3. *Generation of Rayleigh-waves from an Internal Source of Multiplet-type.*[*]

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In a previous paper,[1] one of the present authors had the occasion of studying the problem of the transmission of Rayleigh-waves from an internal source of a doublet-type. Very recently Mr. T. Sakai[2] and Dr. H. Nakano[3] have attacked independently similar problems probably without knowing the paper cited above,[4] though they have given more fully the detailed explanations on the evaluation of the integrals involved in them as H. Lamb[5] had done in his paper on the generation of Rayleigh-waves from an internal singlet. The present paper which was written, rather earlier than even those two other writers, as a continuation of the preceding paper,[6] deals with the more extended case where the source of waves is a multiplet, say n.

Very simple problem where the oscillations of mutiplet-type are transmitted through an elastic body extending to infinity in all directions has already been studied[7] and it was found that the waves generated from such an origin are of two kinds, i.e. dilatational waves and distortional waves. Since we have already seen that both kinds of waves are partitioned at the origin with the prescribed conditions of stresses or displacements there, we can treat of each kind of waves separately for any point outside the origin.

We shall take the free surface of the semi-infinite solid to be $z=0$ and also that the positive sense of the axis of z is directed downwards. In a three dimensional problem the primary waves generated from the multiplet-

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1) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929), 1–17.

2) T. SAKAI's paper read at the meeting of *Math.-Phys. Soc., Tokyo,* in Jan. 1929.

3) H. NAKANO's paper read at the meeting of *Meteorol. Soc., Tokyo,* in Feb. 1929.

4) K. SEZAWA, *l.c.*

5) H. LAMB, *Phil. Trans. Roy. Soc.*, **203** (1904).

6) K. SEZAWA, *l.c.*

7) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **2** (1927), 13–20.

source ($r=0$, $z=\xi$) may be expressed by

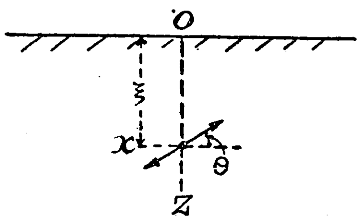
$$\Delta_0' = Ae^{i\sigma t}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)^{n-1}\frac{e^{-ih\sqrt{r^2+(z-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}}, \quad \ldots(1)$$

$$\left.\begin{aligned}2\varpi_y' &= Be^{i\sigma t}\left(\sin\theta\frac{\partial}{\partial x}-\cos\theta\frac{\partial}{\partial z}\right)\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)^{n-1}\frac{e^{-ij\sqrt{r^2+(z-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}},\\ 2\varpi_1' &= Be^{i\sigma t}\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)^{n-1}\frac{e^{-ij\sqrt{r^2+(z-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}},\end{aligned}\right\} \quad \ldots(2)$$

in which $h^2=\rho\sigma^2/(\lambda+2\mu)$, $j^2=\rho\sigma^2/\mu$ and θ is the inclination of the axis of the multiplet, while Δ_0' is the dilatational waves, ϖ_1' is the rotation about the axis perpendicular to the axis of the multiplet in the plane of xz and ϖ_y' is the rotation about the axis perpendicular to xz-plane. The ratio[8] of A to B can be easily obtained in the case of polar coordinates by the conditions of stresses at the origin. Since such determination is not important in this problem as we have explained, we will consider independently the respective cases of the generation of the surface waves due to Δ_0' and due to the resultant of ϖ_y' and ϖ_1'. In the present case we will deal with the problem concerning to Δ_0' [9] given in (1).

Now we know the expression

$$\frac{e^{-ih\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}=\int_0^\infty\frac{e^{-\alpha z}}{\alpha}J_0(kr)\,kdk, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$



which is due to H. Lamb.[10] In this $\alpha=\sqrt{k^2-h^2}$ or $i\sqrt{h^2-k^2}$ according as $k^2\gtrless h^2$.

By the method of induction we can easily find the relations

$$\frac{\partial^n}{\partial x^m\partial z^{n-m}}\int_0^\infty\frac{e^{-\alpha z}}{\alpha}J_0(kr)\,kdk$$

$$=\frac{(-1)^n}{2^{m-1}}\left\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega\right.$$

$$\times\int_0^\infty e^{-\alpha z}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}\,dk$$

$$\left.+(-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty e^{-\alpha z}\frac{J_0(kr)}{2}\alpha^{n-m-1}k^{m+1}\,dk\right\}, \qquad [m:\text{even}]$$

8) K. SEZAWA, *l.c.* p. 41.
9) Hereafter we shall omit the constant A for the sake of simplicity.
10) H. LAMB, *Phil. Trans. Roy. Soc.*, **203** (1904).

$$=\frac{(-1)^n}{2^{m-1}}\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$
$$\times\int_0^\infty e^{-\alpha z}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}\,dk, \qquad [m:\text{odd}]$$
$$\cdots\cdots\cdots\cdots(4)$$

in which ω is the azimuth angle in cylindrical coordinates.

Thus, the primary waves can be expressed by

$$\Delta_0'=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-1}}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\left\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega\int_0^\infty e^{-\alpha z}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}\,dk\right.$$
$$\left.+(-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty e^{-\alpha z}\frac{J_0(kr)}{2}\alpha^{n-m-1}k^{m+1}\,dk\right\}, \qquad [m:\text{even}]$$
$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-1}}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$
$$\times\int_0^\infty e^{-\alpha z}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}\,dk. \qquad [m:\text{odd}]$$
$$\cdots\cdots\cdots\cdots(5)$$

It is convenient to consider the image at the point $r=0$, $z=-\xi$. The dilatation due to the image is expressed by

$$\Delta_0''=e^{i\sigma t}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)^n\frac{e^{-ih\sqrt{r^2+(z+\xi)^2}}}{\sqrt{r^2+(z+\xi)^2}}\cdots\cdots\cdots\cdots(6)$$

Superposing this on Δ_0' in (1) and applying the formulae in (4), we get

$$\Delta_0=\Delta_0'+\Delta_0''$$
$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\left\{\sum_{p=0}^{p=\frac{m-2}{2}}\frac{(-1)^p m!}{(m-p)!\,p!}\cos(m-2p)\,\omega\int_0^\infty \operatorname{ch}\alpha z\, e^{-\alpha\xi}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}\,dk\right.$$

$$+(-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi}\frac{J_0(kr)}{2}\alpha^{n-m-1}k^{m+1}dk\Big\}, \qquad [m:\text{even}]$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(n-p)!\,p!}\cos(m-2p)\,\omega$$
$$\times\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}dk. \qquad [m:\text{odd}]$$
$$\dots\dots\dots(7)$$

The components of displacement in radial, azimuthal and vertical directions can be expressed by

$$u_0=-\frac{1}{h^2}\frac{\partial\Delta_0}{\partial r}$$
$$=-e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\Big\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$
$$\times\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi}\frac{\partial J_{m-2p}(kr)}{\partial r}\alpha^{n-m-1}k^{m+1}dk$$
$$+(-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi}\frac{\partial J_0(kr)}{2\partial r}\alpha^{n-m-1}k^{m+1}dk\Big\}, \qquad [m:\text{even}]$$

$$=-e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$
$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$
$$\times\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi}\frac{\partial J_{m-2p}(kr)}{\partial r}\alpha^{n-m-1}k^{m+1}dk, \qquad [m:\text{odd}]$$

$$v_0=-\frac{1}{h^2 r}\frac{\partial\Delta_0}{\partial\omega}$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}(m-2p)\sin(m-2p)\,\omega$$

$$\times\int_0^\infty \mathrm{ch}\,\alpha z\,e^{-\alpha\xi}\frac{J_{m-2p}(kr)}{r}\alpha^{n-m-1}k^{m+1}\,dk,\qquad [m:\text{even}]$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}(m-2p)\sin(m-2p)\,\omega$$

$$\times\int_0^\infty \mathrm{ch}\,\alpha z\,e^{-\alpha\xi}\frac{J_{m-2p}(kr)}{r}\alpha^{n-m-1}k^{m+1}\,dk,\qquad [m:\text{odd}]$$

$$w_0=-\frac{1}{h^2}\cdot\frac{\partial\Delta_0}{\partial z}$$

$$=-e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\left\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega\right.$$

$$\times\int_0^\infty \mathrm{sh}\,\alpha z\,e^{-\alpha\xi}J_{m-2p}(kr)\,\alpha^{n-m}k^{m+1}\,dk$$

$$\left.+(-1)^{\frac{m}{2}}\frac{m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty \mathrm{sh}\,\alpha z\,e^{-\alpha\xi}\frac{J_0(kr)}{2}\alpha^{n-m}k^{m+1}\,dk\right\},\qquad [m:\text{even}]$$

$$=-e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m!)\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$

$$\times\int_0^\infty \mathrm{sh}\,\alpha z\,e^{-\alpha\xi}J_{m-2p}(kr)\,\alpha^{n-m}k^{m+1}\,dk.\qquad [m:\text{odd}]$$

$$\dots\dots\dots\dots(8)$$

We may superpose on these some free waves[11] accumulated in the

11) The free waves in this case are specified such that they give no any tangential stress, but they give some normal stress.

neighbourhood of the surface to annual the normal stress on that surface. The free waves are given by the solutions of the type, such that

$$\left.\begin{aligned} u_1 &= -\frac{\{(2k^2-j^2)-2\alpha\beta\}}{h^2(2k^2-j^2)}\frac{\partial J_{m-2p}(kr)}{\partial r}\cos(m-2p)\omega\, e^{i\sigma t}, \\ v_1 &= \frac{(m-2p)\{2k^2-j^2-2\alpha\beta\}}{h^2(2k^2-j^2)}\frac{J_{m-2p}(kr)}{r}\sin(m-2p)\omega\, e^{i\sigma t}, \\ w_1 &= -\frac{\alpha^2}{h^2(2k^2-j^2)}J_{m-2p}(kr)\cos(m-2p)\omega\, e^{i\sigma t}, \end{aligned}\right\}[z=0]\,..\,(9)^{12)}$$

in which $\alpha^2=k^2-h^2$, $\beta^2=k^2-j^2$ and $j^2=\rho p^2/\mu$. The above solutions give no any tangential stress, but give us some normal stress on $z=0$.

The condition of no stress on the surface can be satisfied by summing up the solutions of the type (9) in the forms:

$$u_1 = e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\left\{\sum_{p=0}^{p=\frac{m-2}{2}}\frac{(-1)^p m!}{(m-p)!\,p!}\cos(m-2p)\,\omega\right.$$

$$\times\int_0^\infty\frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{F(k)}e^{-\alpha\xi}\frac{\partial J_{m-2p}(kr)}{\partial r}\alpha^{n-m-1}k^{m+1}\,dk$$

$$\left.+\frac{(-1)^{\frac{m}{2}}m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty\frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{2F(k)}e^{-\alpha\xi}\frac{\partial J_0(kr)}{\partial r}\alpha^{n-m-1}k^{m+1}\,dk\right\},$$

[m:even]

$$= e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$

$$\times\int_0^\infty\frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{F(k)}e^{-\alpha\xi}\frac{\partial J_{m-2p}(kr)}{\partial r}\alpha^{n-m-1}k^{m+1}\,dk,\quad [m:\text{odd}]$$

$$v_1 = -e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{m=0}^{p=\frac{m-2}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}(m-2p)\sin(m-2p)\,\omega$$

12) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929), 7–8.

$$\times\int_0^\infty \frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}\,\frac{J_{m-2p}(kr)}{r}\,\alpha^{n-m-1}\,k^{m+1}\,dk, \quad [m:\text{even}]$$

$$=-e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}\,h^2}\,\frac{n\,!}{(n-m)\,!\;m\,!}\,\sin^{n-m}\theta\,\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\,\frac{m\,!}{(m-p)\,!\;p\,!}\,(m-2p)\,\sin\,(m-2)\,\omega$$

$$\times\int_0^\infty \frac{(2k^2-j^2-2\alpha\beta)(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}\,\frac{J_{m-2p}(kr)}{r}\,\alpha^{n-m-1}\,k^{m+1}\,dk, \quad [m:\text{odd}]$$

$$w_1=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}\,h^2}\,\frac{n\,!}{(n-m)\,!\;m\,!}\,\sin^{n-m}\theta\,\cos^m\theta$$

$$\times\Bigg\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\,\frac{m\,!}{(m-p)\,!\,p\,!}\,\cos\,(m-2p)\,\omega$$

$$\times\int_0^\infty \frac{\alpha j^2(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}\,J_{m-2p}(kr)\,\alpha^{n-m-1}\,k^{m+1}\,dk$$

$$+\frac{(-1)^{\frac{m}{2}}\,m\,!}{\left(\frac{m}{2}\,!\right)^2}\int_0^\infty \frac{\alpha j^2(2k^2-j^2)}{2F(k)}\,e^{-\alpha\xi}\,J_0(kr)\,\alpha^{n-m-1}\,k^{m+1}\,dk\Bigg\}, \quad [m:\text{even}]$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}\,h^2}\,\frac{n\,!}{(n-m)\,!\;m\,!}\,\sin^{n-m}\theta\,\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\,\frac{m\,!}{(m-p)\,!\;p\,!}\,\cos\,(m-2p)\,\omega$$

$$\times\int_0^\infty \frac{\alpha j^2(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}\,J_{m-2p}(kr)\,\alpha^{n-m-1}\,k^{m+1}\,dk, \quad [m:\text{odd}]$$

$$\dots\dots\dots\dots(10)$$

provided $$F(k)=(2k^2-j^2)^2-4k^2\alpha\beta. \quad\dots\dots\dots(11)$$

The resultant displacements on $z=0$ are therefore written by

$$u=u_0+u_1$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}\,h^2}\,\frac{n\,!}{(n-m)\,!\;m\,!}\,\sin^{n-m}\theta\,\cos^m\theta$$

$$\times\Bigg\{\sum_{p=0}^{p=\frac{m-2}{2}}(-1)^p\,\frac{m\,!}{(m-p)\,!\;p\,!}\,\cos\,(m-2p)\,\omega$$

$$\times \int_0^\infty \frac{2\beta j^2}{F(k)} e^{-\alpha\xi} \frac{\partial J_{m-2p}(kr)}{\partial r} \alpha^{n-m} k^{m+1} \, dk$$

$$+ \frac{(-1)^{\frac{m}{2}} m!}{\left(\frac{m}{2}!\right)^2} \int_0^\infty \frac{\beta j^2}{F(k)} e^{-\alpha\xi} \frac{\partial J_0(kr)}{\partial r} \alpha^{n-m} k^{m+1} \, dk \Bigg\}, \quad [m:\text{even}]$$

$$= c^{i\sigma t} \sum_{m=0}^{m=n} \frac{(-1)^n}{2^{m-2} h^2} \frac{n!}{(n-m)!\, m!} \sin^{n-m}\theta \cos^m\theta$$

$$\times \sum_{p=0}^{p=\frac{m-1}{2}} (-1)^p \frac{m!}{(m-p)!\, p!} \cos(m-2p)\,\omega$$

$$\times \int_0^\infty \frac{2\beta j^2}{F(k)} e^{-\alpha\xi} \frac{\partial J_{m-2p}(kr)}{\partial r} \alpha^{n-m} k^{m+1} \, dk, \qquad [m:\text{odd}]$$

$$v = v_0 + v_1$$

$$= -e^{i\sigma t} \sum_{m=0}^{m=n} \frac{(-1)^n}{2^{m-2} h^2} \frac{n!}{(n-m)!\, m.} \sin^{n-m}\theta \cos^m\theta$$

$$\times \sum_{p=0}^{p=\frac{m-2}{2}} (-1)^p \frac{m!}{(m-p)!\, p!} (m-2p) \sin(m-2p)\,\omega$$

$$\times \int_0^\infty \frac{2\beta j^2}{F(k)} e^{-\alpha\xi} \frac{J_{m-2p}(kr)}{r} \alpha^{n-m} k^{m+1} \, dk, \qquad [m:\text{even}]$$

$$= -e^{i\sigma t} \sum_{m=0}^{m=n} \frac{(-1)^n}{2^{m-2} h^2} \frac{n!}{(n-m)!\, m!} \sin^{n-m}\theta \cos^m\theta$$

$$\times \sum_{p=0}^{p=\frac{m-1}{2}} (-1)^p \frac{m!}{(m-p)!\, p!} (m-2p) \sin(m-2p)\,\omega$$

$$\times \int_0^\infty \frac{2\beta j^2}{F(r)} e^{-\alpha\xi} \frac{J_{m-2p}(kr)}{r} \alpha^{n-m} k^{m+1} \, dk, \qquad [m:\text{odd}]$$

$$w = w_0 + w_1$$

$$= e^{i\sigma t} \sum_{m=0}^{m=n} \frac{(-1)^n}{2^{m-2} h^2} \frac{n!}{(n-m)!\, m!} \sin^{n-m}\theta \cos^m\theta$$

$$\times \Bigg\{ \sum_{p=0}^{p=\frac{m-1}{2}} (-1)^p \frac{m!}{(m-p)!\, p!} \cos(m-2p)\,\omega$$

$$\times \int_0^\infty \frac{\alpha j^2 (2k^2 - j^2)}{F(k)} e^{-\alpha\xi} J_{m-2p}(kr)\, \alpha^{n-m-1} k^{m+1} \, dk$$

$$+\frac{(-1)^{\frac{m}{2}}m!}{\left(\frac{m}{2}!\right)^2}\int_0^\infty \frac{\alpha j^2(2k^2-j^2)}{2F(k)}e^{-\alpha\xi}J_0(kr)\,\alpha^{n-m-1}k^{m+1}dk\Bigg\},\ [m:\text{even}]$$

$$=e^{i\sigma t}\sum_{m=0}^{m=n}\frac{(-1)^n}{2^{m-2}h^2}\frac{n!}{(n-m)!\,m!}\sin^{n-m}\theta\cos^m\theta$$

$$\times\sum_{p=0}^{p=\frac{m-1}{2}}(-1)^p\frac{m!}{(m-p)!\,p!}\cos(m-2p)\,\omega$$

$$\times\int_0^\infty\frac{\alpha j^2(2k^2-j^2)}{F(k)}e^{-\alpha\xi}J_{m-2p}(kr)\,\alpha^{n-m-1}k^{m+1}dk. \qquad [m:\text{odd}]$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots(12)$$

Since we know the expressions

$$J_{m-2p}(kr)=\frac{2}{\pi}\int_0^\infty\sin\left(kr\,\mathrm{ch}\,f-\frac{m-2p}{2}\pi\right)\mathrm{ch}\,(m-2p)\,f\,df,\text{[13]}\ \dots\dots(13)$$

we find the equations (12) are equivalent to

$$\left.\begin{aligned}u&=e^{i\sigma t}\frac{(-1)^n n!}{\pi h^2}\sum_{m=0}^{m=n}\frac{\sin^{n-m}\theta\cos^m\theta}{2^{m-3}(n-m)!}\frac{\partial\Phi_m}{\partial r},\\ v&=e^{i\sigma t}\frac{(-1)^n n!}{\pi h^2}\sum_{m=0}^{m=n}\frac{\sin^{n-m}\theta\cos^m\theta}{2^{m-3}(n-m)!}\frac{1}{r}\frac{\partial\Phi_m}{\partial\omega},\\ w&=e^{i\sigma t}\frac{(-1)^n n!}{\pi h^2}\sum_{m=0}^{m=n}\frac{\sin^{n-m}\theta\cos^m\theta}{2^{m-3}(n-m)!}\Psi_m,\end{aligned}\right\}\dots\dots\dots\dots(14)$$

in which

$$\left.\begin{aligned}\Phi_m&=i(-1)^{\frac{3m}{2}+1}\sum_{p=0}^{p=q}\gamma\frac{\cos(m-2p)\,\omega}{(m-p)!\,p!}\int_0^\infty\cos(m-2p)\,f\,df\\ &\qquad\times\int_{-\infty}^\infty\frac{\beta j^2}{F(k)}e^{-\alpha\xi+ikr\,\mathrm{ch}\,f}\alpha^{n-m}\kappa^{m+1}d\kappa,\\ \Psi_m&=i(-1)^{\frac{3m}{2}+1}\sum_{p=0}^{p=q}\gamma\frac{\cos(m-2p)\,\omega}{(m-p)!\,p!}\int_0^\infty\cos(m-2p)\,f\,df\\ &\qquad\times\int_{-\infty}^\infty\frac{j^2(2k^2-j^2)}{2F(k)}e^{-\alpha\xi+ikr\,\mathrm{ch}}\alpha^{n-m}k^{m+1}dk,\end{aligned}\right\}\dots(15)$$

where q should be taken at $m/2$ or $(m-1)/2$ according as m is even or odd

13) WATSON, *Theory of Bessel Functions* (1922), 180.

and it is to be remarked that γ is taken to be unity excepting the term, $p=m/2$, in which case $\gamma=1/2$.

The evaluation of the integrals of the forms:

$$\left.\begin{aligned}&\int_{-\infty}^{\infty}\frac{\beta j^2}{F(k)}\,e^{-\alpha\xi+ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,k^{m+1}\,dk,\\&\int_{-\infty}^{\infty}\frac{j^2(2k^2-j^2)}{2F(k)}\,e^{-\alpha\xi+ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,k^{m+1}\,dk,\end{aligned}\right\}\qquad(16)$$

can be performed after H. Lamb's method described in his paper[14] on Rayleigh-waves. We thus obtain

$$\left.\begin{aligned}&\int_{-\infty}^{\infty}\frac{\beta j^2}{F(\kappa)}\,e^{-\alpha\xi}\,e^{ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,\kappa^{m+1}\,d\kappa\\&\qquad=\left.\begin{matrix}i\\-\end{matrix}\right\}4\pi\frac{\beta_1\,j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\\sin\end{matrix}\right\}\kappa r\,\mathrm{ch}\,f\,\alpha_1^{n-m}\,\kappa^{m+1}\\&\qquad\qquad+\text{asymptotically vanishing term},\\&\int_{-\infty}^{\infty}\frac{j^2(2k^2-j^2)}{F(k)}\,e^{-\alpha\xi}\,e^{ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,k^{m+1}\,dk\\&\qquad=\left.\begin{matrix}i\\-\end{matrix}\right\}4\pi\frac{j^2(2\kappa^2-j^2)}{F'(\kappa)}\,e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\\sin\end{matrix}\right\}\kappa r\,\mathrm{ch}\,f\,\alpha_1^{n-m}\,\kappa^{m+1}\\&\qquad\qquad+\text{asymptotically vanishing term}.\end{aligned}\right\}\qquad(17)$$

In these, κ is the real positive root of $F(k)$ and α_1, β_1 are the corresponding values of α, β. (i is for even m and $-$ for odd m.)

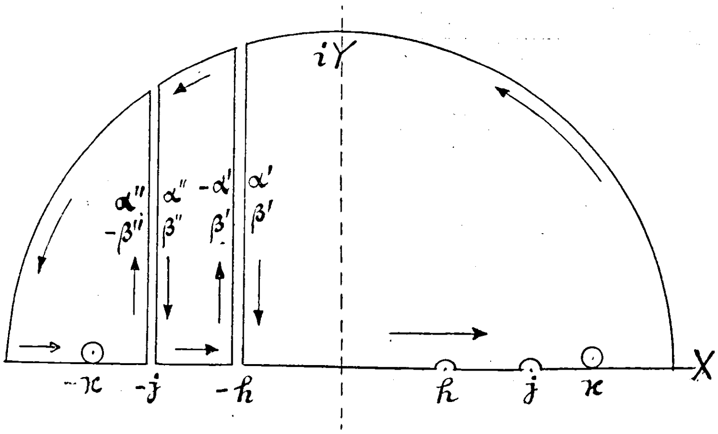
The above evaluations, which are obtained directly from a simple process similar to that of H. Lamb, are sufficient for our purpose. Yet we shall here give the more close investigation of the evaluation to find the residual disturbance, though we owe to H. Lamb the method of investigations thoroughly as described above.

To integrate the left-hand sides of the expressions in (17), we consider the integrals

$$\left.\begin{aligned}&\int\frac{\sqrt{Z^2-j^2}\,j^2}{F(Z)}\,e^{-\sqrt{Z^2-h^2}\,\xi}\,e^{iZr\,\mathrm{ch}\,f}\,(\sqrt{Z^2-h^2})^{n-m}\,Z^{m+1}\,dZ,\\&\int\frac{j^2(2Z^2-j^2)}{F(Z)}\,e^{-\sqrt{Z^2-h^2}\,\xi}\,e^{iZr\,\mathrm{ch}\,f}\,(\sqrt{Z^2-h^2})^{n-m}\,Z^{m+1}\,dZ,\end{aligned}\right\}\qquad(18)$$

taken round the contour annexed in the next page. Though the part enclosed by $(-j\ -h)$ and two grooves directed towards infinitely belongs

14) H. LAMB, *loc. cit.* p. 41.



to a different sheet, we treat of that part independently and adding to the results of two other systems of contour integrations. Hence we find

$$\int_{-\infty}^{\infty}\frac{\beta j^2}{F(k)}\,e^{-\alpha\xi}\,e^{ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,k^{m+1}\,dk=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi\beta_1 j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\,\kappa^{m+1}$$

$$+e^{-ijr\,\mathrm{ch}\,f}\int_0^{\infty}\left\{\frac{j^2\beta''}{(2Z^2-j^2)^2-4Z^2\alpha''\beta''}\right.$$

$$\left.+\frac{j^2\beta''}{(2Z^2-j^2)^2+4Z^2\alpha''\beta''}\right\}Z^{m+1}\,(Z^2-h^2)^{\frac{n-m}{2}}\,e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,idY$$

$$+e^{-ihr\,\mathrm{ch}\,f}\int_0^{\infty}\left\{\frac{j^2\beta'}{(2Z^2-j^2)^2-4Z^2\alpha'\beta'}\right.$$

$$\left.-\frac{j^2\beta'}{(2Z^2-j^2)^2+4Z^2\alpha'\beta'}\right\}Z^{m+1}\,(Z^2-h^2)^{\frac{n-m}{2}}\,e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,idY$$

$$=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi\beta_1 j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\,\kappa^{m+1}$$

$$+8ie^{-ijr\,\mathrm{ch}\,f}\int_0^{\infty}\frac{j^2\beta''\,(2Z^2-j^2)^2\,Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}}{(2Z^2-j^2)^4-16Z^4\alpha''^2\beta''^2}\,e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,dY$$

$$+8ie^{-ihr\,\mathrm{ch}\,f}\int_0^{\infty}\frac{j^2\beta'\,Z^2\alpha'\beta'\,Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}}{(2Z^2-j^2)^4-16Z^4\alpha'^2\beta'^2}\,e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,dY,$$

$$\int_{-\infty}^{\infty}\frac{j^2\,(2k^2-j^2)}{F\,(k)}\,e^{-\alpha\xi}\,e^{ikr\,\mathrm{ch}\,f}\,\alpha^{n-m}\,k^{m+1}\,dk$$

$$=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi ij^2(2\kappa^2-j^2)}{F'(\kappa)}\,e^{-\alpha_1\xi}\cos(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\,\kappa^{m+1}$$

$$+e^{-ijr\,\mathrm{ch}\,f}\int_0^\infty\left\{\frac{2k^2-j^2}{(2Z^2-j^2)^2-4Z^2\alpha''\beta''}\right.$$

$$\left.-\frac{2k^2-j^2}{(2Z^2-j^2)^2+4Z^2\alpha''\beta''}\right\}j^2Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,idY$$

$$+e^{-ihr\,\mathrm{ch}\,f}\int_0^\infty\left\{\frac{2k^2-j^2}{(2Z^2-j^2)^2-4Z^2\alpha'\beta'}\right.$$

$$\left.-\frac{2k^2-j^2}{(2Z^2-j^2)^2+4Z^2\alpha'\beta'}\right\}j^2Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}e^{-\sqrt{Z^2-h^2}\,\xi-Yr\,\mathrm{ch}\,f}\,idY$$

$$=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi j^2(2\kappa^2-j^2)}{F'(\kappa)}e^{-\alpha_1\xi}\cos(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\kappa^{m+1}$$

$$+8ie^{-ijr\,\mathrm{ch}\,f}\int_0^\infty\frac{(2k^2-j^2)Z^2\alpha''\beta''j^2Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}}{(2Z^2-j^2)^4-16Z^4\alpha''^2\beta''^2}e^{-\sqrt{Z^2-h^2}-Yr\,\mathrm{ch}\,f}\,dY$$

$$+8ie^{-ihr\,\mathrm{ch}\,f}\int_0^\infty\frac{(2k^2-j^2)Z^2\alpha'\beta'j^2Z^{m+1}(Z^2-h^2)^{\frac{n-m}{2}}}{(2Z^2-j^2)^4-16Z^4\alpha'^2\beta'^2}e^{-\sqrt{Z^2-h^2}-Yr\,\mathrm{ch}\,f}\,dY.$$

..............................(19)

In each first integral of the right-hand side of these we should put $Z=-j+iY$, and in each second, $Z=-h+iY$. Approximate evaluation of the above integrals can be easily performed. They give the expressions of residual disturbances whose velocities of propagation are limited to two kinds, one of them corresponding to that of dilatational waves and the other to that of distortional waves. It will be noticed that the terms corresponding to these waves quickly disappears in virtue of the factor $e^{-Yr\,\mathrm{ch}\,f}$, so that we disregard these terms hereafter for the investigation of Rayleigh-waves. Thus we arrive at

$$\left.\begin{aligned}&\int_{-\infty}^\infty\frac{\beta j^2}{F(k)}e^{-\alpha\xi}e^{ikr\,\mathrm{ch}\,f}\alpha^{n-m}k^{m+1}\,dk\\&\qquad=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi\beta_1 j^2}{F'(\kappa)}e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\\sin\end{matrix}\right\}(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\kappa^{m+1}\\&\qquad\qquad+\text{asymptotically vanishing terms,}\\&\int_{-\infty}^\infty\frac{j^2(2k^2-j^2)}{F(k)}e^{-\alpha\xi}e^{ikr\,\mathrm{ch}\,f}\alpha^{n-m}k^{m+1}\,dk\\&\qquad=\left.\begin{matrix}i\\-\end{matrix}\right\}\frac{4\pi j^2(2\kappa^2-j^2)}{F'(\kappa)}e^{-\alpha_1\xi}\left.\begin{matrix}\cos\\\sin\end{matrix}\right\}(\kappa r\,\mathrm{ch}\,f)\,\alpha_1^{n-m}\kappa^{m+1}\\&\qquad\qquad+\text{asymptotically vanishing terms.}\end{aligned}\right\}\quad\ldots.(17')$$

Now we return to the original problem. The expressions of Φ_m and Ψ_m which we have written in (15) thus take the forms:

$$
\left.\begin{aligned}
\Phi_m &= i\,(-1)^{\frac{3m}{2}+1}\sum_{p=0}^{p=q}\gamma\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\ \ p\,!} \\
&\quad\times\int_0^\infty \cos\,(m-2p)\,f\left[\begin{Bmatrix} i \\ - \end{Bmatrix}\frac{4\pi\beta_1 j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\kappa r\ \mathrm{ch}\,f)\,\alpha_1^{n-m}\,\kappa^{m+1}\right]df, \\
\Psi_m &= i\,(-1)^{\frac{3m}{2}+1}\sum_{p=0}^{p=q}\gamma\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\,!\,p\,!} \\
&\quad\times\int_0^\infty \cos\,(m-2p)\,f\left[\begin{Bmatrix} i \\ - \end{Bmatrix}\frac{4\pi i j^2(2\kappa^2-j^2)}{2F'(\kappa)}\,e^{-\alpha_1\xi}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\kappa r\ \mathrm{ch}\,f)\,\alpha_1^{n-m}\,\kappa^{m+1}\right]df.
\end{aligned}\right\}\quad(20)
$$

In virtue of (13) and the equation

$$
Y_{m-2p}(\kappa r) = -\frac{2}{\pi}\int_0^\infty \cos\left(\kappa r\ \mathrm{ch}\,f - \frac{m-2p}{2}\,\pi\right)\mathrm{ch}\,(m-2p)\,f\,df,^{15)}\quad\ldots\ldots(21)
$$

we find

$$
\begin{aligned}
\Phi_m &= -\frac{2\pi^2\beta_1 j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\,\alpha_1^{n-m}\,\kappa^{m+1} \\
&\qquad\times\sum_{p=0}^{p=q}\gamma\,(-1)^p\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\,!\,p\,!}\,Y_{m-2p}(\kappa r), \qquad [m:\text{even}] \\
&= +\frac{2\pi^2\beta_1 j^2}{F'(\kappa)}\,e^{-\alpha_1\xi}\,\alpha_1^{n-m}\,\kappa^{m+1} \\
&\qquad\times\sum_{p=0}^{p=q}\gamma\,(-1)^p\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\,!\,p\,!}\,Y_{m-2p}(\kappa r), \qquad [m:\text{odd}] \\
\Psi_m &= -\frac{2\pi^2 j^2\,(2\kappa^2-j^2)}{2F'(\kappa)}\,e^{-\alpha_1\xi}\,\alpha_1^{n-m}\,\kappa^{m+1} \\
&\qquad\times\sum_{p=0}^{p=q}\gamma\,(-1)^p\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\,!\,p\,!}\,Y_{m-2p}(\kappa r), \qquad [m:\text{even}] \\
&= \frac{2\pi^2 j^2\,(2\kappa^2-j^2)}{2F'(\kappa)}\,e^{-\alpha_1\xi}\,\alpha_1^{n-m}\,\kappa^{m+1} \\
&\qquad+\sum_{p=0}^{p=q}\gamma\,(-1)^p\,\frac{\cos\,(m-2p)\,\omega}{(m-p)\,!\,p\,!}\,Y_{m-2p}(\kappa r). \qquad [m:\text{odd}]
\end{aligned}
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Substituting these values of Φ_m and Ψ_m in (14), we find

15) WATSON, *loc. cit.*, p. 180.

$$u=\frac{2e^{i\sigma t}(-1)^n n!\pi}{h^2}\frac{\beta_1 j^2}{F'(\kappa)}e^{-\alpha_1\xi}\sum_{m=0}^{m=n}(-1)^m \sin^{n-m}\theta\cos^m\theta\,\alpha_1{}^{n-m}\kappa^{m+1}$$
$$\times\sum_{m=0}^{p=q}\gamma(-1)^p\frac{\cos(m-2p)\omega}{(m-p)!\,p!}\frac{\partial}{\partial r}\left\{-Y_{m-2p}(\kappa r),\quad Y_{m-2p}(\kappa r)\right\},$$
$$v=\frac{2e^{i\sigma t}(-1)^n n!\pi}{h^2}\frac{\beta_1 j^2}{F'(\kappa)}e^{-\alpha_1\xi}\sum_{m=0}^{m=n}(-1)^m \sin^{n-m}\theta\cos^m\theta\,\alpha_1^{n-m}\kappa^{m+1}$$
$$\times\sum_{p=0}^{p=q}\gamma(-1)^{p+1}\frac{(m-2p)\sin(m-2p)\omega}{(m-p)!\,p!\,r}\left\{-Y_{m-2p}(\kappa r),\quad Y_{m-2p}(\kappa r)\right\},$$
$$w=+\frac{2ie^{i\sigma t}(-1)^n n!\pi}{h^2}\frac{j^2(2\kappa^2-j^2)}{2F'(\kappa)}\sum_{m=0}^{m=n}(-1)^m \sin^{n-m}\theta\cos^m\theta\,\alpha_1^{n-m}\kappa^{m+1}$$
$$\times\sum_{p=0}^{p=q}\gamma(-1)^p\frac{\cos(m-2p)\omega}{(m-p)!\,p!}\left\{-Y_{m-2p}(\kappa r),\quad Y_{m-2p}(\kappa r)\right\},$$
$$\dots\dots\dots(23)$$

where the first term in each bracket should be taken for even value of m and the second for odd value of m. These expressions are suitable to the transmission of waves of a finite extent of disturbances. To get progressive waves of an infinite train, we may combine two systems of different standing types. Though the method is somewhat conventional, it must be borne in mind that, as we know from Fourier's integral theorem, half the amplitude of the standing type is effective for the actual progressive waves. Thus, the final solutions, when r is large, can be expressed by

$$\left.\begin{aligned}
u\approx&\frac{ie^{i\sigma t}(-1)^n n!\pi}{h^2}\frac{\beta_1 j^2}{F'(\kappa)}e^{-\alpha_1\xi}\\
&\times\sum_{m=0}^{m=n}(-1)^m\frac{\sin^{n-m}\theta\cos^m\theta\,\alpha_1^{n-m}\kappa^{m+1}}{2^{m-3}(n-m)!}\\
&\times\sum_{p=0}^{p=q}\gamma(-1)^p\frac{\cos(m-2p)\omega}{(m-p)!\,p!}\frac{\partial H^{(2)}_{m-2p}(\kappa r)}{\partial r},\\
v\approx&\frac{e^{i\sigma t}(-1)^n n!\pi}{h^2}\frac{\beta_1 j^2}{F'(\kappa)}e^{-\alpha_1\xi}\\
&\times\sum_{m=0}^{m=n}(-1)^m\frac{\sin^{n-m}\theta\cos^m\theta\,\alpha_1^{n-m}\kappa^{m+1}}{2^{m-3}(n-m)!}\\
&\times\sum_{p=0}^{p=q}\gamma(-1)^{p+1}\frac{(m-2p)\sin(m-2p)\omega}{(m-p)!\,p!}\frac{H^{(2)}_{m-2p}(\kappa r)}{r},
\end{aligned}\right\}\dots\dots(24)$$

$$
\left.
\begin{aligned}
w \approx & -\frac{ie^{i\sigma t}(-1)^n n!\,\pi}{h^2}\frac{j^2(2\kappa^2-j^2)}{2F'(\kappa)} \\
& \times \sum_{m=0}^{m=n}(-1)^m \frac{\sin^{n-m}\theta\cos^m\theta\,\alpha_1^{n-m}\kappa^{m+1}}{2^{m-3}(n-m)!} \\
& \times \sum_{p=0}^{p=q}\gamma(-1)^p\frac{\cos(m-2p)\,\omega}{(m-p)!\,p!}H_{m-2p}^{(2)}(\kappa r),
\end{aligned}
\right\}
$$

corresponding to the primary disturbance at $(r=0,\ z=\xi)$ in the form:

$$
\Delta_0' = e^{i\sigma t}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial z}\right)^n \frac{e^{-ih\sqrt{r^2+(z-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}}. \quad \cdots\cdots\cdots\cdots\cdots (1')
$$

In the expressions in (24) q should be taken at $m/2$ or $(m-1)/2$ according as m is even or odd and γ is taken to be unity excepting the term $p=m/2$ in which case $\gamma=1/2$.

In a special case where $\theta=0$ and $n=1$, we have

$$
\left.
\begin{aligned}
u &\approx e^{i\sigma t}\frac{4i\pi\cos\omega}{h^2}\frac{j^2\kappa^2\beta_1}{F'(\kappa)}e^{-\alpha_1\xi}\frac{\partial H_1^{(2)}(\kappa r)}{\partial r}, \\
v &\approx -e^{i\sigma t}\frac{4\pi\sin\omega}{h^2}\frac{j^2\kappa^2\beta_1}{F'(\kappa)}e^{-\alpha_1\xi}\frac{H_1^{(2)}(\kappa r)}{r}, \\
w &\approx -e^{i\sigma t}\frac{2\pi\cos\omega}{h^2}\frac{j^2\kappa^2(2\kappa^2-j^2)}{F'(\kappa)}e^{-\alpha_1\xi}H_1^{(2)}(\kappa r),
\end{aligned}
\right\} \cdots\cdots\cdots\cdots (25)
$$

which is the same form as what one[16] of the authors has already obtained in a preceding time, though the negative signs in u and v in this are introduced during the superposing operations.

If we take the case where θ is not zero but n is unity, the expressions of displacement are given by

$$
\left.
\begin{aligned}
u &= -i\,e^{i\sigma t}\frac{4\pi}{h^2}\frac{\beta_1 j^2\kappa}{F'(\kappa)}e^{-\alpha_1\xi}\left\{\alpha_1\sin\theta\frac{\partial H_0^{(2)}(\kappa r)}{\partial r}\right. \\
&\qquad\qquad \left. -\kappa\cos\theta\cos\omega\frac{\partial H_1^{(2)}(\kappa r)}{\partial r}\right\}, \\
v &= -i\,e^{i\sigma t}\frac{4\pi}{h^2}\frac{\beta_1 j^2\kappa^2}{F'(\kappa)}e^{-\alpha_1\xi}\cos\theta\sin\omega\frac{H_1^{(2)}(\kappa r)}{r}, \\
w &= i\,e^{i\sigma t}\frac{2\pi}{h^2}\frac{j^2(2\kappa^2-j^2)\kappa}{F'(\kappa)}e^{-\alpha_1\xi}\left\{\alpha_1\sin\theta\,H_0^{(2)}(\kappa r)\right. \\
&\qquad\qquad \left. -\kappa\cos\theta\cos\omega\,H_1^{(2)}(\kappa r)\right\},
\end{aligned}
\right\} \cdots\cdots (26)
$$

16) K. SEZAWA, *loc. cit.* p. 41.

corresponding to the disturbance at the origin,

$$\Delta_0' = e^{i\sigma t}\left(\sin\theta\frac{\partial}{\partial z} + \cos\theta\frac{\partial}{\partial x}\right)\frac{e^{-ih\sqrt{r^2+(z-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

Let us next consider the case $n=2$. The displacements can be written at once in the forms:

$$\left.\begin{aligned}
u &= i\, e^{i\sigma t}\frac{2\pi}{h^2}\frac{\beta_1 j^2 \kappa}{F'(\kappa)} e^{-\alpha_1\xi}\Big\{(2\alpha_1^2\sin^2\theta - \kappa^2\cos^2\theta)\frac{\partial H_0^{(2)}(\kappa r)}{\partial r} \\
&\quad -4\alpha_1\kappa\sin\theta\cos\theta\cos\omega\frac{\partial H_1^{(2)}(\kappa r)}{\partial r} + \kappa^2\cos^2\theta\cos 2\omega\frac{\partial H_2^{(2)}(\kappa r)}{\partial r}\Big\}, \\
v &= i\, e^{i\sigma t}\frac{2\pi}{h^2}\frac{\beta_1 j^2 \kappa}{F'(\kappa)} e^{-\alpha_1\xi}\Big\{4\alpha_1\sin\theta\cos\theta\sin\omega\frac{H_1^{(2)}(\kappa r)}{r} \\
&\quad -\kappa\cos^2\theta\sin 2\omega\frac{H_2^{(2)}(\kappa r)}{r}\Big\}, \\
w &= -\, c^{i\sigma t}\frac{i\pi}{h^2}\frac{j^2(2\kappa^2-j^2)\kappa}{F'(\kappa)} e^{-\alpha_1\xi}\Big\{(2\alpha_1^2\sin^2\theta - \kappa^2\cos^2\theta)H_0^{(2)}(\kappa r) \\
&\quad -4\alpha_1\kappa\sin\theta\cos\theta\cos\omega H_1^{(2)}(\kappa r) + \kappa^2\cos^2\theta\cos 2\omega H_2^{(2)}(\kappa r)\Big\},
\end{aligned}\right\} \ldots (28)$$

corresponding to

$$\Delta_0' = e^{i\sigma t}\left(\sin\theta\frac{\partial}{\partial z} + \cos\theta\frac{\partial}{\partial x}\right)^2\frac{e^{-ih\sqrt{r^2+(s-\xi)^2}}}{\sqrt{r^2+(z-\xi)^2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

These two cases, where $n=1$ and 2, have been computed in all details and the results are illustrated in Fig. I—IV. In these U is the maximum displacement at the radial distance R from the focus and L is the wave length of the bodily waves in the interior of the solid. Some general notions on these result may be gathered. In the case of a doublet having the horizontal axis, there are maximum displacements of the type of Rayleigh-waves at the azimuth angle $\omega=0$ and π while at $\omega=\pi/2$ and $3\pi/2$ there are no such displacements but the prevalence of the horizontal movement at right angle to the direction of the propagation, indicating the appearance of Love-waves. This azimuthal component of displacement quickly disappears as the distance from the epicentre increases, in spite of the persistence of the nature of the vertical and the horizontal component of displacement in the radial plane. The above facts[17] have already been pointed out by one of the authors in the investigation of Rayleigh-waves due to an internal doublet-source. When the axis of the doublet is inclined, the amplitude of the

17) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929), 12.

surface waves is modified. The chief feature of this case is such that there is superposition of the movements of the type due to the horizontal doublet and those of the ordinary annular Rayleigh-waves. It is to be remarked that the displacements at $\omega=0$ and at $\omega=\pi$ is the same in their magnitudes in spite of the inclined axis of the doublet. If the axis of the doublet is vertical, the movements on the surface are completely symmetrical with respect to the vertical line passing through the origin.

If we proceed to the case due to a quadruplet, more complicated evidence will be observed. In this case the motion at all points of the surface represents no more the type of the pure Love-waves. This is interpreted by the fact that Bessel's functions of zero order are involved in the solutions of u and w. We observe that even at a certain inclination θ of the axis of the quadruplet the amounts of amplitudes at $\omega=0$ and π are the same; it will, however, be understood that this is not unreasonable when the analysis is closely examined.

The foregoing remarks are given for the sake of simplicity on such simple cases as the doublet- or quadruplet-nuclei. Some particular tendencies of them, the authors think, will exist even in more higher multiplets. The fact that the azimuthal component of displacement decreases more quickly than others is valid in any case. It also appears that the azimuthal distribution of the amplitudes (excluding the azimuthal component of the amplitude) is maintained for all radii, so that the disturbances having the azimuthal variation cannot be diffused in their wave fronts progressing outwards. It must be remarked, in addition, that the deeper the origin of the disturbances, the more quickly decrease the amplitudes of the surface waves. The rate of the decreasing amplitudes is proportional to $e^{-\alpha_1\xi}$ and therefore a small increase of the depth of the focus obliges much decrease of the surface displacement, as seen in the appended figures.

The present paper deals merely with the formation of Rayleigh-waves due to dilatational waves from a multiplet-source in the interior of the semi-infinite elastic solid body. This is equivalent to the problem of the generation of the surface waves due to the body waves from a multiplet composed of pure compressional and dilatational sources. The problems concerning distortional waves from multiplets of the type similar to the present case and of the pure distortional type will be left for the future study.

In concluding this paper the authors are indebted to Prof. K. Suyehiro who has allowed them to proceed with this problem.

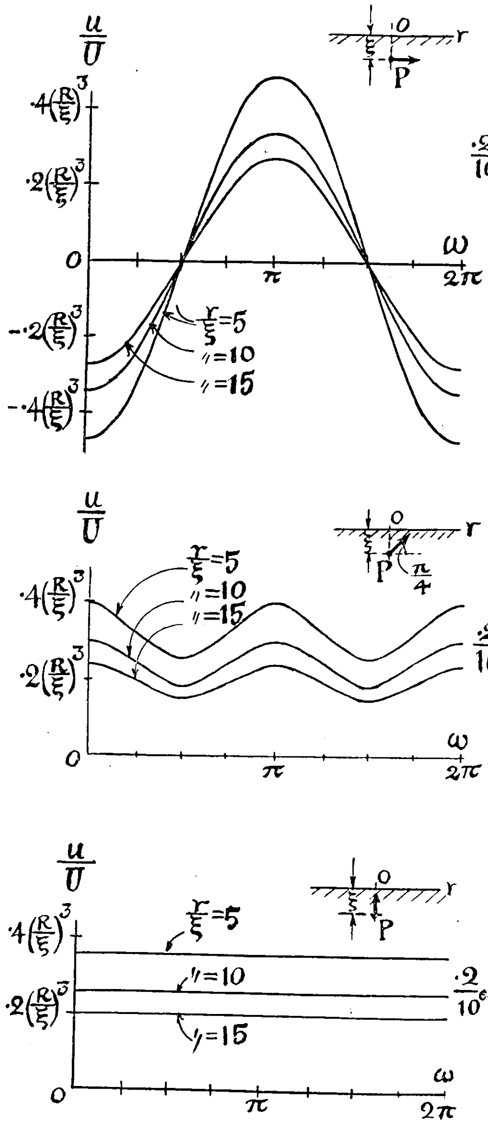


Fig. I$_\alpha$. $n=1$, $\frac{\xi}{L}=1$.

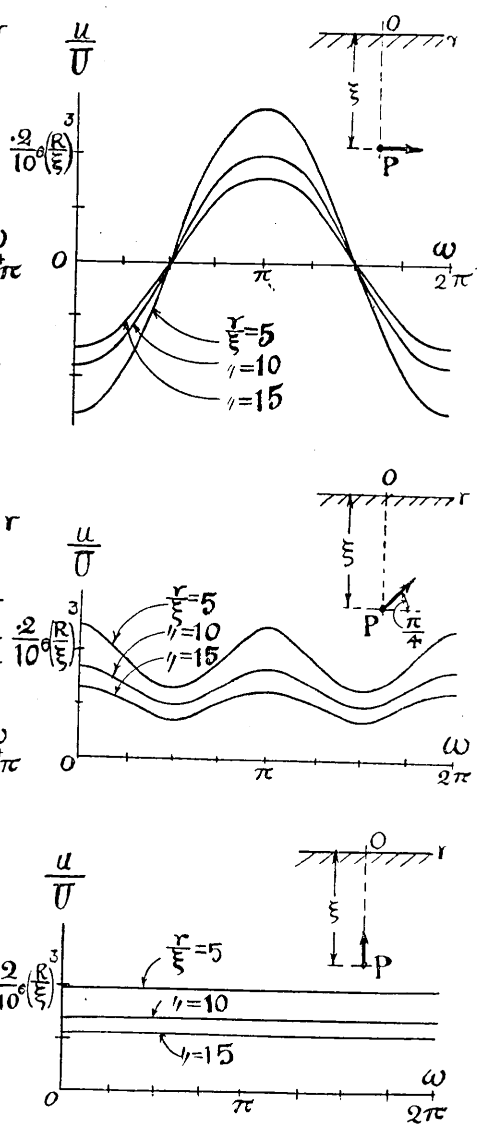


Fig. II$_\alpha$. $n=1$, $\frac{\xi}{L}=5$.

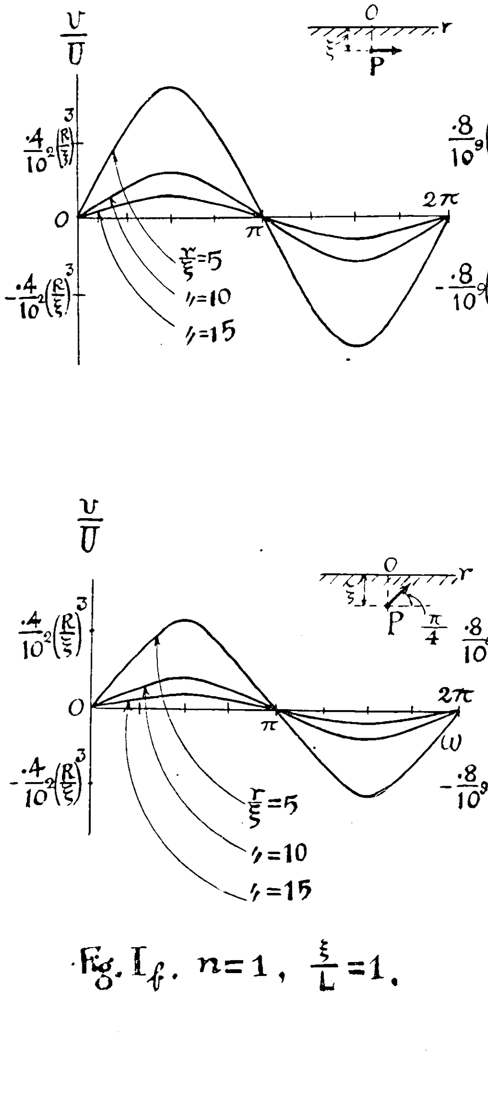


Fig. I_f. $n=1$, $\frac{\xi}{L}=1$.

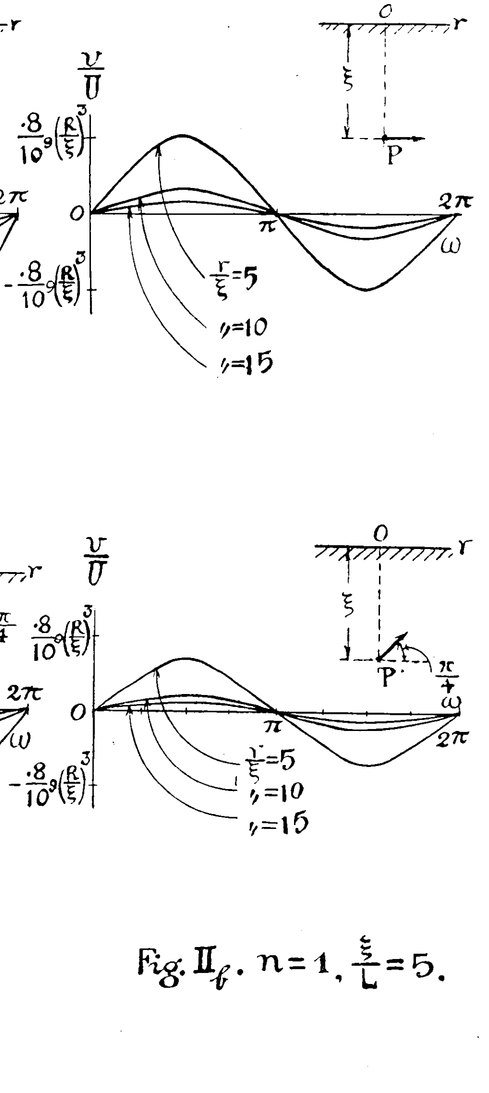


Fig. II_f. $n=1$, $\frac{\xi}{L}=5$.

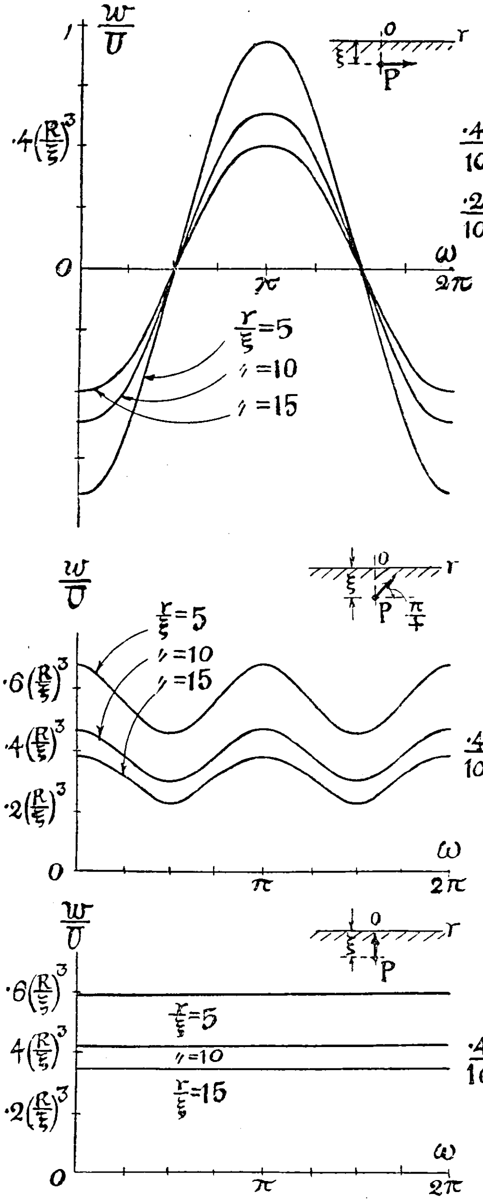


Fig. I_c. $n=1$, $\frac{\xi}{L}=1$.

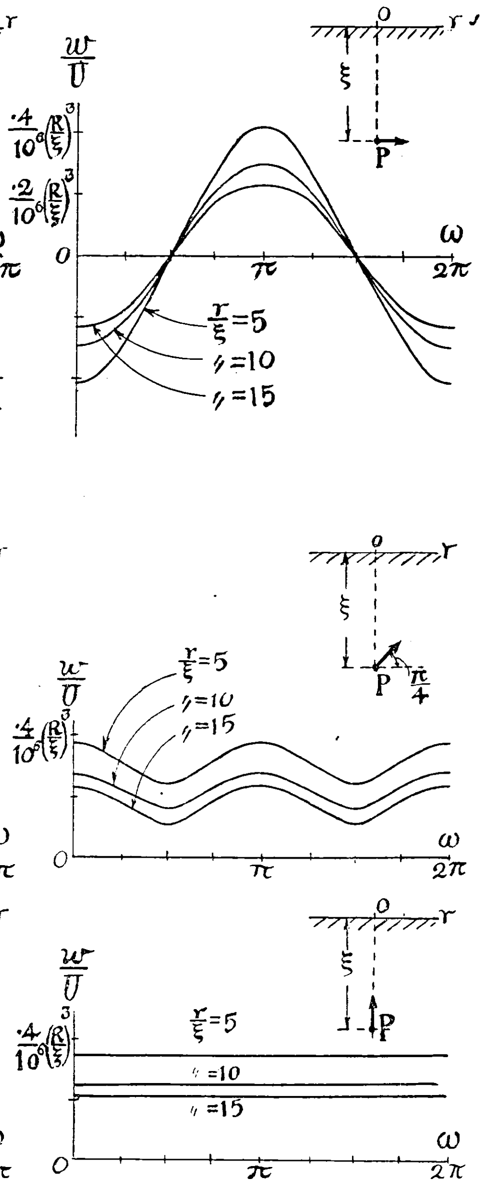


Fig. II_c. $n=1$, $\frac{\xi}{L}=5$.

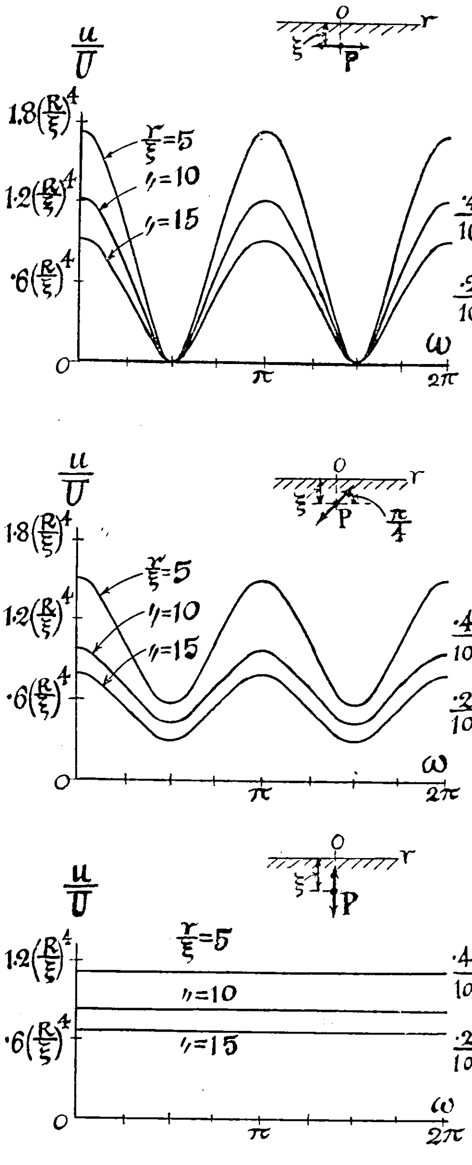


Fig. IIIa. $n=2$, $\frac{\xi}{L}=1$.

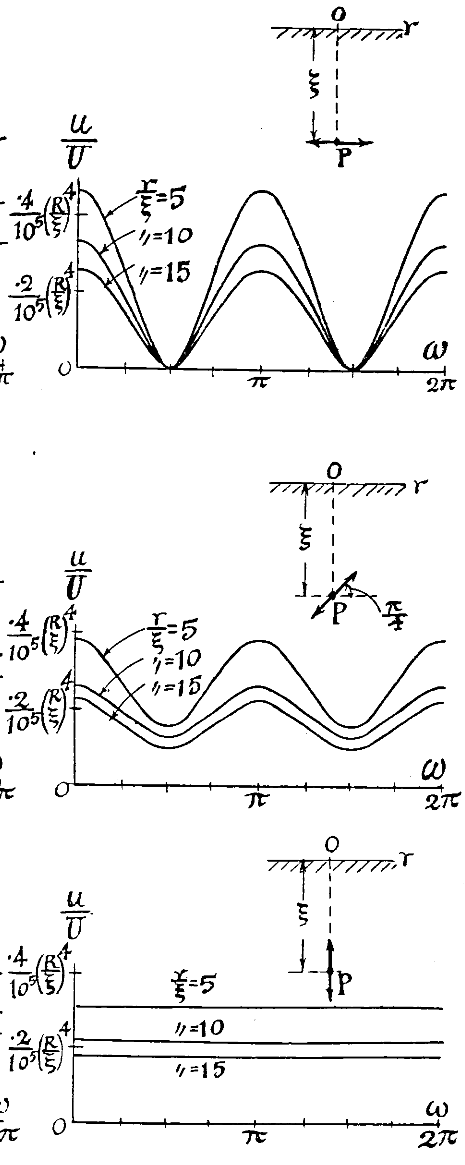


Fig. IVa. $n=2$, $\frac{\xi}{L}=5$.

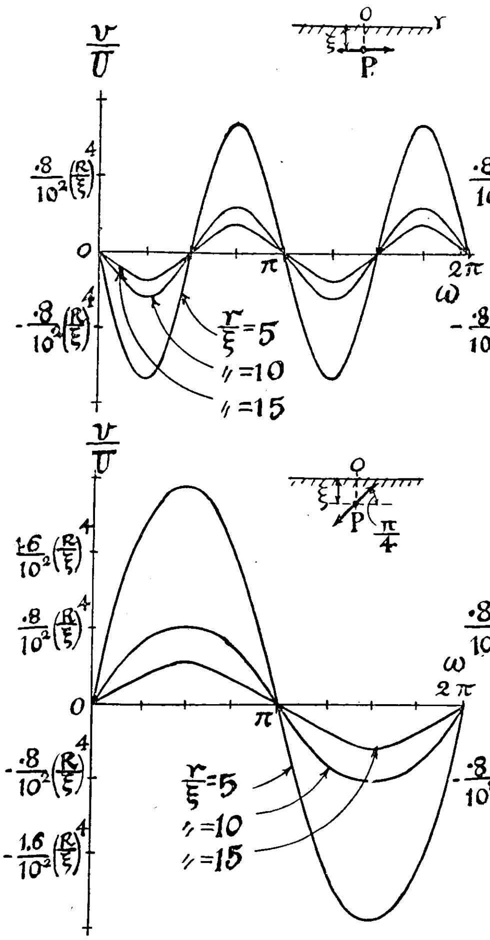


Fig. III$_b$. $n=2$, $\frac{\xi}{L}=1$.

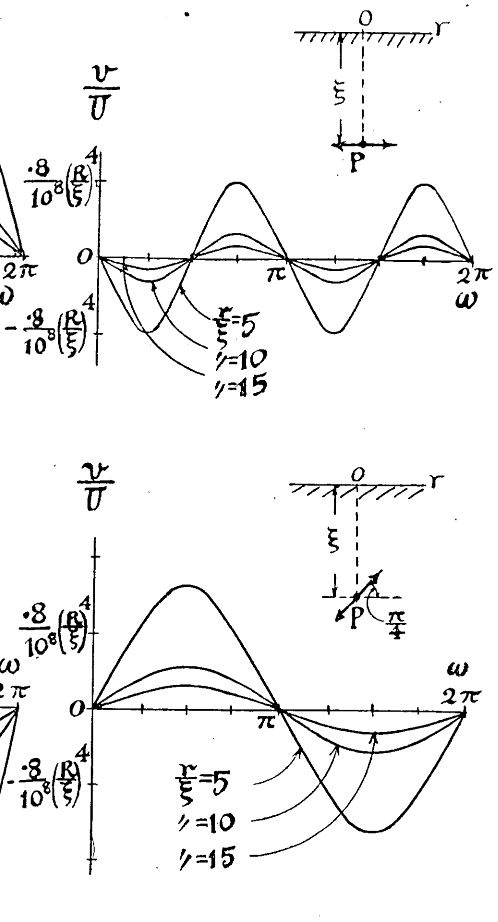


Fig. IV$_b$. $n=2$, $\frac{\xi}{L}=5$.

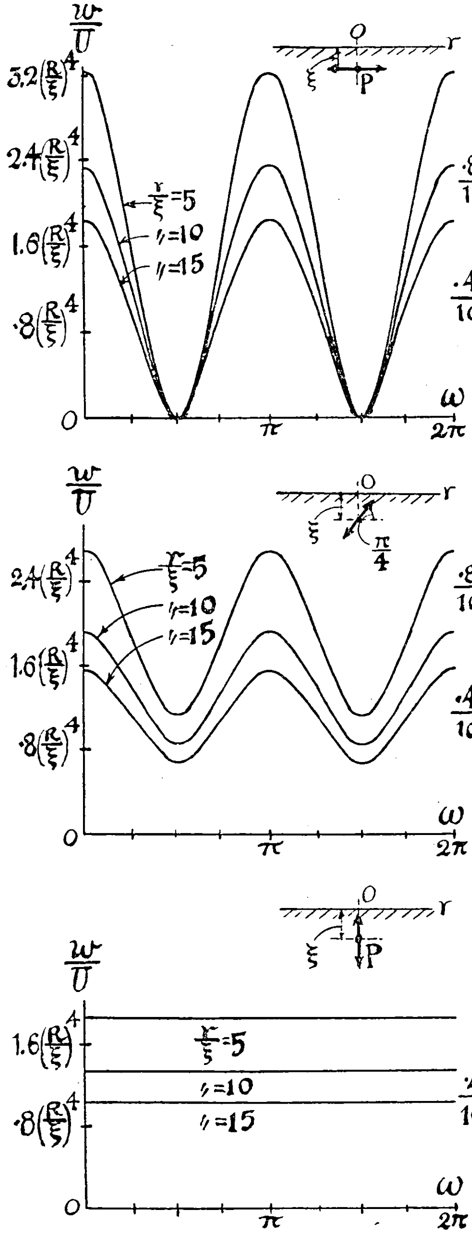


Fig. III$_c$. $n=2$, $\frac{\xi}{L}=1$.

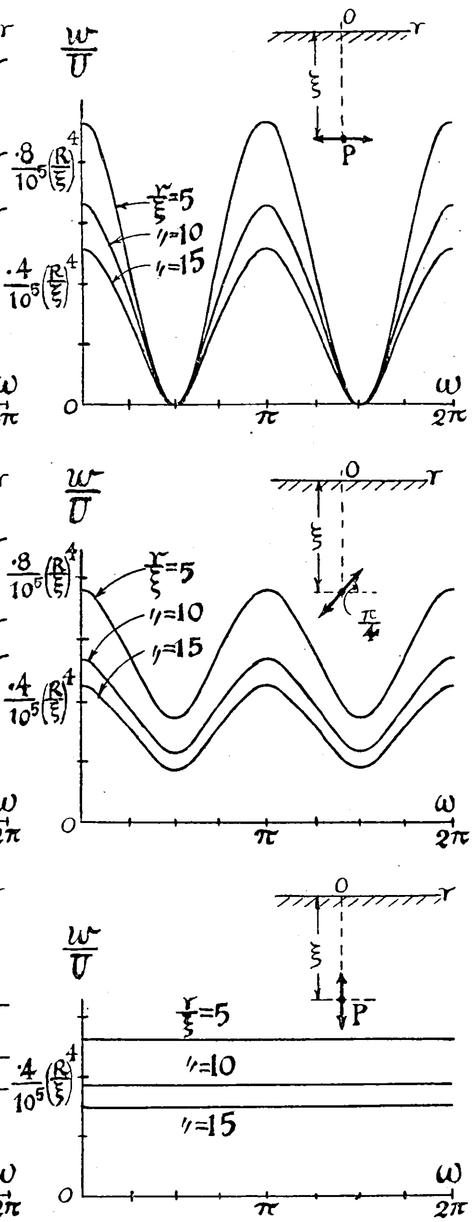


Fig. IV$_c$. $n=2$, $\frac{\xi}{L}=5$.

3. 内部にある多重原によるレーレー波の生成

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この前の機會に著者の一人が半無限固體の内部にある二重原から出る波動によつてレーレー波が誘發される問題を論じて見たが、この論文はその繼續とも見るべきものであつて、こゝには内部の原點が多重點である場合の計算が示してある。しかしこの論文の方が遙に複雜である事は論をまたぬ。而も尚これに直接關聯する問題が殘つて居るが、其は次の時期に殘して置いた。

計算の方法は勝手な n 次の波動原を考へ、之から出る波動を圓壔函數形に變形し、且つ適當な自由波を附加して遠方へ送られるレーレー波を算出した。且つ表面波でなく固體の内部を直接に傳はる縱横波の議論をも述べてある。

應用性のある結果を所々摘錄すれば

1. 前出の事ではあるが震央に近くラブ型波の大なる變位があるが震央から少し遠くなると次第に無くなる。
2. 普通のレーレー波に近い變位の部分は遠方まで送られる。
3. 震央の周圍にあつたレーレー波の振幅の方位的分布は其儘遠方へ送られる。
4. 震原の軸を含む垂直面が地表面を切る線上では震央の兩側に於て同振幅の表面波動が行はれると言ふ一見不思議な事柄がある。
5. 震原の深さが增すに從て深さの負價に比例する自然函數的にレーレー波の振幅の減ずる事は普通のレーレー波の場合に等しい。
6. 多重原の次數が增すに從ひ、誘發される表面波は多重原の次數以下に相當する種々の方位的分布を有するレーレー波の集まりから構成される。
7. 多重原の軸が垂直となれば表面波は單に零次のレーレー波のみとなる。