*On the Damped Transversal Vibration of Prismatic Bars.**

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壔狀體の減衰振動

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架構の組子や塔や煙突などの如き、梁と見做すべきものの振動を論ずるには、其振動の減衰性を考慮せれば、場合によつては誤つた結論に到達する患がないとも限られぬ、本論文では振動に對する抵抗は、主として固體粘性によるものである、といふ說を基礎として梁の減衰振動を論じ、次の事柄を示してゐる、

（イ）振動の次の高き程急激に減衰性が增加すること、

（ロ）隨て或る一定の次以上の振動は生ずる患なきこと、

（ハ）種々の實例を解析して、減衰性を定むる量 ξ/E（ξ は縱粘性係數、E は「ヤング」率）の値を見出し、煉瓦及「コンクリート」造の梁に對しては此値を 10^{-3} の程度に取れば、實際の減衰性を見出し得ること、

（ニ）煉瓦及「コンクリート」造の梁は、一般に三次或は四次以上の自由振動を爲し得る可能性なきこと。

In the study of the vibration of structures, which is one of the important problems of applied seismology, the resistance against the vibration is very seldom, if ever, considered. Hence, in some cases there is a fear that the conclusion arrived at therefrom is incorrect.

In the present paper the author deals with the free transversal vibration of prismatic bars subjected to resistance. The nature of the free vibration of framed structures, towers, chimneys, etc. can well be inferred from the result of the present investigation.

The resistance against the vibration is, as well known, composed of two sorts, one of them being the air resistance and the other the internal

* An outline of this paper was published in Proc. Imp. Aca. 4, 6 (1928).

resistance due to the solid viscosity of the material or to its internal friction. The fact that the former is insignificant compared with the latters was shown by Lord Kelvin[1] and other investigators. Therefore, in the present investigation the air resistance is not taken into consideration.

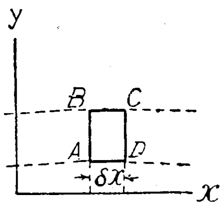
The solid viscosity was first studied by W. Voigt and recently by K. Honda.[2] According to these investigators the viscosity of solid is similar to that of fluid and may be divided into two kinds:—

the tangential viscosity, $\eta \dfrac{\partial \phi}{\partial t}$,

and the normal viscosity, $\xi \dfrac{\partial e}{\partial t}$,

in which t is the time, and ϕ and e are shearing and tensile strains respectively. Although there is every possibility that the internal resistance does not wholly depend upon the rate of change of strain, but also to stress-amplitude,[3] yet the author takes up these investigators' view, as otherwise the problem is not amenable to mathematics, and as the aim of the present paper is merely to study the general nature of the resisted vibration; the investigation of the exact nature of the internal resistance itself being out of place.

To simplify the problem, let us take that the bar is uniform and straight, and has a longitudinal plane of symmetry, and also that the transversal vibration takes place parallel to this plane. Take the axis of the bar in the position of equilibrium as x, and an axis perpendicular to it and parallel to the plane of symmetry as y.



Then the equation of the lateral motion of an elementary slice $ABCD$ taken across the bar and having the thickness δx is

$$\rho A \frac{\partial^2 y}{\partial t^2} \delta x = \frac{\partial F}{\partial x} \delta x + A \eta \frac{\partial^2 \phi}{\partial t \partial x} \delta x,$$

and as

(1) Lord Kelvin's Mathematical and Physical Papers, 3.
(2) K. Honda, Phil. Mag. 42 (1921).
(3) Kimball and Lovell, Phys. Rev. 30, 6 (1927).

$$\phi \doteqdot \frac{F}{A\mu}, \qquad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

we have

$$\rho A \frac{\partial^2 y}{\partial t^2}\delta x = \frac{\partial F}{\partial x}\delta x + \frac{\eta}{\mu}\frac{\partial^2 F}{\partial x \partial t}\delta x, \dots\dots\dots\dots\dots\dots (2)$$

in which A is the sectional area of the bar, ρ its density, μ the modulus of rigidity, and F the shearing force at the section.

Again, if we denote the moment of inertia of the cross section about its neutral axis by I, the Young's modulus by E, and the bending moment at the section by M, the equation of the angular motion of the elementary slice is

$$\rho I \frac{\partial^2}{\partial t^2}\left(\frac{\partial y}{\partial x}\right)\delta x = -\frac{\partial M}{\partial x}\delta x + F\delta x$$
$$-\frac{\partial}{\partial x}\left(\int_A \xi \frac{\partial e}{\partial t}. b. y'. dy'\right)\delta x + A\eta \frac{\partial \phi}{\partial t}\delta x \dots\dots\dots (3)$$

in which y' is the distance of a point on the section from its neutral axis and b the breadth of the section. As

$$e = \frac{M}{E}\frac{y'}{I},$$

we have

$$\int_A \frac{\partial e}{\partial t} by' dy' = \frac{1}{EI}\frac{\partial M}{\partial t}\int_A y' by' dy = \frac{1}{E}\frac{\partial M}{\partial t}.$$

In virtue of this equality and also of (1), (3) can be written

$$\rho I \frac{\partial^2}{\partial t^2}\left(\frac{\partial y}{\partial x}\right)\delta x = -\frac{\partial M}{\partial x}\delta x + F\delta x - \frac{\xi}{E}\frac{\partial^2 M}{\partial x \partial t}\delta x + \frac{\eta}{\mu}\frac{\partial F}{\partial t}\delta x \dots\dots\dots (4)$$

Taking off δx from (2) and (4), and eliminating F between them, we have

$$\rho A \frac{\partial^2 y}{\partial t^2} = \rho I \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 y}{\partial x^2}\right) + \frac{\partial^2 M}{\partial x^2} + \frac{\xi}{E}\frac{\partial^3 M}{\partial x^2 \partial t}.$$

Remembering that $M = -EI\dfrac{\partial^2 y}{\partial x^2}$, we obtain

$$\rho A \frac{\partial^2 y}{\partial t^2} = \rho I \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 y}{\partial x^2}\right) - EI\frac{\partial^4 y}{\partial x^4} - \xi I \frac{\partial^5 y}{\partial x^4 \partial t}. \qquad \dots\dots\dots\dots (5)$$

If we suppose that the transverse dimensions of the bar are small compared with its length, and confine our attention to the waves having the lengths sufficiently larger than the transverse dimensions, the first term of the right-hand member of (5) may be neglected. Then the equation of motion for the resisted vibration of bars takes the form,

$$\rho A\frac{\partial^2 y}{\partial t^2}=-EI\frac{\partial^4 y}{\partial x^4}-\xi I\frac{\partial^5 y}{\partial x^4\partial t}. \qquad \qquad (6)$$

As this equation has already been obtained by T. Sezawa,(4) its deduction is not needed to be described. But as he did neither mention the process of obtaining it, nor show the extent of the applicability of the equation (6), the author describes the procedure in detail.

Now it is well known that the motion expressed by an equation as (6) may be analysed into component motions corresponding to the variation of normal co-ordinate. Accordingly put

$$y=u_r\phi_r,$$

in which u_r is a normal function and ϕ_r the corresponding normal co-ordinate. Substituting this into (6), we obtain

$$\frac{d^4u_r}{dx^4}-m_r^4u_r=0,$$

and

$$\frac{d^2\phi_r}{dt^2}+m_r^4\frac{\xi}{\rho}\kappa^2\frac{d\phi_r}{dt}+m_r^4\frac{E}{\rho}\kappa^2\phi_r=0.$$

where $m_r=\frac{n_r}{l}$, n_r being the number characterizing the rth. order of vibration and l the length of the bar, and κ^2 stands for I/A.

From the latter equation we have

$$\phi_r=a\,e^{-\frac{1}{2}m_r^4\frac{\xi}{\rho}\kappa^2 t}\cos\left(\frac{1}{2}\sqrt{4m_r^4\frac{E}{\rho}\kappa^2-m_r^8\frac{\xi^2}{\rho^2}\kappa^4}\,t-\alpha\right) \qquad \qquad (7)$$

This expression tells that the transverse vibration of a bar is impossible unless

$$m_r^8\frac{\xi^2}{\rho^2}\kappa^4<4m_r^4\frac{E}{\rho}\kappa^2.$$

or

$$\frac{\pi\xi}{E}<\frac{2\pi}{m_r^2\kappa}\sqrt{\frac{\rho}{E}}.$$

(4) K. Sezawa, This Bulletin, **3** (1927).

which can be written

$$\frac{\pi\xi}{E} < T_r, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

T_r being the period of the unresisted vibration of the rth order. This expression gives the upper limit of the frequency of the unresisted vibration and furthermore the upper limit of the order of the vibration. With regard to the same of the actual resisted vibration, it can be shown that the shortest period T_m with which an actual bar can vibrate is given by

$$T_m = \frac{2\pi\xi}{E}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

(not quite this value, but the nearest to it). This can be obtained from (7) in finding the maximum value of $\frac{1}{2}\sqrt{4m_r^4 \frac{E}{\rho} \kappa^2 - m_r^8 \frac{\xi^2}{\rho^2} \kappa^4}$.

For steel $E = 2 \times 10^{12}$, and if we take the vale of ξ given by Honda, namely $\xi = 5 \times 10^8$, we have

$$\frac{\xi}{E} = 2.5 \times 10^{-4}$$

Therefore, from (8) and (9)

$$\frac{\pi\xi}{E} \doteqdot 10^{-3} \text{ sec.} \quad \text{and} \quad T_m \doteqdot \frac{1}{500} \text{ sec.}$$

Thus, if the value of ξ given by Honda is correct, the frequency of the transverse vibration of a steel bar cannot exceed 500 times per sec. whatever scantlings it may have. Evidently our experience cannot justify such a conclusion.

By the way, the values of ξ found by several investigators differ so enormously that even the order of the magnitude cannot be definitely given. As this fact may most likely be attributed to the dissipation of the energy of the vibration of a piece of metal under test through its clamped end, the author experimented with a tuning fork made by a maker having the highest reputation in order to minimize the said effect. The value thus obtained was

$$\xi = 5 \times 10^5$$

Although the author does not pretend that this value is correct, yet the upper limit of the frequency of the vibration found from this value is not against our daily experience.

Thus even the viscous property of the steel is not quite unravelled, and still more obscure is that of ordinary building materials. Indeed, nobody has ever experimented upon it, and accordingly treatment of the damped vibration of a bar made of a building material is hopeless for the present. The following inference deduced from some seismological experiments may, however, gives us some rough idea about it.

Now the intensity of the resistance against the transversal vibration may be found from the exponential term in (7),

$$e^{-\frac{1}{2}m_r^4\frac{\xi}{\rho}\kappa^2 t}=e^{-\frac{1}{2}\frac{\xi}{E}\frac{4\pi^2}{T_r^2}t}=e^{-\frac{1}{2}kt}$$

in which k stands for $\frac{\xi}{E}\frac{4\pi^2}{T_r^2}$. Therefore, if the logarithmic decrement of the amplitude and the natural period of a vibration are given, we can guess the value of $\frac{\xi}{E}$. By the way, as the coeffcient of damping is inversely proportional to the square of the period of vibration, the higher the order of vibration, the quicker the vibration dies away. For instance, in a cantilever the coefficient of damping of the vibration of the second order is

$$\left(\frac{n_2}{n_1}\right)^4=\left(\frac{4.694}{1.875}\right)^4\fallingdotseq 39$$

times as large as that of the first order.

Now making use of the results of several investigators' experiments on the vibration of pillars, chimneys etc., we obtain the following results:—

(1) Experimenter, F. Omori.[5]

A brick column, 4.95 m. high × 0.455 m. broad × 0.225 m. deep.

Period of fundamental vibration = 0.26 sec.

Logarithmic decrement = 0.148

From these data, we obtain

$$\frac{\xi}{E}=1.95\times 10^{-3}$$

Therefore, from (8) and (9)

(5) F. Omori, Bull. Imp. Earthq. Inv. Comm. 2, 3 (1908).

$$T_r > \frac{1}{160} \text{ sec.} \quad \text{and} \quad T_m \doteqdot \frac{1}{80} \text{ sec.}$$

Thus in this case, the actual frequency cannot exceed 80 times per sec.

If we assume that the ratio of the frequencies of the vibrations of different orders is the same as that of a uniform prismatic cantilever, the upper limit of the frequency of the unresisted vibration is less than that of the 5th. order. In the remaining examples the upper limit of the frequency is not shown, as its calculation can easily be done. But it is worth especially mentioning that in all cases the vibration of the order higher than, say, the third or fourth cannot actually takes place, so that in dealing with the vibrations of a structure those of high orders should not be taken into consideration.

(2) Experimenter, F. Omori.(6)

A brick chimney, 15.45 m. high × 1.32 m. square.

Period of fundamental vibration = 1.03 sec.

Logarithmic decrement = 0.06

From these data we get

$$\frac{\xi}{E} = 3.13 \times 10^{-3}$$

(3) Experimenter, T. Taniguti.(7)

A reinforced concrete square frame:

Height, 200 cm.

Breadth, 200 cm.

Section, Rectangle, 10 cm. in depth (across the plane of the frame), and 8 cm. in breadth (in the plane of it).

Period of fundamental vibration = 0.14 sec.

Logarithmic decrement = 0.26

From these values we obtain

$$\frac{\xi}{E} = 1.85 \times 10^{-3}$$

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(6) F. Omori, Pub. Imp. Earthq. Inv. Comm. 12 (1902).

(7) T. Taniguti, Jour. Jap. Soc. Arch. 506 (1928).

(4) Experimenter, T. Tanakadate.[8]

A brick chimney, 5.9 m. high × 1.06 m. square with two flues.

(a) Vibration in the direction of the least moment of inertia of section.

Period of fundamental vibration = 0.333 sec.

Logarithmic decrement = 0.253

Whence we have

$$\frac{\xi}{E} = 4.28 \times 10^{-3}$$

(b) Vibration in the direction of the largest moment of inertia of section.

Period of fundamental vibration = 0.450 sec.

Logarithmic decrement = 0.210

From these data we obtain

$$\frac{\xi}{E} = 4.78 \times 10^{-3}$$

It should be kept in mind that in all cases some portion of the energy of vibration might have dissipated through the fixed end. Therefore, the values of $\frac{\xi}{E}$ given in the above examples may have little value for finding ξ as a physical constant. However, for practical purpose of estimating the resistance against the vibration of an actual structural member the informations given above may be of some value.

(8) A Tanakadate, Rep. Imp. Earthq. Inv. Comm. (in Jap.) 21 (1898).