Rayleigh-type Waves propagated along an Inner Stratum of a Body.

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彈性體の內部層に傳はるレーレー型波

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表面層に傳はるレーレー型波に就ては種々の場合の計算が著者の一人に依て試みてある。又ラム教授は十年程前に板に傳はる短波長の彈性波の計算を行ひ、次に數年前にはストーンレー氏が彈性體の內部層に傳はるラブ型波を研究した。最近には松澤助教授が地殼內に弱い層の存在する事を指摘した。卽ち內部層に傳はるレーレー型波は早晚誰かの手によつて計算さるべき問題であつたのであるが、地震學上でも斯く重要な意味を持つ樣になつたから著者等は勇氣を得て、この研究を行つた譯である。

計算法は表面層の場合と同樣であるが波型が層の中間面に對して相對稱の場合と斜對稱の場合とについて研究し種々の結果を得た。その重要な項目を擧れば

(1) 中間層に傳はるレーレー波は分散性を現はす。

(2) 或 L/H に對して斜對稱のものが相對稱のものよりも高傳播速度を持つ。

(3) 波動速は層に特有な橫波の速度と外部媒體に特有な橫波の速度との間にある。

(4) $\frac{\mu'}{\mu}$ が小になるに從て、可なり長い L/H の間略一定の速度を保つ。

(5) $\frac{\mu'}{\mu}$ が非常に小になる時はこの一定に近い速度は半無限體上のレーレー波の速度に近づく。

(6) 短波長の波動勢力は弱い中間層に集積する。

It has already been studied mathematically by one[(1)] of the present authors how Rayleigh-type waves propagated on the surface of stratified bodies are dispersed. The results of calculation for various ratio of wave length and thickness of the layer, besides for different ratio between the elasticity of the layer and that of the subjacent material, have also been given. Not so long ago, Professor Lamb[(2)] investigated the manner, in which waves of various length

(1) "Dispersion of Elastic Waves & etc.," *Bull. of the Inst.* III. (1927).

(2) Lamb, "On Waves in an Elastic Plate," *Proc. Roy. Soc.*, (1916).

are transmitted over a plate. Stoneley[(1)] later showed, together with some problems of elastic waves, the propagation of Love-type waves along an inner medium of a body. The hypothesis of the existence of a stratum in the earth crust has recently been shown by Mr. Matuzawa[(2)] to accord with the results of seismic observations. Thus by the theoretical need of obtaining Rayleigh-type waves along a stratum, which are left to be discovered by any student, and by the practical importance of seismology to explore the structure of the earth crust, the authors have been encouraged to study the present problem.

The method employed in this paper resembles to that in the investigation of Rayleigh-waves on a stratified body. It is, however, noted that the authors have considered in this paper both cases of symmetric and asymmetric waves with respect to the central plane of the stratum. The intermediate cases have, of course, been imagined. In the case of Rayleigh-waves on a stratified surface of a body one type only was to be investigated.

Solutions for Symmetrical Case.

When the axis of x is taken coincident with the centre line of a stratum lying in an infinite body and the axis of y is directed perpendicular to x-axis, the equations of motion, in two dimensions, of the outer medium are expressed by

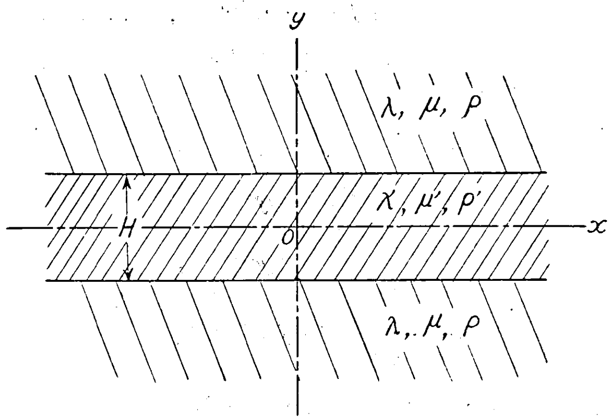
$$\left.\begin{aligned} \rho\frac{\partial^2 \Delta}{\partial t^2} &= (\lambda+2\mu)\left(\frac{\partial^2 \Delta}{\partial x^2}+\frac{\partial^2 \Delta}{\partial y^2}\right), \\ \rho\frac{\partial^2 \varpi}{\partial t^2} &= \mu\left(\frac{\partial^2 \varpi}{\partial x^2}+\frac{\partial^2 \varpi}{\partial y^2}\right), \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (1)$$

in which

$$\left.\begin{aligned} \Delta &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \\ 2\varpi &= \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \\ u,v &= \text{components of displacement parallel to } x\text{- and } y\text{-axes respectively}, \\ \rho &= \text{density}, \\ \lambda,\mu &= \text{Lamé's elastic constants}. \end{aligned}\right\} \cdots (2)$$

(1) Stoneley, "Elastic Waves at Surface of Separation...," *Proc. Roy. Soc.*, (1924).

(2) Matuzawa, "Propagation of Love-waves along Trebly Stratified Body," *Proc. Physico-Math. Soc. of Japan*, March 1928.



The solutions of (1) for the outer medium can be expressed by

$$\Delta = Ae^{\pm ry + i(pt - fx)}, \quad \dots\dots \quad (3)$$

$$2\varpi = Be^{\pm sy + i(pt - fx)}, \quad \dots\dots \quad (4)$$

$\pm$ being taken for each outer medium on the negative and positive sides of y-axis. While it is restricted that

$$\left.\begin{aligned} r^2 &= f^2 - h^2, & s^2 &= f^2 - k^2, \\ h^2 &= \frac{\rho p^2}{\lambda + 2\mu}, & k^2 &= \frac{\rho p^2}{\mu}. \end{aligned}\right\} \quad \dots\dots \quad (5)$$

Displacement (u_1, v_1) answering to Δ in (3) and satisfying $\varpi = 0$ is given by

$$\left.\begin{aligned} u_1 &= \frac{if}{h^2} Ae^{\pm ry + i(pt - fx)}, \\ v_1 &= -\frac{r}{h^2} Ae^{\pm ry + i(pt - fx)}. \end{aligned}\right\} \quad \dots\dots \quad (6)$$

Displacement (u_2, v_2) obtained from the value of ϖ in (4) and fulfilling the condition, $\Delta = 0$, is expressed by

$$\left.\begin{aligned} u_2 &= \frac{s}{k^2} Be^{\pm sy + i(pt - fx)}, \\ v_2 &= \frac{if}{k^2} Be^{\pm sy + i(pt - fx)}. \end{aligned}\right\} \quad \dots\dots \quad (7)$$

The equations of motion of the stratum are expressed by

$$\left.\begin{aligned} \rho' \frac{\partial^2 \Delta'}{\partial t^2} &= (\lambda' + 2\mu')\left(\frac{\partial^2 \Delta'}{\partial x^2} + \frac{\partial^2 \Delta'}{\partial y^2}\right), \\ \rho' \frac{\partial^2 \varpi'}{\partial t^2} &= \mu'\left(\frac{\partial^2 \varpi'}{\partial x^2} + \frac{\partial^2 \varpi'}{\partial y^2}\right). \end{aligned}\right\} \quad \dots\dots \quad (8)$$

where ρ', λ', μ' are density and Lamé's elastic constants of this stratum.

The solutions of (8) can be written by

$$\Delta' = C \cos r'y e^{i(pt-fx)}, \quad \dots\dots\dots\dots\dots\dots \quad (9)$$

$$2\varpi' = D \sin s'y e^{i(pt-fx)}, \quad \dots\dots\dots\dots\dots \quad (10)$$

in which

$$r'^2 = h'^2 - f^2, \qquad s'^2 = k'^2 - f^2.$$

Displacement proper to Δ' is given by

$$\left.\begin{aligned} u_1' &= \frac{if}{h^2} C \cos r'y e^{i(pt-fx)}, \\ v_1' &= \frac{r'}{h^2} C \sin r'y e^{i(pt-fx)}. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots \quad (11)$$

Displacement peculiar to ϖ' is expressed by

$$\left.\begin{aligned} u_2' &= \frac{s'}{k^2} D \cos s'y e^{i(pt-fx)}, \\ v_2' &= \frac{if}{k^2} D \sin s'y e^{i(pt-fx)}. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots \quad (12)$$

Solutions for Asymmetric Case.

In like manner the solutions for the asymmetric case can be obtained. The expressions of the dilatation, distortion and displacements of the outer medium are of the same forms as in the former case. Those of the stratum are expressed by

$$\left.\begin{aligned} \Delta' &= C \sin r'y e^{i(pt-fx)}, \\ 2\varpi' &= D \cos s'y e^{i(pt-fx)}, \\ u_1' &= \frac{if}{h^2} C \sin r'y e^{i(pt-fx)}, \\ v_1' &= -\frac{rC}{h^2} \cos r'y e^{i(pt-fx)}, \\ u_2' &= -\frac{s'D}{h^2} \sin s'y e^{i(pt-fx)}, \\ v_2' &= \frac{if}{h^2} D \cos s'y e^{i(pt-fx)}. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots \quad (13)$$

Boundary Conditions.

Now the boundary conditions are given by the following equations,

$$u_1 + u_2 = u_1' + u_2',$$

$$v_1 + v_2 = v_1' + v_2',$$

$$\left.\begin{aligned} \lambda\Delta+2\mu\frac{\partial}{\partial y}(v_1+v_2)&=\lambda'\Delta'+2\mu'\frac{\partial}{\partial y}(v_1'+v_2'), \\ \mu\left[\frac{\partial}{\partial y}(u_1+u_2)+\frac{\partial}{\partial x}(v_1+v_2)\right]&=\mu'\left[\frac{\partial}{\partial y}(u_1'+u_2')+\frac{\partial}{\partial x}(v_1'+v_2')\right] \end{aligned}\right\} \quad \ldots (14)$$

at $y=\pm\frac{H}{2}$, where H is thickness of the layer.

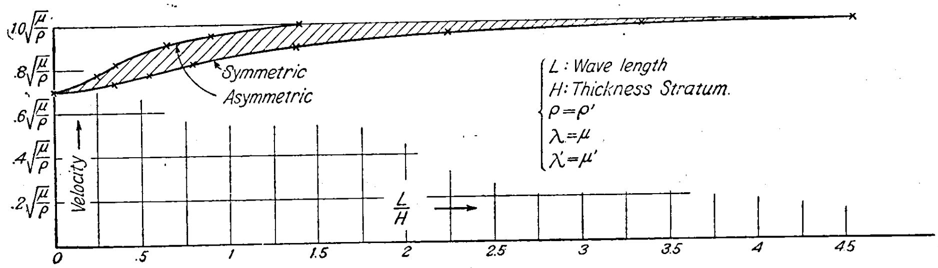
Putting the values of $\Delta, u_1, u_2, v_1, v_2, \Delta', u_1', u_2', v_1', v_2'$ from (3), (6), (7), (9), (11), (12), (13) in the conditions (14) at the boundaries and eliminating A, B, C, and D, it follows,

$$\begin{vmatrix} i, & \frac{s}{f}, & i, & \frac{s'}{f} \\ \frac{r}{f}, & -i, & \frac{r'}{f}X, & iY \\ \left(\frac{\lambda}{\mu}\frac{h^2}{f^2}-2\frac{r^2}{f^2}\right), & 2i\frac{s}{f}, & \left(\frac{\lambda'}{\mu}\frac{h'^2}{f^2}-\frac{2\mu'}{\mu}\frac{r'^2}{f^2}\right), & 2i\frac{\mu'}{\mu}\frac{s'}{f} \\ 2i\frac{r}{f}, & \left(1+\frac{s^2}{f^2}\right), & 2i\frac{\mu'}{\mu}\frac{r'}{f}X, & -\frac{\mu'}{\mu}\left(1-\frac{s'^2}{f^2}\right)Y \end{vmatrix}=0, \quad \ldots (15)$$

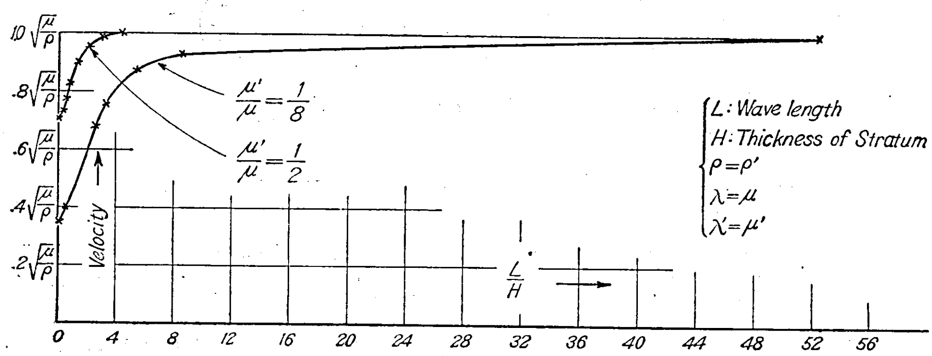
where $X=\tan\frac{r'H}{2}$ and $Y=\tan\frac{s'H}{2}$ for symmetric type,

$X=-\cot\frac{r'H}{2}$ and $Y=-\cot\frac{s'H}{2}$ for asymmetric type.

Solving the above determinantal equation by tentative methods for the cases $\mu=2\mu'$ and $\mu=8\mu'$, in which $\lambda=\mu$, $\lambda'=\mu'$ and $\rho=\rho'$, the authors have obtained the relations between the velocity and $\frac{L}{H}$, L being the wave length. The illustrated diagrams on the next page give these relations. In (A) the domain indicated by the hatched lines give the velocity of propagation of waves for $\frac{\mu'}{\mu}=\frac{1}{2}$ and for different ratios of L/H. This shows that waves of the asymmetric type are propagated more quickly than those of the symmetric type. Each waves of the intermediate types have the velocity which is represented by a certain point of the hatched part. Waves, whose velocity is greater than $\sqrt{\frac{\mu}{\rho}}$, cannot exist, because of the nature that very large amplitudes of the motion at infinite distance from the stratum must be then induced and they must require an infinite energy. A significant fact is that the velocity of propagation of every case is limited between those of distortional



(A) $\left(\frac{\mu'}{\mu}=\frac{1}{2}\right)$



(B) (Symmetrical Case)

[The results in the upper diagrams, together with those results written in Bull. III, has been computed in very strict manners. The application of a tentative method to the numerical solutions of complicated equations involving implicit quantities gives very often much better results than those obtained by the use of the ordinary method of treating algebraic equations.]

waves in the stratum and the outer medium. Not less important is the fact that smaller ratio of $\frac{\mu'}{\mu}$ gives longer range of L/H, in which approximately constant velocities of propagation are obtained. It is imagined from the figure that, when $\frac{\mu'}{\mu}$ tends to be very small, the constant value approaches the velocity of Rayleigh-waves on a semi-infinite solid body.

The diagram and the foregoing solutions will, again, tell us, though implicitly, that the energy of waves of short length accumulates in the vicinity of such a weak stratum; this may have some importance on the problem of seismic radiation, as well as on the problem of vibration of constructive materials.

Résumé.

(1) Rayleigh-type waves propagated along an inner stratum of an elastic body can be classified into two elementary types. According as the wave forms are symmetrical or asymmetric about the central plane of the stratum, the waves may be named symmetric or asymmetric type.

(2) Rayleigh-type waves propagated along an inner stratum show generally dispersive nature.

(3) The asymmetric type has velocity higher than that of the symmetrical type for some values of L/H.

(4) Velocities of propagation of the waves along a stratum are limited between those of distortional waves in the stratum and the outer medium.

(5) Smaller ratio of $\frac{\mu'}{\mu}$ gives longer range of L/H, which keeps approximately constant values of the velocity of propagation.

(6) As $\frac{\mu'}{\mu}$ tends to be very small, the constant value above mentioned approaches the velocity of propagation of Rayleigh-waves on a semi-infinite solid body.

(7) Energy of waves of short length accumulates in the vicinity of a weak stratum.

In conclusion, the authors are indebted to Professor Suyehiro, the Director of the Institute, who has allowed them to complete this investigation at leisure.

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