On the Diffraction of Elastic Waves.

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彈性波の廻折に就て

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間隙による音波の廻折に就てはレーレー卿などによつて其の解を與へられてゐるが、彈性波に於ける同問題は今日に到るまで遂に提出を見なかつた樣である。近年マシュー函數の發展は可なり著しい樣に思はれるが、それを適當に利用して、著者は彈性波が障碍の間隙を通つてから廻折する理論を計算した。マシュー函數が未だ極く完全な發達の領域に達して居らぬから、多少の欠陥はあるが、大體其の廻折の性質を極め得る樣に見える。

理論的に面白い結果を摘錄すれば

(1) 間隙を通しての廻折は種々の場合があり、波長、間隙の幅、入射波の性質、障碍の應力狀態によつて其の性質を異にする。

(2) 入射波が縱波とし、障碍が自由な爲に應力を受けぬ時は、間隙からの廻折波は縱橫二種があり、且つ其の二種の勢力は同じ程度のものである。

(3) 障碍が固定されて居る時は廻折波は主として縱波のみから成る。

The theory of the scattering of elastic waves by some kinds of obstacles was given by the present author[1] in this Bull. III. Although these obstacles were of some idealized kinds, yet it seems that the theory was an important guidance for the confirmation of the behaviours of seismic waves. This can be clearly known in the light of other branches of physical sciences, in which highly idealized models contribute very often to the determination of the natures.

An important problem arising in connection with the theory of scattering is the treatment of the diffraction of elastic waves by an aperture of a screen. So far as the author is aware, this problem has not yet been studied for the reason that it involves mathematical difficulty. The diffraction of sound waves by an apperture of a screen was successfully investigated by Lord Rayleigh[2];

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(1) *Bull.* III *of the Institute* (1927), p. 19.

(2) Lord Rayleigh, *Scientific Papers*, Vol. 4, p. 283.

the method employed by him, however, can never be applied to the problem of elastic waves. No effort, that might have been made by means of ordinary mathematics to the analysis of the diffraction of elastic waves by an aperture of a screen, would have given any successful solution of the problem.

As the recent development(1) of Mathieu's functions enables us that a few complicated problems in physics are attacked analytically, it reveals that Mathieu's functions will also be applied to the problem of the diffraction of elastic waves. The author's attempt to apply Mathieu's functions to such a problem is not, of course, sufficient for the complete solutions; but it appears that the investigation by means of such functions has answered the purpose of the elucidation of the nature of the present problem.

In this paper an attempt is made to study the way in which the waves are diffracted by an apperture of an infinitely extended screen, when the primary waves are of plane type with wave fronts parallel to the screen. The primary waves, which are transmitted in a purely elastic medium, are reflected at the screen, while the diffracted waves are formed at the aperture so as to make the stresses and the displacements continuous there. As to the breadth of the aperture, we consider it very narrow, because of the easy application of Mathieu's functions. The secondary waves are composed of two kinds of waves, that is, diffracted dilatational and distortional waves. The effects of barriers, which form the screen, on the secondary waves are neglected for the reason that their energy is very small compared with that of the normally reflected waves.

General Solutions.

1. When the axis of x is taken coincident with the screen in an elastic body and the axis of y, passing through the middle point of the slit, is directed perpendicularly to the x-axis, the equations of motion of the body are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2 \Delta}{\partial t^2} &= (\lambda+2\mu)\left(\frac{\partial^2 \Delta}{\partial x^2}+\frac{\partial^2 \Delta}{\partial y^2}\right), \\ \rho\frac{\partial^2 \varpi}{\partial t^2} &= \mu\left(\frac{\partial^2 \varpi}{\partial x^2}+\frac{\partial^2 \varpi}{\partial y^2}\right), \end{aligned}\right\} \quad \cdots\cdots(1)$$

(1) H. Jeffreys, *London Math. Soc. Proc.* (1925).
Humbert, *Fonctions de Mathieu et de Lamé* (1926).
Goldstein, *Cambridge Phil. Soc. Trans.* Vol. 23 (1927).
M. J. O. Strutt, *Ann. d. Phys. & Phil. Mag.* (1927 & 1928).

where

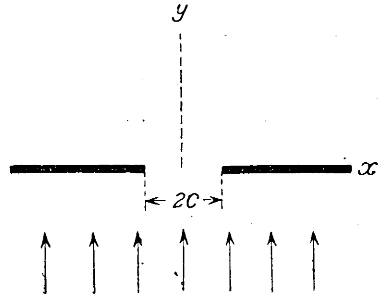
$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

$$2\varpi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

u, v = components of displacement parallel to x- and y-axes respectively,

ρ = density of the medium,

λ, μ = Lamé's elastic constants.



The expression of the primary waves, when they are dilatational, and directed parallel to y-axis, are deduced from the equation (1) as follows:—

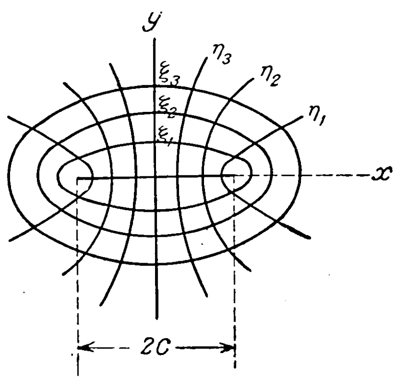
$$\Delta_0' = Ae^{i(hy-pt)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which

$$h^2 = \frac{\rho p^2}{\lambda + 2\mu},$$

$$\frac{2\pi}{p} = \text{period of waves},$$

$$\frac{2\pi}{h} = \text{wave length}.$$



The equation (2), when we refer to the orthogonal curvilinear co-ordinates constituting a series of confocal ellipses and hyperbolas, is written by

$$\Delta_0' = Ae^{i(hc\,\mathrm{sh}\xi\sin\eta - pt)}$$

$$= e^{-ipt}\left[\sum_{n=0}^{\infty} A_n Ce_n(\xi, q)\, ce_n(\eta, q) + \sum_{1=n}^{\infty} A_n' Se_n(\xi, q)\, se_n(\eta, q)\right], \quad \ldots (3)$$

where $2c$=breadth of the apperture and

$$q^2 = \frac{h^2c^2}{32}.$$

In the equation (3), $ce_n(\eta, q)$, $se_n(\eta, q)$ are Mathieu's functions that reduce to constant multiples of $\cos nx$, $\sin nx$ when $q \to 0$. $Ce_n(\xi, q)$, $Se_n(\xi, q)$ are associated[1] Mathieu's functions, such that

$$\left.\begin{aligned} Ce_n(\xi, q) &= ce_n(i\xi, q), \\ Se_n(\xi, q) &= se_n(i\xi, q). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

2. Mathieu's functions have the integral properties which are given by

$$\left.\begin{aligned} &\int_0^{2\pi} ce_n(x, q)\, ce_m(x, q)\, dx = 0, \qquad [n \neq m] \\ &\int_0^{2\pi} se_n(x, q)\, se_m(x, q)\, dx = 0, \qquad [n \neq m] \\ &\int_0^{2\pi} ce_n(x, q)\, se_m(x, q)\, dx = 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5)$$

and, when $q \to 0$,

$$\left.\begin{aligned} &\int_0^{2\pi} [ce_0(x, q)]^2 dx = 2\pi, \qquad \int_0^{2\pi} [ce_n(x, q)]^2 dx = \pi, \\ &\int_0^{2\pi} [se_n(x, q)]^2 dx = \pi. \end{aligned}\right\} \quad \ldots\ldots\ldots (6)$$

Now $ce_n(\eta, q)$, $se_n(\eta, q)$ can be expanded in the forms,[2]

$$\begin{aligned} ce_0(\eta, q) &= 1 + 4q\cos 2\eta + 2q^4\cos 4\eta \\ &\quad + 8^3q^6\left\{\frac{\cos 6\eta}{2^7.3^2} - \frac{7\cos 2\eta}{3^2}\right\} + 8^4q^8\left\{\frac{\cos 8\eta}{2^{13}.3^2} - \frac{5\cos 4\eta}{2^7.3^2}\right\} + \cdots, \\ ce_1(\eta, q) &= \cos\eta + q^2\cos 3\eta + q^4\left(\frac{1}{3}\cos 5\eta - \cos 3\eta\right) \\ &\quad + q^6\left(\frac{1}{18}\cos 7\eta - \frac{4}{9}\cos 5\eta + \frac{1}{3}\cos 3\eta\right) \\ &\quad + q^8\left(\frac{1}{180}\cos 9\eta - \frac{1}{12}\cos 7\eta + \frac{1}{6}\cos 5\eta + \frac{11}{9}\cos 3\eta\right) \\ &\quad + \cdots, \end{aligned}$$

(1) H. Jeffreys, *London Math. Soc. Proc.* (1925).
(2) Whittaker, *Modern Analysis*, p. 427.

$$\left.\begin{aligned}
se_1(\eta, q) &= \sin\eta + q^2\sin 3\eta + q^4\left(\frac{1}{3}\sin 5\eta + \sin 3\eta\right)\\
&\quad + q^6\left(\frac{1}{18}\sin 7\eta + \frac{4}{9}\sin 5\eta + \frac{1}{3}\sin 3\eta\right)\\
&\quad + q^8\left(\frac{1}{180}\sin 9\eta + \frac{1}{12}\sin 7\eta + \frac{1}{6}\sin 5\eta - \frac{11}{9}\sin 3\eta\right)\\
&\quad + \cdots,\\
c_2(\eta, q) &= \cos 2\eta + q^2(\cos 4\eta - 2) + \frac{1}{6}q^4\cos 6\eta\\
&\quad + q^6\left(\frac{1}{45}\cos 8\eta + \frac{43}{27}\cos 4\eta + \frac{40}{3}\right)\\
&\quad + q^8\left(\frac{1}{540}\cos 10\eta + \frac{293}{540}\cos 6\eta\right) + \cdots.
\end{aligned}\right\} \quad (7)$$

3. In the case of expansion in (3), the approximate values of the associated Mathieu's functions with constant coefficients are given by the forms[1] below:—

$$\left.\begin{aligned}
&[A_0 Ce_0(\xi, q)]_{q\to 0} = AJ_0(hc\,\mathrm{ch}\xi),\\
&A_1 = 0,\\
&[A_2 Ce_2(\xi, q)]_{q\to 0} = 2AJ_2(hc\,\mathrm{ch}\xi),\\
&A_3 = 0,\\
&[A_4 Ce_4(\xi, q)]_{q\to 0} = 2AJ_4(hc\,\mathrm{ch}\xi),\\
&A_5 = 0,\\
&\cdots\cdots
\end{aligned}\right\} \quad \cdots\cdots\cdots (8)$$

$$\left.\begin{aligned}
&[A_1' Se_1(\xi, q)]_{q\to 0} = 2iAJ_1(hc\,\mathrm{ch}\xi),\\
&A_2' = 0,\\
&[A_3' Se_3(\xi, q)]_{q\to 0} = 2iAJ_3(hc\,\mathrm{ch}\xi),\\
&A_4' = 0,\\
&\cdots\cdots
\end{aligned}\right\} \quad \cdots\cdots\cdots (9)$$

4. Returning to the original solution (3), the displacement corresponding to Δ_0' is expressed by

$$\left.\begin{aligned}
u_0' = -e^{-ipt}\Bigg[&\sum_{n=0}^{\infty} \frac{A_n}{h^2 c\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \frac{\partial Ce_n(\xi, q)}{\partial\xi} ce_n(\eta, q)\\
&+ \sum_{n=1}^{\infty} \frac{A_n'}{h^2 c\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \frac{\partial Se_n(\xi, q)}{\partial\xi} se_n(\eta, q)\Bigg],
\end{aligned}\right\} \quad \cdots (10)$$

(1) Heine, *Kugelfunktionen*, Bd. I, § 106.

$$v_0' = -e^{-ipt}\left[\sum_{n=0}^{\infty}\frac{A_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}Ce_n(\xi,q)\frac{\partial ce_n(\eta,q)}{\partial\eta}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{A_n'}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}Se_n(\eta,q)\frac{\partial se_n(\eta,q)}{\partial\eta}\right].$$

5. The reflected waves at the screen are readily obtained in the following forms:—

When the screen is free from stress, the dilatation is expressed by

$$\Delta_0'' = -Ae^{-i(hy+pt)}, \qquad\qquad (11)$$

so that the resultant dilatation composed of Δ_0' and Δ_0'' is expressed by

$$\Delta_0 = \Delta_0' + \Delta_0'' = 2\sum_{n=1}^{\infty}A_n'Se_n(\xi,q)se_n(\eta,q)e^{-ipt}, \qquad\qquad (12)$$

The corresponding displacement is expressed by

$$\left.\begin{aligned} u_0 &= -\sum_{n=1}^{\infty}\frac{2A_n'}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial Se_n(\xi,q)}{\partial\xi}se_n(\eta,q)e^{-ipt},\\ v_0 &= -\sum_{n=1}^{\infty}\frac{2A_n'}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}Se_n(\xi,q)\frac{\partial se_n(\eta,q)}{\partial\eta}e^{-ipt}, \end{aligned}\right\} \qquad (13)$$

When the screen is composed of infinitely rigid barriers, at which $u_0'+u_0''=v_0'+v_0''=0$ must satisfy, the dilatation of the reflected waves is expressed by

$$\Delta_0'' = Ae^{-i(hy+pt)}, \qquad\qquad (14)$$

so that

$$\Delta_0 = \Delta_0' + \Delta_0'' = 2\sum_{n=0}^{\infty}A_nCe_n(\xi,q)ce_n(\eta,q)e^{-ipt}, \qquad\qquad (15)$$

and the corresponding displacement is written by

$$\left.\begin{aligned} u_0 &= -\sum_{n=0}^{\infty}\frac{2A_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial Ce_n(\xi,q)}{\partial\xi}ce_n(\eta,q)e^{+ipt},\\ v_0 &= -\sum_{n=0}^{\infty}\frac{2A_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}Ce_n(\xi,q)\frac{\partial ce_n(\eta,q)}{\partial\eta}e^{-ipt}. \end{aligned}\right\} \qquad (16)$$

6. The equations of motion of the secondary waves generated from the aperture, where the elastic bodies on both sides of the screen has a narrow communication, are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2} &= \frac{\lambda+2\mu}{c^2(\mathrm{ch}^2\xi-\cos^2\eta)}\left(\frac{\partial^2\Delta}{\partial\xi^2}+\frac{\partial^2\Delta}{\partial\eta^2}\right),\\ \rho\frac{\partial^2\varpi}{\partial t^2} &= \frac{\mu}{c^2(\mathrm{ch}^2\xi-\cos^2\eta)}\left(\frac{\partial^2\varpi}{\partial\xi^2}+\frac{\partial^2\varpi}{\partial\eta^2}\right), \end{aligned}\right\} \qquad (17)$$

where

$$\left.\begin{aligned} \Delta &= h_1^2\left\{\frac{\partial}{\partial\xi}\left(\frac{u}{h_1}\right)+\frac{\partial}{\partial\eta}\left(\frac{v}{h_1}\right)\right\}, \\ 2\varpi &= h_1^2\left\{\frac{\partial}{\partial\xi}\left(\frac{v}{h_1}\right)+\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right\}, \\ \frac{1}{h_1^2} &= c^2(\mathrm{ch}^2\xi-\cos^2\eta). \end{aligned}\right\} \qquad (18)$$

Writing

$$\Delta=\Delta_1 e^{-ipt}, \qquad \varpi=\varpi_1 e^{-ipt},$$

we get

$$\left.\begin{aligned} \frac{\partial^2\Delta_1}{\partial\xi^2}+\frac{\partial^2\Delta_1}{\partial\eta^2}+h^2c^2(\mathrm{ch}^2\xi-\cos^2\eta)\,\Delta_1=0, \\ \frac{\partial^2\varpi_1}{\partial\xi^2}+\frac{\partial^2\varpi_1}{\partial\eta^2}+k^2c^2(\mathrm{ch}^2\xi-\cos^2\eta)\,\varpi_1=0, \end{aligned}\right\} \qquad (19)$$

in which

$$\frac{\rho p^2}{\lambda+2\mu}=h^2, \qquad \frac{\rho p^2}{\mu}=k^2.$$

The solutions of (17) are, thus, given by

$$\left.\begin{aligned} \Delta &= \sum_{n=0}^{\infty} B_n He_n^{(1)}(\xi,q)\left\{\begin{matrix} ce_n(\eta,q) \\ se_n(\eta,q)\end{matrix}\right\}e^{-ipt}, \\ 2\varpi &= \sum_{n=0}^{\infty} C_n He_n^{(1)}(\xi,q')\left\{\begin{matrix} ce_n(\eta,q') \\ se_n(\eta,q')\end{matrix}\right\}e^{-ipt}, \end{aligned}\right\} \qquad (20)$$

where $He_n^{(1)}(\xi, q)$ are solutions of Mathieu's differential equations in the forms of Hankel's type and

$$q'^2=\frac{k^2c^2}{32},$$

$$\left.\begin{aligned} [He_0^{(1)}(\xi,q)]_{q\to 0} &= H_0^{(1)}(hc\,\mathrm{ch}\xi), \\ [He_1^{(1)}(\xi,q)]_{q\to 0} &= H_1^{(1)}(hc\,\mathrm{ch}\xi), \\ [He_2^{(1)}(\xi,q)]_{q\to 0} &= H_2^{(1)}(hc\,\mathrm{ch}\xi), \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \qquad (21)$$

Displacement (u_1, v_1) answering to Δ in (20) and satisfying, $\varpi=0$, is expressed by the forms,

$$\left.\begin{aligned} u_1 &= -\sum_{n=0}^{\infty}\frac{B_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial He_n^{(1)}(\xi,q)}{\partial\xi}\left\{\begin{matrix} ce_n(\eta,q) \\ se_n(\eta,q)\end{matrix}\right\}e^{-ipt}, \\ v_1 &= -\sum_{n=1}^{\infty}\frac{B_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}He_n^{(1)}(\xi,q)\left\{\begin{matrix} \dfrac{\partial ce_n(\eta,q)}{\partial\eta} \\ \dfrac{\partial se_n(\eta,q)}{\partial\eta}\end{matrix}\right\}e^{-ipt}. \end{aligned}\right\} \qquad (22)$$

Displacement (u_2, v_2) derived from the value of ϖ in (20) under the condition, $\Delta=0$, is expressed by

$$\left.\begin{aligned} u_2 &= \sum_{n=1}^{\infty} \frac{C_n}{k^2 c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}} He_n^{(1)}(\xi, q') \begin{Bmatrix} \dfrac{\partial ce_n(\eta, q')}{\partial \eta} \\ \dfrac{\partial se_n(\eta, q')}{\partial \eta} \end{Bmatrix} e^{-ipt} \\ v_2 &= -\sum_{n=1}^{\infty} \frac{C_n}{k^2 c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}} \frac{\partial He_n^{(1)}(\xi, q')}{\partial \xi} \begin{Bmatrix} ce_n(\eta, q') \\ se_n(\eta, q') \end{Bmatrix} e^{-ipt} \end{aligned}\right\} \quad \ldots (23)$$

Free Barrier.

7. At the aperture of the screen the stresses and the displacements must be continuous. These conditions are given by the following four relations.

When $\xi=0$, we must have

$$\left.\begin{aligned} &(a)\quad u_0 = u_1+u_2, \\ &(b)\quad v_0 = v_1+v_2, \\ &(c)\quad \lambda\Delta_0 + 2\mu\left[\frac{1}{c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial u_0}{\partial \xi} + \frac{v_0}{c^2(\mathrm{ch}^2\xi-\cos^2\eta)}\frac{\partial}{\partial\eta}(c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta})\right] \\ &\qquad = \lambda\Delta + 2\mu\left[\frac{1}{c\sqrt{\mathrm{ch}^2\xi-\cos^2\xi}}\frac{\partial(u_1+u_2)}{\partial\xi}\right. \\ &\qquad\qquad \left. + \frac{(v_1+v_2)}{c^2(\mathrm{ch}^2\xi-\cos^2\eta)}\frac{\partial}{\partial\eta}(c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta})\right], \\ &(d)\quad \frac{\partial}{\partial\xi}\left[\frac{v_0}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\right] + \frac{\partial}{\partial\eta}\left[\frac{u_0}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\right] \\ &\qquad = \frac{\partial}{\partial\xi}\left[\frac{v_1+v_2}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\right] + \frac{\partial}{\partial\eta}\left[\frac{u_1+u_2}{\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\right]. \end{aligned}\right\} \quad (24)$$

8. When the screen is free from stress and when we superpose some stationary vibration on the side of the secondary waves, we have to apply only the relations of (a) and (b). Thus we get,

$$\left.\begin{aligned} &-\sum_{n=1}^{\infty}\frac{2A_n'}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial Se_n(\xi, q)}{\partial\xi}se_n(\eta, q) \\ &\quad = -\sum_{n=1}^{\infty}\frac{B_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial He_n^{(1)}(\xi, q)}{\partial\xi}se_n(\eta, q) \\ &\qquad + \sum_{n=1}^{\infty}\frac{C_n}{k^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}He_n^{(1)}(\xi, q')\frac{\partial ce_n(\eta, q')}{\partial\eta}, \\ &-\sum_{n=1}^{\infty}\frac{2A_n'}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}Se_n(\xi, q)\frac{\partial se_n(\eta, q)}{\partial\eta} \end{aligned}\right\} \quad \ldots\ldots (25)$$

$$
\left.\begin{aligned}
&=-\sum_{n=1}^{\infty}\frac{B_n}{h^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}He_n^{(1)}(\xi,q)\frac{\partial se_n(\eta,q)}{\partial\eta}\\
&-\sum_{n=1}^{\infty}\frac{C_n}{k^2c\sqrt{\mathrm{ch}^2\xi-\cos^2\eta}}\frac{\partial He_n^{(1)}(\xi,q')}{\partial\xi}ce_n(\eta,q')
\end{aligned}\right\}
$$

at $\xi=0$.

Solving (25) for $q\to0$, we get approximately

$$
\left.\begin{aligned}
B_1&=-4iA\begin{vmatrix}\frac{H_1^{(1)}(kc)}{c}, & \frac{\partial J_1(hc)}{\partial c}\\ \frac{\partial H_1^{(1)}(kc)}{\partial c}, & \frac{J_1(kc)}{c}\end{vmatrix}\Bigg/\begin{vmatrix}\frac{\partial H_1^{(1)}(hc)}{\partial c}, & \frac{H_1^{(1)}(kc)}{c}\\ \frac{H_1^{(1)}(hc)}{c}, & \frac{\partial H_1^{(1)}(kc)}{\partial c}\end{vmatrix},\\
C_1&=4i\frac{k^2}{h^2}A\begin{vmatrix}\frac{\partial H_1^{(1)}(hc)}{\partial c}, & \frac{\partial J_1(hc)}{\partial c}\\ \frac{H_1^{(1)}(hc)}{c}, & \frac{J_1(hc)}{c}\end{vmatrix}\Bigg/\begin{vmatrix}\frac{\partial H_1^{(1)}(hc)}{\partial c}, & \frac{H_1^{(1)}(kc)}{c}\\ \frac{H_1^{(1)}(hc)}{c}, & \frac{\partial H_0^{(1)}(kc)}{\partial c}\end{vmatrix},
\end{aligned}\right\}\quad\ldots(26)
$$

and generally for odd n,

$$
\left.\begin{aligned}
B_n&=-4iA\begin{vmatrix}\frac{nH_n^{(1)}(kc)}{c}, & \frac{\partial J_n(hc)}{\partial c}\\ \frac{\partial H_n^{(1)}(kc)}{\partial c}, & \frac{nJ_n(hc)}{c}\end{vmatrix}\Bigg/\begin{vmatrix}\frac{\partial H_n^{(1)}(hc)}{\partial c}, & \frac{nH_n^{(1)}(kc)}{c}\\ \frac{nH_n^{(1)}(hc)}{c}, & \frac{\partial H_n^{(1)}(kc)}{\partial c}\end{vmatrix},\\
C_n&=4iA\frac{k^2}{h^2}A\begin{vmatrix}\frac{\partial H_n^{(1)}(hc)}{\partial c}, & \frac{\partial J_n(hc)}{\partial c}\\ \frac{nH_n^{(1)}(hc)}{c}, & \frac{nJ_n(hc)}{c}\end{vmatrix}\Bigg/\begin{vmatrix}\frac{\partial H_n^{(1)}(hc)}{\partial c}, & \frac{nH_n^{(1)}(kc)}{c}\\ \frac{nH_n^{(1)}(hc)}{c}, & \frac{\partial H_n^{(1)}(kc)}{\partial c}\end{vmatrix}.
\end{aligned}\right\}\quad(27)
$$

For relatively large values of hc and kc, we have

$$
\left.\begin{aligned}
B_1&=4iA\frac{\left\{\frac{1}{c^2}\cos\left(hc-\frac{3\pi}{4}\right)+ihk\sin\left(hc-\frac{3\pi}{4}\right)\right\}e^{i\left(kc-\frac{3\pi}{4}\right)}}{\left(hk-\frac{1}{c^2}\right)e^{i\left(hc+kc-\frac{3\pi}{2}\right)}},\\
C_1&=-4i\frac{k^2}{h^2}A\frac{\frac{h}{c}\left\{i\cos\left(hc-\frac{3\pi}{4}\right)+\sin\left(hc-\frac{3\pi}{4}\right)\right\}e^{i\left(hc-\frac{3\pi}{4}\right)}}{\left(hk-\frac{1}{c^2}\right)e^{i\left(hc+kc-\frac{3\pi}{2}\right)}},
\end{aligned}\right\}\quad\ldots(26')
$$

and generally for $n=$odd,

$$
B_n=4iA\frac{\left\{\frac{n^2}{c^2}\cos\left(hc-\frac{2n+1}{4}\pi\right)+ihk\sin\left(hc-\frac{2n+1}{4}\pi\right)\right\}e^{i\left(kc-\frac{2n+1}{4}\pi\right)}}{\left(hk-\frac{n^2}{c^2}\right)e^{i\left(hc+kc-\frac{2n+1}{2}\pi\right)}},\quad\ldots(27')
$$

$$C_n=-4i\frac{k^2}{h^2}A\frac{\frac{hn}{c}\left\{i\cos\left(hc-\frac{2n+1}{4}\pi\right)+\sin\left(hc-\frac{2n+1}{4}\pi\right)\right\}e^{i\left(hc-\frac{2n+1}{4}\pi\right)}}{\left(hk-\frac{n^2}{c^2}\right)e^{i\left(hc+kc-\frac{2n+1}{2}\pi\right)}}.$$

B_n and C_n for even values of n vanish.

The diffracted waves are, thus, given by

$$\left.\begin{aligned}\Delta&=\sum_{n=1}^{\infty}B_nHe_n^{(1)}(\xi,q)se_n(\eta,q)e^{-ipt},\\ 2\varpi&=\sum_{n=1}^{\infty}C_nHe_n^{(1)}(\xi,q')ce_n(\eta,q')e^{-ipt},\end{aligned}\right\}\qquad\qquad(28)$$

$$n=1,3,5,\ldots,$$

together with the coefficients written in (26) and (27).

These solutions tell the fact that, when the barriers are free from stress, the diffracted waves are composed of dilatational and distortional waves of the first order distribution and those of higher orders. The principal waves are seen from the determinantal forms of B_n and C_n in (26) and (27) to be of the types,

$$\left.\begin{aligned}\Delta&=B_1He_1^{(1)}(\xi,q)se_1(\eta,q)e^{-ipt},\\ 2\varpi&=C_1He_1^{(1)}(\xi,q)ce_1(\eta,q')e^{-ipt}.\end{aligned}\right\}\qquad\qquad(29)$$

These two kinds of waves are, thus, principally diffracted from an aperture of a screen.

Rigid Barrier.

9. When the screen is composed of very rigid barriers and when we superpose some stationary vibration on the side of the secondary waves, we have to apply the relations (c) and (d). As these relations, however, are very complicated, we may be contented with the following simplified formulae:—

When $q\to 0$, $\xi=0$,

$$\left.\begin{aligned}&\lambda\Delta_0+2\mu\frac{\partial u_0}{\partial\xi_1}=\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial\xi_1},\\ &\frac{\partial v_0}{\partial\xi_1}-\frac{v_0}{\xi_1}+\frac{1}{\xi_1}\frac{\partial u_0}{\partial\eta}=\frac{\partial(v_1+v_2)}{\partial\xi_1}-\frac{1}{\xi_1}(v_1+v_2)+\frac{1}{\xi_1}\frac{\partial(u_1+u_2)}{\partial\eta},\end{aligned}\right\}\qquad(30)$$

in which $\xi_1=\text{ch}\xi$.

These give us

$$\left.\begin{aligned} B_0 &= \frac{\left[\dfrac{\lambda}{2\mu}J_0(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 J_0(hc)}{\partial c^2}\right]2A}{\dfrac{\lambda}{2\mu}H_0^{(1)}(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 H_0^{(1)}(hc)}{\partial c^2}}, \\ &\text{or} \\ B_0 &= \cos\left(hc - \frac{\pi}{4}\right)e^{-\left(hc-\frac{\pi}{4}\right)}2A \end{aligned}\right\} \qquad \text{(31)}$$

for relatively large value of hc; and generally for even n we have

$$B_n = 4A\frac{\begin{vmatrix} \dfrac{n}{k^2}\dfrac{\partial}{\partial c}\dfrac{H_n^{(1)}(kc)}{c}, & \dfrac{\lambda}{2\mu}J_n(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 J_n(hc)}{\partial c^2} \\ -\dfrac{n^2}{k^2c^2}H_n^{(1)}(kc) - \dfrac{1}{k^2}\dfrac{\partial^2 H_n^{(1)}(kc)}{\partial c^2} + \dfrac{1}{k^2c}\dfrac{\partial H_n^{(1)}(kc)}{\partial c}, & \dfrac{n}{h^2c}\dfrac{\partial J_n(hc)}{\partial c} + \dfrac{n}{h^2}\dfrac{\partial}{\partial c}\dfrac{J_n(hc)}{c} - \dfrac{n}{h^2}\dfrac{J_n(hc)}{c^2} \end{vmatrix}}{\begin{vmatrix} \dfrac{\lambda}{2\mu}H_n^{(1)}(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 H_n^{(1)}(hc)}{\partial c^2}, & \dfrac{n}{k^2}\dfrac{\partial}{\partial c}\dfrac{H_n^{(1)}(kc)}{c} \\ \dfrac{n}{h^2c}\dfrac{\partial H_n^{(1)}(hc)}{\partial c} + \dfrac{n}{h^2}\dfrac{\partial}{\partial c}\dfrac{H_n^{(1)}(hc)}{c} - \dfrac{n}{h^2}\dfrac{H_n^{(1)}(hc)}{c^2}, & -\dfrac{n^2}{k^2c^2}H_n^{(1)}(kc) - \dfrac{1}{k^2}\dfrac{\partial^2 H_n^{(1)}(kc)}{\partial c^2} + \dfrac{1}{k^2c^2}\dfrac{\partial H_n^{(1)}(hc)}{\partial c} \end{vmatrix}},$$

$$C_n = -4A\frac{\begin{vmatrix} \dfrac{\lambda}{2\mu}H_n(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 H_n(hc)}{\partial c^2}, & \dfrac{\lambda}{2\mu}J_n(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 J_n(hc)}{\partial c^2} \\ \dfrac{n}{h^2c}\dfrac{\partial H_n^{(1)}(hc)}{\partial c} + \dfrac{n}{h^2}\dfrac{\partial}{\partial c}\dfrac{H_n^{(1)}(hc)}{c} - \dfrac{n}{h^2}\dfrac{H_n^{(1)}(hc)}{c^2}, & \dfrac{n}{h^2c}\dfrac{\partial J_n(hc)}{\partial c} + \dfrac{n}{h^2}\dfrac{\partial}{\partial c}\dfrac{J_n(hc)}{c} - \dfrac{n}{h^2}\dfrac{J_n(hc)}{c^2} \end{vmatrix}}{\begin{vmatrix} \dfrac{\lambda}{2\mu}H_n^{(1)}(hc) - \dfrac{1}{h^2}\dfrac{\partial^2 H_n^{(1)}(hc)}{\partial c^2}, & \dfrac{n}{k^2}\dfrac{\partial}{\partial c}\dfrac{H_n(kc)}{c} \\ \dfrac{n}{h^2c}\dfrac{\partial H_n^{(1)}(hc)}{\partial c} + \dfrac{n}{h^2}\dfrac{\partial}{\partial c}\dfrac{H_n^{(1)}(hc)}{c} - \dfrac{n}{h^2}\dfrac{H_n^{(1)}(hc)}{c^2}, & -\dfrac{n^2}{k^2c^2}H_n^{(1)}(kc) - \dfrac{1}{k^2}\dfrac{\partial^2 H_n^{(1)}(kc)}{\partial c^2} + \dfrac{1}{k^2c}\dfrac{\partial H_n^{(1)}(kc)}{\partial c} \end{vmatrix}}. \qquad \text{(32)}$$

In these $n = 2, 4, 6, \ldots$

The diffracted waves are, thus, given by

$$\left.\Delta = \sum_{n=0}^{\infty} B_n He_n^{(1)}(\xi, q)\, ce_n(\eta, q)e^{-ipt},\right\} \qquad \text{(33)}$$

$$\left. 2\bar{\omega}=\sum_{n=1}^{\infty} C_n H_n^{(1)}(\xi,\, q')se_n(\eta,\, q')e^{-ipt}, \right\}$$

together with coefficients written in (31) and (32).

The determinantal forms of B_n and C_n show that the principal diffracted waves from an aperture of a rigid screen are given by

$$\varDelta=B_0 He_0^{(1)}(\xi,\, q)ce_0(\eta,\, q)e^{-ipt} \quad \ldots\ldots\ldots\ldots\ldots\ldots(34)$$

The principal waves diffracted from an aperture of a rigid screen are, thus, purely dilatational waves. We have in this case a fair resemblance to the sound waves.

Summary.

We have thus obtained some results having theoretical interest on seismology by the use of Mathieu's functions: the principal results are enumerated as follows:—

1. Elastic waves passing through an aperture of a screen are diffracted in complex manners.

2. When the barriers are free from stress and the primary waves, propagated perpendicularly to the barriers, are purely dilatational, the diffracted waves are of two kinds, that is, dilatational and distortional types, energy of each kind of waves being of comparable magnitude.

3. When the barriers are very rigid and the primary waves are of the same nature as the above case, the principal diffracted waves are of the dilatational kind, resembling to the diffraction phenomena of sound waves.

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