The Reflection of the Elastic Waves generated from an Internal Point of a Sphere.

By

Katsutada SEZAWA.

Earthquake Research Institute.

球の內部の點から起る彈性波の反射

所　員　妹　澤　克　惟

平面彈性波が地表面で反射する理論はケルビンやノットによつて既に論じられて居るが、この論文では其延長として球の內部の一點から起つた彈性波が其曲面で反射する場合を考へた。

計算の方法として調和函數の級數を用ひ、又問題を簡易にする爲に重力の影響を看過し、彈性及び密度を均等として取扱つた。重力の影響を看過する事は地球の場合にも大して問題にならぬであらうが、物質の性質を均等と考へる事は、事實上不合理であるかも知れぬ。しかしこの樣な單純な場合から計算を推し進める事は理論上の研究としては反て好都合であり、又彈性論其れ自身の進步にも幾分效果があるであらう。

問題は內部の點が縱波のみ送り出す場合と、橫波のみ送り出す場合の二種類が論じてある。理論的に面白い事項を摘錄すれば

1. 調和函數の級數を用ひる事によつて複雜な問題が比較的簡單に取扱はれる。
2. 球の內部の點から起る彈性波が球面で反射する性質は、平面境界の場合と同じである。反射波が一般に橫波と縱波とから成る事も同樣である。
3. 波長が極めて小さい時は球面の變位は簡單な形で表はされる。
4. 橫波が反射して橫波のみを生ずる場合は、極大變位の球帶が波動源央に對して或餘緯を持ち、且つ源の深さによつて其位置が變る。
5. 橫波の反射の爲に起る表面の變位はラブ波の傳播に非常に類似する。

The theory of the reflection of plane elastic waves on the flat surface of solid bodies has been investigated very exhaustively by Lord Kelvin and C. G. Knott.[1] The extended case, that the elastic waves generated from an internal point of a sphere are reflected on on its surface, has not yet been studied. As the recent observations reveals that the focal depths of earthquakes are sometimes very large, it becomes urgent to discover the extension of the existing theory of reflection to that on a spherical surface. Although the

(1) Phil. Mag. July 1899.

variations of the nature of the materials composing the Earth crust and of the intensity of gravity with the radial distance from the centre of the Earth are more or less effective, yet in the present problem a non-gravitating and isotropic sphere has been taken for the sake of simplicity. Such a simplified investigation is worth while in the sense that it tells how the reflection depends upon the geometrical forms of the elastic bodies. Moreover, as the effect of the gravity on the propagation of elastic waves in the Earth is insignificant and the variation of the materials is slight, the result obtained by the present investigation may not be far from the truth. Further, apart from its importance in connection with the seismology, the problem itself has an important bearing on the theory of elasticity.

The present investigation consists of two sections: the first deals with the reflection of dilatational waves and the second gives that of a certain type of distortional waves generated from a point.

I. Dilatational Waves from an Internal Point.

The expressions for primary waves, when they are dilatational, are given by

$$\Delta=\frac{A}{\sqrt{h}}\,\frac{H^{(1)}_{\frac{1}{2}}(hR)}{\sqrt{R}}\,e^{-ipt} \tag{1}$$

where $\Delta=\frac{1}{R}\frac{\partial(RU)}{\partial R}$, $\frac{2\pi}{p}$=period of waves,

$h^2=\frac{\rho p^2}{\lambda+2\mu}$,

$\frac{2\pi}{h}$=wave length,

R=radial distance from the origin O_1,

U=radial displacement relative to O_1.

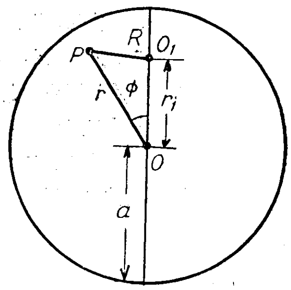

Fig. a.

Now

$$\frac{H^{(1)}_{\frac{1}{2}}(hR)}{\sqrt{hR}}=\sqrt{2\pi}\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)\frac{H^{(1)}_{m+\frac{1}{2}}(hr)}{\sqrt{hr}}\,\frac{J_{m+\frac{1}{2}}(hr_1)}{\sqrt{hr_1}}\,P_m(\cos\varphi) \tag{2}$$

where $R^2=r^2+r_1^2-2rr_1\cos\varphi$

so that $$\Delta=\frac{A\sqrt{2\pi}}{\sqrt{h}}\sum_{m=0}^{\infty}\frac{H^{(1)}_{m+\frac{1}{2}}(hr)}{\sqrt{r}}\,\frac{J_{m+\frac{1}{2}}(hr_1)}{\sqrt{hr_1}}\,P_m(\cos\varphi)\,e^{-ipt}$$

and the radial and colatitudinal displacements relative to the centre of the sphere, 0, are expressed by

$$
\left.\begin{aligned}
u_0 &= -\frac{A\sqrt{2\pi}}{h^{\frac{5}{2}}}\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)\frac{d}{dr}\frac{H^{(1)}_{m+\frac{1}{2}}(hr)}{\sqrt{r}}\frac{J_{m+\frac{1}{2}}(hr_1)}{\sqrt{hr_1}}P_m(\cos\varphi)\,e^{-ipt}\\
v_0 &= -\frac{A\sqrt{2\pi}}{h^{\frac{5}{2}}}\sum_{m=1}^{\infty}\left(m+\frac{1}{2}\right)\frac{H^{(1)}_{m+\frac{1}{2}}(hr)}{r^{\frac{3}{2}}}\frac{J_{m+\frac{1}{2}}(hr_1)}{\sqrt{hr_1}}\frac{dP_m(\cos\varphi)}{d\varphi}\,e^{-ipt}
\end{aligned}\right\}\quad(3)
$$

The expressions of the reflected waves are given by

$$
\left.\begin{aligned}
\Delta' &= \sum_{m=0}^{\infty}\frac{B_m}{\sqrt{hr}}H^{(2)}_{m+\frac{1}{2}}(hr)P_m(\cos\varphi)\,e^{-ipt}\\
2\bar{\omega}' &= \sum_{m=1}^{\infty}\frac{C_m}{\sqrt{kr}}H^{(2)}_{m+\frac{1}{2}}(kr)\frac{dP_m(\cos\varphi)}{d\varphi}\,e^{-ipt}\\
u_1 &= -\sum_{m=0}^{\infty}\frac{B_m}{h^{\frac{5}{2}}}\frac{d}{dr}\frac{H^{(2)}_{m+\frac{1}{2}}(hr)}{\sqrt{r}}P_m(\cos\varphi)\,e^{-ipt}\\
v_1 &= -\sum_{m=1}^{\infty}\frac{B_m}{h^{\frac{5}{2}}r^{\frac{3}{2}}}H^{(2)}_{m+\frac{1}{2}}(hr)\frac{dP_m(\cos\varphi)}{d\varphi}\,e^{-ipt}\\
u_2 &= -\sum_{m=1}^{\infty}\frac{C_m m(m+1)}{k^{\frac{5}{2}}r^{\frac{3}{2}}}H^{(2)}_{m+\frac{1}{2}}(kr)P_m(\cos\varphi)\,e^{-ipt}\\
v_2 &= -\sum_{m=1}^{\infty}\frac{C_m}{k^{\frac{5}{2}}r}\frac{d}{dr}\sqrt{r}\,H^{(2)}_{m+\frac{1}{2}}(kr)\frac{dP_m(\cos\varphi)}{d\varphi}\,e^{-ipt}
\end{aligned}\right\}\quad(4)
$$

where $k^2=\frac{\rho p^2}{\mu}$

Δ', $\bar{\omega}'$=reflected dilatational and distortional waves,

u_1, v_1=radial and colatitudinal components of displacement corresponding to Δ',

u_2, v_2=radial and colatitudinal components of displacement corresponding to $\bar{\omega}'$.

On the surface, $r=a$, we have

$$
\left.\begin{aligned}
&\lambda(\Delta+\Delta')+2\mu\frac{\partial}{\partial a}(u_0+u_1+u_2)=0\\
&\frac{\partial}{\partial a}(v_0+v_1+v_2)-\frac{1}{a}(v_0+v_1+v_2)+\frac{1}{a}\frac{\partial}{\partial\theta}(u_0+u_1+u_2)=0
\end{aligned}\right\}\quad(5)
$$

From (3), (4) and (5), we get

$$
\left.
\begin{aligned}
& A\sqrt{2\pi}\left(m+\frac{1}{2}\right)\frac{J_{m+\frac{1}{2}}(hr_1)}{\sqrt{hr_1}}\left[\lambda\frac{H^{(1)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}-2\mu\frac{d^2}{d(ha)^2}\frac{H^{(1)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}\right] \\
& +B_m\left[\lambda\frac{H^{(2)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}-2\mu\frac{d^2}{d(ha)^2}\frac{H^{(2)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}\right] \\
& \qquad\qquad -C_m m(m+1)2\mu\frac{d}{d(ka)}\frac{H^{(1)}_{m+\frac{1}{2}}(ka)}{(ka)^{\frac{3}{2}}}=0 \\
& A\sqrt{2\pi}\left(m+\frac{1}{2}\right)\left[\frac{d}{d(ha)}\frac{H^{(1)}_{m+\frac{1}{2}}(ha)}{(ha)^{\frac{3}{2}}}-\frac{H^{(1)}_{m+\frac{1}{2}}(ha)}{(ha)^{\frac{5}{2}}}\right. \\
& \qquad\qquad \left.+\frac{1}{(ha)}\frac{d}{d(ha)}\frac{H^{(1)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}\right] \\
& +B_m\left[\frac{d}{d(ha)}\frac{H^{(2)}_{m+\frac{1}{2}}(ha)}{(ha)^{\frac{3}{2}}}-\frac{H^{(2)}_{m+\frac{1}{2}}(ha)}{(ha)^{\frac{5}{2}}}+\frac{1}{(ha)}\frac{d}{d(ha)}\frac{H^{(2)}_{m+\frac{1}{2}}(ha)}{\sqrt{ha}}\right] \\
& +C_m\left[\frac{d}{d(ka)}\frac{1}{(ka)}\frac{d}{d(ka)}\sqrt{ha}\,H^{(2)}_{m+\frac{1}{2}}(ka)\right. \\
& \qquad \left.-\frac{1}{(ka)^2}\frac{d}{d(ka)}\sqrt{ka}\,H^{(2)}_{m+\frac{1}{2}}(ka)+\frac{m(m+1)}{(ka)^{\frac{5}{2}}}H^{(2)}_{m+\frac{1}{2}}(ka)\right]=0
\end{aligned}
\right\} \quad (6)
$$

The expressions (4) together with the values of constants, B_m and C_m, determined by the above equations (6), give both kinds of reflected waves.

For relatively large values of ha, ka and hr_1, we get approximately,

$$
\left.
\begin{aligned}
B_m &= -\frac{2}{hr_1}A\cos\left(hr_1-\frac{m+1}{2}\pi\right)\left(m+\frac{1}{2}\right)e^{i(2ha-\overline{m+1}\pi)} \\
C_m &= \frac{4ika}{hr_1(ha)^2}A\cos\left(hr_1-\frac{m+1}{2}\pi\right)\left(m+\frac{1}{2}\right)e^{i(ha+ka-\overline{m+1}\pi)}
\end{aligned}
\right\} \quad (7)
$$

and, from (3), (4) and (7), the solutions of displacement on $r=a$ is given by

$$
\left.
\begin{aligned}
u &\doteqdot -\frac{2A}{h^{\frac{5}{2}}}\frac{d}{da}\frac{H^{(1)}_{\frac{1}{2}}(hR)}{\sqrt{R}}e^{-ipt} \\
v &\doteqdot \frac{2A}{(ka)h^{\frac{5}{2}}}\frac{d^2}{da\,d\varphi}\frac{H^{(1)}_{\frac{1}{2}}(hR)}{\sqrt{R}}e^{-ipt}
\end{aligned}
\right\} \quad (8)
$$

where $R=\sqrt{a^2+r_1^2-2ar_1\cos\varphi}$

This solution seems to be easily evaluated from the expression of the primary waves given by (1); but, without the expressions of the reflected waves, the analytical determination of the surface displacement will be impossible.

The distribution of u and v on the surface of the sphere thus determined for the case $r_1=a/2$ is illustrated in Fig. I. The results well conform with those for the reflection of the plane waves at the plane boundary.

It is to be noted that, though the problem of the reflection of waves should be attacked by assuming the surface of virtual origins[1] at the external region of the sphere, the author has studied the problem in a form consisting of series of harmonic functions. The application of virtual origins in the present problem is very difficult as the reflected waves are composed of dilatational and distortional waves.

II. Distortional Waves from an Internal Point.

The equation of motion of the primary waves are expressed by

$$\frac{\rho}{\mu}\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial R^2}+\frac{2}{R}\frac{\partial w}{\partial R}-\frac{2}{R^2}w+\frac{1}{R^2}\left\{\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial w}{\partial \theta}\cot\theta+w(1-\cot^2\theta)\right\}=0 \qquad (1)$$

where w is the azimuthal displacement.

The distortional waves, which are generated from a point oscillating about a line passing through that point and the centre of the sphere, are given by putting $n=1$ in the solution of (1) as follows:—

$$w=\frac{A}{\sqrt{R}}H^{(1)}_{\frac{3}{2}}(kR)\sin\theta\, e^{-ipt} \qquad (2)$$

where $R^2=r^2+r_1^2-2rr_1\cos\varphi$

or
$$w=\frac{A}{R^{\frac{3}{2}}}H^{(1)}_{\frac{3}{2}}(kR)r\sin\varphi\, e^{-ipt} \qquad (3)$$

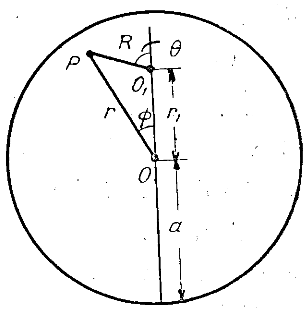

Fig. b.

Now

$$\frac{H^{(1)}_{\frac{3}{2}}(kR)}{(kR)^{\frac{3}{2}}}=\sqrt{2\pi}\sum_{m=0}^{\infty}\left(m+\frac{3}{2}\right)\cdot\frac{H^{(1)}_{m+\frac{3}{2}}(kr)}{(kr)^{\frac{3}{2}}}\,\frac{J_{m+\frac{1}{2}}(kr_1)}{(kr_1)^{\frac{3}{2}}}\,\frac{dP_m(\cos\varphi)}{-\sin\varphi d\varphi} \qquad (4)$$

(1) Professor Nagaoka's advice.

so that

$$w=-A\sqrt{2\pi}\sum_{m=0}^{\infty}\left(m+\frac{3}{2}\right)\frac{H^{(1)}_{m+\frac{3}{2}}(kr)}{\sqrt{r}}\frac{J_{m+\frac{3}{2}}(kr_1)}{(kr_1)^{\frac{3}{2}}}\frac{dP_m(\cos\varphi)}{d\varphi}e^{-ipt} \tag{5}$$

The reflected waves are purely distortional and expressed by

$$w'=\sum_{m=0}^{\infty}\frac{B_m}{\sqrt{r}}H^{(2)}_{m+\frac{1}{2}}(kr)\frac{dP_m(\cos\varphi)}{d\varphi}e^{-ipt} \tag{6}$$

The boundary condition on $r=a$ is written by

$$\frac{\partial(w+w')}{\partial a}-\frac{w+w'}{a}=0 \tag{7}$$

From (5), (6) and (7), we get

$$B_m=A\sqrt{2\pi}\left(m+\frac{3}{2}\right)\frac{J_{m+\frac{3}{2}}(kr_1)}{(kr_1)^{\frac{3}{2}}}\frac{d}{da}\frac{H^{(1)}_{m+\frac{3}{2}}(ka)}{a^{\frac{3}{2}}}\Big/\frac{d}{da}\frac{H^{(2)}_{m+\frac{3}{2}}(ka)}{a^{\frac{3}{2}}} \tag{8}$$

The approximate solution for relatively large values of ka and kr_1, is expressed by

$$B_m\doteqdot A\,(2m+3)(-i)\frac{\cos\left(kr_1-\frac{m+2}{2}\pi\right)}{(kr_1)^2} \tag{9}$$

and the surface displacement by

$$\left.\begin{aligned} &w_{r=a}\doteqdot\frac{2A}{R^{\frac{3}{2}}}H^{(1)}_{\frac{3}{2}}(kR)\,a\sin\varphi\, e^{-ipt}\\ \text{where}\quad &R=\sqrt{a^2+r_1^2-2ar_1\cos\varphi}\end{aligned}\right\} \tag{10}$$

As an example, the case for $r_1=a/2$ is illustrated in Fig. II. The results show that, the zone of the maximum displacement of the surface is situated at a certain colatitudinal distance from the epicentre, though this distance varies as the depth of the origin.

A noteworthy fact is that the surface movement by such incidental and reflected waves well resembles that of Love-waves. Indeed, those waves which are taken to be Love-waves, may in somes cases be this surface movement.

Summary.

(1) When the primary waves are generated from an internal point of an elastic sphere, the nature of the reflection on the spherical surface is the same

as at the plane boundary. The reflected waves are, of course, composed of both dilatational and distortortional waves, even when the primary waves are purely dilatational.

(2) The expression of surface displacement are very simple if the wave length is sufficiently small compared with the radius of the Earth.

(3) In the case of the reflection of purely distortional waves, the zone of maximum displacement of the surface is situated at a certain colatitudinal distance from the epicentre, the distance varying as the depth of the origin.

(4) The reflection of the distortional waves has a close resemblance to the propagation of Love-waves.

In conclusion the author's thanks are due to Professor Nagaoka for his kind advice and suggestion, though the solution has not yet been extended to follow his advice through the inability of the author.

October 1927.

FIG. 1.

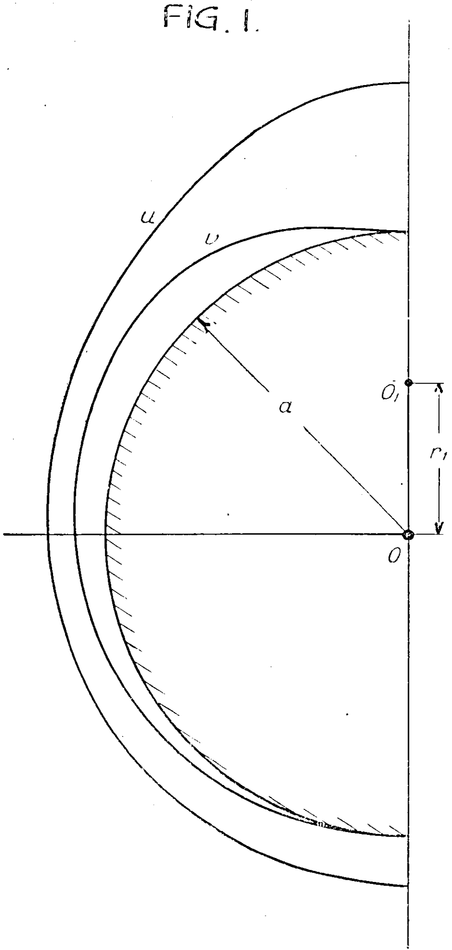

FIG. 2.