpha z}]\,(2k^2-j^2)}{F(k)} \\ &\qquad\qquad\times e^{-\beta\xi+ik(x-\xi\cot\theta)}\,dk, \\ w_1 &= \frac{2ie^{i\sigma t}}{\pi j^2}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{0}^{\infty}\frac{[(2k^2-j^2)\,e^{-\beta z}-2\alpha\beta\,e^{-\alpha z}]\,(2k^2-j^2)}{\beta F(k)} \\ &\qquad\qquad\times e^{-\beta\xi+ik(x-\xi\cot\theta)}\,k\,dk. \end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

The composed waves gives the surface displacement of the type

$$\left.\begin{aligned} u &= u_0+u_1 = -\frac{2e^{i\sigma t}}{\pi}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\frac{(2k^2-j^2)}{F(k)}\,e^{-\beta\xi+ik(x-\xi\cot\theta)}\,dk, \\ w &= w_0+w_1 = \frac{4ie^{i\sigma t}}{\pi}\int_0^\infty f(\xi)\,d\xi\,\operatorname{cosec}\theta\int_{-\infty}^{\infty}\frac{\alpha k}{F(k)}\,e^{-\beta\xi+ik(x-\xi\cot\theta)}\,dk. \end{aligned}\right\}(28)$$

Proceeding in the same manner as before, we arrive at

$$\left.\begin{aligned} u &\approx \frac{8e^{i\sigma t}(2\kappa^2-j^2)}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\beta_1\xi}\sin\kappa\,(x-\xi\cot\theta)\,d\xi, \\ v &\approx -\frac{16ie^{i\sigma t}\,\alpha_1\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,e^{-\beta_1\xi}\cos\kappa\,(x-\xi\cot\theta)\,d\xi, \end{aligned}\right\}\quad\ldots\ldots(29)$$

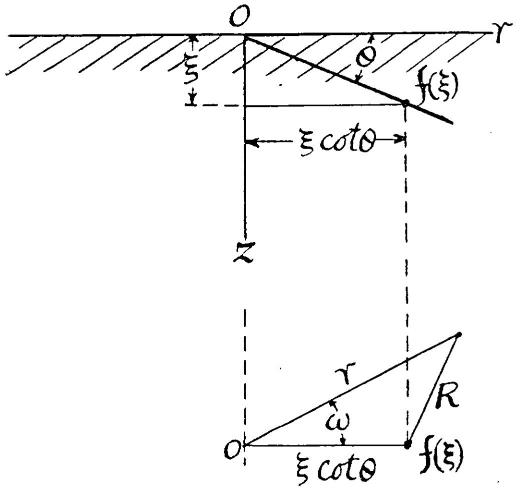
which are, too, of the same forms as those of the splitting origin, except the interchange of α_1 and β_1. Though the compiling of the above equations has been omitted, the generated surface waves will shew the more pronounced characters in the direction of the case $\lambda=\mu$ than that of $\lambda=\infty$ in the splitting disturbances.

II. Three-dimensional Propagation.

9. We shall take the free surface of a semi-infinite solid to be $z=0$, and its positive sense downwards. Suppose that the sheet of the dilatational sources lies in a straight line in a three-dimensional space. Then the element of the sheet is expressed by

$$\frac{e^{i\sigma t}\, e^{-ih\sqrt{R^2+z^2}}}{\sqrt{R^2+z^2}}, \quad\cdots\cdots\cdots\cdots(30)$$

where $R^2=r^2+(\xi\cot\theta)^2-2r\xi\cot\theta\cos\omega$.



Now by means of Lamb's formula:[3)]

$$\frac{e^{-ih\sqrt{R^2+z^2}}}{\sqrt{R^2+z^2}}=\int_0^\infty \frac{e^{-\alpha z}}{\alpha} J_0(kR)\, kdk, \quad\cdots\cdots\cdots\cdots(31)$$

we get the total dilatational waves from the splitting sources in the form:

$$\Delta_0'=e^{i\sigma t}\int_0^\infty f(\xi)\, d\xi \operatorname{cosec}\theta \int_0^\infty \frac{e^{\alpha(z-\xi)} J_0(kR)\, kdk}{\alpha}, \quad\cdots\cdots(32)$$

Adding the effect of the image to (32), we get

$$\Delta_0=\Delta_0'+\Delta_0''=2e^{i\sigma t}\int_0^\infty f(\xi)\, d\xi \operatorname{cosec}\theta \int_0^\infty \frac{\operatorname{ch}\alpha z\, e^{-\alpha\xi} J_0(kR)\, kdk}{\alpha} \cdots\cdots(33)$$

The corresponding values of the components of the displacement in radial, azimuthal and vertical directions are expressed by

$$\left.\begin{aligned} u_0&=-\frac{2e^{i\sigma t}}{h^2}\frac{\partial}{\partial r}\int_0^\infty f(\xi)\, d\xi \operatorname{cosec}\theta\int_0^\infty \frac{\operatorname{ch}\alpha z}{\alpha} e^{-\alpha\xi} J_0(kR)\, kdk,\\ v_0&=-\frac{2e^{i\sigma t}}{h^2 r}\frac{\partial}{\partial \omega}\int_0^\infty f(\xi)\, d\xi \operatorname{cosec}\theta\int_0^\infty \frac{\operatorname{ch}\alpha z}{\alpha} e^{-\alpha\xi} J_0(kR)\, kdk,\\ w_0&=-\frac{2e^{i\sigma t}}{h^2}\frac{\partial}{\partial z}\int_0^\infty f(\xi)\, d\xi \operatorname{cosec}\theta\int_0^\infty \frac{\operatorname{ch}\alpha z}{\alpha} e^{-\alpha\xi} J_0(kR)\, kdk. \end{aligned}\right\}\cdots\cdots(34)$$

3) H. LAMB, *Phil. Trans. Roy. Soc.*, **203** (1904).

The expressions of the superposed waves are given by

$$\left.\begin{aligned} u_1&=\frac{2e^{i\sigma t}}{h^2}\frac{\partial}{\partial r}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta \\ &\qquad\times\int_0^\infty \frac{(2k^2-j^2-2\alpha\beta)}{\alpha F(k)}(2k^2-j^2)\,e^{-\alpha\xi}J_0(kR)\,kdk, \\ v_1&=\frac{2e^{i\sigma t}}{h^2 r}\frac{\partial}{\partial\omega}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta \\ &\qquad\times\int_0^\infty \frac{(2k^2-j^2-2\alpha\beta)}{\alpha F(k)}(2k^2-j^2)\,e^{-\alpha\xi}J_0(kR)\,kdk, \\ w_1&=\frac{2e^{i\sigma t}}{h^2}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\int_0^\infty \frac{j^2(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}J_0(kR)\,kdk. \end{aligned}\right\}\quad\ldots(35)$$

The resultant displacement on $z=0$ is denoted by

$$\left.\begin{aligned} u&=u_0+u_1=\frac{4e^{i\sigma t}}{h^2}\frac{\partial}{\partial r}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\int_0^\infty \frac{\beta j^2}{F(k)}\,e^{-\alpha\xi}J_0(kR)\,kdk, \\ v&=v_0+v_1=\frac{4e^{i\sigma t}}{h^2 r}\frac{\partial}{\partial\omega}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\int_0^\infty \frac{\beta j^2}{F(k)}\,e^{-\alpha\xi}J_0(kR)\,kdk, \\ w&=w_0+w_1=\frac{2e^{i\sigma t}}{h^2}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\int_0^\infty \frac{j^2(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}J_0(kR)\,kdk. \end{aligned}\right\}\quad(36)$$

The method of contour integrations applied on the above integrals gives us

$$\left.\begin{aligned} u&\approx\frac{8\pi e^{i\sigma t}}{h^2}\frac{\partial}{\partial r}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\,\frac{\beta_1 j^2\kappa}{F'(\kappa)}\,e^{-\alpha_1\xi}\,Y_0(\kappa R), \\ v&\approx\frac{8\pi e^{i\sigma t}}{h^2 r}\frac{\partial}{\partial\omega}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\,\frac{\beta_1 j^2\kappa}{F'(\kappa)}\,e^{-\alpha_1\xi}\,Y_0(\kappa R), \\ w&\approx\frac{4\pi e^{i\sigma t}}{h^2}\int_0^\infty f(\xi)\,d\xi\ \mathrm{cosec}\,\theta\,\frac{j^2(2\kappa^2-j^2)\kappa}{F'(\kappa)}\,e^{-\alpha_1\xi}\,Y_0(\kappa R). \end{aligned}\right\}\quad\ldots.(37)$$

Now we know the relation

$$Y_0(\kappa R)=Y_0(\kappa r)\,J_0(\kappa\xi\cot\theta)+2\sum_{n=1}^{\infty}Y_n(\kappa r)\,J_n(\kappa\xi\cot\theta)\cos n\omega,\qquad [r>\xi\cot\theta]\qquad\ldots(38)$$

where

$$R=\sqrt{r^2+(\xi\cot\theta)^2-2r\xi\cot\theta\cos\omega}.\qquad\ldots(39)$$

Substituting from (38) in (37), we get

$$u=\frac{8\pi e^{i\sigma t}}{h^2}\frac{\beta_1 j^2\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}\left[\frac{\partial Y_0(\kappa r)}{\partial r}J_0(\kappa\xi\cot\theta)\right.$$
$$\left.+2\sum_{n=1}^{\infty}\frac{\partial Y_n(\kappa r)}{\partial r}J_n(\kappa\xi\cot\theta)\cos n\omega\right],$$
$$v=-\frac{16\pi\, e^{i\sigma t}\,\beta_1\, j^2\,\kappa}{h^2\, r\, F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}$$
$$\times\sum_{n=1}^{\infty} nY_n(\kappa r)\,J_n(\kappa\xi\cot\theta)\sin n\omega,$$
$$w=\frac{4\pi e^{i\sigma t}}{h^2}\frac{j^2(2\kappa^2-j^2)\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}\Big[Y_0(\kappa r)\,J_0(\kappa\xi\cot\theta)$$
$$+2\sum_{n=1}^{\infty}Y_n(\kappa r)\,J_n(\kappa\xi\cot\theta)\cos n\omega\Big].$$
$$\dots\dots\dots\dots(40)$$

To get the progressive waves, we superpose on the above type of solutions Bessel's function of the first kind, and after the same operation as in the two-dimensional case, we get finally

$$u=\frac{4\pi e^{i\sigma t}}{h^2}\frac{\beta_1 j^2\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}\left[\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}J_0(\kappa\xi\cot\theta)\right.$$
$$\left.+2\sum_{n=1}^{\infty}\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}J_n(\kappa\xi\cot\theta)\cos n\omega\right],$$
$$v=-\frac{8\pi e^{i\sigma t}}{h^2 r}\frac{\beta_1 j^2\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}$$
$$\times\sum_{n=1}^{\infty} nH_n^{(2)}(\kappa r)\,J_n(\kappa\xi\cot\theta)\sin nw,$$
$$w=\frac{2\pi e^{i\sigma t}}{h^2}\frac{j^2(2\kappa^2-j^2)\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\alpha_1\xi}\Big[H_0^{(2)}(\kappa r)\,J_0(\kappa\xi\cot\theta)$$
$$+2\sum_{n=1}^{\infty}H_n^{(2)}(\kappa r)\,J_n(\kappa\xi\cot\theta)\cos n\omega\Big].$$
$$\dots\dots\dots\dots(41)$$

These equations give us the excited Rayleigh-waves on a boundary plane of a semi-infinite solid, when the sources are distributed in any assigned manner on a straight line.

10. For an example, suppose that

$$f(\xi)=\text{constant}=c, \quad\dots\dots\dots\dots(42)$$

then the surface displacement is given by

$$u=\frac{4\pi e^{i\sigma t}\beta_1 j^2 c}{h^2 F'(\kappa)}\left[\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}}\left\{\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}\right.\right.$$

$$\left.\left.+2\sum_{n=1}^{\infty}\frac{\partial H_n^{(2)}(\kappa r)}{\partial r}\left(\frac{\sqrt{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}-\frac{\alpha_1}{\kappa}}{\cot\theta}\right)^n\cos n\omega\right\}\right],$$

$$v=-\frac{8\pi e^{i\sigma t}\beta_1 j^2 c}{h^2 rF'(\kappa)}\left[\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}}\right.$$

$$\left.\times\sum_{n=1}^{\infty}nH_n^{(2)}(\kappa r)\left(\frac{\sqrt{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}-\frac{\alpha_1}{\kappa}}{\cot\theta}\right)^n\sin n\omega\right],$$

$$w=\frac{2\pi e^{i\sigma t}j^2(2\kappa^2-j^2)c}{h^2F'(\kappa)}\left[\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\alpha_1}{\kappa}\right)^2+\cot^2\theta}}\left\{H_0^{(2)}(\kappa r)\right.\right.$$

$$\left.\left.+2\sum_{n=1}^{\infty}H_n^{(2)}(\kappa r)\left(\frac{\sqrt{\left(\frac{\alpha^2}{\kappa}\right)^2+\cot^2\theta}-\frac{\alpha_1}{\kappa}}{\cot\theta}\right)^n\cos n\omega\right\}\right].$$

$$\dots\dots\dots\dots(43)$$

The absolute values of [ ] in the equation of w multiplied by $\sqrt{\frac{\pi\kappa r}{2}}$ for different angles of θ are expressed in Fig. 5 and Fig. 6. In these figures the curves marked $a/\kappa=0$ give the result of the present examples.

11. Next, let us take

$$f(\xi)=ce^{-a\xi},\dots\dots\dots\dots(44)$$

then we obtain

$$w=\frac{2\pi e^{i\sigma t}j^2(2\kappa^2-j^2)\kappa}{h^2F'(\kappa)}\left[\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\alpha_1+a}{\kappa}\right)^2+\cot^2\theta}}\left\{H_0^{(2)}(\kappa r)\right.\right.$$

$$\left.\left.+2\sum_{n=1}^{\infty}H_n^{(2)}(\kappa r)\left(\frac{\sqrt{\left(\frac{\alpha_1+a}{\kappa}\right)^2+\cot^2\theta}-\frac{\alpha_1}{\kappa}}{\cot\theta}\right)^n\cos n\omega\right\}\right]$$

$$\dots\dots\dots\dots(45)$$

and similar expressions for u and v. The absolute values of [ ] in this equation multiplied by $\sqrt{\frac{\pi\kappa r}{2}}$ for various angles θ are plotted in Fig. 5 and Fig. 6. Fig. 5 gives us the case $\lambda=\mu$ and Fig. 6 gives us that of $\lambda=\infty$. It will be noticed that for a greater values of a/κ the amplitude of the waves takes a smaller value. As already stated, this is self-evident from the nature of the assumed distribution of the sources. A most noteworthy fact is that there is a miximum value of the displacement at a certain angle between 0 and $\pi/2$ for each value of a/κ. It will also be seen that the angle θ which gives the maximum of the displacement changes for different values of a/κ. The maximum of the displacement when $a/\kappa=0$ and $\lambda=\mu$ takes place at $\theta\fallingdotseq 65°$ and the similar maximum in the case $\lambda=\infty$ occure at $\theta\fallingdotseq 45°$, while for a larger values of a/κ the position of the maximum shifts more and more in the sense of the smaller angle θ.

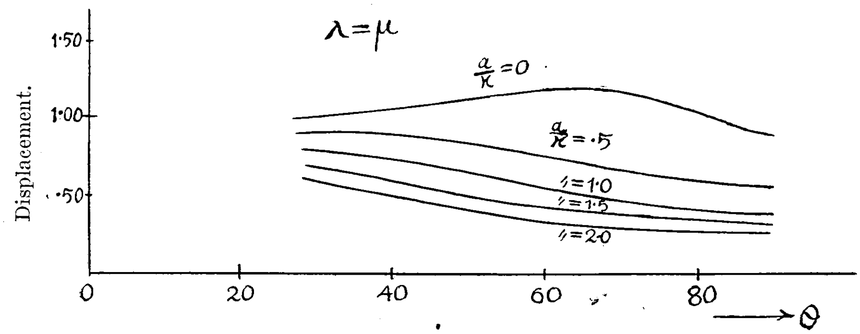


Fig. 5.

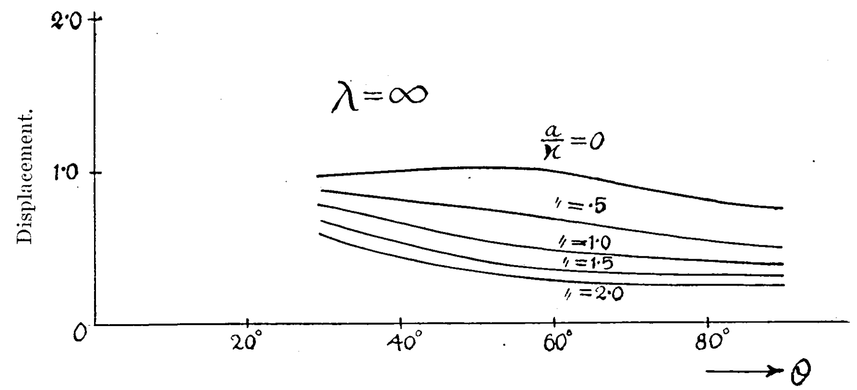


Fig. 6.

12. Let us take

$$f(\xi)=c^m\,\xi^{m-1}, \qquad\qquad (46)$$

then, by means of the formula

$$\int_0^\infty e^{-\alpha_1\xi} J_n(\kappa\xi\cot\theta)\,\xi^{m-1}\,d\xi$$

$$=\frac{\left(\frac{1}{2}\frac{\kappa\cot\theta}{\alpha_1}\right)^n\Gamma(m+n)}{\alpha_1{}^m\,\Gamma(n+1)}F\left(\frac{m+n}{2},\ \frac{m+n+1}{2},\ n+1,\ -\left(\frac{\kappa\cot\theta}{\alpha_1}\right)^2\right),$$

$$[R(\alpha_1\pm i\kappa\cot\theta)>0] \qquad\qquad (47)[4]$$

where

$$F(\alpha,\beta,\gamma,x)=1+\sum_{s=1}^{\infty}\frac{\alpha(\alpha+1)..(\alpha+s-1)\,\beta(\beta+1)..(\beta+s-1)\,x^s}{s!\;\gamma(\gamma+1)..(\gamma+s-1)}, \qquad\qquad (48)$$

we find

$$u=\frac{4\pi e^{i\sigma t}\beta_1 j^2\kappa}{h^2F'(\kappa)}\left(\frac{c}{\alpha_1}\right)^m\left(\frac{\kappa\cot\theta}{2\alpha_1}\right)^n\operatorname{cosec}\theta$$

$$\times\left[\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}\Gamma(m)\,F\left(\frac{m}{2},\ \frac{m+1}{2},\ n+1,\ -\left(\frac{\kappa\cot\theta}{\alpha_1}\right)^2\right)\right.$$

$$+2\sum_{n=1}^{\infty}\frac{\partial H_n^{(2)}(\kappa r)}{\partial r}\frac{\Gamma(m+n)}{\Gamma(n+1)}$$

$$\left.\times F\left(\frac{m+n}{2},\ \frac{m+n+1}{2},\ n+1,\ -\left(\frac{\kappa\cot\theta}{\alpha_1}\right)^2\right)\cos n\omega\right]$$

$$[R(\alpha_1\pm i\kappa\cot\theta)>0] \qquad\qquad (49)$$

and similar expressions for v and w.

13. When the sheet of sources forms a plane surface, the problem can be analysed in the same manner. The process is, however, too complex and we will leave the problem to the future occasion.

14. The case where the source of the disturbance is of a sliding type, the direction of the slip being in the sense of the line of the sheet, can be treated first by solving the problem of a point nucleous and then superposing a number of such solutions.

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4) WATSON, *Theory of Bessel Functions*, (1922), 385.

Let the intensity of an element be

$$\frac{e^{i\sigma t}\,e^{-ij\sqrt{R^2+z^2}}}{\sqrt{R^2+z^2}}f(\xi), \qquad \text{(50)}$$

then the general distortional waves generated from the sliding line

$$z=x\tan\theta, \qquad \text{(2'')}$$

may be expressed by

$$2\varpi_0'=e^{i\sigma t}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\int_0^\infty \frac{e^{\beta(z-\xi)}\,J_0(kR)}{\beta}\,kdk. \qquad \text{(51)}$$

Superposing the effect of the image, we get

$$\left.\begin{aligned}
u_0&=2\frac{e^{i\sigma t}}{j^2}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\cos\omega\int_0^\infty \sin\beta z\,e^{-\beta\xi}J_0(kR)\,kdk,\\
v_0&=-2\frac{e^{i\sigma t}}{j^2}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\sin\omega\int_0^\infty \sin\beta z\,e^{-\beta\xi}J_0(kR)\,kdk,\\
w_0&=-2\frac{e^{i\sigma t}}{j^2}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\cos\omega\int_0^\infty \frac{\mathrm{ch}\,\beta z}{\beta}\,e^{-\beta\xi}\frac{\partial J_0(kR)}{\partial r}\,kdk.
\end{aligned}\right\} \qquad \text{(52)}$$

Adding to these displacement some free displacement (u_1, v_1, w_1) accumulated on the surface of the solid to annul the surface stress, we find

$$\left.\begin{aligned}
u&=u_0+u_1=-2e^{i\sigma t}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\cos\omega\int_0^\infty \frac{(2k^2-j^2)}{F(k)}\,e^{-\beta\xi}J_0(kR)\,kdk,\\
v&=v_0+v_1=2e^{i\sigma t}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\sin\omega\int_0^\infty \frac{(2k^2-j^2)}{F(k)}\,e^{-\beta\xi}J_0(kR)\,kdk,\\
w&=w_0+w_1=4e^{i\sigma t}\int_0^\infty f(\xi)\,d\xi\,\mathrm{cosec}\,\theta\cos\omega\int_0^\infty \frac{\alpha}{F(k)}\,e^{-\beta\xi}\frac{\partial J_0(kR)}{\partial r}\,kdk,
\end{aligned}\right\} \qquad \text{(53)}$$

at the surface $z=0$.

Proceeding in the same manner as before, we get finally

$$u=\frac{2\pi e^{i\sigma t}(2\kappa^2-j^2)\kappa}{F'(\kappa)}\,\mathrm{cosec}\,\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\beta_1\xi}\Big[H_0^{(2)}(\kappa r)\,J_0(\kappa\xi\cot\theta)\cos\omega$$

$$+2\sum_{n=1}^{\infty}H_n^{(2)}(\kappa r)\,J_n(\kappa\xi\cot\theta)\cos n\omega\cos\omega\Big],$$

$$v=-\frac{2\pi e_{i\sigma t}(2\kappa^2-j^2)\kappa}{F'(\kappa)}\,\mathrm{cosec}\,\theta\int_0^\infty f(\xi)\,d\xi\,e^{-\beta_1\xi}\Big[H_0^{(2)}(\kappa r)\,J_0(\kappa\xi\cot\theta)\sin\omega$$

$$+2\sum_{n=1}^{\infty} H_n^{(2)}(\kappa r)\, J_n(\kappa\xi\cot\theta)\cos n\omega\sin\omega\Big],$$

$$w=-\frac{4\pi e^{i\sigma t}\alpha_1\kappa}{F'(\kappa)}\operatorname{cosec}\theta\int_0^{\infty} f(\xi)\,d\xi\, e^{-\beta_1\xi}\left[\frac{\partial H_0^{(2)}(\kappa r)}{\partial r} J_0(\kappa\xi\cot\theta)\cos\omega\right.$$

$$\left.+2\sum_{n=1}^{\infty}\frac{\partial H_0^{(2)}(\kappa r)}{\partial r} J_n(\kappa\xi\cot\theta)\cos n\omega\cos\omega\right]. \quad \ldots\ldots (54)$$

These integrals can be evaluated in a very simple manner.

15. Thus, in the case

$$f(\xi)=\text{constant}=c, \quad \ldots\ldots (55)$$

the results of evaluation are of the forms:

$$u=\frac{2\pi e^{i\sigma t}(2\kappa^2-j^2)c}{F'(\kappa)}\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}}\left\{H_0^{(2)}(\kappa r)\cos\omega\right.$$

$$\left.+2\sum_{n=1}^{\infty} H_n^{(2)}(\kappa r)\left(\frac{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}-\frac{\beta_1}{\kappa}}{\cot\theta}\right)^n\cos n\omega\cos\omega\right\}.$$

$$v=-\frac{2\pi e^{i\sigma t}(2\kappa^2-j^2)c}{F'(\kappa)}\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}}\left\{H_0^{(2)}(\kappa r)\sin\omega\right.$$

$$\left.+2\sum_{n=1}^{\infty} H_n^{(2)}(nr)\left(\frac{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}-\frac{\beta_1}{\kappa}}{\cot\theta}\right)^n\cos\omega\sin\omega\right\},$$

$$w=-\frac{4\pi e^{i\sigma t}\alpha_1 c}{F'(\kappa)}\frac{\operatorname{cosec}\theta}{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}}\left\{\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}\cos\omega\right.$$

$$\left.+2\sum_{n=1}^{\infty}\frac{\partial H_n^{(2)}(\kappa r)}{\partial r}\left(\frac{\sqrt{\left(\frac{\beta_1}{\kappa}\right)^2+\cot^2\theta}-\frac{\beta_1}{\kappa}}{\cot\theta}\right)^n\cos n\omega\cos\omega\right\}. \quad \ldots\ldots (56)$$

From these it may be concluded that Rayleigh-waves propagated from a sliding source of the present type have not the component which is symmetrical about a vertical line $r=0$, but they manifest the nature of the polarity with respect to the azimuthal angle ω.

Concluding Remarks.

16. We may now review some results of this investigation from the general consideration. Although the sliding or splitting sources are of sheets of nuclei, yet we may believe that the theory is some guidance for the acknowledgment of the generation of Rayleigh-waves from a sliding or a splitting origin.

The principal results obtained in this paper are enumerated as follows:

1. Generated surface waves are bestowed with more energy in a compressible solid than in an incompressible solid.

2. In a two-dimensional motion, the amplitude of the surface waves generated from a line source of increasing intensity downwards reaches its maximum value at the upright position of the line of the source.

3. When the intensity of the element of the source is constant at any depth on the line of source, the displacement of the surface waves is constant for any angle of inclination of the above line.

4. When the intensity of the line source decreases with the depth, the amplitude takes its maximum value at a certain angle θ between 0 and $\pi/2$.

5. The angle θ of the maximum displacement in the above example changes with the rate of the decrease of the intensity in the downward direction.

6. In a three-dimensional motion, there is a certain angle θ corresponding to the maximum displacement even in the case of the constant intensity of the elementary portion of the source for all depths.

7. In spite of the existence of the component of displacement which is symmetrical about a vertical line in the case of the splitting sources of any inclination, there is no such component in the case of the sliding source.

In conclusion I am much indebted to Mr. G. Nishimura for his laborious assistance in all the numerical calculations and the preparation of this paper.

22. 内部の層源によるレーレー波の生成

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この論文は半無限體の內部の斷層面が辷る場合や切離される場合に出るレーレー波を研究する事を目的としたものである。內部に幾何學的の點があつて之れから出る波動を研究する方が寧ろ學理的であり又普遍性もある筈であるが、多くの地震學者があまり之を認めて呉れぬ樣に思はれたから、態々この論文を書いた次第である。

研究の方法は點源を種々組合せて層面を作り、これから出る彈性波が細長い切れ目から出る波動と結果に於て一致する樣に工夫をした。

研究の主な結果を拔萃すれば、

1. 固體が非變容性の時よりも變容性がある時の方が餘計に表面波の勢力を與へる。
2. 層の中の蓄積力が地下に進む程大きいものでは層面が垂直の時に表面波に一番大なる勢力を與へる。
3. 非變容性固體中にある層面の中の蓄積力が深さに無關係の時は、表面波の振幅は層の傾斜に無關係となる。
4. 層面の中の蓄積力が下程小になる場合に層面の傾斜の零度と 90 度との中間に極大表面波を與へる角度がある。
5. この角度は層の深さに從ひ蓄積力が減少する割合によつて異つて居る。
6. 三次元の場合には非變容性固體中にある層の中の蓄積力が深さに無關係の時でも層面の或角度の傾が極大表面波を與へる。
7. 層面が切離される時は、層面の如何なる傾斜を持つ場合でも表面波が垂直線に對して對稱をなす成分が存在するが、層が辷る場合には決して斯る成分が現はれる樣な事がない。