Elastic Equilibrium of a Spherical Body under Surface Tractions of a Certain Zonal and Azimuthal Distribution.

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方位的及帶狀的に分布せる表面應力を受ける彈性球體の平衡

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半無限體が一點の周圍に對稱をなさぬ荷重を受ける時の彈性平衡は寺澤博士などによつて既に解かれてゐるが、表面が球狀をなす場合は未だ考へられて居らぬ樣に思はれる。この問題が地形の複雜な部分例へばパナマの周圍の如き場所の潮汐荷重問題に大切な事は勿論であるが、尙海岸線の水陸分布による地殼の應力狀態の想像をも與へる。

この硏究は完全な彈性體の場合に極めて嚴密な計算を施したものであるから、地球物理學の問題としては重力の影響（荷重自身の引力でなく）や、地球內部の物質分布及び物質の時間による變形の議論の必要があるが、前二者は計算の結果から見て唯其補正計算を別に行へば充分らしく、又時間による影響に就ては未だ一般物質の場合でさへ明瞭でないから猶硏究の餘地があると思はれる。然し或る瞬間に於ける應力分布に對しては本文の結果が可成適用されるであらう。

計算の方法は球座標による彈性體平衡式から變容歪方程式と非變容歪方程式とを導出し、次に此等各に特有なる變位式を見出し、之に變容歪にも非變容歪にも無關係な變位式を加へて一般解式とする。上述の解式を境界條件によつて適當に調整し且調和分析によつて既與の荷重壓の爲めに生ずる內力の分布を得た次第である。

主要な結果を摘錄すれば、

（１）　球の一部に加へられた荷重が如何に集中する場合でも、球の內部の應力は全體に均一される傾向があり、これだけでも充分靜水壓の性質を示す。時によつて同じ半徑中にあつて異なる符號の應力の現はれる事もある。

（２）　表面では剪應力がなく、中心に向ふ垂直應力のみを與へても、他の直角方向の垂直應力が前述の垂直應力と略同じ値に達し、これが水底の固體應力の靜水平衡にある事と一致する傾向を示す。

（3） 本論文の第一章にある一般解式は其儘球の外部問題に應用され得る形、即ち張力、曲力、剪力を受ける無限彈性體に球窩や充塡物のある問題の解となる。但し具體的に係數等を決定する事は他の機會に殘した。

The theory of elastic equilibriurn of a semi-infinite solid under a normal boundary pressure, which is distributed unsymmetrically about a point on the surface, was given by Prof. K. Terazawa,(1) leading to important results in regard to the geophysical problem such as the deflection of the vertical,(2) both in the tilting of the surface and in the sense of gravity, due to the tidal loading of the earth's surface. The present problem, which is somewhat similar to that of Prof. Terazawa, is related to the more extended case of the elastic equilibrium of a spherical body under surface tractions of a certain zonal and azimuthal distribution. Such a problem on the sphere in a simplest case, where there is symmetry about a point at the pole, was studied by many investigators, such as Lord Kelvin,(3) G. H. Darwin,(4) A. E. H. Love;(5) the case, in which the normal boundary pressure is not distributed symmetrically, has not yet been studied as far as the authors are aware.

The importance of dealing with the problem in the case, where there is no symmetry about a point on the surface, arises from the tidal loading in the aqueous region of the irregular form on the earth's surface, for instance, in the vicinity of Panama where there is a certain azimuthal distribution of land and sea. In fact, such distribution will be observed at various portions of the margin lines separating the continental block and th ocean basin or at the mountain ranges including volcanoes. Besides this, the equivalent stress distribution at some local point in the earth crust appears to exist everywhere, as supposed from the records of the

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(1) K. Terazawa, *Phil Trans. Roy. Soc.* (A) **217** (1916); and also *J. Coll. Sci., Tokyo*, **37** (1916).

(2) H Lamb, "On the Deflection of the Vertical by Tidal Loading of the Earth's Surface," *Roy. Soc. Proc.* (A) **93** (1917).

(3) Kelvin and Tait, *Nat. Phil.*, Part 2.

(4) G. H. Darwin, *Phil. Trans. Roy. Soc.*, **173** (1882).

(5) A. E. H. Love, *Some Problems of Geodynamics* (1911).

observation of the earthquake movements at different stations in the vicinity of the seismic origin.

The present paper treats of the problem of the elastic equilibrium of a non-gravitating spherical solid, whose elasticities and density are uniform. In the first section the general solutions of the problem have been obtained in the form of harmonic functions, while the second section deals with a certain example of polar loads of some azimuthal distribution.

The method of analysis employed in this paper is somewhat similar to that of the wave motion on the surface of an elastic sphere; in this case, however, a different component of displacement having a special meaning has been introduced. Moreover the azimuthal variation of the boundary pressure makes the treatment much difficult, yet the problem has been studied strictly throughout the whole paper.

In order to investigate the manner in which the elastic sphere is in equilibrium under the applied stress at the boundary, it will be necessary to consider the problem under certain simplifying assumptions. It will be assumed here that the earth may be treated as a homogeneous elastic body. The body will be supposed to be free from the effects of gravity and the rotation of the earth, as these effects are insignificant when the normal pressure on the boundary surface is not so small. The present study is some extension of the plane problem to the case where the effect of the curvature of the surface of the earth is taken into account; and the problems involving every geophysical element will be discussed in the further study.

1. General Solutions.

We shall use the spherical polar coordinates r, θ, ϕ, the origin being at the center of the sphere, the axis of $\theta=0$ being the radial line connecting the centre of the sphere and the poles, in the vicinity of which the normal pressure is distributed. We shall denote u, v, w as the components of displacement in the direction of the radius, colatitude and azimuth and λ, μ Lamé's elastic constants. Then the equations of equilibrium of the body are expressed by

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\frac{\partial(\varpi_\phi\sin\theta)}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial\varpi_\theta}{\partial\phi}=0,\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-\frac{2\mu}{r\sin\theta}\frac{\partial\varpi_r}{\partial\phi}+\frac{2\mu}{r}\frac{\partial(\varpi_\phi r)}{\partial r}=0,\\
&(\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}-\frac{2\mu}{r}\frac{\partial(\varpi_\theta r)}{\partial r}+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}=0,
\end{aligned}\right\}\qquad\qquad (1)$$

where

$$\left.\begin{aligned}
&\Delta=\frac{1}{r^2\sin\theta}\left[\frac{\partial(ur^2\sin\theta)}{\partial r}+\frac{\partial(vr\sin\theta)}{\partial\theta}+\frac{\partial(wr)}{\partial\phi}\right],\\
&2\varpi_r=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(wr\sin\theta)-\frac{\partial}{\partial\phi}(vr)\right],\\
&2\varpi_\theta=\frac{1}{r\sin\theta}\left[\frac{\partial u}{\partial\phi}-\frac{\partial(wr\,\mathrm{ins}\,\theta)}{\partial r}\right],\\
&2\varpi_\phi=\frac{1}{r}\left[\frac{\partial(vr)}{\partial r}-\frac{\partial u}{\partial\theta}\right].
\end{aligned}\right\}\qquad\qquad (2)$$

The elimination of u, v, w among (1) and (2) gives us that

$$\left.\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Delta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Delta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Delta}{\partial\phi^2}=0,\\
&\frac{\partial^2\varpi_r}{\partial r^2}+\frac{4}{r}\frac{\partial\varpi_r}{\partial r}+\frac{2}{r^2}\varpi_r+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varpi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_r}{\partial\phi^2}=0,\\
&\frac{1}{r}\frac{\partial^2(\varpi_\theta r)}{\partial r^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_\theta}{\partial\phi^2}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi(_\phi\sin\theta)}{\partial\phi\partial\theta}-\frac{1}{r}\frac{\partial^2\varpi_r}{\partial r\partial\theta}=0,\\
&\frac{1}{r}\frac{\partial^2(\varpi_\phi r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial(\varpi_\phi\sin\theta)}{\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\varpi_\theta}{\partial\phi}-\frac{1}{r\sin\theta}\frac{\partial^2\varpi_r}{\partial r\partial\phi}=0.
\end{aligned}\right\}$$

$$\qquad\qquad (3)$$

These give us

$$\left.\begin{aligned}
&\Delta=\left(A_{mn}r^n+\frac{A'_{mn}}{r^{n+1}}\right)P_n^m(\cos\theta)\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,\\
&2\varpi_r=\left(B_{mn}r^{n-1}+\frac{B'_{mn}}{r^{n+2}}\right)P_n^m(\cos\theta)\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi,\\
&2\varpi_\theta=\left[\left(D_{mn}mr^n+\frac{D'_{mn}m}{r^{n+1}}\right)\frac{P_n^m(\cos\theta)}{\sin\theta}\right.\\
&\qquad\left.+\left(\frac{B_{mn}}{n}r^{n-1}-\frac{B'_{mn}}{(n+1)r^{n+2}}\right)\frac{dP_n^m(\cos\theta)}{d\theta}\right]\left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\}m\phi,
\end{aligned}\right\}\qquad (4)$$

$$2\varpi_{\phi}=\left[\left(D_{mn}r^{n}+\frac{D'_{mn}}{r^{n+1}}\right)\frac{dP_{n}^{m}(\cos\theta)}{d\theta}+\left(\frac{B_{mn}}{n}r^{n-1}-\frac{B'_{mn}}{(n+1)\,r^{n+2}}\right)\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\phi,$$

in which A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn}, C'_{mn} are arbitrary constants.

Displacement (u_1, v_1, w_1) answering to Δ in (4) and satisfying $\varpi_r=\varpi_\theta=\varpi_\phi=0$ is expressed by

$$\left.\begin{aligned}u_1&=\left[\frac{A_{mn}(n+2)}{2(2n+3)}r^{n+1}+\frac{A'_{mn}(n-1)}{2(2n-1)\,r^{n}}\right]P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ v_1&=\left[\frac{A_{mn}}{2\,(2n+3)}r^{n+1}-\frac{A'_{mn}}{2\,(2n-1)\,r^{n}}\right]\frac{dP_n^m(\cos\theta)}{d\theta}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ w_1&=-\left[\frac{A_{mn}m}{2\,(2n+3)}r^{n+1}-\frac{A'_{mn}}{2\,(2n-1)\,r^{n}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\}m\phi.\end{aligned}\right\}\quad\ldots\ldots(5)$$

Displacement (u_2, v_2, w_2) answering to ϖ_r together with the second terms in the expression of ϖ_θ, ϖ_ϕ, all given in (4) under the condition that $\Delta=0$, is expressed by

$$\left.\begin{aligned}u_2&=0,\\ v_2&=\left[\frac{mB_{mn}}{n\,(n+1)}r^{n}+\frac{mB_{mn}}{n\,(n+1)\,r^{n+1}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ w_2&=-\left[\frac{B_{mn}}{n\,(n+1)}r^{n}+\frac{B'_{mn}}{n\,(n+1)\,r^{n+1}}\right]\frac{dP_n^m(\cos\theta)}{d\theta}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\}m\phi.\end{aligned}\right\}\quad\ldots\ldots\ldots(6)$$

Displacement (u_4, v_4, w_4) derived from the values of the first terms of ϖ_θ, ϖ_ϕ, in (4) and fulfilling the conditions, $\Delta=\varpi_r=0$, is written by

$$\left.\begin{aligned}u_4&=\left[\frac{D_{mn}n\,(n+1)}{2(2n+3)}r^{n+1}+\frac{D'_{mn}\,n\,(n+1)}{2\,(2n-1)\,r^{n}}\right]P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ v_4&=\left[\frac{D_{mn}(n+3)}{2(2n+3)}r^{n+1}+\frac{D'_{mn}(n-2)}{2\,(2n-1)\,r^{n}}\right]\frac{1}{m}\,\frac{dP_n^m(\cos\theta)}{d\,\theta}\begin{matrix}\cos\\ \sin\end{matrix}\Big\}m\phi,\\ w_4&=-\left[\frac{D_{mn}\,m\,(n+3)}{2\,(2n+3)}r^{n+1}+\frac{D'_{mn}\,m\,(n-2)}{2\,(2n-1)\,r^{n}}\right]\frac{P_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\sin\\ -\cos\end{matrix}\Big\}m\phi.\end{aligned}\right\}\quad\ldots(7)$$

Displacement (u_3, v_3, w_3), which satisfies $\Delta=\varpi_r=\varpi_\theta=\varpi_\phi=0$, is expressed by

$$\left.\begin{aligned} u_3 &= \left[C_{mn}\, n r^{n-1} - \frac{C'_{mn}(n+1)}{r^{n+2}}\right] P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \\ v_3 &= \left[C_{mn}\, r^{n-1} + \frac{C'_{mn}}{r^{n+2}}\right] \frac{dP_n^m(\cos\theta)}{d\theta} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \\ w_3 &= -m\left[C_{mn}\, r^{n-1} + \frac{C'_{mn}}{r^{n+2}}\right] \frac{P_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\phi. \end{aligned}\right\} \quad \cdots\cdots\cdots (8)$$

From the equations in (1) we have the identical relations such that

$$\frac{D_{mn}}{A_{mn}} = -\frac{(\lambda+2\mu)}{\mu(n+1)}, \quad \frac{D'_{mn}}{A'_{mn}} = \frac{\lambda+2\mu}{\mu n}. \quad \cdots\cdots\cdots (9)$$

Writing

$$\alpha = \frac{\lambda+2\mu}{\mu}, \quad \cdots\cdots\cdots (10)$$

and substituting the relations of (9) in (7), we find the general expressions of the spherical problem of elasticity as follows:

$$\left.\begin{aligned} \Delta &= \left(A_{mn} r^n + \frac{A_{mn}}{r^{n+1}}\right) P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \\ u_1 &= u_1 \text{ in } (5) + u_4 = \left[\frac{A_{mn}\{(n+2)-\alpha n\}}{2(2n+3)} r^{n+1} \right. \\ &\qquad \left. + \frac{A'_{mn}\{(n-1)-\alpha(n+1)\}}{2(2n-1)\, r^n}\right] P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \\ v_1 &= v_1 \text{ in } (5) + v_4 = \left[\frac{A_{mn}\left\{1-\alpha\dfrac{n+3}{n+1}\right\}}{2(2n+3)} r^{n+1} \right. \\ &\qquad \left. - \frac{A'_{mn}\left\{1-\alpha\dfrac{n-2}{n}\right\}}{2(2n-1)\, r^n}\right] \frac{dP_n^m(\cos\theta)}{d\theta} \begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \\ w_1 &= w_1 \text{ in } (5) + w_4 = \left[-\frac{m A_{mn}\left\{1-\alpha\dfrac{n+3}{n+1}\right\}}{2(2n+3)} r^{n+1} \right. \\ &\qquad \left. + \frac{m A'_{mn}\left\{1-\alpha\dfrac{n-2}{n}\right\}}{2(2n-1)\, r^n}\right] \frac{P_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin\\ -\cos\end{matrix}\Big\} m\phi, \end{aligned}\right\} \quad \cdots\cdots (11)$$

together with the expressions of u_2, v_2, w_2 in (6) and u_3, v_3, w_3 in (8).

Applying in the boundary conditions of stresses or of displacements the values of Δ, u, v, w, in which $u=u_1+u_2+u_3$, $v=v_1+v_2+v_3$, $w=w_1$

$+w_2+w_3$, we can determine various cases of physical problems. Some problems of practical importance, such as the present investigation, the stress-distribution in an elastic solid having cavities or imbedded spheres under certain boundary tractions and like problems, can be solved by means of the foregoing solutions.

2. Spherical Body under Surface Tractions of a Certain Zonal and Azimuthal Distribution.

We shall suppose the body to be strained by the application of the normal forces in the vicinity of a pole and at the antipode, where the forces have a certain zonal and azimuthal distribution. Thus we may put the boundary conditions in the forms:

$$\left.\begin{aligned} \widehat{r\theta}_{r=a}&=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)_{r=a}=0,\\ \widehat{r\phi}_{r=a}&=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial r}+\frac{\partial v}{\partial \phi}-\frac{v}{r}\right)_{r=a}=0,\\ \widehat{rr}_{r=a}&=\left(\lambda\Delta+2\mu\frac{\partial u}{\partial r}\right)_{r=a}=\sin^2\theta\cos^q\theta\cos m\phi. \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(12)$$

Disregarding terms which approach to infinity as r tends to zero, we get from the first or the second equation in (12) the following relation:

$$\frac{C_{mn}}{A_{mn}}=\frac{\left\{\left(1-\alpha\frac{n+1}{n+1}\right)(1+n)-\left(1-\alpha\frac{n+3}{n+1}\right)+(n+2)-\alpha n\right\}a^2}{4(1-n)(2n+3)},\quad\ldots\ldots(13)$$

the terms corresponding (6), which have the arbitrary constants B_{mn}, being omitted as they are not suited for the present case.

It remains now that, when C_{mn} are replaced by A_{mn} by means of (13), we should determine A_{mn} from the condition expressed by the third of the relations in (12). Thus we find the expression of the element $\widehat{rr}$ in the form:

$$A_{mn}\left[\lambda+\frac{(n+2-\alpha n)(n+1)}{2n+3}\mu\right.$$
$$\left.-\frac{n\left\{\left(1-\alpha\frac{n+3}{n+1}\right)(n+1)-\left(1-\alpha\frac{n+3}{n+1}\right)+(n+2)-\alpha\right\}}{2(n+3)}\mu\right].\quad\ldots\ldots(14)$$

Now we may expand $\sin^2\theta\cos^q\theta\cos m\phi$ in series(1) of harmonic functions as below:

$$\sin^2\theta\cos^q\theta\cos m\phi=\sum_m\sum_n{}' A_n^m a^n P_n^m(\cos\theta)\cos m\phi, \qquad \ldots\ldots\ldots(15)$$

where

$$A_n^m=\frac{1}{\pi a^n}\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}$$
$$\int_{-\pi}^{\pi}\int_0^{\infty}\sin^2\theta\cos^q\theta\cos m\phi\, P_n^m(\cos\theta)\cos m\phi\sin\theta\, d\theta\, d\phi, \qquad \ldots\ldots(16)$$

besides

$$A_n^m=A_{mn}\Bigg[\lambda+\frac{(n+2-\alpha n)(n+1)}{2n+3}\mu$$
$$-\frac{n\left\{1-\alpha\frac{n+3}{n+1}\right)(n+1)-\left(1-\alpha\frac{n+3}{n+1}\right)+(n+2)-\alpha n\right\}}{2(2n+3)}\mu\Bigg] \qquad \ldots\ldots(17)$$

From (16) and (17), we can determine the values of A_{mn}. Substituting these values of A_{mn} in the expressions of $\widehat{rr}, \widehat{\theta\theta}, \widehat{\phi\phi}, \widehat{r\theta}, \widehat{r\phi}, \widehat{\theta\phi}$, such that,

$$\left.\begin{aligned}
&\widehat{rr}=\lambda\Delta+2\mu\frac{\partial u}{\partial r}, \qquad \widehat{\theta\theta}=\lambda\Delta+2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right),\\
&\widehat{\phi\phi}=\lambda\Delta+2\mu\left(\frac{v}{r}\cot\theta+\frac{u}{r}+\frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi}\right),\\
&\widehat{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right), \qquad \widehat{r\phi}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\frac{\partial w}{\partial r}-\frac{w}{r}\right),\\
&\widehat{\theta\phi}=\mu\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi}-\frac{w}{r}\cot\theta\right),
\end{aligned}\right\} \qquad \ldots\ldots(18)$$

and taking the case in which $q=8$, $m=2$, we obtain the stress-distributions of all the components of stresses as in the following forms:

$$\widehat{rr}_{r=a}=\sin^2\theta\cos^8\theta\cos 2\phi, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

$$\widehat{rr}=\sum_{n=2}^{10}{}'\frac{1}{a^n}\frac{2n+1}{2}\frac{(n-2)!}{(n+2)!}\frac{(n+1)\{3+n-n^2)\lambda+(2+n-n^2)\mu\}r^n+n\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu}$$
$$\times\int_0^{\pi}\sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times P_n^2(\cos\theta)\cos 2\phi, \qquad \ldots\ldots\ldots(20)$$

(1) Whittaker, *Modern Analysis*, p. 393.

$$\widehat{\theta\theta}=\sum_{n=2}^{10}\frac{1}{a^n}\frac{2n+1}{2}\frac{(n-2)!}{(n+2)!}\frac{(1-n^2)\{(n+3)\lambda-(n-2)\mu\}r^n-n\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times P_n^2(\cos\theta)\cos 2\phi$$

$$+\sum_{n=2}^{10}\frac{1}{a^n}\frac{2n+1}{2}\frac{(n-2)!}{(n+2)!}\frac{(n-1)\{(n+3)\lambda+(n+5)\mu\}r^n-\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times\frac{d^2P_n^2(\cos\theta)}{d\theta^2}\cos 2\phi,\qquad\qquad(21)$$

$$\widehat{\phi\phi}=\sum_{n=2}^{10}\frac{1}{a^n}\frac{2n+1}{2}\frac{(n-2)!}{(n+2)!}\frac{(1-n^2)\{(n+3)\lambda-(n-2)\mu\}r^n-\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times P_n^2(\cos\theta)\cos 2\phi$$

$$+\sum_{n=2}^{10}\frac{1}{a^n}\frac{(2n+1)}{2}\frac{(n-2)!}{(n+2)!}\frac{(n-1)\{(n+3)\lambda+(n+5)\mu\}r^n-\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times\cot\theta\frac{dP_n^2(\cos\theta)}{d\theta}\cos 2\phi$$

$$-4\sum_{n=2}^{10}\frac{1}{a^n}\frac{(2n+1)}{2}\frac{(n-2)!}{(n+2)!}\frac{(n-1)\{(n+3)\lambda+(n+5)\mu\}r^n-\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times\frac{P_n^2(\cos\theta)}{\sin^2\theta}\cos 2\phi,\qquad\qquad(22)$$

$$\widehat{r\theta}=\sum_{n=2}^{10}\frac{1}{a^n}\frac{(2n+1)}{2}\frac{(n-2)!}{(n+2)!}\frac{\{n(n+2)\lambda+(n^2+2n-1)\mu\}(a^2r^{n-2}-r^n)}{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times\frac{dP_n^2(\cos\theta)}{d\theta}\cos 2\phi\qquad\qquad(23)$$

$$\widehat{r\phi}=-2\sum_{n=2}^{10}\frac{1}{a^n}\frac{(2n+1)}{2}\frac{(n-2)!}{(n+2)!}\frac{\{n(n+2)\lambda+(n^2+2n-1)\mu\}(a^2r^{n-2}-r^n)}{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta\times\frac{P_n^2(\cos\theta)}{\sin\theta}\sin 2\phi,\ \dots\dots\dots\dots(24)$$

$$\widehat{\theta\phi}=\sum_{n=2}^{10}\frac{(2n+1)}{a^n}\frac{(n-2)!}{(n+2)!}\frac{(n-1)\{(n+3)\lambda+(n+5)\mu\}r^n-\{n(n+2)\lambda+(n^2+2n-1)\mu\}a^2r^{n-2}}{(1-n)\{(2n^2+4n+3)\lambda+2(n^2+n+1)\mu\}}$$

$$\times\int_0^\pi \sin^3\theta\cos^8\theta P_n^2(\cos\theta)\,d\theta$$

$$\times\left\{\frac{\cos\theta}{\sin^2\theta}P_n^2(\cos\theta)-\frac{1}{\sin\theta}\frac{dP_n^2(\cos\theta)}{d\theta}\right\}\sin 2\phi\dots\dots(25)$$

As the above expressions in series are of finite number of terms, we may rewrite them as in the forms below:

$$\widehat{rr}=\left\{\frac{5}{9.\,11.\,13}\frac{3\lambda r^2+2(8\lambda+7\mu)a^2}{(19\lambda+14\mu)a^2}P_2^2(\cos\theta)\right.$$

$$+\frac{8}{11.\,13.\,15}\frac{4(24\lambda+23\mu)a^2r^2-5(9\lambda+10\mu)r^4}{(51\lambda+42\mu)a^4}P_4^2(\cos\theta)$$

$$+\frac{16}{15.\,17.\,33}\frac{6(48\lambda+47\mu)a^2r^4-7(27\lambda+28\mu)r^6}{(99\lambda+86\mu)a^6}P_6^2(\cos\theta)$$

$$+\frac{64}{13.\,15.\,19.\,33}\frac{8(80\lambda+79\mu)a^2r^6-9(53\lambda+54\mu)r^8}{(163\lambda+146\mu)a^8}P_8^2(\cos\theta)$$

$$+\frac{128}{13.\,15.\,17.\,19.\,33}$$

$$\left.\frac{10(120\lambda+119\mu)a^2r^8-11(87\lambda+88\mu)r^{10}}{(243\lambda+222\mu)a^{10}}P_{10}^2(\cos\theta)\right\}\cos 2\phi,\ \dots(20')$$

$$\widehat{\theta\theta}=\left\{\frac{5}{9.\,11.\,13}\frac{3.\,5.\,\lambda r^2+2(8\lambda+7\mu)a^2}{(19\lambda+14\mu)a^2}P_2^2(\cos\theta)\right.$$

$$+\frac{8}{11.\,13.\,15}\frac{15(7\lambda-2\mu)r^4+4(24\lambda+23\mu)a^2r^2}{3(51\lambda+42\mu)a^4}P_4^2(\cos\theta)$$

$$+\frac{16}{15.\,17.\,33}\frac{35(9\lambda-4\mu)r^6+6(48\lambda+47\mu)a^2r^4}{5(99\lambda+86\mu)a^6}P_6^2(\cos\theta)$$

$$+\frac{64}{13.\,15.\,19.\,33}\frac{63(11\lambda-6\mu)r^8+8(80\lambda+79\mu)a^2r^6}{7(163\lambda+146\mu)a^8}P_8^2(\cos\theta)$$

$$+\frac{128}{13.15.17.19.33}\frac{99(13\lambda-8\mu)r^{10}+10(120\lambda+119\mu)a^2r^8}{9(243\lambda+222\mu)a^{10}}P_8^2(\cos\theta)\Big\}\cos 2\phi$$

$$+\Big\{\frac{5}{9.11.13}\frac{(8\lambda+7\mu)a^2-(5\lambda+7\mu)r^2}{(19\lambda+14\mu)a^2}\frac{d^2P_2^2(\cos\theta)}{d\theta^2}$$

$$+\frac{8}{11.13.15}\frac{(24\lambda+23\mu)a^2r^2-3(7\lambda+9\mu)r^4}{3(51\lambda+42\mu)a^4}\frac{d^2P_4^2(\cos\theta)}{d\theta^2}$$

$$+\frac{16}{15.17.33}\frac{(48\lambda+47\mu)a^2r^4+5(9\lambda+11\mu)r^6}{5(99\lambda+86\mu)a^6}\frac{d^2P_6^2(\cos\theta)}{d\theta^2}$$

$$+\frac{64}{13.15.19.33}\frac{(80\lambda+79\mu)a^2r^6-7(11\lambda+13\mu)r^8}{7(163\lambda+146\mu)a^8}\frac{d^2P_8^2(\cos\theta)}{d\theta^2}$$

$$+\frac{128}{13.15.17.19.33}$$

$$\frac{(120\lambda+119\mu)a^2r^8-9(13\lambda+15\mu)r^{10}}{9(243\lambda+222\mu)a^{10}}\frac{d^2P_{10}^2(\cos\theta)}{d\theta^2}\Big\}\cos 2\phi, \quad \ldots (21')$$

$$\widehat{\phi\phi}=\Big\{\frac{5}{9.11.13}\frac{3.5.\lambda r^2+2(8\lambda+7\mu)a^2}{(19\lambda+14\mu)a^2}P_2^2(\cos\theta)$$

$$+\frac{8}{11.13.15}\frac{15(7\lambda-2\mu)r^4+4(24\lambda+23\mu)a^2r^2}{3(51\lambda+42\mu)a^4}P_4^2(\cos\theta)$$

$$+\frac{16}{15.17.33}\frac{35(9\lambda-4\mu)r^6+6(48\lambda+47\mu)a^2r^4}{5(99\lambda+86\mu)a^6}P_6^2(\cos\theta)$$

$$+\frac{64}{13.15.19.33}\frac{63(11\lambda-6\mu)r^8+8(80\lambda+79\mu)a^2r^6}{7(163\lambda+146\mu)a^8}P_8^2(\cos\theta)$$

$$+\frac{128}{13.15.17.19.33}\frac{99(13\lambda-8\mu)r^{10}+10(120\lambda+119\mu)a^2r^8}{9(243\lambda+222\mu)a^{10}}P_{10}^2(\cos\theta)\Big\}\cos 2\phi$$

$$+\Big\{\frac{5}{9.11.13}\frac{(8\lambda+7\mu)a^2-(5\lambda+7\mu)r^2}{(19\lambda+14\mu)a^2}\left(\frac{dP_2^2(\cos\theta)}{d\theta}\cot\theta-4\frac{P_2^2(\cos\theta)}{\sin^2\theta}\right)$$

$$+\frac{8}{11.13.15}\frac{(24\lambda+23\mu)a^2r^2-3(7\lambda+9\mu)r^4}{3(51\lambda+42\mu)a^4}\left(\frac{dP_4^2(\cos\theta)}{d\theta}\cot\theta-4\frac{P_4^2(\cos\theta)}{\sin^2\theta}\right)$$

$$+\frac{16}{15.17.33}\frac{(48\lambda+47\mu)a^2r^4-5(9\lambda+11\mu)r^6}{5(99\lambda+86\mu)a^6}\left(\frac{dP_6^2(\cos\theta)}{d\theta}\cot\theta-4\frac{P_6^2(\cos\theta)}{\sin^2\theta}\right)$$

$$+\frac{64}{13.15.19.33}\frac{(80\lambda+79\mu)a^2r^6-7(11\lambda+13\mu)r^8}{7(163\lambda+146\mu)a^8}\left(\frac{dP_8^2(\cos\theta)}{d\theta}\cot\theta\right.$$

$$\left.-4\frac{P_8^2(\cos\theta)}{\sin^2\theta}\right)$$

$$+\frac{128}{13.15.17.19.33}\frac{(120\lambda+119\mu)a^2r^8-9(13\lambda+15\lambda)r^{10}}{9(243\lambda+222\mu)a^{10}}\left(\frac{dP_{10}^2(\cos\theta)}{d\theta}\cot\theta\right.$$

$$\left.-4\frac{P_{10}^2(\cos\theta)}{\sin^2\theta}\right)\Bigg\}\cos 2\phi,\ldots\ldots(22')$$

$$\widehat{r\theta}=\Bigg\{\frac{5}{9.11.13}\frac{(8\lambda+7\mu)(a^2-r^2)}{(19\lambda+14\mu)a^2}\frac{dP_2^2(\cos\theta)}{d\theta}$$

$$+\frac{8}{11.13.15}\frac{(24\lambda+23\mu)(a^2r^2-r^4)}{(51\lambda+42\mu)a^4}\frac{dP_4^2(\cos\theta)}{d\theta}$$

$$+\frac{16}{15.17.33}\frac{(48\lambda+47\mu)(a^2r^4-r^6)}{(99\lambda+86\mu)a^6}\frac{dP_6^2(\cos\theta)}{d\theta}$$

$$+\frac{64}{13.15.19.33}\frac{(80\lambda+79\mu)(a^2r^6-r^8)}{(163\lambda+146\mu)a^8}\frac{dP_8^2(\cos\theta)}{d\theta}$$

$$+\frac{128}{13.15.17.19.33}\frac{(120\lambda+119\mu)(a^2r^8-r^{10})}{(243\lambda+222\mu)a^{10}}\frac{dP_{10}^2(\cos\theta)}{d\theta}\Bigg\}\cos 2\phi,\ldots(23')$$

$$\widehat{r\phi}=-2\Bigg\{\frac{5}{9.11.13}\frac{(8\lambda+7\mu)(a^2-r^2)}{(19\lambda+14\mu)a^2}\frac{P_2^2(\cos\theta)}{\sin\theta}$$

$$+\frac{8}{11.13.15}\frac{(24\lambda+23\mu)(a^2r^2-r^4)}{(51\lambda+42\mu)a^4}\frac{P_4^2(\cos\theta)}{\sin\theta}$$

$$+\frac{16}{15.17.33}\frac{(48\lambda+47\mu)(a^2r^4-r^6)}{(99\lambda+86\mu)a^6}\frac{P_6^2(\cos\theta)}{\sin\theta}$$

$$+\frac{64}{13.15.19.33}\frac{(80\lambda+79\mu)(a^2r^6-r^8)}{(163\lambda+146\mu)a^8}\frac{P_8^2(\cos\theta)}{\sin\theta}$$

$$+\frac{128}{13.15.17.19.33}\frac{(120\lambda+119\mu)(a^2r^8-r^{10})}{(243\lambda+222\mu)a^{10}}\frac{P_{10}^2(\cos\theta)}{\sin\theta}\Bigg\}\sin 2\phi,\ldots\ldots(24')$$

$$\widehat{\theta\phi}=2\Bigg\{\frac{5}{9.11.13}\frac{(8\lambda+7\mu)a^2-(5\lambda+7\mu)r^2}{(19\lambda+14\mu)a^2}\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}P_2^2(\cos\theta)\right.$$

$$\left.-\frac{dP_2^2(\cos\theta)}{d\theta}\right)$$

$$+\frac{8}{11.13.15}\frac{(24\lambda+23\mu)a^2r^2-3(7\lambda+9\mu)r^4}{3(51\lambda+42\mu)a^4}\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}P_4^2(\cos\theta)\right.$$

$$\left.-\frac{dP_4^2(\cos\theta)}{d\theta}\right)$$

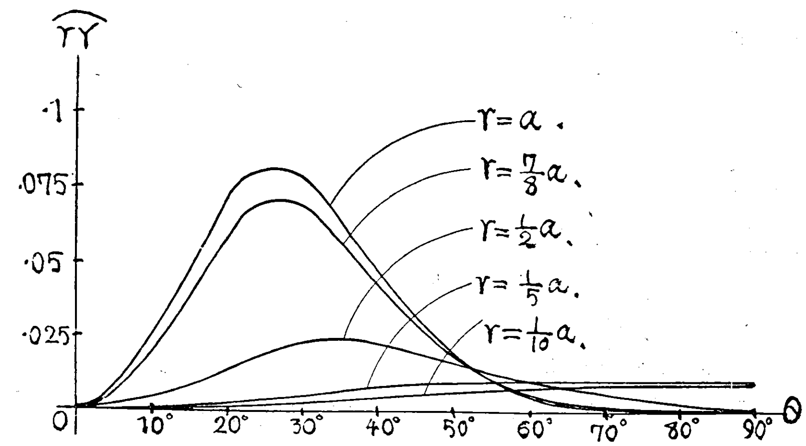
$$+\frac{16}{15.17.33}\frac{(48\lambda+47\mu)a^2r^4-5(9\lambda+11\mu)r^6}{5(99\lambda+86\mu)a^6}\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}P_6^2(\cos\theta)\right.$$

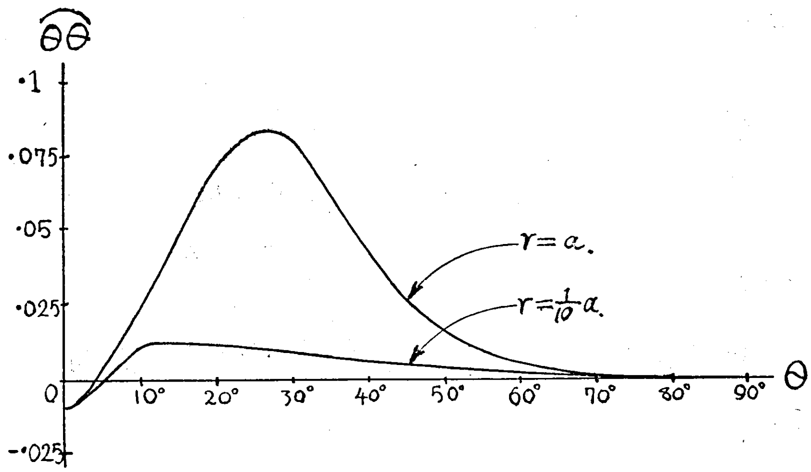
$$\left.-\frac{dP_6^2(\cos\theta)}{d\theta}\right)$$

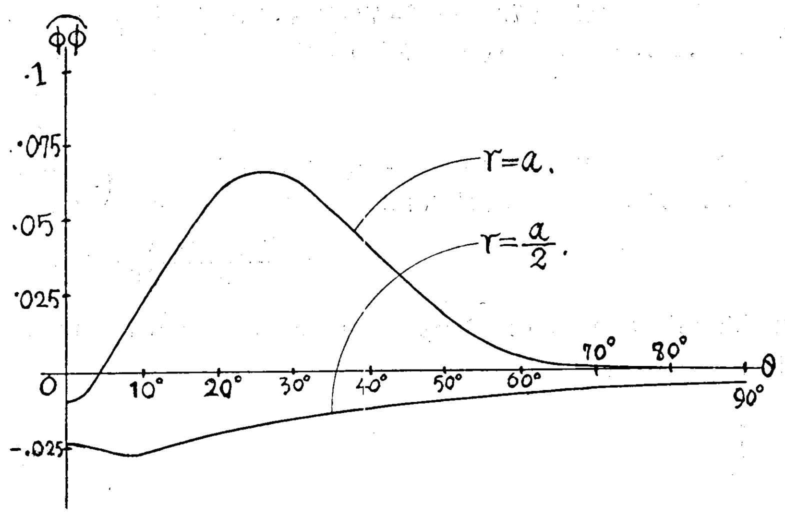
$$+\frac{64}{13.\,15.\,19.\,33}\frac{(80\lambda+79\mu)\,a^2r^6-7(11\lambda+13\mu)\,r^8}{7(163\lambda+146\mu)\,a^8}\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}P_8^2(\cos\theta)-\frac{dP_8^2(\cos\theta)}{d\theta}\right)$$

$$+\frac{128}{13.\,15.\,17.\,19.\,33}\frac{(120\lambda+119\mu)\,a^2r^8-9(13\lambda+15\mu)\,r^{10}}{9(243\lambda+222\mu)\,a^{10}}\frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}P_{10}^2(\cos\theta)-\frac{dP_{10}^2(\cos\theta)}{d\theta}\right)\Bigg\}\sin 2\phi\ldots\ldots(25')$$

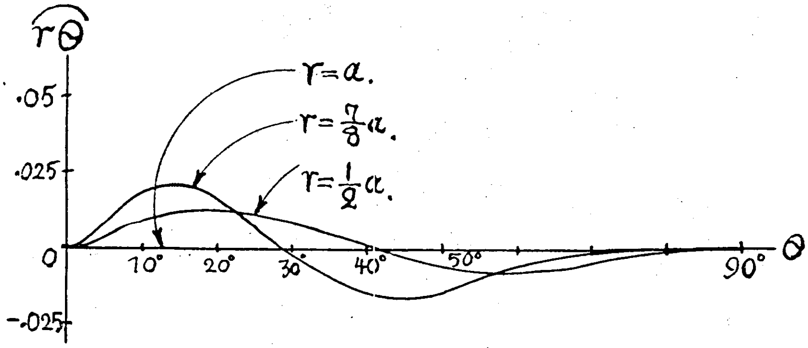
Compiling these in the case $\lambda=\mu$, the results are plotted in the annexed drawing.



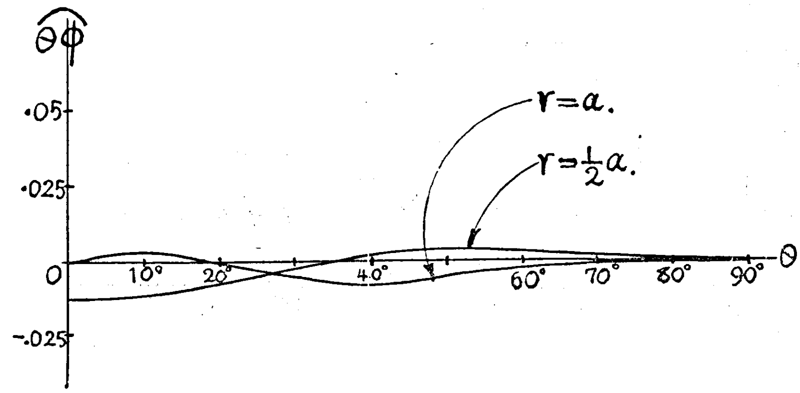




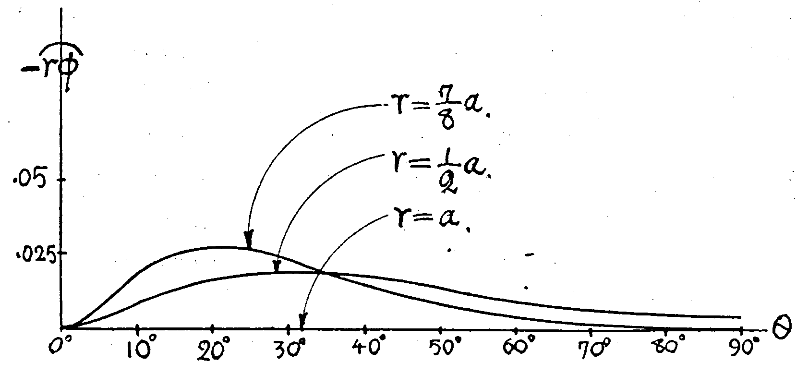
$\widehat{\phi\phi}$
·1
·075
·05
·025
0
-.025
10° 20° 30° 40° 50° 60° 70° 80° 90°
θ
r=a.
r=a/2.



$\widehat{r\theta}$
·05
·025
0
-.025
10° 20° 30° 40° 50° 90°
θ
r=a.
r=7/8 a.
r=1/2 a.



$\widehat{\theta\phi}$
·05
·025
0
-.025
10° 20° 40° 60° 70° 80° 90°
θ
r=a.
r=1/2 a.



From these diagrams it appears that, however sharply concentrated the stress on the surface may be, the distribution of stresses in the interior of the sphere has the tendency of uniformization; sometimes it occurs that stresses of contrary signs can exist at a certain radius smaller than the radius of the sphere. It reveals that the apparently hydrostatic pressure in the solid body may be explained by the above uniformising property of stresses in the interior of the solid. Again, it will not be difficult to prove such property in the case of a semi-infinite solid body; this special nature is, however, conspicuous in the case of a solid sphere. The fact, that the components of stress, such as $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$ are of comparable magnitude as that of $\widehat{rr}$, has some importance on break-down problem of the earth crust. If the earth could be treated as homogeneous, to a depth considerably greater than the scale of the tracted region on the surface, the results of the present study will be valid for the practical problem. The modification necessitated by gravity and some other causes is nothing more than a correction, which will be made to the principal results.

Résumé.

We have now obtained, by the application of harmonic functions, some results having interests on the geophysical problem, and shall enumerate the principal results as follows:

(1) However sharp the concentrated stress on the surface of the sphere may be, the stresses in the interior of the sphere have the tendency

of approaching the uniformized distribution, showing the nature of the apparently hydrostatic pressure beneath the surface of the earth. Sometimes it occurs that the same component of the stress having contrary signs can exist at different points of a given radius of the sphere.

(2) Even though the radial stress (excepting zero tangential stresses) is merely specified, the azimuthal and colatitudinal components of the stress are of comparable magnitude with the radial component; this conforms with the nature of the bottom of the sea, in which the surface of the solid earth is in a state of nearly so called "hydrostatic equilibrium" under the surface loading.

(3) Besides the problem of tidal loading in the equeous regions of the irregular form on the earth's surface, as in Panama, some portions of the margin lines separating the continental block and the oceanic basin, where the isostatic equilibrium of the crusts cannot hold perfectly and the state of stressed equilibrium is maintained, may be regarded as examples which will be treated of by the present analysis. The problem of apparently isostatic support, which holds for somewhat greater deepth of the earth crust, can be explained by the results of this paper without considering the liquidous basin for such small portions.

In concluding this the authors are indebted to Professor Terada for his kind advice.

October, 1928.