On the Propagation of the Leading and Trailing Parts of a Train of Elastic Waves.

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弾性列波の首部及び尾部の傳播に就て

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弾性列波が一定の長さを持つ時、其が如何に傳播するかは、理論上にも實際上にも重要な問題であらう。著者はフーリエの二重積分の方法により、三つの簡單な場合、即ち媒體が完全な彈性體である時の傳播、分散性媒體の中の彈性波及び粘性を含む彈性體中の波動を研究して種々の結論を得た。其重要なる項目を列擧すれば、

1. 完全な彈性體中の有限長波動は何等の變化なく傳播される。
2. 分散性媒體中の有限長波動は必ず變形を伴ふ。
3. 分散性媒體中を唯一個の群速を持つ有限長波動が傳はる時、波動列波の主な部分は其全長を變ずる事なく、其群速を以て送られる。
4. 上述の波動列波中の振動形は其形に基づく特有な速度を以て傳はり、且つ列波の兩端に於て振動形が消失する。
5. 分散性媒體中に於ける有限長の波動部に先立つて極めて弱いがしかし可成長い傾斜波動が傳播する場合がある。
6. 地震記錄の各位相は、分散性媒體中の有限波動を考へて決定すべき可能性がある。
7. 地震學に於ける「押しと引き」の法則は、分散性媒體中では成立し難い。
8. 波源の働きが如何に銳い性質のものでも、遠方へ行くに從ひ、漸次鈍くなり、波動形の長さは次第に長くなる。

The elementary theories on the propagation of elastic waves with infinitely extended disturbances enable us to find the various natures of the waves, provided they are deduced from the differential equations of elastic waves and satisfy the initial and boundary conditions. This can be seen from some results of experimental investigation(1) carried out by Professor Terada and Mr. Tsuboi,

(1) Terada & C. Tsuboi: Experimental Studies on Elastic Waves (Bull. of the Institute Vol. III) and C. Tsuboi: On the Velocity of Elastic Waves along the Surface of Stratified Layer (Proc. Physico-Math. Soc. Japan Vol. 9 No. 5).

well conforming with the author's mathematical theories which were reported in this Bulletin Vol. II and Vol. III. It must, however, be understood that certain natures cannot be found from the elementary theories; the propagation of elastic waves with a finite train is an instance, which must be dealt with specially.

Although the proper meaning of the leading part of each stage of disturbances recorded in seismograph is now clearly known in the light of increasing knowledge of seismology, yet the exact nature of these parts in transmission over a medium, which is regarded as elastic, viscous and dispersive, is not completely studied. The necessity for a further development of the analysis of waves thus arising, the author has obtained some theoretical results which might explain the important features involved in seismic waves, such as the difference between the group and phase velocities (due to Professor Gutenberg[2]) or long waves leading earthquake motion[3] found by Professor Imamura.

The method brought in this paper is merely the application of Fourier's double integral, which seems to be adequate to the investigation of the propagation of an arbitrary disturbance occurring in a limited region of the medium. An inevitable difficulty, which comes out as a result of this mathematical treatment is an occurrence of the discontinuous displacements at certain parts of the medium. Though such a discontinuity is allowed mathematically, yet the discontinuous displacement in real substances is impossible. This may, however, be avoided by assuming a sharp continuous displacement existing from the start at these parts.

The paper consists of three sections: the first is on the waves in perfectly elastic bodies, the second some examples for the case in a dispersive medium and the third on that in visco-elastic solid bodies.

I. Waves in Elastic Bodies.

The elementary solutions of surface waves, when $\lambda=\mu$, are expressed by

$$\left.\begin{aligned} v&=(-3.01\,e^{.847fy}+5.21\,e^{.393fy})\,e^{i(pt-fx)} \\ u&=i\,(3.55\,e^{.847fy}-2.04\,e^{.393fy})\,e^{i(pt-fx)} \end{aligned}\right\}$$

where

(1) Gutenberg (Phys. Zeits. 1926).

(2) Imamura: Long Waves leading Earthquake Motion (Proc. Imp. Acad. Vol. 2. No. 3).

$$h^2=\frac{\rho p^2}{\lambda+2\mu} \qquad k^2=\frac{\rho p^2}{\mu}$$

u, v = horizontal and vertical components of displacement respectively.

Generalising this, we obtain for the initial disturbance $v=\varphi(x)$ on $y=0$,

$$\left.\begin{aligned} v&=\frac{1}{\pi}\int_0^\infty df\cos fv_2t(-1.35\,e^{.847fy}+2.35\,e^{.393fy})\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-\lambda)d\lambda \\ u&=\frac{1}{\pi}\int_0^\infty df\cos fv_2t(1.61\,e^{.847fy}-0.92\,e^{.393fy})\int_{-\infty}^{\infty}\varphi(\lambda)\sin f(x-\lambda)d\lambda \end{aligned}\right\}$$

in which v_2 is the velocity of Rayleigh-waves.

If $\varphi(\lambda)=C$ for $a>x>-a$, then

$$\left.\begin{aligned} v&=\frac{C}{2\pi}\int_0^\infty\frac{(-1.35\,e^{.847fy}+2.35\,e^{.393fy})}{f}\{\sin f(a+x+v_2t)+\sin(a+x-v_2t) \\ &\qquad -\sin f(x-a-v_2t)+\sin f(x-a+v_2t)\}df \\ u&=\frac{C}{2\pi}\int_0^\infty\frac{(-1.61\,e^{.847fy}+0.92\,e^{.393fy})}{f}\{\cos f(a+x+v_2t)+\cos f(a+x-v_2t) \\ &\qquad -\cos f(x-a-v_2t)-\cos f(x-a+v_2t)\}df \end{aligned}\right\}$$

Using the relations

$$\int_0^\infty e^{-ax}\frac{\sin px}{x}dx=\tan^{-1}\frac{p}{a}$$

$$\int_0^\infty e^{-ax}\frac{\cos px-\cos qx}{x}dx=\frac{1}{2}\log\frac{a^2+q^2}{a^2+p^2}$$

we get the integrals

$$\begin{aligned} v=\frac{C}{2\pi}\Big[&-1.35\Big\{\tan^{-1}\frac{a+x+v_2t}{.847y}+\tan^{-1}\frac{a+x-v_2t}{.847y}-\tan^{-1}\frac{x-a-v_2t}{.847y} \\ &-\tan^{-1}\frac{x-a+v_2t}{.847y}\Big\}+2.35\Big\{\tan^{-1}\frac{a+x+v_2t}{.393y}+\tan^{-1}\frac{a+x-v_2t}{.393y} \\ &-\tan^{-1}\frac{x-a-v_2t}{.393y}-\tan^{-1}\frac{x-a+v_2t}{.393y}\Big\}\Big] \end{aligned}$$

$$\begin{aligned} u=\frac{C}{4\pi}\Big[&-1.35\log\frac{\{(.847y)^2+(x-a+v_2t)^2\}\{(.847y)^2+(x-a-v_2t)^2\}}{\{(.847y)^2+(x+a+v_2t)^2\}\{(.847y)^2+(x+a-v_2t)^2\}} \\ &+0.92\log\frac{\{(.393y)^2+(x-a+v_2t)^2\}\{(.393y)^2+(x-a-v_2t)^2\}}{\{(.393y)^2+(x+a+v_2t)^2\}\{(.393y)^2+(x+a-v_2t)^2\}} \end{aligned}$$

By computing this we find that the waves are propagated without change of forms as illustrated in Fig. 1.

If $\varphi(\lambda)=\cos k\lambda$ for $a>x>-a$, then

$$v=\frac{1}{4\pi}\int_0^\infty(-1.35e^{.847fy}+2.35e^{.393fy})\left\{\frac{\sin(f\overline{x+a+v_2t}+ka)}{f+k}+\frac{\sin(f\overline{x+a-v_2t}+ka)}{f+k}\right.$$

$$+\frac{\sin(f\overline{a-x+v_2t}+ka)}{f+k}+\frac{\sin(f\overline{a-x-v_2t}+ka)}{f+k}+\frac{\sin(f\overline{a-x+v_2t}-ka)}{f+k}$$

$$\left.+\frac{\sin(f\overline{a-x-v_2t}-ka)}{f-k}+\frac{\sin(f\overline{x+a+v_2t}-ka)}{f-k}+\frac{\sin(f\overline{a+x-v_2t}-ka)}{f-k}\right\}df$$

$$u=\frac{1}{4\pi}\int_0^\infty(1.61e^{.847fy}-0.92e^{.393fy})\left\{-\frac{\cos(f\overline{x+a+v_2t}+ka)}{f+k}-\frac{\cos(f\overline{x+a-v_2t}+ka)}{f+k}\right.$$

$$+\frac{\cos(f\overline{a-x+v_2t}+ka)}{f+k}+\frac{\cos(f\overline{a-x-v_2t}+ka)}{f+k}-\frac{\cos(f\overline{a-x+v_2t}-ka)}{f-k}$$

$$\left.-\frac{\cos(f\overline{a-x-v_2t}-ka)}{f-k}+\frac{\cos(f\overline{x+a+v_2t}-ka)}{f-k}+\frac{\cos(f\overline{a+x-v_2t}-ka)}{f-k}\right\}df$$

By contour integrations, we have the following relations.

When $a>0$

$$\int_0^\infty\frac{e^{-bx}\cos(ax-d)}{x-k}dx=-\pi e^{-bk}\sin(ak-d)+\cos d\int_0^\infty\frac{e^{-ay}(y\cos by-k\sin by)}{y^2+k^2}dy$$
$$-\sin d\int_0^\infty\frac{e^{-ay}(y\sin by+k\cos by)}{y^2+k^2}dy$$

$$\int_0^\infty\frac{e^{-bx}\sin(ax-d)}{x-k}dx=\pi e^{-bk}\cos(ak-d)-\cos d\int_0^\infty\frac{e^{-ay}(y\sin by+k\cos by)}{y^2+k^2}dy$$
$$-\sin d\int_0^\infty\frac{e^{-ay}(y\cos by-k\sin by)}{y^2+k^2}dy$$

$$\int_0^\infty\frac{e^{-bx}\cos(ax+d)}{x+k}dx=\cos d\int_0^\infty\frac{e^{-ay}(y\cos by+k\sin by)}{y^2+k^2}dy$$
$$+\sin d\int_0^\infty\frac{e^{-ay}(y\sin by-k\cos by)}{y^2+k^2}dy$$

$$\int_0^\infty\frac{e^{-bx}\sin(ax+d)}{x+k}dx=-\cos d\int_0^\infty\frac{e^{-ay}(y\sin by-k\cos by)}{y^2+k^2}$$
$$+\sin d\int_0^\infty\frac{e^{-ay}(y\cos by+k\sin by)}{y^2+k^2}dy$$

When $a<0$

$$\int_0^\infty\frac{e^{-bx}\cos(ax-d)}{x-k}dx=\pi e^{-bk}\sin(ak-d)+\cos d\int_0^\infty\frac{e^{ay}(y\cos by-k\sin by)}{y^2+k^2}dy$$
$$+\sin d\int_0^\infty\frac{e^{ay}(y\sin by+k\cos by)}{y^2+k^2}dy$$

$$\int_0^\infty \frac{e^{-bx}\sin(ax-d)}{x-k}dx = -\pi e^{-bk}\cos(ak-d)+\cos d\int_0^\infty \frac{e^{ay}(y\sin by+k\cos by)}{y^2+k^2}dy$$
$$-\sin d\int_0^\infty \frac{e^{ay}(y\cos by-k\sin by)}{y^2+k^2}dy$$

$$\int_0^\infty \frac{e^{-bx}\cos(ax+d)}{x+k}dx = \cos d\int_0^\infty \frac{e^{ay}(y\cos by+k\sin by)}{y^2+k^2}dy$$
$$-\sin d\int_0^\infty \frac{e^{ay}(y\sin by-k\cos by)}{y^2+k^2}dy$$

$$\int_0^\infty \frac{e^{-bx}\sin(ax+d)}{x+k}dx = \cos d\int_0^\infty \frac{e^{ay}(y\sin by-k\cos by)}{y^2+k^2}dy$$
$$+\sin d\int_0^\infty \frac{e^{ay}(y\cos by+k\sin by)}{y^2+k^2}dy$$

Using the above relation, we get, finally, at $y=0$

$$\begin{array}{ll}
x>v_2t+a & v=0 \\
v_2t+a>x>v_2t-a & v=\dfrac{1}{2}\cos k(x-v_2t) \\
v_2t-a>x>a>0 & v=0 \\
0>-a>x>-(v_2t-a) & v=0 \\
-(v_2t-a)>x>-(v_2t+a) & v=\dfrac{1}{2}\cos k(x-v_2t) \\
-(v_2t+a)>x & v=0
\end{array}$$

In this case, too, no change of wave forms can take place. This is illustrated in Fig. 2.

II. Waves in Dispersive Medium.

The true nature of the propagation of elastic waves in a dispersive medium is not yet completely known. We may expect that heterogeneity of the earth crust, gravitation, initial stress and stratification of layers cause dispersions. Bromwich was the first to point out the way in which the waves are dispersed, though slightly, in gravitating elastic medium. According to his investigation the velocity of propagation of the elastic waves in such a medium is approximately given by

$$V=V_0\left(1+0.213\frac{\lambda}{R}\right)$$

where

V_0 = the velocity in a non-gravitating medium,

λ = wave length,

R = radius of the Earth.

Now the author's intention is not to study the nature of the dispersion; but he aims to know the behaviour of waves in the dispersive medium. Thus, for the sake of simplicity, adopting Bromwich's dispersion formula,[1] the propagation of a disturbance in dispersive medium is dealt with in the following examples.

Accordingly the velocity of progagation, throughout this section, is written by.

$$v_2 = v_0\left(1 + \frac{\alpha}{f}\right)$$

where v_0 = the velocity of propagation of very small wave length,

$\frac{2\pi}{f}$ = wave length

α = a constant for a given medium.

The elementary solution of the wave motion, when the principal part is selected, is written by the type

$$\cos(fx - fv_2t)$$

The group velocity of this system is expressed by

$$\frac{d(fv_2)}{df} = v_0$$

The general expressions for the components of displacement of the surface waves are given

$$v = \frac{1}{\pi}\int_0^\infty df \cos fv_2t\,(-1.35e^{.847fy} + 2.35e^{.393fy})\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-\lambda)\,d\lambda$$

$$u = \frac{1}{\pi}\int_0^\infty df \cos fv_2t\,(1.61e^{.847fy} - 0.92e^{.393fy})\int_{-\infty}^{\infty}\varphi(\lambda)\sin f(x-\lambda)\,d\lambda$$

in which $\varphi(\lambda)$ is the distribution of the vertical displacement of the surface at $t=0$.

If $\varphi(\lambda) = C$ for $a > x > -a$, we get after a few operations,

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(1) Proc. London Math. Soc. Vol. 30 (1898)

$$
\begin{aligned}
v=\frac{C}{2\pi}\Bigg[&-1.35\cos\alpha v_0t\left\{\tan^{-1}\frac{a+x+v_0t}{.847y}+\tan^{-1}\frac{a+x-v_0t}{.847y}\right.\\
&\qquad\qquad\left.-\tan^{-1}\frac{x-a-v_0t}{.847y}-\tan^{-1}\frac{x-a+v_0t}{.847y}\right\}\\
&+2.35\cos\alpha v_0t\left\{\tan^{-1}\frac{a+x+v_0t}{.393y}+\tan^{-1}\frac{a+x-v_0t}{.393y}\right.\\
&\qquad\qquad\left.-\tan^{-1}\frac{x-a-v_0t}{.393y}-\tan^{-1}\frac{x-a+v_0t}{.393y}\right\}\\
&-\frac{1.35}{2}\sin\alpha v_0t\log\frac{\{(.847y)^2+(a+x-v_0t)^2\}\{(.847y)^2+(x-a+v_0t)^2\}}{\{(.847y)^2+(a+x+v_0t)^2\}\{(.847y)^2+(x-a-v_0t)^2\}}\\
&+\frac{0.92}{2}\sin\alpha v_0t\log\frac{\{(.393y)^2+(a+x-v_0t)^2\}\{(.393y)^2+(x-a+v\,t)^2\}}{\{(.393y)^2+(a+x+v_0t)^2\}\{(.393y)^2+(x-a-v_0t)^2\}}
\end{aligned}
$$

$$
\begin{aligned}
u=\frac{C}{2\pi}\Bigg[&-1.35\frac{\cos\alpha v_0t}{2}\log\frac{\{(.847y)^2+(x-a+v_0t)^2\}\{(.847y)^2+(x-a-v_0t)^2\}}{\{(.847y)^2+(x+a+v_0t)^2\}\{(.847y)^2+(x+a-v_0t)^2\}}\\
&+0.92\frac{\cos\alpha v_0t}{2}\log\frac{\{(.393y)^2+(x-a+v_0t)^2\}\{(.393y)^2+(x-a-v_0t)^2\}}{\{(.393y)^2+(x+a+v_0t)^2\}\{(.393y)^2+(x+a-v_0t)^2\}}\\
&-1.35\sin\alpha v_0t\left\{\tan^{-1}\frac{a+x+v_0t}{.847y}-\tan^{-1}\frac{a+x-v_0t}{.847y}\right.\\
&\qquad\qquad\left.-\tan^{-1}\frac{a-x+v_0t}{.847y}+\tan^{-1}\frac{a-x-v_0t}{.847y}\right\}\\
&+0.92\sin\alpha v_0t\left\{\tan^{-1}\frac{a+x+v_0t}{.393y}-\tan^{-1}\frac{a+x-v_0t}{.393y}\right.\\
&\qquad\qquad\left.-\tan^{-1}\frac{a-x+v_0t}{.393y}+\tan^{-1}\frac{a-x-v_0t}{.393y}\right\}\Bigg]
\end{aligned}
$$

By computing this we find that the waves are gradually deformed in propagation through the medium. An example is illustrated in Fig. 3.

If $\varphi(\lambda)=\cos k\lambda$ for $a>x>-a$, we have

$$
\begin{aligned}
v=\frac{1}{4\pi}\int_0^\infty&\cos\alpha v_0t(-1.35e^{.847fy}+2.35e^{.393fy})\left\{\frac{\sin(f\overline{x+a+v_0t}+ka)}{f+k}\right.\\
&+\frac{\sin(f\overline{x+a-v_0t}+ka)}{f+k}+\frac{\sin(f\overline{a-x+v_0t}+ka)}{f+k}+\frac{\sin(f\overline{a-x-v_0t}+ka)}{f+k}\\
&+\frac{\sin(f\overline{a-x+v_0t}-ka)}{f-k}+\frac{\sin(f\overline{a-x-v_0t}-ka)}{f-k}+\frac{\sin(f\overline{x+a+v_0t}-ka)}{f-k}\\
&\left.+\frac{\sin(f\overline{a+x-v_0t}-ka)}{f-k}\right\}df
\end{aligned}
$$

$$+\frac{1}{4\pi}\int_0^\infty \sin\alpha v_0t(-1.35e^{.847fy}+2.35e^{.393fy})\Big\{\frac{\cos(f\,\overline{x+a+v_0t}+ka)}{f+k}$$

$$-\frac{\cos(f\,\overline{x+a-v_0t}+ka)}{f+k}+\frac{\cos(f\,\overline{a-x+v_0t}+ka)}{f+k}-\frac{\cos(f\,\overline{a-x-v_0t}+ka)}{f+k}$$

$$+\frac{\cos(f\,\overline{a-x+v_0t}-ka)}{f-k}-\frac{\cos(f\,\overline{a-x-v_0t}-ka)}{f-k}+\frac{\cos(f\,\overline{a+x+v_0t}-ka)}{f-k}$$

$$-\frac{\cos(f\,\overline{a+x-v_0t}-ka)}{f-k}\Big\}df$$

$$u=\frac{1}{4\pi}\int_0^\infty \cos\alpha v_0t(1.61e^{.847fy}-0.92e^{.393fy})\Big\{-\frac{\cos(f\,\overline{x+a+v_0t}+ka)}{f+k}$$

$$+\frac{\cos(f\,\overline{a+x-v_0t}+ka)}{f+k}+\frac{\cos(f\,\overline{a-x+v_0t}+ka)}{f+k}+\frac{\cos(f\,\overline{a-x-v_0t}+ka)}{f+k}$$

$$-\frac{\cos(f\,\overline{a+x+v_0t}-ka)}{f-k}-\frac{\cos(f\,\overline{a-x-v_0t}-ka)}{f-k}+\frac{\cos(f\,\overline{x+a+v_0t}-ka)}{f-k}$$

$$+\frac{\cos(f\,\overline{a+x-v_0t}-ka)}{f-k}\Big\}df$$

$$+\frac{1}{4\pi}\int_0^\infty \sin\alpha v_0t(1.61e^{.847fy}-0.92e^{.393fy})\Big\{\frac{\sin(f\,\overline{a+x+v_0t}+ka)}{f+k}$$

$$-\frac{\sin(f\,\overline{a+x-v_0t}+ka)}{f+k}-\frac{\sin(f\,\overline{a-x+v_0t}+ka)}{f+k}-\frac{\sin(f\,\overline{a-x-v_0t}+ka)}{f+k}$$

$$+\frac{\sin(f\,\overline{a+x+v_0t}-ka)}{f-k}-\frac{\sin(f\,\overline{a+x-v_0t}-ka)}{f-k}-\frac{\sin(f\,\overline{a-x+v_0t}-ka)}{f-k}$$

$$+\frac{\sin(f\,\overline{a-x-v_0t}-ka)}{f-k}\Big\}df$$

By applying the formulae in pages 110 and 111 and putting $y=0$, we get

i) $x>v_0t+a$

$$v=\frac{\sin\alpha v_0t}{2\pi}[-\mathrm{Ci}\,kz_1\cos k(z_1-a)+\mathrm{Ci}\,kz_2\cos k(z_2-a)$$
$$-\mathrm{Ci}\,k\bar{z}_3\cos k(\bar{z}_3+a)+\mathrm{Ci}\,k\bar{z}_4\cos k(\bar{z}_4+a)$$
$$-\mathrm{Si}\,kz_1\sin k(z_1-a)+\mathrm{Si}\,kz_2\sin k(z_2-a)$$
$$-\mathrm{Si}\,k\bar{z}_3\sin k(\bar{z}_3+a)+\mathrm{Si}\,k\bar{z}_4\sin k(\bar{z}_4+a)]$$

where $z_1=x+v_0t+a,\quad z_2=x-v_0t+a,\quad \bar{z}_3=x-v_0t-a,\quad \bar{z}_4=x+v_0t-a$

and $$\mathrm{Ci}\,u=-\int_u^\infty\frac{\cos u}{u}du\qquad \mathrm{Si}\,u=\int_0^\infty\frac{\sin u}{u}du.$$

ii) $v_0t+a>x>vt-a$

$$v=\frac{1}{2}\cos(kx-\overline{k+\alpha v_0t})$$

$$+\frac{\sin\alpha v_0t}{2\pi}[-\mathrm{Ci}\,kz_1\cos k(z_1-a)+\mathrm{Ci}\,kz_2\cos k(z_2-a)$$

$$-\mathrm{Ci}\,kz_3\cos k(z_3-a)+\mathrm{Ci}\,k\bar{z}_4\cos k(\bar{z}_4+a)$$

$$-\mathrm{Si}\,kz_1\sin k(z_1-a)+\mathrm{Si}\,kz_2\sin k(z_2-a)$$

$$-\mathrm{Si}\,kz_3\sin k(z_3-a)+\mathrm{Si}\,k\bar{z}_4\sin k(\bar{z}_4+a)]$$

where $z_1=x+v_0t+a,\quad z_2=x-v_0t+a,\quad z_3=v_0t-x+a,\quad \bar{z}_4=x+v_0t-a$

iii) $v_0t-a>x$

$$v=\frac{\sin\alpha v_0t}{2\pi}\cos ka\Big[-\mathrm{Ci}\,kz_1\cos kz_1+\mathrm{Ci}\,k\bar{z}_2\cos k\bar{z}_2-\mathrm{Ci}\,kz_3\cos kz_3+\mathrm{Ci}\,k\bar{z}_4\cos k\bar{z}_4$$

$$+\left(\frac{\pi}{2}-\mathrm{Si}\,kz_1\right)\sin kz_1-\left(\frac{\pi}{2}-\mathrm{Si}\,k\bar{z}_2\right)\sin k\bar{z}_2$$

$$+\left(\frac{\pi}{2}-\mathrm{Si}\,kz_3\right)\sin kz_3-\left(\frac{\pi}{2}-\mathrm{Si}\,k\bar{z}_4\right)\sin k\bar{z}_4\Big]$$

where $z_1=x+v_0t+a,\quad \bar{z}_2=v_0t-x-a,\quad z_3=v_0t-x+a,\quad \bar{z}_4=v_0t+x-a$

These solutions tell the fact that, while the disturbed portion is propagated with the group velocity without the change in its length, the harmonic form of the oscillatory profile proceeds with the velocity peculiar to the apparent length of the waves. Fig. 4 is illustrated as an example. This is an important fact for practical seismology, as suggested by Gutenberg.[1] Each phase observed in seismogram may be due to the group velocity above given. In our case the equation to determine the group velocity has one derivative, so that only one group is possible; when the equation in question has a number of derivatives, we obtain many phases as often found in seismic obervations. As the waves with the oscillatory nature progress with a certain velocity different from that of the disturbed portion and disappear at the leading part of that portion, the law of "pull and push" in seismology is not applicable to the waves propagated in a dispersive medium. Apart from these facts, an equally important fact is the existence of very feeble but long tilting waves at the front of seismic waves, as observed by Professor Imamura,[2] though such waves

(1) loc. cit. p. 108. In the analysis by the present author, only the case, that the group velocity is less than the phase velocity, is taken.

(2) loc. cit., ante p. 108.

given by the mathematics are not of just the same nature as those actually observed.

III. Waves in a Visco-Elastic Medium.

The propagation of waves in visco-elastic bodies has been investigated in the preceding Bulletin, and we shall be contented here with a simple example of a discontinuous disturbance propagated in an infinitely extended visco-elastic medium.

The expression of dilatational waves is given by

$$u=\frac{1}{\pi}\int_0^\infty df\cos fv_1t\, e^{-w_1f^2t}\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-\lambda)d\lambda$$

where v_1 = velocity of dilatational waves and $w_1=\dfrac{\lambda'+2\mu'}{2\rho}$

If the initial disturbance is given by $\varphi(\lambda)=C$ for $a>x>-a$, we have

$$u=\frac{1}{\pi}\int_0^\infty df\cos fv_1t\, e^{-w_1f^2t}\int_{-a}^{\infty}C\cos f(x-\lambda)d\lambda$$

From this, we get

$$u=\frac{C}{2\pi}\int_0^\infty\frac{e^{-w_1f^2t}}{f}\{\sin f(x+a-v_1t)-\sin f(x-a-v_1t)$$
$$+\sin f(x+a+v_1t)-\sin f(x-a+v_1t)\}df$$

Now

$$\int_0^\infty e^{-hx^2}\frac{\sin kx}{x}dx=\frac{1}{2}\sqrt{\frac{\pi}{h}}\int_0^k e^{-\frac{k^2}{4h}}dk$$
$$=\sqrt{\pi}\int_0^{\frac{k}{2\sqrt{h}}}e^{-\left(\frac{k}{2\sqrt{h}}\right)^2}d\left(\frac{k}{2\sqrt{h}}\right)$$
$$=\sqrt{\pi}\int_0^x e^{-x^2}dx \quad \text{where } x=\frac{k}{2\sqrt{h}}$$

so that

$$u=\frac{C}{2\sqrt{\pi}}\left[\int_0^{x_1}e^{-\alpha^2}d\alpha-\int_0^{x_2}e^{-\alpha^2}d\alpha+\int_0^{x_3}e^{-\alpha^2}d\alpha-\int_0^{x_4}e^{-\alpha^2}d\alpha\right]$$

where

$$x_1=\frac{x+a-v_1t}{2\sqrt{w_1t}}\quad x_2=\frac{x-a-v_1t}{2\sqrt{w_1t}}\quad x_3=\frac{x+a+v_1t}{2\sqrt{w_1t}}\quad x_4=\frac{x-a+v_1t}{2\sqrt{w_1t}}$$

The result of evaluation is illustrated in Fig. 5. This shows that, however sharp the initial form of disturbance may be, its forms become gradually flat and long; thereby the condition of continuity is naturally satisfied.

A more complicated case in which many of the simple types as given in the above example are combined, is shown in Fig. 6.

Résumé.

We have thus obtained some results having practical importance on the seismology by the use of Fourier's double integral theorems; the principal results are enumerated as follows:—

1. Waves with limited extent in a perfectly elastic body are propagated without change of the type.

2. Waves with limited extent in a dispersive body are propagated with modification of the type.

3. Waves having a group velocity in a dispersive medium are propagated with this velocity, keeping the original extent of the disturbance constant.

4. The oscillatory forms in the above problem are propagated with a velocity peculiar to the wave length of the oscillatory part. The oscillatory nature disappears at the extreme ends of the disturbed portion.

5. In some dispersive waves very feeble but long tilting waves lead the train of oscillatory waves.

6. In analysing the phases in seismic records the nature of waves with a limited extent in dispersive solid body should be taken into consiedration.

7. The law of "pull and push" in seismology is not applicable to the propagation of waves in a dispersive medium.

8. However sharp the initial form of disturbance may be, the pulses in visco-elastic bodies gradually flatten.

In concluding this paper the author's sincere thanks are due to Professors Suyehiro and Fujiwhara for their kind suggestions and advices.

October 1927.

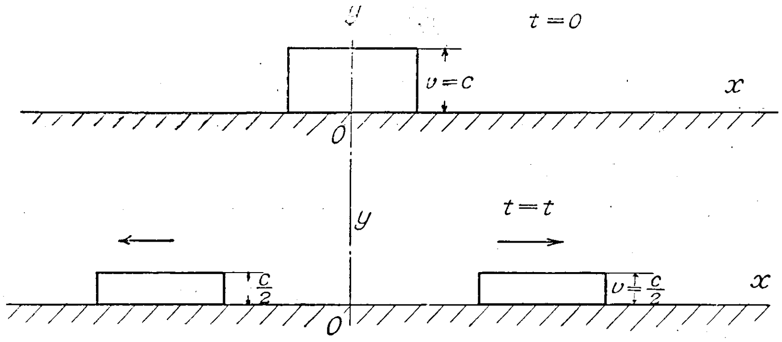


Fig. 1.

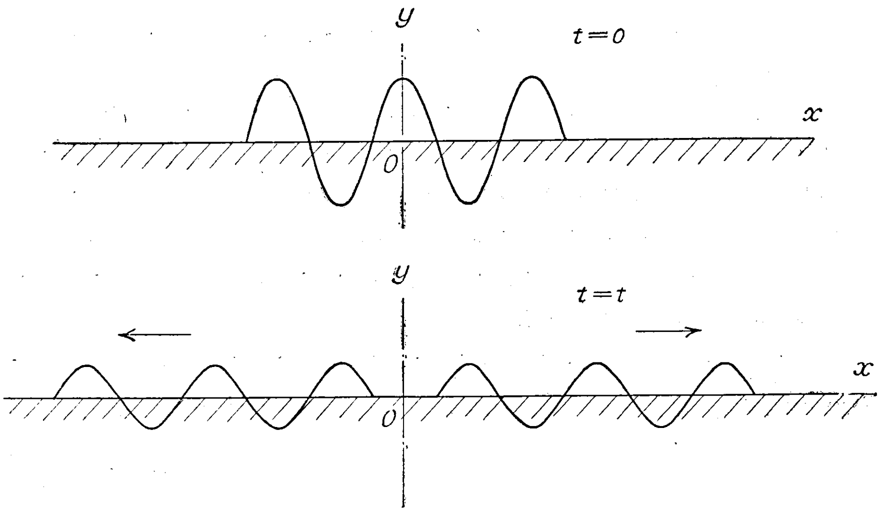
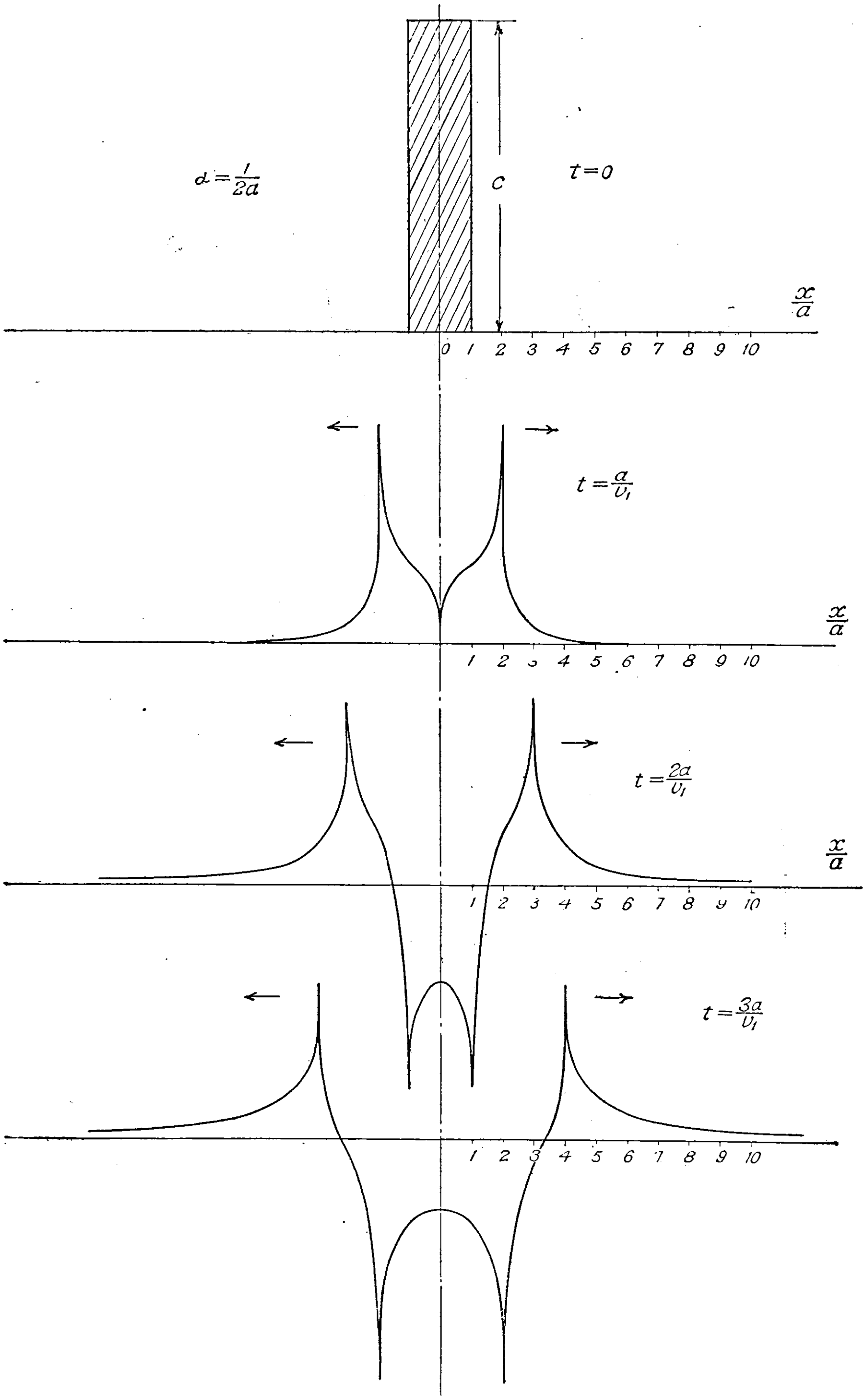
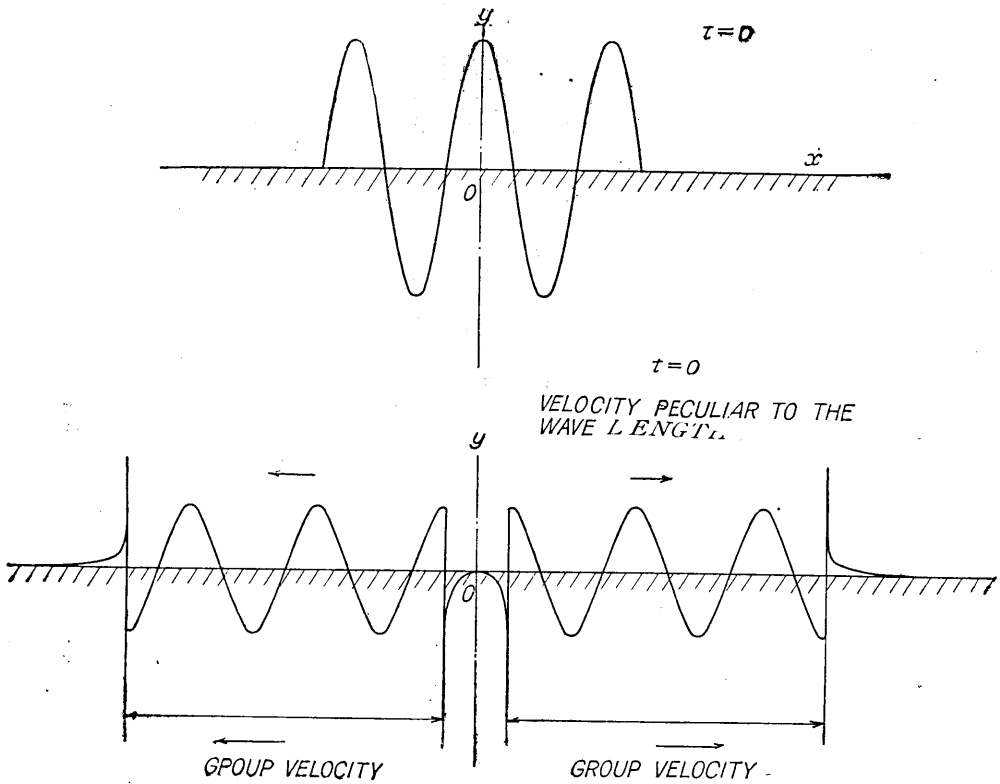


Fig. 2.



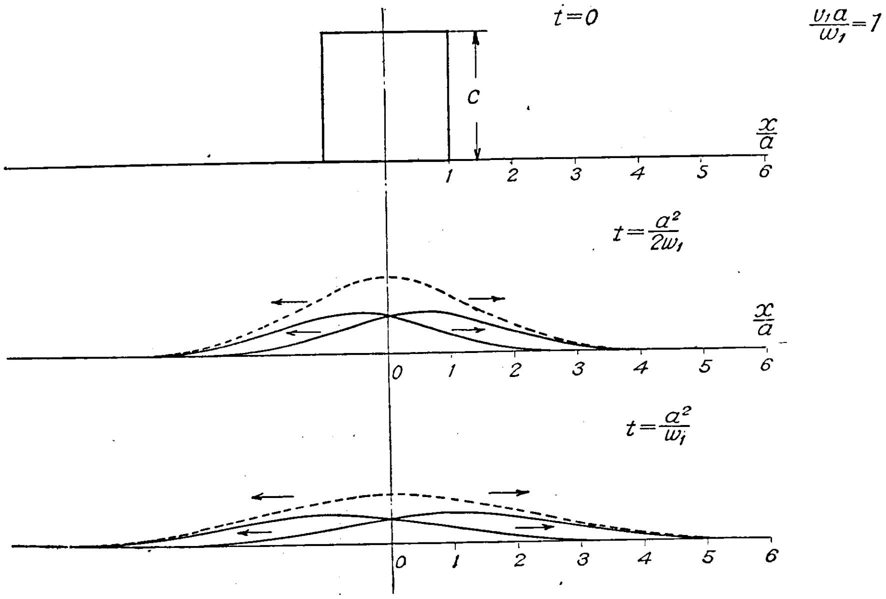
$\alpha = \frac{1}{2a}$
c
$t=0$
$\frac{x}{a}$
0 1 2 3 4 5 6 7 8 9 10
$t = \frac{a}{v_1}$
$\frac{x}{a}$
1 2 3 4 5 6 7 8 9 10
$t = \frac{2a}{v_1}$
$\frac{x}{a}$
1 2 3 4 5 6 7 8 9 10
$t = \frac{3a}{v_1}$
1 2 3 4 5 6 7 8 9 10

Fig. 3.



y
t=0
x
0
t=0
VELOCITY PECULIAR TO THE WAVE LENGTH
y
0
GPOUP VELOCITY
GROUP VELOCITY

Fig. 4.



$t=0$
$\frac{v_1 a}{w_1}=1$
c
$\frac{x}{a}$
1 2 3 4 5 6
$t=\frac{a^2}{2w_1}$
$\frac{x}{a}$
0 1 2 3 4 5 6
$t=\frac{a^2}{w_1}$
0 1 2 3 4 5 6

Fig. 5.

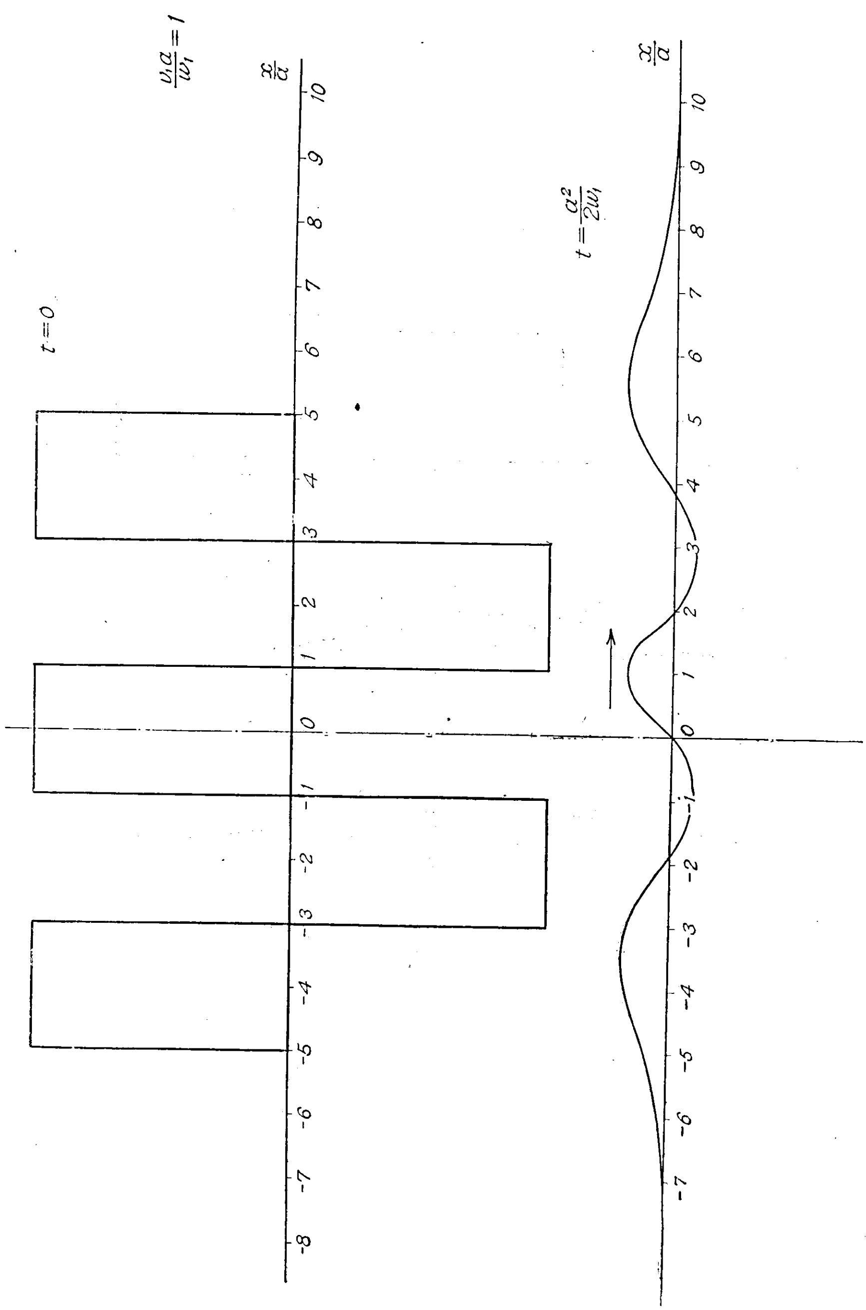


Fig. 6.