Formation of Deep-water Waves due to Subaqueous Shocks.

By **Katsutada SEZAWA,**

Member of the Institute.

水中衝撃による深海波の生成

所員　妹澤克惟

海底地震がある場合に如何なる海嘯が起るであらうかといふ問題は、其の海底の變動の如きものを推定する爲めに大切なわけであるが、其第一歩として深海中の適當な深さで衝撃が働く場合に表面に現はれる波動を算定してみた。勿論次の機會（彙報第七號）には淺海の底に衝撃が働く場合の研究を試みようと思ふがこの論文は寧ろ其準備である。

さて波動の概念は、水中の一點で働く調和振動が壓縮波として水に傳へられ、表面に達した部分が反射する時に表面に壓力の不平衡が起り、其を消す爲めに、其所に重力表面波が現はれ、これが水面上の遠距離へ送られるものとした。但し注意すべきは、この重力波は主として振動原より餘り遠からぬ上部の水面で發生する事と、始めの衝撃による波動は波長が相當に長い爲、多くの場合には其慣性を看過してもよい事とである。しかしこの論文では水深の大なる場合を考慮して慣性を其儘保存して置いた。

論文は六節に分れ、始めの三節では、二次元問題に於て原點が單原、水平に振動する複原、及び上下に振動する複原がある場合、他の三節はこれ等が三次元問題で行はれる場合とする。計算の重要な結果を拔萃すれば、

（１）　原點の變位が極めて僅かであるにも拘らず水面に起る波は可なり大なる變位を持つ。

（２）　表面に現はれる波動は主として普通の重力波であつて、原點の周期に相當した重力波の波長を持ち、且つ水點は圓形軌道を畫く。

（３）　表面波の方位的分布は大體原點の運動に應ずるが、三次元の場合に原點が水平に振動するものでは、遠方の横變位が消滅する。

The problem of the propagation of deep-water waves in the case in which the waves are caused by some disturbing shocks acting on an interior point of the water, has the important bearings on the *tsunamis* as well as on the general oceanography. Yet this problem has not been much studied for the reason that it involves mathematical difficulty and more probably has received very little attention from mathematical physicists in general. The problem itself, however, has been occasioned to be considered by a few students such as Prof. K. Terazawa,[1] H. Lamb,[2] K. Sano,[3] and R. Nomitsu,[4] though it is difficult to say that they have given the full satisfaction both in process and in result. The methods of analysis employed by Prof. K. Terazawa, H. Lamb and K. Sano seem to be some extensions of the problem of the ordinary water waves travelling horizontally from the start, while that appeared in Nomitsu's paper may be regarded as giving the flow of water from a source of stream. A conceivable problem arising in connection with the present investigation is on the wavy surface of water due to a submerged obstacle in running stream or conversely on the formation of water waves propagated with the same velocity as that of a submerged body moving uniformly. The case of the running stream attracted Lamb's attention,[5] and in a recent paper by T. H. Havelock,[6] the wave resistance of submerged moving body was considered from the same line of argument. Although it appears that our problem is related to these papers, they cannot be applied entirely to the present case. Because the nature of the original disturbance in this is quite different from those of Lamb and Havelock and accordingly the problem has been treated *ab initio* by a different method.

In the present analysis, an attempt is made to study the way in which the waves of the disturbing shocks acting on an interior point of the water are transmitted in all directions and in which these primary waves tend to excite the gravity waves travelling on the surface of the deep-water.

(1) Prof. K. Terazawa, "On Deep-sea Water Waves....," *Proc. Roy. Soc.*, **92** (1916).

(2) H. Lamb, *London Math. Soc. Proc.*, **21** (1922).

(3) Keizô Sano & Ken Hasegawa, "On the Waves produced by the Sudden Depression......of the Bottom of the Sea.....", *Proc. Tokyo Math.-Phys. Soc.*, Ser. II **8** (1915–16).

(4) 理學博士 野滿隆治「水中爆發の理論的研究」火兵學會誌、第十二卷（大正七年）。

(5) Prof. H. Lamb, "Waves due to Submerged Cylinder" *Ann. di math.*, **21**; also his *Hydrodynamics*, Chap. IX.

(6) Prof. T. H. Havelock, "Some Cases of Wave Motion due to a Submerged Obstacle." *Proc. Roy. Soc.*, **93** (1917).

The paper consists of two parts; the first part is on the propagation of shocks in two dimensions and the remaining part on that in three dimensions.

The method of analysis employed in this is to apply thoroughly the formulae which the eminent mathematician H. Lamb originally obtained by means of Green's theorem for the purpose of solving some problem of seismic waves. When the depth[1] of the point on which the disturbance acts is very small, the solution may otherwise be obtained approximately; when the depth is relatively large, it will be found that the transmission of shocks in the water wholly depends upon the waves of compression of the water and in effect the formulae quoted above play their important *roles*.

Part I. Two-dimensional Problems.

1. Among different types of motion at the origin special importance attaches to the simple harmonic type, by means of which the mathematical treatments are easily carried out. Generalising this type using Fourier's double integral theorem, complex cases such as the wave motion due to some arbitrary disturbances at the origin are obtained without difficulty. We shall take the free surface of deep water to be $y=0$, and shall suppose that the positive sense of the axis of y is directed towards the interior of the water.

Again, in two-dimensional problems where there is symmetry about a point, we must first find the solution of

$$\rho\frac{\partial^2\psi}{\partial t^2}=\kappa\left(\frac{\partial^2\psi}{\partial R^2}+\frac{1}{R}\frac{\partial\psi}{\partial R}\right), \quad\dots\dots\dots\dots\dots\dots (1)$$

where ψ is the dilatation, ρ is the active density of the water and κ is the volume elasticity of the water such that

$$\kappa=\rho_0\left(\frac{dp}{d\rho}\right)_{\rho=\rho_0}, \qquad \kappa\psi=p_0 \quad\dots\dots\dots\dots\dots\dots (23)$$

For the ordinary water at 15°C the value of κ is 2.046×10^{10} dyne/cm^2.

In the case of simple-harmonic waves, the time factor being e^{ipt}, the

(1) The case of shallow-water waves due to bottom-shocks, in which the effect of the inertia of water is neglected or is not neglected, will appear in Bull. 7.

equation (1) takes the form

$$\nabla^2\psi+h^2\psi=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

provided

$$h^2=\frac{\rho p^2}{\kappa} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The expression of the primary waves are, thus, expressed by

$$\psi=-ie^{ipt}H_0^{(2)}(hR), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where

$$R=\sqrt{x^2+y^2}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The radial displacement near the origin is given by

$$u_R=-\frac{1}{h^2}\frac{\partial\psi}{\partial R}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

so that the real form of displacement is approximately written by

$$u_R \doteqdot \frac{1}{h}\sqrt{\frac{2}{\pi hR}}\cos\left(pt-hR+\frac{\pi}{4}\right) \ldots\ldots\ldots\ldots\ldots (8)$$

Now we know the expression

$$H_0^{(2)}(hR)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha y}e^{\mp ifx}}{\alpha}df, \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

which is first discovered by H. Lamb[2] using Green's theorem and recently renewed by H. Nakano[3] in proving this expression by means of the contour integration.

In the expression (9), $\mp$ of $e^{\mp ifx}$ should be taken for x-positive and. α is taken such that

$$\alpha=\sqrt{f^2-h^2}, \quad \text{or} \quad i\sqrt{h^2-f^2} \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

according as $f^2 \gtrless h^2$, the radical being taken positively.

In the case of an internal source of disturbance, resident at the point $x=0$, $y=\xi$, we may conveniently consider an image at $x=0$ and $y=-\xi$ The dilatation is therefore expressed by

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(1) Exact solution at the origin will be seen in Lamb's paper, "The Early Stages of a Submarine Explosion," *Phil. Mag.*, **45** (1923).

(2) Prof. H. Lamb, "On the Propagation of Tremors over the Surface of an Elastic Solid", *Phil. Trans. Roy. Soc.*, **203** (1904).

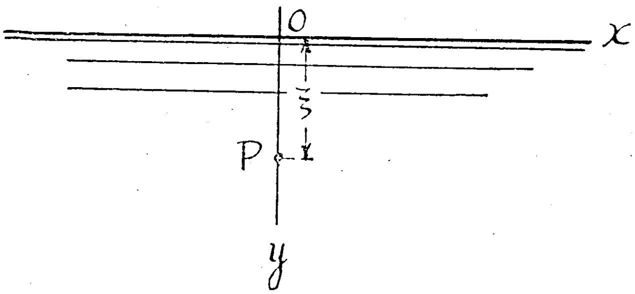
(3) Dr. H. Nakano, "On Rayleigh Waves", *Jap. Journ. Astr. & Geophys.* **2** (1925).

$$\psi=\frac{e^{ipt}}{\pi}\left[\int_{-\infty}^{\infty}\frac{e^{\alpha(y-\xi)}\,e^{\mp ifx}}{\alpha}df+\int_{-\infty}^{\infty}\frac{e^{-\alpha(y+\xi)}\,e^{\mp ifx}}{\alpha}df\right]$$

$$=\frac{2e^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha y}{\alpha}e^{-\alpha\xi}e^{\mp ifx}df, \qquad [y<\xi] \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

and similarly

$$=\frac{2e^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha\xi}{\alpha}e^{-\alpha y}e^{\mp ifx}df. \qquad [y>\xi] \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$



The expression of displacement is thus written by

$$\left.\begin{aligned} u_0&=-\frac{1}{h^2}\frac{\partial\psi}{\partial x}=\pm\frac{2ie^{ipt}}{\pi h^2}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha y}{\alpha}e^{-\alpha\xi}e^{\mp ifx}f\,df,\\ v_0&=-\frac{1}{h^2}\frac{\partial\psi}{\partial y}=-\frac{2e^{ipt}}{\pi h^2}\int_{-\infty}^{\infty}\operatorname{sh}\alpha y\, e^{-\alpha\xi}e^{\mp ifx}df,\end{aligned}\right\}[y<\xi] \quad \ldots\ldots (13)$$

where u_0, v_0 are components of displacement in x- and y-directions.

The pressure-components at the surface are expressed by

$$[p_{xy}]_{y=0}=0, \quad [p_0]_{y=0}=\kappa\psi=\frac{2\kappa e^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha\xi}\,e^{\mp ifx}}{\alpha}df \ldots\ldots\ldots (14)$$

Now, for gravity waves, the velocity potential ϕ will satisfy the equation of continuity such as

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

the solution of which may be written by

$$\phi=Ce^{-fy}e^{i(pt\mp fx)}, \quad \ldots\ldots\ldots\ldots\ldots (16)$$

the upper and the lower signs being taken for x-positive and -negative respectively.

As the displacement (u_1, v_1) is connected with ϕ in the near forms,

$$\frac{\partial u_1}{\partial t}=-\frac{\partial \phi}{\partial x}, \quad \frac{\partial v_1}{\partial t}=-\frac{\partial \phi}{\partial y}, \qquad (17)$$

we find

$$\left.\begin{aligned} u_1&=\pm\frac{f}{p}Ce^{-fy}e^{i(pt\mp fx)}, \\ v_1&=-\frac{if}{p}Ce^{-fy}e^{i(pt\mp fx)}. \end{aligned}\right\} \qquad (18)$$

The integral of the equation of motion of the fluid is expressed by

$$\frac{p_1}{\rho}=\frac{\partial \phi}{\partial t}-gv_1-F(t), \qquad (19)$$

where p_1 is the pressure and $F(t)$ is the integration constant which may be supposed merged in the value of $\frac{\partial \phi}{\partial t}$. At the surface, we have

$$\frac{p_1}{p}=\left(\frac{\partial \phi}{\partial t}\right)_{y=0}+gv_1, \qquad (20)$$

in which v_1 is the surface elevation.

The kinematical relation at the surface is given by

$$\frac{\partial v_1}{\partial t}=-\left(\frac{\partial \phi}{\partial y}\right)_{y=0}, \qquad (21)$$

with the condition that the normal to the free surface makes very small angle with the vertical.

Differentiating (20) with respect to t and subsituting from (21), we obtain

$$\frac{1}{\rho}\frac{\partial p_1}{\partial t}=(-p^2+fg)C[e^{-y}]_{y=0}e^{i(pt\mp fx)}, \qquad (22)$$

the integral of which is written by

$$[p_1]_{y=0}=i\rho C\frac{p^2-fg}{p}e^{i(pt\mp fx)}. \qquad (23)$$

The surface $y=0$ being free from traction, we have

$$[\text{the element of } p_0]_{y=0}+[p_1]_{y=0}=0, \qquad (24)$$

from which we find

$$C=-\frac{2\kappa}{\pi}\frac{e^{-\alpha\xi}p}{\alpha i\rho(p^2-fg)}. \qquad (25)$$

Thus u_1 and v_1 at the surface in the expressions of (18) can be written by

$$\left.\begin{aligned} u_1 &= \mp\frac{2\kappa f}{\pi}\,\frac{e^{-\alpha\xi}}{\alpha i\rho(p^2-fg)}\,e^{i(pt\mp fx)},\\ v_1 &= \frac{2\kappa f}{\pi}\,\frac{e^{-\alpha\xi}}{\alpha\rho\,(p^2-fg)}\,e^{i(pt\mp fx)}. \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (26)$$

Multiplying these by df, integrating from $-\infty$ to ∞ and afterwards adding to u_0 and v_0, we get

$$\left.\begin{aligned} u &= u_0+u_1 = \pm\frac{2ie^{ipt}}{\pi h^2\rho}\int_{-\infty}^{\infty}\frac{(\rho p^2-fg)+\kappa h^2}{p^2-fg}\,e^{-\alpha\xi\mp ifx}\,\frac{f}{\alpha}\,df,\\ v &= v_0+v_1 = \frac{2\kappa e^{ipt}}{\pi\rho}\int_{-\infty}^{\infty}\frac{e^{-\alpha\xi\mp ifx}}{p^2-fg}\,\frac{f}{\alpha}\,df. \end{aligned}\right\} \cdots\cdots (27)$$

These expressions may be written by

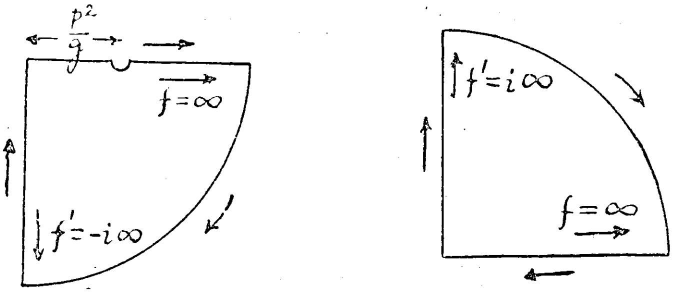
$$\left.\begin{aligned} u &= \mp\frac{2ie^{ipt}}{\pi h^2\rho}\Bigg[\int_0^{\infty}\frac{\rho(p^2-fg)+\kappa h^2}{p^2-fg}\,e^{-\alpha\xi\mp ifx}\,\frac{f}{\alpha}\,df\\ &\qquad -\int_0^{\infty}\frac{\rho(p^2+fg)+\kappa h^2}{p^2+fg}\,e^{-\alpha\xi\pm ifx}\,\frac{f}{\alpha}\,df\Bigg],\\ v &= \frac{2\kappa e^{ipt}}{\pi\rho}\Bigg[\int_0^{\infty}\frac{e^{-\alpha\xi\mp ifx}}{p^2-fg}\,\frac{f}{\alpha}\,df-\int_0^{\infty}\frac{e^{-\alpha\xi\pm ifx}}{p^2+fg}\,\frac{f}{\alpha}\,df\Bigg]. \end{aligned}\right\} \cdots\cdots\cdots (28)$$

These are the expressions of the wave motion on the surface of the water, when symmetrical shocks having the displacement expressed by (8) act on the origin ($x=0$, $y=\xi$).

To integrate the expressions in (28), we consider the integrals,

$$\left.\begin{aligned} &\int_c\frac{\rho(p^2-gZ)+\kappa h^2}{p^2-gZ}\,e^{-\xi\sqrt{Z^2-h^2}\mp iZx}\,\frac{Z}{\sqrt{Z^2-h^2}}\,dZ,\\ &\int_c\frac{e^{-\xi\sqrt{Z^2-h^2}\mp iZx}}{p^2-gZ}\,\frac{Z}{\sqrt{Z^2-h^2}}\,dZ,\\ &\int_c\frac{\rho(p^2+gZ)+\kappa h^2}{p^2+gZ}\,e^{-\xi\sqrt{Z^2-h^2}\pm iZx}\,\frac{Z}{\sqrt{Z^2-h^2}}\,dZ,\\ &\int_c\frac{e^{-\xi\sqrt{Z^2-h^2}\pm iZx}}{p^2+gZ}\,\frac{Z}{\sqrt{Z^2-h^2}}\,dZ, \end{aligned}\right\} \qquad \cdots\cdots\cdots (29)$$

taken round some closed contours. In the present case, the first two integrals in (29) should be taken round a contour represented by the left of the annexed drawings, while the remaining two should be taken round that of the right-hand side.



Equating each sum of the integrals taken round the contour to zero,(1) we find

$$\left.\begin{aligned} u&=\mp\frac{2k}{g\rho}e^{ipt}\left[\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2\mp\frac{ip^2}{g}}x}\right.\\ &\qquad\left.-\frac{2g}{\pi h^2\kappa}\int_0^\infty\frac{\rho(p^2+igf)+\kappa h^2}{(p^2+igf)}e^{-i\xi\sqrt{f^2+h^2}\mp fx}\frac{f}{\sqrt{f^2+h^2}}df\right],\\ v&=\frac{2i\kappa}{g\rho}e^{ipt}\left[\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2\mp\frac{ip^2}{g}}x}\right.\\ &\qquad\qquad\left.-\frac{2g}{i\pi}\int_0^\infty e^{-i\xi\sqrt{f^2+h^2}\mp fx}\frac{f}{\sqrt{f^2+h^2}}df\right]. \end{aligned}\right\}\quad\ldots\ldots(30)$$

in which the upper and the lower signs are taken according as $x\gtrless 0$.

In this the "principal values" due to Cauchy are taken for the integration of the form,

$$\int_0^\infty\frac{F(f)}{p^2-fg}df,\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

which is indeterminate in the ordinary sense of integration. In such a case the indeterminateness is avoided by inserting in the equations of motion small frictional terms proportional to the velocity and finally making the coefficients due to these terms vanish. In the present investigation this procedure has been omitted.

As the second terms on the right-hand sides of the expressions of u and

(1) Correction due to the branch point has been omitted in this deep-water problem; such effect will be considered in Bull. 7.

v in (30) tend to vanish as the distance on the surface from $x=0$ is increased in virtue of the factor $e^{\mp fx}$, we may disregard these second terms for large distances.

Thus we find the expressions of the horizontal and the vertical components of displacement of the surface of the water in the forms:

$$\left.\begin{aligned} u &\fallingdotseq \mp\frac{2\kappa}{g\rho}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\cos\left(pt\mp\frac{p^2}{g}x\right),\\ v &\fallingdotseq -\frac{2\kappa}{g\rho}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\sin\left(pt\mp\frac{p^2}{g}x\right), \end{aligned}\right\}[p^2>hg) \quad \ldots\ldots(32)$$

and

$$\left.\begin{aligned} u &\fallingdotseq \pm\frac{2\kappa}{g\rho}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\sin\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right),\\ v &\fallingdotseq \frac{2\kappa}{g\rho}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\cos\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right), \end{aligned}\right\}[p^2<hg] \quad \ldots\ldots(33)$$

corresponding to the displacement near the origin

$$u_R=\frac{1}{h}\sqrt{\frac{2}{\pi hR}}\cos\left(pt-hR+\frac{\pi}{4}\right). \quad [R^2=x^2+(y-\xi)^2] \ldots\ldots(34)$$

These give us the nature that the surface waves are propagated from the disturbed portion. The motion of the water particle on the surface is parfectly circular. The amplitude depends on the period and the depth of the focus. The figure in next page shews the ratio of the amplitude at x and that near the disturbed origin.

2. In two-dimensional motion where there is not symmetry about a point, but where the origin of the disturbance is a doublet oscillating horizontally, we must find the solution of the equation

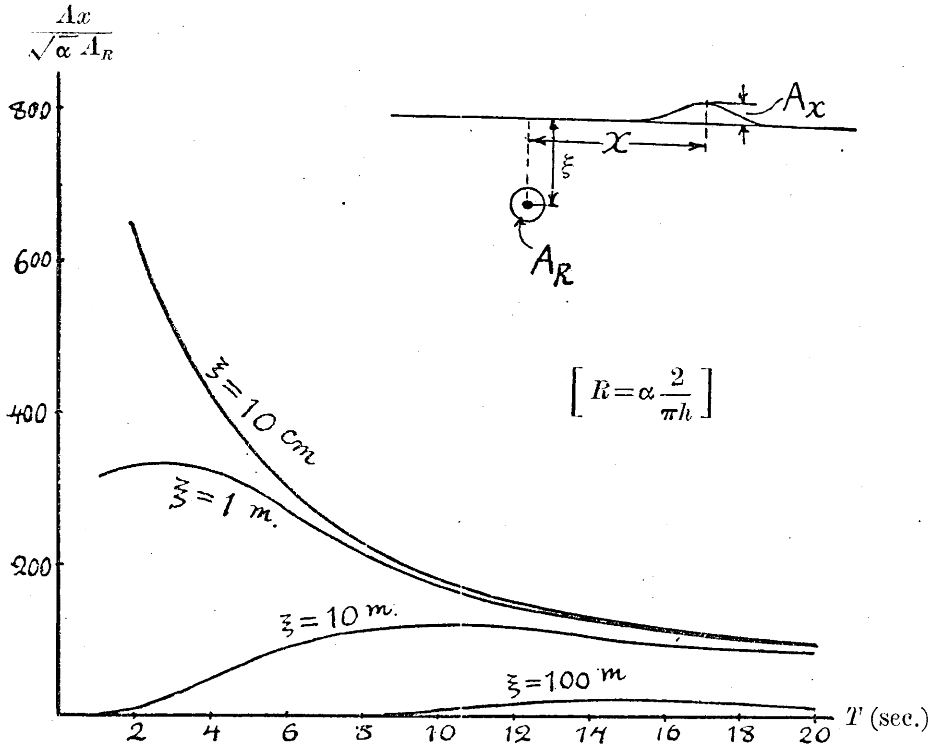
$$\rho\frac{\partial^2\psi}{\partial t^2}=\kappa\left(\frac{\partial^2\psi}{\partial R^2}+\frac{1}{R}\frac{\partial\psi}{\partial R}+\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}\right), \quad \ldots\ldots\ldots\ldots(35)$$

where θ is the agimuth at the origin of shocks.

In the case of a doublet oscillating horizontally, the primary waves may be written by

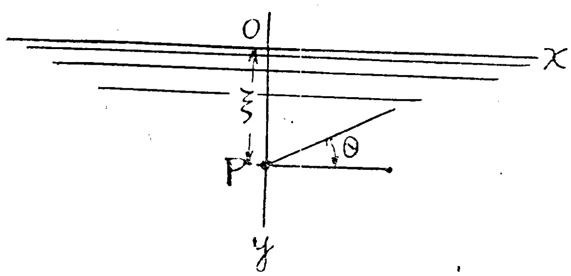
$$\psi = -ie^{ipt}\frac{\partial H_0^{(2)}(hR)}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

$$= ihe^{ipt}H_1^{(2)}(hR)\cos\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (36')$$



The radial displacement in real form is approximately writted by

$$u_R = -\sqrt{\frac{2}{\pi hR}}\cos\left(pt - hR + \frac{3\pi}{4}\right)\cos\theta. \quad \ldots\ldots\ldots\ldots (37)$$



Modifying the relation written in (9), we find

$$ihH_1^{(2)}(hR)\cos\theta = \mp\int_{-\infty}^{\infty}\frac{e^{-\alpha y \mp ifx}}{\alpha}fdf. \qquad\qquad (38)$$

Superposing the effect of the image for the source ($x=0$, $y=\xi$), we obtain

$$\left.\begin{aligned}\psi &= \mp\frac{2ie^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha y}{\alpha}e^{-\alpha\xi\mp ifx}fdf. & [y<\xi]\\ \psi &= \mp\frac{2ie^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha\xi}{\alpha}e^{-\alpha y\mp ifx}fdf. & [y>\xi]\end{aligned}\right\} \qquad (39)$$

The expressions of displacement are

$$\left.\begin{aligned}u_0 &= -\frac{1}{h^2}\frac{\partial\psi}{\partial x} = \frac{2e^{ipt}}{\pi h^2}\int_{-\infty}^{\infty}\frac{\operatorname{ch}\alpha y}{\alpha}e^{-\alpha\xi\mp ifx}f^2df,\\ v_0 &= -\frac{1}{h^2}\frac{\partial\psi}{\partial y} = \pm\frac{2ie^{ipt}}{\pi h^2}\int_{-\infty}^{\infty}\operatorname{sh}\alpha y e^{-\alpha\xi\mp ifx}fdf.\end{aligned}\right\}[y<\xi]. \qquad (40)$$

The stress-components are given by

$$[p_{xy}]_{y=0}=0. \quad [p_0]_{y=0}=\kappa\psi = \mp\frac{2i\kappa e^{ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha\xi\mp ifx}}{\alpha}fdf, \qquad (41)$$

the upper and the lower signs being taken for x-positive and -negative respectively.

Now, for gravity waves, a similar treatment as in section 1 being carried out, the results of the calculation of the pressure on the surface can be written by

$$[p_1]_{y=0} = i\rho C\frac{p^2-fg}{p}e^{i(pt\mp fx)}. \qquad\qquad (42)$$

Putting (41) and (42) in the surface conditions, we find

$$C = \pm\frac{2\kappa}{\pi}\frac{e^{-\alpha\xi}pf}{\alpha\rho(p^2-fg)}. \qquad\qquad (43)$$

The displacement due to the gravity waves is therefore expressed by

$$\left.\begin{aligned}u_1 &= \frac{2\kappa}{\pi}\frac{e^{-\alpha\xi}f^2}{\alpha\rho(p^2-fg)}e^{i(pt\mp fx)},\\ v_1 &= \mp\frac{2i\kappa}{\pi}\frac{e^{-\alpha\xi}f^2}{\alpha\rho(p^2-fg)}e^{i(pt\mp fx)}.\end{aligned}\right\} \qquad (44)$$

Multiplying these by df, integrating from $-\infty$ to ∞ and afterwards adding to u_0 and v_0, we get

$$\left.\begin{aligned} u&=\frac{2e^{ipt}}{\pi h^2\rho}\int_{-\infty}^{\infty}\frac{\rho(p^2-fg)+\kappa h^2}{p^2-fg}e^{-\alpha\xi\mp ifx}\frac{f^2}{\alpha}df,\\ v&=\mp\frac{2i\kappa e^{ipt}}{\pi\rho}\int_{-\infty}^{\infty}\frac{e^{-\alpha\xi\mp ifx}}{p^2-fg}\frac{f^2}{\alpha}df. \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(45)$$

Following the same process of integration as in the former case, we find

$$\left.\begin{aligned} u&=\frac{2i\kappa}{g\rho}e^{ipt}\left[\frac{p^2}{g}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}\mp i\frac{p^2}{g}x}\right.\\ &\qquad\left.-\frac{2g}{\pi h^2\kappa}\int_0^{\infty}\frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}e^{-i\xi\sqrt{f^2+h^2}\mp ifx}\frac{f^2}{\sqrt{f^2+h^2}}df\right].\\ v&=\pm\frac{2\kappa}{g\rho}e^{ipt}\left[\frac{p^2}{g}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}\mp i\frac{p^2}{g}x}\right.\\ &\qquad\left.-\frac{2g}{i\pi}\int_0^{\infty}\frac{e^{-i\xi\sqrt{f^2+h^2}+ifx}}{p^2+igf}\frac{f^2}{\sqrt{f^2+h^2}}df\right]. \end{aligned}\right\}\quad\ldots.(46)$$

in which the upper and the lower signs are taken according as $x\gtrless 0$. The principal values have been taken as in the former case. As the second terms in (46) become negligibly small as x tends to $\pm\infty$, we obtain at such a point,

$$\left.\begin{aligned} u&\doteqdot-\frac{2\kappa}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\sin\left(pt\mp\frac{p^2}{g}x\right),\\ v&=\pm\frac{2\kappa}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\cos\left(pt\mp\frac{p^2}{g}x\right), \end{aligned}\right\}[p^2>hg]\quad\ldots.(47)$$

and

$$\left.\begin{aligned} u&\doteqdot\frac{2\kappa}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\cos\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right),\\ v&=\doteqdot\frac{2\kappa}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\sin\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right), \end{aligned}\right\}[p^2<hg]\quad\ldots(48)$$

corresponding to the displacement near the origin,

$$u_R = -\sqrt{\frac{2}{\pi h R}} \cos\left(pt - hR + \frac{3\pi}{4}\right) \cos\theta. \quad [R^2 = x^2 + (y-\xi)^2]. \dots (49)$$

These give us the nature that the surface waves are generated from the disturbed portion. The motion of the water particle in this case, too, is on a circular orbit. The amplitude at a large distance is p^2/gh times that of the former case, when the horizontal displacements at the origin are taken the same for both cases.

3. In two-dimensional motion where the disturbing origin is a doublet oscillating vertically, we find the solution of (35) for the primary waves in the following form:

$$\psi = -ie^{ipt} \frac{\partial H_0^{(2)}(hR)}{\partial y} \quad \dots\dots\dots\dots\dots\dots\dots (50)$$

$$= ihe^{ipt} H_0^{(2)}(hR) \sin\theta \quad \dots\dots\dots\dots\dots\dots (50')$$

The radical displacement in real form is approximately written by

$$u_R = -\sqrt{\frac{2}{\pi h R}} \cos\left(pt - hR + \frac{3\pi}{4}\right) \text{ in } \theta. \quad \dots\dots\dots\dots (51)$$

Considering the effect of the image for the source ($x=0,\ y=\xi$), we find

$$\psi = -\frac{2}{\pi} e^{ipt} \int_{-\infty}^{\infty} \operatorname{ch} \alpha y\, e^{-\alpha\xi \mp ifx} df, \quad [y < \xi] \quad \dots\dots\dots\dots (52)$$

so that the expressions of the displacement are given by

$$\left.\begin{aligned} u_0 &= \mp \frac{2i}{\pi h^2} e^{ipt} \int_{-\infty}^{\infty} \operatorname{ch} \alpha y\, e^{-\alpha\xi \mp ifx} f df, \\ v_0 &= \frac{2}{\pi h^2} e^{ipt} \int_{-\infty}^{\infty} \alpha \operatorname{sh} \alpha y\, e^{-\alpha\xi \mp ifx} df. \end{aligned}\right\} [y > \xi] \dots\dots\dots\dots (53)$$

The stress-components are

$$[p_{xy}]_{y=0} = 0, \quad [p_0]_{y=0} = -\frac{2\kappa}{\pi} \int_{-\infty}^{\infty} e^{-\alpha\xi \mp ifx} df. \dots\dots\dots\dots (54)$$

Again, pursuing the same line of argument for the gravity waves as before, we find the value of the constant in (16) in the form:

$$C = \frac{2\kappa}{\pi} e^{-\alpha\xi} \frac{p}{i\rho(p^2 - fg)} \dots\dots\dots\dots\dots\dots\dots\dots (55)$$

The surface displacement due to gravity waves is, thus, expressed by

$$\left.\begin{aligned} u_1&=\pm\frac{2\kappa}{\pi}\frac{e^{-\alpha\xi}f}{i\rho(p^2-fg)}e^{i(pt\mp fx)},\\ v_1&=-\frac{2\kappa}{\pi}\frac{e^{-\alpha\xi}f}{\rho(p^2-fg)}e^{i(pt\mp fx)}. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(56)$$

The resultant displacement, when the contour integrations are performed, is expressed by

$$\left.\begin{aligned} u&=\mp\frac{2\kappa}{g\rho}e^{ipt}\Bigg[\frac{p^2}{g}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}\mp i\frac{p^2}{g}x}\\ &\qquad-\frac{2g}{\pi h^2\kappa}\int_0^\infty\frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}e^{-i\xi\sqrt{f^2+h^2}\mp fx}fdf\Bigg],\\ v&=-\frac{2i\kappa}{g\rho}e^{ipt}\Bigg[\frac{p^2}{g}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}\mp i\frac{p^2}{g}x}\\ &\qquad-\frac{2g}{i\pi}\int_0^\infty e^{-i\xi\sqrt{f^2+h^2}\mp fx}fdf\Bigg]. \end{aligned}\right\}\ldots\ldots\ldots(57)$$

Neglecting the second terms for a large distance, we find

$$\left.\begin{aligned} u&\fallingdotseq\pm\frac{2\kappa}{g\rho}\frac{p^2}{g}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\cos\left(pt\mp\frac{p^2}{g}x\right),\\ v&\fallingdotseq\frac{2\kappa}{g\rho}\frac{p^2}{g}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\sin\left(pt\mp\frac{p^2}{g}x\right), \end{aligned}\right\}[p^2>hg] \quad\ldots\ldots(58)$$

and

$$\left.\begin{aligned} u&\fallingdotseq\pm\frac{2\kappa}{g\rho}\frac{p^2}{g}\cos\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{\frac{p^4}{g^2}-h^2}\right),\\ v&\fallingdotseq\frac{2\kappa}{g\rho}\frac{p^2}{g}\sin\left(pt\mp\frac{p^2}{g}x-\xi\sqrt{\frac{p^4}{g^2}-h^2}\right), \end{aligned}\right\}[p^2<hg] \quad\ldots\ldots(59)$$

corresponding to the displacement in the neighbourhood of the origin,

$$u_R=-\sqrt{\frac{2}{\pi hR}}\cos\left(pt-hR+\frac{3\pi}{4}\right)\sin\theta. \quad [R^2=x^2+(y-\xi)^2]\ldots\ldots(60)$$

We may conclude that the amplitude of waves at a large distance, when the origin is a doublet oscillating vertically, is $\sqrt{1-\left(\frac{hg}{p^2}\right)^2}$ times that of former case where the doublet prosecutes the horizontal oscillations. In these both cases the absolute amplitudes at the origins are taken the same.

Part II. Three-dimensional Problems.

4. We shall take the free surface of the water to be $z=0$, and shall

suppose that the positive sense of the axis of z is directed towards the interior of the water.

Again, in three-dimensional problems where there is symmetry about a point, we must first find the solution of

$$\rho\frac{\partial^2\psi}{\partial t^2}=\kappa\left(\frac{\partial^2\psi}{\partial R^2}+\frac{2}{R}\frac{\partial\psi}{\partial R}\right). \qquad (61)$$

When the motion at the origin is of a simple-harmonic type, the equation (61) takes the form,

$$\frac{\partial^2\psi}{\partial R^2}+\frac{2}{R}\frac{\partial\psi}{\partial R}+h^2\psi=0, \qquad (62)$$

provided

$$h^2=\frac{\rho p^2}{\kappa}. \qquad (63)$$

The expression of the primary waves are, thus, given by

$$\psi=e^{ipt}\frac{e^{-ihR}}{R}, \qquad (64)$$

where $\quad R=\sqrt{x^2+y^2+z^2}$.

The radial displacement near the origin is written by

$$u_R=-\frac{1}{h^2}\frac{\partial\psi}{\partial R}, \qquad (65)$$

so that the real form of displacement is approximately expressed by

$$u_R=-\frac{1}{hR}\sin(pt-hR). \qquad (66)$$

Now we know the expression

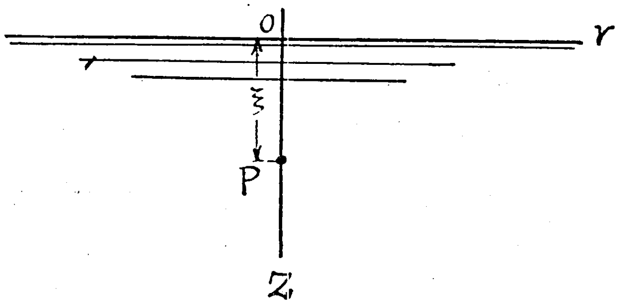
$$\frac{e^{-ihR}}{R}=\int_0^\infty\frac{e^{-\alpha z}}{\alpha}J_0(fr)f\,df, \qquad (67)$$

which is due to H. Lamb.(1) In this $R\doteq\sqrt{r^2+z^2}$ and α has the same meaning as in (10), i.e.

$$\alpha=\sqrt{f^2-h^2}, \quad \text{or} \quad i\sqrt{h^2-f^2}, \qquad (68)$$

according as $f^2\gtrless h^2$, the radical being taken positively.

(1) Prof. H. Lamb, *loc. cit.* p. 22; also *Math. Tripos Paper* (1905).



In the case of an internal source of disturbance, resident at the point $r=0$ and $z=\xi$, the dilatation is expressed by

$$\psi=e^{ipt}\left[\int_0^\infty \frac{e^{\alpha(z-\xi)}}{\alpha}J_0(fr)fdf+\int_0^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha}J_0(fr)fdf\right]$$

$$=e2^{ipt}\int_0^\infty \frac{\mathrm{ch}\,\alpha z}{\alpha}e^{-\alpha\xi}J_0(fr)fdf, \qquad [z<\xi] \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

and similarly

$$\psi=2e^{ipt}\int_0^\infty \frac{\mathrm{ch}\,\alpha\xi}{\alpha}e^{-\alpha z}J_0(fr)fdf. \qquad [z>\xi] \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

The expressions of displacement are given by

$$\left.\begin{aligned} u_0&=-\frac{1}{h^2}\frac{\partial\psi}{\partial R}=\frac{2e^{ipt}}{h^2}\int_0^\infty \frac{\mathrm{ch}\,\alpha z}{\alpha}e^{-\alpha\xi}J_1(fr)f^2df,\\ v_0&=-\frac{1}{h^2r}\frac{\partial\psi}{\partial\omega}=0,\\ w_0&=-\frac{1}{h^2}\frac{\partial\psi}{\partial z}=-\frac{2e^{ipt}}{h^2}\int_0^\infty \mathrm{sh}\,\alpha z e^{-\alpha\xi}J_0(fr)fdf, \end{aligned}\right\} \ldots\ldots\ldots\ldots (70)$$

where u_0, v_0, w_0 are radial, azimuthal and vertical components of displacement respectively.

The stress-components at the surface are expressed by

$$[p_{zr}]_{z=0}=0, \qquad [p_{zw}]_{z=0}=0,$$

$$[p_0]_{z=0}=\kappa\psi=2\kappa e^{ipt}\int_0^\infty \frac{e^{-\alpha\xi}}{\alpha}J_0(fr)fdf. \ldots\ldots\ldots\ldots\ldots\ldots (71)$$

Now, for gravity waves, the velocity potential ϕ will satisfy the equation of continuity such that

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0, \qquad (72)$$

the solution of which may be written by

$$\phi=Ce^{-fz}e^{ipt}J_0(fr). \qquad (73)$$

As the displacement (u_1, v_1, w_1) is connected with ϕ in the near forms,

$$\frac{\partial u_1}{\partial t}=-\frac{\partial\phi}{\partial r},\quad \frac{\partial v_1}{\partial t}=-\frac{\partial\phi}{\partial\omega},\quad \frac{\partial w_1}{\partial t}=-\frac{\partial\phi}{\partial z}, \qquad (74)$$

we find

$$\left.\begin{aligned} u_1&=-\frac{if}{p}Ce^{-fz+ipt}J_1(fr),\\ v_1&=0,\\ w_1&=-\frac{if}{p}Ce^{-fz+ipt}J_0(fr). \end{aligned}\right\} \qquad (75)$$

The integral of the equations of motion of the fluid is expressed by

$$\frac{p_1}{\rho}=\frac{\partial\phi}{\partial t}-gz-F(t), \qquad (76)$$

where p_1 is the pressure and $F(t)$ is an integration constant which can be merged in the value of $\frac{\partial\phi}{\partial t}$. At the surface of the water we have

$$\frac{p_1}{\rho}=\left(\frac{\partial\phi}{\partial t}\right)_{z=0}+gw_1, \qquad (77)$$

in which w_1 is the surface elevation.

The kinematical relation at the surface is given by

$$\frac{\partial w_1}{\partial t}=-\left(\frac{\partial\phi}{\partial z}\right)_{z=0}, \qquad (78)$$

with the condition that the normal to the free surface makes very small angle with the vertical.

Differentiating (77) with respect t and substituting from (78), we obtain

$$\frac{1}{\rho}\frac{\partial p_1}{\partial t}=(-p^2+fg)C[e^{-fz}]_{z=0}e^{ipt}J_0(fr), \qquad (79)$$

the integral of which is written by

$$[p_1]_{z=0}=i\rho C\frac{p^2-fg}{p}e^{ipt}J_0(fr) \qquad (80)$$

The surface $z=0$ being free from traction, the equation,

$$(\text{element cf } p_0)+p_1=0 \qquad (81)$$

must hold on that surface. Thus we find

$$C=-2\kappa\frac{e^{-\alpha\xi}}{\alpha}\frac{pf}{i\rho(p^2-fg)}. \qquad (82)$$

The displacement given in (75) becomes

$$\left.\begin{aligned} u_1&=\frac{2\kappa f^2 e^{-\alpha\xi}}{\alpha\rho(p^2-fg)}e^{ipt}J_1(fr),\\ v_1&=0,\\ w_1&=\frac{2\kappa f^2 e^{-\alpha\xi}}{\alpha\rho(p^2-fg)}e^{ipt}J_0(fr). \end{aligned}\right\} \qquad (83)$$

Multiplying these by df, integrating from 0 to ∞ and afterwards adding to u_0 and w_0, we get

$$\left.\begin{aligned} u&=u_0+u_1=\frac{2e^{ipt}}{h^2\rho}\int_0^\infty\frac{\rho(p^2-fg)+\kappa h^2}{p^2-fg}e^{-\alpha\xi}J_1(fr)\frac{f^2}{\alpha}df,\\ w&=w_0+w_1=\frac{2\kappa e^{ipt}}{\rho}\int_0^\infty\frac{e^{-\alpha\xi}}{p^2-fg}J_0(fr)\frac{f^2}{\alpha}df. \end{aligned}\right\} \qquad (84)$$

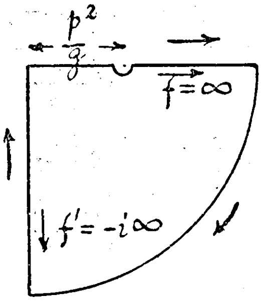
These give us the oscillatory motion of a standing type at infinite distances, so that we may be justified if, following H. Lamb, we superpose on these some similar formulae of vibratory motion, whose phase only differs a quarter period from that of the expression in (84). This procedure, when r is increased, is not unreasonable. Thus we take the following formulae, in which a system of standing vibrations is superposed:

$$\left.\begin{aligned} u&=\frac{2e^{ipt}}{h^2\rho}\int_0^\infty\frac{\rho(p^2-fg)+\kappa h^2}{p^2-fg}e^{-\alpha\xi}H_1^{(2)}(fr)\frac{f^2}{\alpha}df,\\ w&=\frac{2\kappa e^{ipt}}{\rho}\int_0^\infty\frac{e^{-\alpha\xi}}{p^2-fg}H_0^{(2)}(fr)\frac{f^2}{\alpha}df. \end{aligned}\right\} \qquad (85)$$

To integrate the expressions in (85), we consider the integrals,

$$\left.\begin{aligned} &\int_c\frac{\rho(p^2-gZ)+\kappa h^2}{p^2-gZ}e^{-\xi\sqrt{Z^2-h^2}}H_1^{(2)}(rZ)\frac{Z^2}{\sqrt{Z^2-h^2}}dZ,\\ &\int_c\frac{e^{-\xi\sqrt{Z^2-h^2}}}{p^2-gZ}H_0^{(2)}(rZ)\frac{Z^2}{\sqrt{Z^2-h^2}}dZ, \end{aligned}\right\} \qquad (86)$$

taken round some closed contours. In the present case the integration is to be taken round the contour below.



Equating each sum of the integrals taken round the contour to zero,[1] we find

$$\left.\begin{aligned}
u &= \frac{2i\kappa e^{ipt}}{g\rho}\left[\frac{p^2}{g}\frac{\pi}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}\,e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\,H_1^{(2)}\left(\frac{p^2}{g}r\right)\right.\\
&\qquad\left.+\frac{g}{ih^2\kappa}\int_0^\infty\frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}\,e^{-i\xi\sqrt{f^2+h^2}}\,H_1^{(2)}(-ifr)\,\frac{f^2}{\sqrt{f^2+h^2}}\,df\right],\\
w &= \frac{2i\kappa e^{ipt}}{g\rho}\left[\frac{p^2}{g}\frac{\pi}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}\,e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\,H_0^{(2)}\left(\frac{p^2}{g}r\right)\right.\\
&\qquad\left.+\frac{g}{i}\int_0^\infty\frac{e^{i\xi\sqrt{f^2+h^2}}}{p^2+igf}\,H_0^{(2)}(-ifr)\,\frac{f^2}{\sqrt{f^2+h^2}}\,df.\right.
\end{aligned}\right\}\quad(87)$$

In this the "principal values" has been taken in the process of integrations. As the second terms on the right-hand sides of the expressions of u and w in (87) become negligible, as r is increased, in virtue of $H_0^{(2)}(-ifr)$ and $H_1^{(2)}(-ifr)$, we may disregard these second terms.

Thus we find

$$\left.\begin{aligned}
u &\fallingdotseq -\frac{2\kappa\pi}{g\rho}\,\frac{p^2}{g}\,\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}\,e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\,\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}},\\
w &\fallingdotseq \frac{2\kappa\pi}{g\rho}\,\frac{p^2}{g}\,\frac{1}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}\,e^{-\xi\sqrt{\frac{p^2}{g^2}-h^2}}\,\frac{\cos\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}},
\end{aligned}\right\}[p^2>hg]\quad(88)$$

(1) Correction due to branch point has been omitted; such effect will be considered in Bull. 7.

and

$$\left.\begin{aligned} u &\doteqdot \frac{2\kappa\pi}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\frac{\cos\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}},\\ w &\doteqdot \frac{2\kappa\pi}{g\rho}\frac{p^2}{g}\frac{1}{\sqrt{\left(\frac{hg}{p^2}\right)^2-1}}\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}-\xi\sqrt{h^2-\frac{p^4}{g^2}}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}, \end{aligned}\right\}[p^2<hg] \quad (89)$$

corresponding to the displacement in the vicinity of the origin as written by

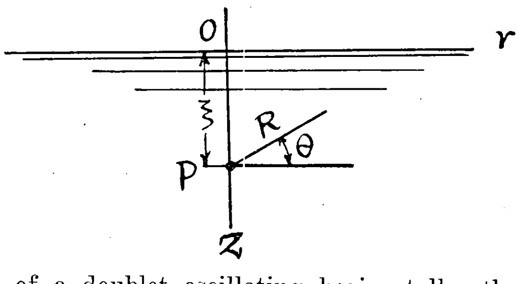
$$u_R \doteqdot -\frac{1}{hR}\sin(pt-hR), \quad [R^2=r^2+(z-\xi)^2]. \quad \ldots\ldots\ldots\ldots (90)$$

The motion of the water particle is perfectly circular. The amplitude depends on the period of oscillations at the origin and the depth of that origin beneath the water level.

5. In three-dimensional motion where there is not symmetry about a point but where the disturbing origin is a doublet oscillating horizontally, we must find the solution of

$$\rho\frac{\partial^2\psi}{\partial t^2}=\kappa\left(\frac{\partial^2\psi}{\partial R^2}+\frac{2}{R}\frac{\partial\psi}{\partial R}+\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{R^2}\frac{\partial\psi}{\partial\theta}\cot\theta\right); \quad \ldots\ldots (91)$$

where θ is the co-latitude at the origin.



In the case of a doublet oscillating horizontally, the primary waves may be written by

$$\psi=e^{ipt}\frac{\partial}{\partial x}\frac{e^{-ihR}}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

$$=e^{ipt}\cos\theta\frac{\partial}{\partial R}\frac{e^{ihR}}{R}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (92')$$

The radial displacement in real form is approximately written by

$$u_R=\frac{1}{R}\cos(pt-hR)\cos\theta. \qquad \cdots\cdots(93)$$

Modifying the relation in (67), we find

$$\cos\theta\frac{\partial}{\partial R}\frac{e^{-ihR}}{R}=-\cos\omega\int_0^\infty\frac{e^{-\alpha z}}{\alpha}J_1(fr)\,f^2\,df, \qquad \cdots\cdots(94)$$

where ω is the azimuth with respect to z-axis.

Supposing the effect of the image for the source $(r=0,\ z=\xi)$, we get

$$\left.\begin{aligned}\psi&=-2e^{ipt}\cos\omega\int_0^\infty\frac{\operatorname{ch}\alpha z}{\alpha}e^{-\alpha\xi}J_1(fr)\,f^2\,df, && [z<\xi]\\ \psi&=-2e^{ipt}\cos\omega\int_0^\infty\frac{\operatorname{ch}\alpha\xi}{\alpha}e^{-\alpha z}J_1(fr)\,f^2\,df. && [z>\xi]\end{aligned}\right\} \qquad \cdots\cdots(95)$$

The expressions of the displacement are given by

$$\left.\begin{aligned}u_0&=-\frac{1}{h^2}=\frac{2e^{ipt}\cos\omega}{h^2}\int_0^\infty\frac{\operatorname{ch}\alpha z}{\alpha}e^{-\alpha\xi}\frac{\partial J_1(fr)}{\partial r}f^2\,df,\\ v_0&=-\frac{1}{h^2 r}\frac{\partial\Delta_0}{\partial\omega}=-\frac{2e^{ipt}\sin\omega}{h^2}\int_0^\infty\frac{\operatorname{ch}\alpha z}{\alpha}e^{-\alpha\xi}\frac{J_1(fr)}{r}f^2\,df,\\ w_0&=-\frac{1}{h^2}\frac{\partial\Delta_0}{\partial z}=\frac{2e^{ipt}\cos\omega}{h^2}\int_0^\infty\operatorname{sh}\alpha z\,e^{-\alpha\xi}J_1(fr)f^2\,df.\end{aligned}\right\}[z<\xi] \qquad \cdots\cdots(96)$$

The stress-components at the surface are expressed by

$$\left.\begin{aligned}&[p_{zr}]_{z=0}=0, \qquad [p_{z\omega}]_{z=0}=0,\\ &[p_0]_{z=0}=-2\kappa e^{ipt}\cos\omega\int_0^\infty\frac{e^{-\alpha\xi}}{\alpha}J_1(fr)\,f^2\,df.\end{aligned}\right\} \qquad \cdots\cdots(97)$$

Now, for gravity waves, the velocity potential ϕ will satisfy the equation of continuity such as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\omega^2}=0, \qquad \cdots\cdots(98)$$

the solution of which may be written by

$$\phi=-Ce^{-fz+ipt}J_1(fr)\cos\omega. \qquad \cdots\cdots(99)$$

The displacement corresponding to this is expressed by

$$\left.\begin{aligned} u_1 &= -\frac{i}{p} C e^{-fz+ipt} \frac{\partial J_1(fr)}{\partial r} \cos\omega, \\ v_1 &= \frac{i}{p} C e^{-fz+ipt} \frac{J_1(fr)}{r} \sin\omega, \\ w_1 &= \frac{if}{p} C e^{-fz+ipt} J_1(fr) \cos\omega. \end{aligned}\right\} \qquad \text{(100)}$$

From the integral of the equations of equilibrium and the kinematical condition at the surface, we find the surface pressure as below:

$$p_1 = -i\rho C \frac{p^2 - fg}{p} e^{-fz+ipt} J_1(fr) \cos\omega. \qquad \text{(101)}$$

The condition at the surface gives us

$$C = -2\kappa \frac{e^{-\alpha\xi}}{\alpha} \frac{pf^2}{i\rho(p^2 - fg)}. \qquad \text{(102)}$$

The displacement in (100) is therefore written by

$$\left.\begin{aligned} u_1 &= \frac{2\kappa f^2 e^{-\alpha\xi}}{\alpha\rho(p^2-fg)} e^{ipt} \frac{\partial J_1(fr)}{\partial r} \cos\omega, \\ v_1 &= -\frac{2\kappa f^2 e^{-\alpha\xi}}{\alpha\rho(p^2-fg)} e^{ipt} \frac{J_1(fr)}{r} \sin\omega, \\ w^1 &= -\frac{2\kappa f^3 e^{-\alpha\xi}}{\alpha\rho(p^2-fg)} e^{ipt} J_1(fr) \cos\omega. \end{aligned}\right\} \qquad \text{(103)}$$

Multiplying these by df, integrating from 0 to ∞ and afterwards adding to u_0, v_0 and w_0, besides the superposition of the terms due to Bessel's function of the second kind, we get

$$\left.\begin{aligned} u &= \frac{2\kappa e^{ipt} \cos\omega}{h^2 \rho} \int_0^\infty \left[\frac{\rho}{\kappa} + \frac{h^2}{p^2 - fg}\right] e^{-\alpha\xi} \frac{\partial H_1^{(2)}(fr)}{\partial r} \frac{f^2}{\alpha} df, \\ v &= -\frac{2\kappa e^{ipt} \sin\omega}{h^2 \rho} \int_0^\infty \left[\frac{\rho}{\kappa} + \frac{h^2}{p^2 - fg}\right] e^{-\alpha\xi} \frac{H_1^{(2)}(fr)}{r} \frac{f^2}{\alpha} df, \\ w &= -\frac{2\kappa e^{ipt} \cos\omega}{\rho} \int_0^\infty \frac{e^{-\alpha\xi}}{p^2 - fg} H_1^{(2)}(fr) \frac{f^3}{\alpha} df. \end{aligned}\right\} \qquad \text{(104)}$$

Considering the contour integrations similar to those in (86), we find

$$
\left.
\begin{aligned}
u &= \frac{2i\kappa e^{ipt}\cos\omega}{g\rho}\Bigg[\frac{p^2}{g}\frac{\pi}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\partial H_1^{(2)}\left(\frac{p^2}{g}r\right)}{\partial r} \\
&\quad + \frac{g}{ih^2\kappa}\int_0^\infty \frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}e^{-i\xi\sqrt{f^2+h^2}}\frac{\partial H_1^{(2)}(-ifr)}{\partial r}\frac{f^2}{\sqrt{f^2+h^2}}df\Bigg], \\
v &= -\frac{2i\kappa e^{ipt}\sin\omega}{g\rho r}\Bigg[\frac{p^2}{g}\frac{\pi}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}H_1^{(2)}\left(\frac{p^2}{g}r\right) \\
&\quad + \frac{g}{ih^2\kappa}\int_0^\infty \frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}e^{-i\xi\sqrt{f^2+h^2}}H_1^{(2)}(-ifr)\frac{f^2}{\sqrt{f^2+h^2}}df\Bigg], \\
w &= -\frac{2i\kappa e^{ipt}\cos\omega}{g\rho}\Bigg[\frac{p^4}{g^2}\frac{\pi}{\sqrt{1-\left(\frac{hg}{p^2}\right)^2}}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}H_1^{(2)}\left(\frac{p^2}{g}r\right) \\
&\quad + \frac{g}{i}\int_0^\infty \frac{e^{i\xi\sqrt{f^2+h^2}}}{p^2+igf}H_0^{(2)}(-ifr)\frac{f^3}{\sqrt{f^2+h^2}}df\Bigg].
\end{aligned}
\right\}
\quad (105)
$$

Thus we find the components of displacement on the surface of the water in the forms:

$$
\left.
\begin{aligned}
u &\doteqdot \frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\cos\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\cos\omega, \\
v &\doteqdot \frac{2\kappa\pi}{g\rho r}\frac{p^2}{g}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\sin\omega, \\
w &\doteqdot \frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\cos\omega.
\end{aligned}
\right\}
[p^2>hg] \quad \ldots\ldots(106)
$$

and

$$
u \doteqdot \frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}-\xi\sqrt{\frac{p^4}{g^2}-h^2}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\cos\omega,
$$

$$\left.\begin{aligned} v &\fallingdotseq -\frac{2\kappa\pi}{g\rho r}\frac{p^2}{g}\frac{\cos\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}-\xi\sqrt{\frac{p^4}{g^2}-h^2}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\sin\omega, \\ w &\fallingdotseq -\frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}\frac{\cos\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}-\xi\sqrt{\frac{p^4}{g^2}-h^2}\right)}{\sqrt{\frac{\pi p^2 r}{2g}}}\cos\omega, \end{aligned}\right\}[p^2<hg] \quad (107)$$

corresponding to the displacement near the origin as written by

$$u_R \fallingdotseq \frac{1}{R}\cos(pt-hR)\cos\theta. \quad (108)$$

It is worthy of notice that in spite of the maintenance of the natures of the vertical and the horizontal components of displacement in wave profile and in azimuthal distribution, the azimuthal component of displacement quickly disappears as the distance from the origin increases.

6. In three-dimensional motion in which the origin of the disturbance is a doublet oscillating vertically, we find the solution of (91) in the following form:

$$\psi = e^{ipt}\frac{\partial}{\partial z}\frac{e^{-ihR}}{R}. \quad (109)$$

The radial displacement is approximately written by

$$u_R = \frac{1}{R}\cos(pt-hR)\sin\theta, \quad (110)$$

where θ is the latitude at the origin concerning the line of the oscillations.

Considering the effect of the image for the source $(r=0,\ z=\xi)$, we find

$$\psi = -2e^{ipt}\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi} J_0(fr) f\,df, \quad [y<\xi] \quad (111)$$

so that the expressions of the displacement are givien by

$$\left.\begin{aligned} u_0 &= -\frac{2e^{ipt}}{h^2}\int_0^\infty \mathrm{ch}\,\alpha z\, e^{-\alpha\xi} J_1(fr) f^2\,df, \\ v_0 &= 0, \\ w_0 &= \frac{2e^{ipt}}{h^2}\int_0^\infty \alpha\,\mathrm{sh}\,\alpha z\, e^{-\alpha\xi} J_0(fr) f\,df. \end{aligned}\right\}[y<\xi] \quad (112)$$

The stress-components are expressed by

$$\left.\begin{aligned}&[p_{zr}]_{z=0}=0, \qquad [p_{z\omega}]_{z=0}=0,\\ &[p_0]_{z=0}=-2\kappa e^{ipt}\int_0^\infty e^{-\alpha\xi}J_0(fr)fdf.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(113)$$

Again, pursuing the same line of treatment as before, we find

$$C=2\kappa e^{-\alpha\xi}\frac{pf}{i\rho(p^2-fg)}.\quad\ldots\ldots\ldots\ldots\ldots\ldots(114)$$

The resultant displacement on the surface is expressed by

$$\left.\begin{aligned}u&=-\frac{2\kappa e^{ipt}}{h^2\rho}\int_0^\infty\left[\frac{\rho}{\kappa}+\frac{h^2}{p^2-fg}\right]e^{-\alpha\xi}J_1(fr)f^2df,\\ w&=-\frac{2\kappa e^{ipt}}{\rho}\int_0^\infty\frac{e^{-\alpha\xi}}{p^2-fg}J_0(fr)f^2df.\end{aligned}\right\}\quad\ldots\ldots\ldots(115)$$

Thus the results, when the contour integrations for the adjusted integrals are performed by means of the formulae similar to (86), are written by

$$\left.\begin{aligned}u=&-\frac{2i\kappa e^{ipt}}{g\rho}\left[\frac{p^4}{g^2}\pi e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}H_1^{(2)}\left(\frac{p^2}{g}r\right)\right.\\ &\left.+\frac{g}{ih^2\kappa}\int_0^\infty\frac{\rho(p^2+igf)+\kappa h^2}{p^2+igf}e^{-i\xi\sqrt{f^2+h^2}}H_1^{(2)}(-ifr)f^2df\right],\\ w=&-\frac{2i\kappa e^{ipt}}{g\rho}\left[\frac{p^4}{g^2}\pi e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}H_0^{(2)}\left(\frac{p^2}{g}r\right)\right.\\ &\left.+\frac{g}{i}\int_0^\infty\frac{e^{-i\xi\sqrt{f^2+h^2}}}{p^2-fg}H_0^{(2)}(-ifr)f^2df\right],\end{aligned}\right\}\quad\ldots(116)$$

Neglecting the second terms for a large distance, we obtain

$$\left.\begin{aligned}u&=\frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\sin\left(pt-\frac{p^2}{g}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2r}{2g}}},\\ w&=-\frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}e^{-\xi\sqrt{\frac{p^4}{g^2}-h^2}}\frac{\cos\left(pt-\frac{p^2}{g^2}r+\frac{3\pi}{4}\right)}{\sqrt{\frac{\pi p^2r}{2g}}},\end{aligned}\right\}[p^2>hg]\quad\ldots(117)$$

and

$$\left.\begin{aligned} u&=\frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}\frac{\sin\left(pt-\dfrac{p^2}{g}r+\dfrac{3\pi}{4}-\xi\sqrt{h^2-\dfrac{p^4}{g^2}}\right)}{\sqrt{\dfrac{\pi p^2 r}{2g}}},\\ w&=-\frac{2\kappa\pi}{g\rho}\frac{p^4}{g^2}\frac{\cos\left(pt-\dfrac{p^2}{g}r+\dfrac{3\pi}{4}-\xi\sqrt{h^2-\dfrac{p^4}{g^2}}\right)}{\sqrt{\dfrac{\pi p^2 r}{2g}}}, \end{aligned}\right\}[p^2<hg] \ldots\ldots (118)$$

corresponding to the displacement in the neighbourhood of the origin,

$$u_R=\frac{1}{R}\cos(pt-hR)\sin\theta, \quad [R^2=r^2+(z-\xi)^2] \ldots\ldots\ldots (119)$$

Some notices which should be added here seem very like those in the preceding sections, so that we will complete this section without adding any remark.

Résumé.

The formation of deep water waves, when some disturbances act on a point in the interior of the water, has been studied. The compressional disturbances of the simple-harmonic type being propagated radially from the origin, a succession of gravity waves is gradually developed by the surface conditions of the water.

The method of analysis employed in this is, in the first place, to apply on the primary waves (the primary pressure having inertia effect) the mathematical results due to H. Lamb, i.e.

$$H_0^{(2)}(hR)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha y}e^{\mp ifx}}{\alpha}df, \quad [R^2=x^2+y^2]$$

and

$$\frac{e^{-ihR}}{R}=\int_0^{\infty}\frac{e^{-\alpha z}}{\alpha}J_0(fr)fdf, \quad [R^2=x^2+y^2+z^2]$$

where

$$\alpha=\sqrt{f^2+h^2} \quad \text{or} \quad i\sqrt{h^2-f^2},$$

according as $f^2\gtrless h^2$. Some modified formulae have also been employed. In the second step, gravity waves, which are to be superposed on the primary waves, have been formulated to satisfy the boundary conditions.

The paper consists of six sections: the beginning three sections deal with the cases in which the origin is either a singlet, a doublet oscillating horizontally or a doublet moving vertically all in two dimensions, while the remaining three treat of the similar cases in three dimensions.

The principal results obtained by this investigation are enumerated as follows:

1. In spite of very small displacements of the compressional waves in the neighbourhood of the origin in the interior of the water, the excited surface waves have relatively large amplitudes.

2. The generated surface waves are chiefly the ordinary gravity waves having the same frequency as that of the origin together with their wave length proper to the period.

3. The distribution of the wave motion on the surface of water always conspires with the modes of oscillation at the origin.

4. This fails in a three-dimensional case where a doublet oscillates horizontally. In this, notwithstanding the maintenance of the natures of the vertical and the horizontal components of displacement in wave profile and in azimuthal distribution, the azimuthal component of displacement quickly disappears as the distance from the disturbed portion is increased.

Concluding Remarks.

We may now review some results of this mathematical investigation from the general consideration. Although the examples in the present paper are of some idealized kinds, yet we may say that the theory is an important guidance for the confirmation of the behaviours of the formation of deep-water waves under certain possible conditions. This can be clearly known in the light of other branches of siences in which highly idealized models contribute very often to the determination of the natures.

It has been necessary to idealize this problem in different manners to obtain the regorous solutions of the mathematical equations and to find the most important natures not in combined forms but in pure states. In the first place, the motion of both compressional and gravity waves has been taken to be small. The "small motion" is not meant that the amplitude of the motion is small, but it defines that the slope or the gradient of the motion is small. Secondly, the type of the motion is assumed to be simple-harmonic. Generalising this type by means of Fourier's double integral theorem the complicated cases such as the wave motion due to some arbitrary disturbances at the origin may be obtained without difficulty. This, however, has been omitted in this occasion and left for the future study.

The preceding solutions tell the fact that, in spite of very small displacements of the compressional waves in water, the generated surface waves proceed with large amplitudees. The figure in p. 28 is illustrated as an example. This example also shows that the deeper the origin of the disturbance and the quicker the period of the oscillations, the more quiescent become the surface waves at a certain distance. Not less important is the fact that at a large distance from the origin the generated surface waves due the compressional disturbances acting on the interior of the water are chiefly the ordinary gravity waves having the same frequency as that of the origin together with their wave length proper to the period. The orbital motion of the water particle is perfectly circular. An equally significant nature is that the distribution of the wave motion on the surface always conspires with the modes of the oscillations at the origin. An exceptional case is the three-dimensional motion due to a doublet oscillating horizontally. In this case, notwithstanding the maintenance of the natures of the vertical and the horizontal components of displacement in wave profile and in azimuthal distribution, the azimuthal component of displacement quickly disappears as the distance from the origin is increased.

The extended case of the present problem, in which the shallow-water waves are formed by the shocks at the bottom of the sea, is more significant on the practical seismology; this, however, will be left for future studies.(1)

In conclusion the author must express his sincere thanks to Professors Suyehiro and Terada for their kind advices.

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(1) The problem of shallow-water waves due to bottom shocks will be published in Bull. 7 of the Institute. The case of the pressure-distribution, in which the effect of inertia is neglected, will also be involved in that paper, because such a case is rather important on shallow-water problem.