東京帝國大學
地震研究所彙報
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Dispersion of Elastic Waves propagated on the Surface of Stratified Bodies and on Curved Surfaces

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表面層及び曲面に傳播する彈性波の分散

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地震波が震源から遠ざかるに從て繰替の振動性が增加する傾向は彈性波の分散によつて可なり說明出來るが、其には種々の原因がある。この論文では境界面の狀況によるものを主として取扱ひ、其を表面の層に起因するものと、表面の曲率によるものとに分類して考へた。

問題を二部門に分け、第一部門では種々の波長の調和波が表面層に沿うて傳播する場合を考へ、各分波に相當する速度を算出する時、各異なる値を有し、從て波の分散性を確定する事が出來た。倘斯る問題の特別な場合はブロムウキツチ及びラブ兩氏によつて夫々特種の方法で手を著けられて居るが（第二頁脚註）、層厚、速度、彈性間の一般關係は全く發見されて居らぬ。この論文は斯る關係を充分に攻究して地震學者に確實なる（三桁の數字迄）材料を提供したばかりでなく前記諸氏とは全く別の計算法によつて普通のレーレー波と一致する理論を與へた。

第二部門では表面に曲率がある時のレーレー波、ラブ波等を計算した。而して何れの場合にも波長によつて傳播速度が異なり、分散の原因を作る事を述べ、合せて一般の彈性體に於て高周振動の勢力が其表面に集まる事をも示した。

倘本論文中第一圖の曲線は（$\mu=2\mu'$ を除き）地震研究所の宮崎武平氏によつて主として算出されたるものである事を附加へる。

The earthquake motion at a distant point from the origin is oscillatory, though it is not necessarily oscillatory in the neighbourhood of the source.

The farther the epicentral distance, the more grows the oscillation. This is due to the fact that a shock is composed of an aggregate of simple harmonic trains of waves, each travelling with different velocity corresponding to its wave length. The proper velocity of each train of waves depends not only on the surface conditions of the solid, but on the physical nature of the medium, such as the effect of gravity or of heterogeneity of elastic and dissipation constants. In this paper only the former will be dealt with.

The investigation of the effect of the surface conditions on the velocity of propagation is limited to two simple cases, namely the effect of surface layers on the propagation of Rayleigh-type waves and the other the propagation of waves on curved surfaces. Although special problems on the propagation of waves on a stratified surface have already been taken up by eminent mathematicians such as Bromwich(1) and Love(2), yet the general relation between the thickness of the layers and the velocity of waves has not completely been studied. As to the propagation of waves on curved surfaces, it seems that it has not yet been much studied, excepting those on a spherical surface, in spite of its importance.

The present investigation consists of two parts: the first part relates to the dispersion of elastic waves on a solid body with surface layers and the second to the waves propagated over a curved surface.

I. Dispersion of Elastic Waves on a Stratified Surface.

1. When the axis of x is taken coincident with the lower boundary of a layer lying on the semi-infinite solid body and the axis of y is directed vertically upwards, the equations of motion of the lower medium are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2}&=(\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right)\\ \rho\frac{\partial^2\varpi}{\partial t^2}&=\mu\left(\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}\right)\end{aligned}\right\}\qquad(1)$$

where

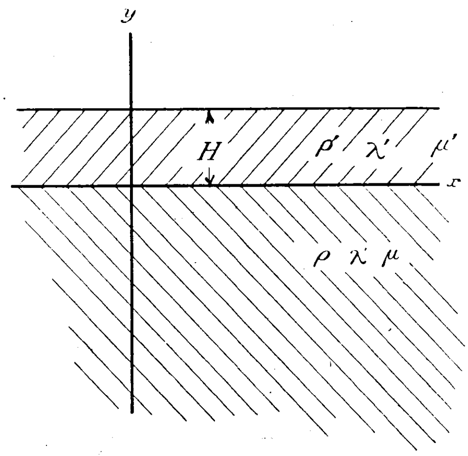
(1) London Math. Soc. Proc., vol. 30 (1898).
(2) Some Problems of Geodynamics (1911).

$$\left.\begin{aligned} &\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \\ &2\varpi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \end{aligned}\right\}$$

u, v = horizontal and vertical components of displacement respectively,

ρ = density

and λ, μ = Lame's elastic constants. (2)



The solutions of (1) can be written in the forms:—

$$\Delta=Ae^{ry}\,e^{i(pt-fx)} \tag{3}$$

$$2\varpi - Be^{sy}\,c^{i(pt-fx)} \tag{4}$$

where

$$\left.\begin{aligned} &r^2=f^2-h^2, \qquad s^2=f^2-k^2, \\ &h^2=\frac{\rho p^2}{\lambda+2\mu} \qquad k^2=\frac{\rho p^2}{\mu} \end{aligned}\right\} \tag{5}$$

Displacement (u_1, v_1) answering to Δ in (3) and satisfying $\varpi=0$ is given by

$$\left.\begin{aligned} u_1&=\frac{if}{h^2}Ae^{ry+i(pt-fx)} \\ v_1&=-\frac{r}{h^2}Ae^{ry+i(pt-fx)} \end{aligned}\right\} \tag{6}$$

Displacement (u_2, v_2) derived from the value of ϖ in (4) with the condition, $\Delta=0$, is expressed by

$$\left.\begin{aligned} u_2&=\frac{s}{k^2}Be^{sy+i(pt-fx)} \\ v_2&=\frac{if}{k^2}Be^{sy+i(pt-fx)} \end{aligned}\right\} \tag{7}$$

In like manner the equations of motion of the layer together with their solutions are expressed by

$$\left.\begin{aligned}
&\rho' \frac{\partial^2 \Delta'}{\partial t^2}=(\lambda'+2\mu')\left(\frac{\partial^2 \Delta'}{\partial x^2}+\frac{\partial^2 \Delta'}{\partial y^2}\right)\\
&\rho' \frac{\partial^2 \varpi'}{\partial t^2}=\mu'\left(\frac{\partial^2 \varpi'}{\partial x^2}+\frac{\partial^2 \varpi'}{\partial y^2}\right)\\
&\quad \Delta'=(c \cosh r'y+D \sinh r'y)\, e^{i(pt-fx)}\\
&\quad \varpi'=(E \cos s'y+F \sin s'y)\, e^{i(pt-fx)}\\
&\quad u_1'=\frac{if}{h'^2}(c \cosh r'y+D \sinh r'y)\, e^{i(pt-fx)}\\
&\quad v_1'=-\frac{r'}{h'^2}(C \sinh r'y+D \cosh r'y)\, e^{i(pt-fx)}\\
&\quad u_2'=-\frac{s'}{k'^2}(E \sin s'y-F \cos s'y)\, e^{i(pt-fx)}\\
&\quad v_2'=\frac{if}{k'^2}(E \cos s'y+F \sin s'y)\, e^{i(pt-fx)}
\end{aligned}\right\} \quad (8)$$

in which

$$\left.\begin{aligned}
&\Delta'=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\\
&2\varpi'=\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\\
&u',\ v'=\text{horizontal and vertical components of displacement respectively,}\\
&\rho',\ \lambda'\ \mu'=\text{density and Lame's elastic constants for the layer,}\\
&r'^2=f^2-h'^2 \gtreqless 0, \qquad s'^2=f^2-k'^2 \gtreqless 0\\
&h'^2=\frac{\rho' p^2}{\lambda'+2\mu'}, \qquad k'^2=\frac{\rho' p^2}{\mu'}
\end{aligned}\right\} \quad (9)$$

Now the boundary conditions are given by the following equations:—

$$\left.\begin{array}{l}\left.\begin{array}{l} u_1+u_2=u_1'+u_2' \\ v_1+v_2=v_1'+v_2' \\ \lambda\Delta+2\mu\dfrac{\partial}{\partial y}(v_1+v_2)=\lambda'\Delta'+2\mu'\dfrac{\partial}{\partial y}(v_1'+v_2') \\ \mu\left[\dfrac{\partial(u_1+u_2)}{\partial y}+\dfrac{\partial(v_1+v_2)}{\partial x}\right]=\mu'\left[\dfrac{\partial(u_1'+u_2')}{\partial y}+\dfrac{\partial(v_1'+v_2')}{\partial x}\right] \end{array}\right\} \text{at } y=0 \\ \left.\begin{array}{l} \lambda'\Delta'+2\mu'\dfrac{\partial}{\partial y}(v_1'+v_2')=0 \\ \dfrac{\partial(u_1'+v_2')}{\partial y}+\dfrac{\partial(v_1'+v_2')}{\partial x}=0 \end{array}\right\} \text{at } y=H \end{array}\right\} \quad (10)$$

where H is the thickness of the layer.

Putting the values of Δ, u_1, u_2, v_1, v_2, Δ', u_1', u_2', v_1' and v_2' from (3), (6), (7) and (8) in the conditions of (10) at the boundaries, and eliminating A, B, C, D, E and F, we have

$$\begin{vmatrix} -\dfrac{(f^2-s'^2)}{k'^2}Y_2, & 2\dfrac{\mu'}{\mu}\dfrac{ifs'}{k'^2}Y_1, & 0, & -2\dfrac{\mu'}{\mu}\dfrac{ifs'}{k'^2}, & 0, & \dfrac{s'f}{k'^2} \\ -\dfrac{(f^2-s'^2)}{k'^2}Y_1, & -2\dfrac{\mu'}{\mu}\dfrac{ifs'}{k'^2}Y_2, & \dfrac{\mu'}{\mu}\dfrac{(f^2-s'^2)}{k'^2}, & 0, & \dfrac{if^2}{k'^2}, & 0 \\ 2\dfrac{ifr'}{h'^2}X_1, & \left(2\dfrac{\mu'}{\mu}\dfrac{r'^2}{h'^2}-\dfrac{\lambda'}{\mu}\right)X_2, & -\dfrac{\mu'}{\mu}\dfrac{2ifr'}{h'^2}, & 0, & \dfrac{r'f}{k'^2} & 0 \\ 2\dfrac{ifr'}{h'^2}X_2, & \left(2\dfrac{\mu'}{\mu}\dfrac{r'^2}{h'^2}-\dfrac{\lambda'}{\mu}\right)X_1, & 0, & \dfrac{\lambda'}{\mu}-2\dfrac{\mu'}{\mu}\dfrac{r'^2}{h'^2}, & 0, & -\dfrac{if^2}{h'^2} \\ 0, & 0, & -\dfrac{(f^2+s^2)}{k^2}, & 2\dfrac{ifs}{k^2}, & \dfrac{if^2}{k^2}, & -\dfrac{sf}{k^2} \\ 0, & 0, & \dfrac{2ifr}{h^2}, & 2\dfrac{r^2}{h^2}-\dfrac{\lambda}{\mu}, & -\dfrac{rf}{h^2}, & \dfrac{if^2}{h^2} \end{vmatrix}=0 \quad (11)$$

where $X_1=\cosh r'H$, $X_2=\sinh r'H$, $Y_1=\cos s'H$ and $Y_2=\sin s'H$.

Solving the above determinantal equation by tentative methods for the cases, $\mu=2\mu'$, $\mu=3\mu'$, $\mu=4\mu'$ and $\mu=5\mu'$, in which $\lambda=\mu$, $\lambda'=\mu'$ and $\rho=\rho$, the author has obtained, by the aid of Mr. Miyazaki, the relation between the velocity and L/H, L being the wave length. Fig. I. gives this relation and shows the nature that longer waves have higher velocities with an asymptotic limiting value. Thus it will be seen that elastic surface waves on a stratified body are dispersive. In general, if $\mu'<\mu$, the velocity of such waves lies between the velocities of waves proper to the two materials, the upper layer and the subjacent materials, when they take the semi-infinite extension.

2. Following the same line of argument for two dimensional propagation of waves started from an origin and also the same on a spherical surface, we arrive at the following determinantal equations:—

for the former

$$\begin{vmatrix}
\frac{k^2}{k^2-\alpha^2}, & \frac{-\alpha k}{k^2-\alpha_2}, & \frac{\alpha^2}{k^2-\alpha^2}-\frac{\lambda}{\mu}, & \frac{2\alpha k}{k^2-\alpha^2}, & 0, & 0, \\
\frac{-\beta k}{k^2-\beta^2}, & \frac{k^2}{k^2-\beta^2}, & \frac{-2k\beta}{k^2-\beta^2}, & \frac{-(\beta^2+k^2)}{k^2-\beta^2}, & 0, & 0, \\
-\frac{k^2}{k^2-\alpha'^2}, & 0, & \frac{\mu'}{\mu}\frac{\alpha'^2}{k^2-\alpha'^2}-\frac{\lambda'}{\mu}, & -\frac{\mu'}{\mu}\frac{2\alpha' k}{k^2-\alpha'^2}, & \left(2\frac{\mu'}{\mu}\frac{\alpha'^2}{k^2-\alpha'^2}-\frac{\lambda'}{\mu}\right)\cosh\alpha' H, & \frac{2k\alpha'}{k^2-\alpha'^2}\cosh\alpha' H \\
0, & \frac{\alpha' k}{k^2-\alpha'^2}, & 0, & 0, & \left(2\frac{\mu'}{\mu}\frac{k\alpha'}{k^2-\alpha'^2}-\frac{\lambda'}{\mu}\right)\sinh\alpha' H. & \frac{2k\beta'}{k^2-\beta'^2}\sinh\alpha' H \\
0, & \frac{-k^2}{k^2-\beta'^2}, & 0, & \frac{\mu'}{\mu}\frac{\beta'^2+k^2}{k^2-\beta'^2}, & 2\frac{\mu'}{\mu}\frac{k\beta'}{k^2-\beta'^2}\sinh\beta' H, & \frac{-(\beta'^2+k^2)}{k^2-\beta'^2}\cos\beta' H \\
-\frac{\beta' k}{k^2-\beta'^2}, & 0, & 2\frac{\mu'}{\mu}\frac{\beta' k}{k^2-\beta'^2}, & 0, & -2\frac{\mu'}{\mu}\frac{k\beta'}{k^2-\beta'^2}\cosh\beta' H, & \frac{-(\beta'^2+k^2)}{k^2-\beta'^2}\sin\beta' H
\end{vmatrix}=0 \qquad \text{.....................(12)}$$

where

$$
\left.\begin{aligned}
k_2&=\alpha^2+\frac{\rho p^2}{\lambda+2\mu}=\beta^2+\frac{\rho p^2}{\mu}\\
&=\alpha'^2+\frac{\rho' p^2}{\lambda'+2\mu'}=\beta'^2+\frac{\rho' p^2}{\mu'}\\
\frac{2\pi}{k}&=\text{wave length}
\end{aligned}\right\}\qquad(13)
$$

and for the latter,

$$
\begin{vmatrix}
\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{a}}, & \frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{a}}, & (k^2-2h^2)\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{a}}-2\frac{d^2}{da^2}\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{a}},\\
n(n+1)\frac{J_{n+\frac{1}{2}}(ka)}{a^{3/2}}, & \frac{d}{da}\left(\sqrt{a}\,J_{n+\frac{1}{2}}(ka)\right), & -2(n+1)\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ka)}{a^{3/2}},\\
\frac{d}{da}\frac{J_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}, & \frac{J_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}, & \frac{\mu'}{\mu}(k'^2-h'^2)\frac{J_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}-2\frac{d^2}{da^2}\frac{J_{n+\frac{1}{2}}(h'a)}{\sqrt{a}},\\
\frac{d}{da}\frac{Y_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}, & \frac{Y_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}, & \frac{\mu'}{\mu}(k'^2-h'^2)\frac{Y_{n+\frac{1}{2}}(h'a)}{\sqrt{a}}-2\frac{d^2}{da^2}\frac{Y_{n+\frac{1}{2}}(h'a)}{\sqrt{a}},\\
n(n+1)\frac{J_{n+\frac{1}{2}}(k'a)}{a^{3/2}}, & \frac{d}{da}\left(\sqrt{a}\,J_{n+\frac{1}{2}}(k'a)\right), & -2\frac{\mu'}{\mu}\frac{d}{da}\frac{J_{n+\frac{1}{2}}(k'a)}{a^{3/2}},\\
n(n+1)\frac{Y_{n+\frac{1}{2}}(k'a)}{a^{3/2}}, & \frac{d}{da}\left(\sqrt{a}\,Y_{n+\frac{1}{2}}(k'a)\right), & -2\frac{\mu'}{\mu}\frac{d}{da}\frac{Y_{n+\frac{1}{2}}(k'a)}{a^{3/2}},
\end{vmatrix}
$$

$$
\begin{aligned}
&\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ha)}{a^{3/2}},\\
&\frac{d^2}{da^2}\frac{J_{n+\frac{1}{2}}(ka)}{\sqrt{a}}-\frac{k^2}{2}\frac{J_{n+\frac{1}{2}}(ka)}{\sqrt{a}},\\
&\frac{\mu'}{\mu}\frac{d}{da}\frac{J_{n+\frac{1}{2}}(h'a)}{a^{3/2}},\\
&\frac{\mu'}{\mu}\frac{d}{da}\frac{Y_{n+\frac{1}{2}}(h'a)}{a^{3/2}},\\
&\frac{\mu'}{\mu}\frac{d^2}{da^2}\frac{J_{n+\frac{1}{2}}(k'a)}{\sqrt{a}}-\frac{k'^2}{2}\frac{J_{n+\frac{1}{2}}(k'a)}{\sqrt{a}},\\
&\frac{\mu'}{\mu}\frac{d^2}{da^2}\frac{Y_{n+\frac{1}{2}}(k'a)}{\sqrt{a}}-\frac{k'^2}{2}\frac{Y_{n+\frac{1}{2}}(k'a)}{\sqrt{a}},
\end{aligned}
$$

$$
\begin{matrix}
0 & 0\\
0 & 0\\
\frac{\mu'}{\mu}(k'^2-h'^2)\frac{J_{n+\frac{1}{2}}(h'b)}{\sqrt{b}}-2\frac{d^2}{db^2}\frac{J_{n+\frac{1}{2}}(h'b)}{\sqrt{b}}, & \frac{d}{db}\frac{J_{n+\frac{1}{2}}(h'b)}{b^{3/2}}
\end{matrix}\;\Bigg|=0
$$

$$\left.\begin{matrix} \dfrac{\mu'}{\mu}(k'^2-h'^2)\dfrac{Y_{n+\frac{1}{2}}(h'b)}{\sqrt{b}}-2\dfrac{d^2}{db^2}\dfrac{Y_{n+\frac{1}{2}}(h'b)}{\sqrt{b}}, & \dfrac{d}{db}\dfrac{Y_{n+\frac{1}{2}}(h'b)}{b^{3/2}} \\ -2\dfrac{\mu'}{\mu}\dfrac{d}{db}\dfrac{J_{n+\frac{1}{2}}(k'b)}{b^{3/2}}, & \dfrac{d^2}{db^2}\dfrac{J_{n+\frac{1}{2}}(k'b)}{\sqrt{b}}-\dfrac{k'^2}{2}\dfrac{J_{n+\frac{1}{2}}(k'b)}{\sqrt{b}} \\ -2\dfrac{\mu'}{\mu}\dfrac{d}{db}\dfrac{Y_{n+\frac{1}{2}}(k'b)}{b^{3/2}}, & \dfrac{d^2}{db^2}\dfrac{Y_{n+\frac{1}{2}}(k'b)}{\sqrt{b}}-\dfrac{k'^2}{2}\dfrac{Y_{n+\frac{1}{2}}(k'b)}{\sqrt{b}} \end{matrix}\right| \quad (14)$$

where $h^2=\dfrac{\rho p^2}{\lambda+2\mu}$, $k^2=\dfrac{\rho p^2}{\mu}$, $h'^2=\dfrac{\rho' p^2}{\lambda'+2\mu'}$, $k'^2=\dfrac{\rho' p^2}{\mu'}$

a=radius of inner circle,

b= ,, ,, outer ,, ,

n=number of waves on the whole spherical surface.

These determinantal equations are too complicated to study the nature of motion directly therefrom; so that they are left for further study.

II. Dispersion of Elastic Waves on Curved Surface.

4. We proceed to investigate the propagation of Rayleigh-type waves on a cylindrical surface parallel to its generating lines, when the cylinder is circular and has a central core. The equations of motion of the core in cylindrical coordinates, when the circumferential component is omitted, are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2}&=(\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{\partial^2\Delta}{\partial z^2}\right)\\ \rho\frac{\partial^2\varpi}{\partial t^2}&=\mu\left(\frac{\partial^2\varpi}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi}{\partial r}-\frac{\varpi}{r^2}+\frac{\partial^2\varpi}{\partial z^2}\right) \end{aligned}\right\} \quad (1)$$

where

$$\Delta=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}$$

$$2\varpi=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}$$

u, w=radial and vertical components of displacement respectively,

ρ=density,

and λ, μ=Lame's elastic constants. (2)

The solutions of (1) can be written in the forms:—

$$\Delta = A J_0(hr)\, e^{i(pt-fz)} \qquad (3)$$

$$\varpi = B J_1(kr)\, e^{i(pt-fz)} \qquad (4)$$

where

$$\left.\begin{aligned} h^2 &= \frac{\rho p^2}{\lambda+2\mu} - f^2 \gtrless 0 \\ k^2 &= \frac{\rho p^2}{\mu} - f^2 \gtrless 0 \end{aligned}\right\} \qquad (5)$$

Displacement (u_1, w_1) answering to Δ in (2) and satisfying $\varpi=0$ is given by

$$\left.\begin{aligned} u_1 &= \frac{Ah}{h^2+f^2} J_1(hr)\, e^{i(pt-fz)} \\ w_1 &= \frac{Aif}{h^2+f^2} J_0(hr)\, e^{i(pt-fz)} \end{aligned}\right\} \qquad (6)$$

Displacement (u_2, w_2) derived from the value of ϖ in (4) with the condition $\Delta=0$, is expressed by

$$\left.\begin{aligned} u_2 &= \frac{Bif}{k^2+f^2} J_1(kr)\, e^{i(pt-fz)} \\ w_2 &= \frac{Bf}{k^2+f^2} J_0(kr)\, e^{i(pt-fz)} \end{aligned}\right\} \qquad (7)$$

In like manner the equations of motion of the layer and their solutions are expressed by

$$\left.\begin{aligned} \rho' \frac{\partial^2 \Delta'}{\partial t^2} &= (\lambda'+\mu')\left(\frac{\partial^2 \Delta'}{\partial r^2} + \frac{1}{r}\frac{\partial \Delta'}{\partial r} + \frac{\partial^2 \Delta'}{\partial z^2}\right) \\ \rho' \frac{\partial^2 \varpi'}{\partial r^2} &= \mu'\left(\frac{\partial^2 \varpi'}{\partial r^2} + \frac{1}{r}\frac{\partial \varpi'}{\partial r} - \frac{\varpi'}{r^2} + \frac{\partial^2 \varpi'}{\partial z^2}\right) \\ \Delta' &= [C J_0(h'r) + D Y_0(h'r)]\, e^{i(pt-fz)} \\ \varpi' &= [E J_1(k'r) + F Y_1(k'r)]\, e^{i(pt-fz)} \\ u_1' &= \frac{h'}{h'^2+f^2}[C J_1(h'r) + D Y_1(h'r)]\, e^{i(pt-fz)} \\ w_1' &= \frac{if}{h'^2+f^2}[C J_0(h'r) + D Y_0(h'r)]\, e^{i(pt-fz)} \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} u_2' &= -\frac{if}{k'^2+f^2}[EJ_1(k'r)+FY_1(k'r)]\,e^{i(pt-fz)} \\ w_2' &= \frac{k'}{k'^2+f^2}[EJ_0(k'r)+FY_0(k'r)]\,e^{i(pt-fz)} \end{aligned}\right\}$$

in which

$$\left.\begin{aligned} &\Delta' = \frac{\partial(ru')}{\partial r} + \frac{\partial w'}{\partial z} \\ &2\varpi' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r} \\ &\rho', \lambda', \mu' = \text{density and Lame's elastic constants for the layer,} \\ &h'^2 = \frac{\rho' p^2}{\lambda'+2\mu'} - f^2 \gtreqless 0 \\ &k'^2 = \frac{\rho' p^2}{\mu'} - f^2 \gtreqless 0 \end{aligned}\right\} \quad (9)$$

Now the boundary conditions are given by the following equations:—

$$\left.\begin{aligned} &\left.\begin{aligned} &u_1+u_2 = u_1'+u_2' \\ &w_1+w_2 = w_1'+w_2' \\ &\lambda\Delta+2\mu\frac{\partial}{\partial r}(u_1+u_2) = \lambda'\Delta'+2\mu'\frac{\partial}{\partial r}(u_1'+u_2') \\ &\mu\left[\frac{\partial}{\partial z}(u_1+u_2)+\frac{\partial}{\partial r}(w_1+w_2)\right] \\ &\qquad = \mu'\left[\frac{\partial}{\partial z}(u_1'+u_2')+\frac{\partial}{\partial r}(w_1'+w_2')\right] \end{aligned}\right\} \text{on } r=a \\ &\left.\begin{aligned} &\lambda'\Delta'+2\mu'\frac{\partial}{\partial r}(u_1'+u_2') = 0 \\ &\frac{\partial}{\partial z}(u_1'+u_2')+\frac{\partial}{\partial r}(w_1'+w_2') \end{aligned}\right\} \text{on } r=b \end{aligned}\right\} \quad (10)$$

where a and b are the inner and outer radii respectively.

Putting the values of Δ, u_1, u_2, w_1, w_2, Δ', u_1', u_2', w_1' and w_2' from (3), (6), (7) and (8) in (10) and eliminating A, B, C, D, E and F, we have

$$
\left|\begin{array}{llll}
\frac{hf}{h_1^2}J_1(ha), & \frac{if^2}{h_1^2}J_0(ha), & -J_0(ha)-\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dJ_1(ha)}{da}, & \frac{2ihf}{h_1^2}J_1(ha),\ 0,\ 0, \\
-\frac{if^2}{k_1^2}J_1(ka), & \frac{kf}{k_1^2}J_0(ka), & \frac{2\mu}{\lambda}\frac{if}{k_1^2}\frac{d}{da}J_1(ka). & J_1(ka),\ 0,\ 0, \\
-\frac{h'f}{h_1'^2}J_1(h'a), & -\frac{if^2}{h_1'^2}J_0(h'a), & \frac{\lambda'}{\lambda}J_0(h'a)+\frac{2\mu'}{\lambda}\frac{h'}{h_1'^2}\frac{dJ_1(h'a)}{da}, & \\
-\frac{h'f}{h_1'^2}Y_1(h'a), & -\frac{if^2}{h_1'^2}Y_0(h'a), & \frac{\lambda'}{\lambda}Y_0(h'a)+\frac{2\mu'}{\lambda}\frac{h'}{h_1'^2}\frac{dY_1(h'a)}{da}, & \\
-\frac{if^2}{k_1'^2}J_1(k'a), & \frac{k'f}{k_1'^2}J_0(k'a), & \frac{2\mu'}{\lambda}\frac{f}{k_1'^2}\frac{dJ_1(k'a)}{da}, & \\
-\frac{if^2}{k_1'^2}Y_1(k'a), & \frac{k'f}{k_1'^2}Y_0(k'a), & \frac{2\mu'}{\lambda}\frac{f}{k_1'^2}\frac{dY_1(k'a)}{da}, &
\end{array}\right.
$$

$$
\left.\begin{array}{lll}
-\frac{\mu'}{\mu}\frac{2ih'f}{h_1'^2}J_1(h'a), & J_0(h'b)+\frac{2\mu'}{\lambda'}\frac{h'}{h_1'^2}\frac{dJ_1(h'b)}{db}, & \frac{2ih'}{h_1'^2}fJ_1(h'b) \\
\frac{\mu'}{\mu}\frac{2ih'f}{h_1'^2}Y_1(h'a), & Y_0(h'b)+\frac{2\mu'}{\lambda'}\frac{h'}{h_1'^2}\frac{dY_1(h'b)}{db}, & \frac{2ih'}{h_1'^2}fY_1(h'b) \\
\frac{\mu'}{\mu}J_1(k'a), & \frac{2\mu'}{\lambda'}\frac{f}{k_1'^2}\frac{dJ_1(k'b)}{db}, & -J_1(k'b) \\
\frac{\mu'}{\mu}Y_1(k'a), & \frac{2\mu'}{\lambda'}\frac{f}{k_1'^2}\frac{dY_1(k'b)}{db}, & -Y_1(k'b)
\end{array}\right| = 0 \qquad (11)
$$

where $h_1^2=\frac{\rho p^2}{\lambda+2\mu}$, $k_1^2=\frac{\rho p^2}{\mu^2}$, $h_1'^2=\frac{\rho' p^2}{\lambda'+2\mu'}$ and $k_1'^2=\frac{\rho' p^2}{\mu'}$.

Solving the above determinantal equation by tentative methods for the case, $\rho=\rho'$, $\lambda=\mu$, $\lambda'=\mu'$ and $\mu=2\mu'$, the author obtained the relation between the velocity of propagation and L/R, L and R being the wave length and the radius of curvature respectively. Fig. II. shows this relation. It will be seen that longer waves have larger velocities with asymptotic limiting value. In this case too the waves are dispersed on account of the reason that the velocities of propagation are different for different lengths of harmonic waves. It will be worthy of notice that as equations (5), (6) and (7) show the energy of short waves is accumulated on the surface of the solid, while that of long waves is distributed approximately uniformly in the cylindrical section.

5. If the form of the surface in the above problem is concave, we obtain other types of solutions. For example, for a homogeneous solid, the determinant takes such a form as:—

$$\begin{vmatrix} H_0^{(1)}(iha)+\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dH_1^{(1)}(iha)}{da}, & \frac{2\mu}{\lambda}\frac{if}{k_1^2}\frac{dH_1(ika)}{da} \\ \frac{2ihf}{h_1^2}H_1(iha) & -H_1(ika) \end{vmatrix}=0 \qquad (12)$$

6. In a special case, when the inner core is removed, we have the propagation of Rayleigh-waves along a hollow cylinder. The determinantal equation giving the velocity is expressed by

$$\begin{vmatrix} J_0(ha)+\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dJ_1(ha)}{da}, & Y_0(ha)+\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dY_1(ha)}{da}, & \frac{2\mu}{\lambda}\frac{f}{k_1^2}J_1(ka), & \frac{2\mu}{\lambda k_1^2}Y_1(ka) \\ \frac{2ihf}{h_1^2}J_1(ha), & \frac{2ihf}{h_1^2}Y_1(ha), & -J_1(ka), & -Y_1(ka) \\ J_0(hb)+\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dJ_1(hb)}{db}, & Y_0(hb)+\frac{2\mu}{\lambda}\frac{h}{h_1^2}\frac{dY_1(hb)}{db}, & \frac{2\mu}{\lambda}\frac{f}{k_1^2}J_1(kb), & \frac{2\mu}{\lambda k_1^2}Y_1(kb) \\ \frac{2ihf}{h_1^2}J_1(hb), & \frac{2ihf}{h_1^2}Y_1(hb), & -J_1(kb), & -Y_1(kb) \end{vmatrix}=0 \qquad (13)$$

the results of which is shown in Fig. III. These waves have a close connection to those propagated in plates, which was investigated by Lamb[1] in 1916.

7. When Rayleigh-waves are propagated along the circumferential direction of a cylindrical surface, the equations of motion and their solutions are to be modified as shown below. Taking the simple case of a circular cylinder, the equations of motion in cylindrical co-ordinates, when the axial component of the motion is omitted, are expressed by

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2}&=(\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}\right) \\ \rho\frac{\partial^2\varpi}{\partial t^2}&=\mu\left(\frac{\partial^2\varpi}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varpi}{\partial\theta^2}\right) \end{aligned}\right\} \qquad (14)$$

where

(1) Roy. Soc. Proc. (1917).

$$\left.\begin{aligned}\Delta&=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}\\ 2\varpi&=\frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial \theta}\\ &u,\ v=\text{radial and circumferential components of displacement respectively.}\end{aligned}\right\}\tag{15}$$

The solutions of (14) can be written in the forms:—

$$\Delta=AJ_{fa}(hr)\,e^{i(pt+fa\theta)}\tag{16}$$

$$\varpi=BJ_{fa}(kr)\,e^{i(pt+fa\theta)}\tag{17}$$

where

$$h^2=\frac{\rho p^2}{\lambda+2\mu},\qquad k^2=\frac{\rho p^2}{\mu},\qquad \frac{2\pi}{f}=\text{wave length}\tag{18}$$

Displacement (u_1, v_1) derived from the value of Δ in (16) with the condition, $\varpi=0$, is expressed by

$$\left.\begin{aligned}u_1&=-\frac{A}{h^2}\frac{\partial J_{fa}(hr)}{\partial r}e^{i(pt+fa\theta)}\\ v_1&=-\frac{A}{h^2}\frac{ifa}{r}J_{fa}(hr)\,e^{i(pt+fa\theta)}\end{aligned}\right\}\tag{19}$$

Displacement (u_2, v_2) answering to ϖ in (17) with the condition, $\Delta=0$, is expressed by

$$\left.\begin{aligned}u_2&=\frac{B}{k^2}\frac{ifa}{r}J_{fa}(kr)\,e^{i(pt+fa\theta)}\\ v_2&=-\frac{B}{k^2}\frac{\partial J_{fa}(kr)}{\partial r}e^{i(pt+fa\theta)}\end{aligned}\right\}\tag{20}$$

The surface $r=a$ being free from traction, the equations,

$$\left.\begin{aligned}&\lambda\Delta+2\mu\frac{\partial u}{\partial r}=0\\ &\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}=0\end{aligned}\right\}\tag{21}$$

in which $u=u_1+u_2$ and $v=v_1+v_2$, must hold on that surface.

Putting the values of Δ, u_1, u_2, v_1 and v_2 in (16), (19) and (20) in (21), we obtain the following equation to determine the velocity of propagation.

$$\left[(k^2-2h^2)\, J_{fa}(ha)-2\,\frac{d^2J_{fa}(ha)}{da^2}\right](k^2-2f^2)\, J_{fa}(ka)$$

$$-4f\left\{\frac{dJ_{fa}(ha)}{da}-\frac{J_{fa}(ha)}{a}\right\}\frac{dJ_{fa}(ka)}{da}=0 \qquad (22)$$

Solving the above equation in the case, $\lambda=\mu$, we obtain the relation between the velocity and L/R, in which L and R are the wave length and the radius of curvature respectively. Fig. IV. gives this relation and indicates the dispersive nature of a transmitted wave system.

8. We now proceed to study the propagation of Love-type waves on a circular cylindrical surface having a central core parallel to its generating lines. The velocity of propagation is given by the following determinant:—

$$\begin{vmatrix} J_1(ka), & J_1(k'a), & Y_1(k'a) \\ \frac{d}{da}\frac{J_1(ka)}{a}, & \frac{\mu'}{\mu}\frac{d}{da}\frac{J_1(k'a)}{a}, & \frac{\mu'}{\mu}\frac{d}{da}\frac{Y_1(k'a)}{a} \\ 0, & \frac{d}{db}\frac{J_1(k'b)}{b}, & \frac{d}{db}\frac{Y_1(k'b)}{b} \end{vmatrix}=0 \qquad (23)$$

where

$$\left.\begin{array}{l} k^2=\frac{\rho p^2}{\mu}-f_2 \gtreqless 0, \qquad k'^2=\frac{\rho' p^2}{\mu'}-f^2 \gtreqless 0 \\ \frac{2\pi}{f}=\text{wave length}, \\ a,\, b=\text{radius of the core and the layer} \\ \qquad \text{respectively} \end{array}\right\} \qquad (24)$$

Solving the above determinant in the case, $\lambda=\mu$, $\lambda/=\mu'$, $\mu=2\mu'$ and $\rho=\rho'$, we obtain the results as shown in Fig. V.

9. When Love-waves propagate along the circumferential direction of a circular cylinder, we have to solve the determinant expressed by

$$\begin{vmatrix} J_{fa}(ka), & J_{fa}(k'a), & Y_{fa}(k'a) \\ \frac{d}{da}J_{fa}(ka), & \frac{\mu'}{\mu}\frac{d}{da}J_{fa}(k'a), & \frac{\mu'}{\mu}\frac{d}{da}Y_{fa}(k'a) \\ 0, & \frac{d}{db}J_{fa}(k'b), & \frac{d}{db}Y_{fa}(k'b) \end{vmatrix}=0 \qquad (25)$$

where

$$\left.\begin{aligned}&k^2=\frac{\rho p^2}{\mu} \qquad k'^2=\frac{\rho' p^2}{\mu'}\\&\frac{2\pi}{f}=\text{wave length},\\&a,\ b=\text{inner and outer radii respectively.}\end{aligned}\right\}\qquad(26)$$

The equation (25), however, has not yet been solved by the author.

Concluding Remarks.

We have seen that, if an elastic body is covered with superficial layers, or if the surface has a curvature, convex or concave, as mountains or lakes, the velocity of elastic surface waves depends on their wave lengths, that is to say, the aggregate of trains of waves is dispersive. The fact, that the energy of waves of short length accumulates on the surface of a body may have some important bearings on the relation between surface and non-surface seismic waves as well as on the problem of the vibration of constructive materials. In conclusion the auther wishes to express his indebtness to Professor Nagaoka and Professor Suyehiro for valuable advices and suggestions and at the same time to Mr. B. Miyazaki who assisted him in computation to obtain the important results in Fig. I.

February, 1927.

Fig. I. Dispersion of Elastic Waves over Stratified Surface.

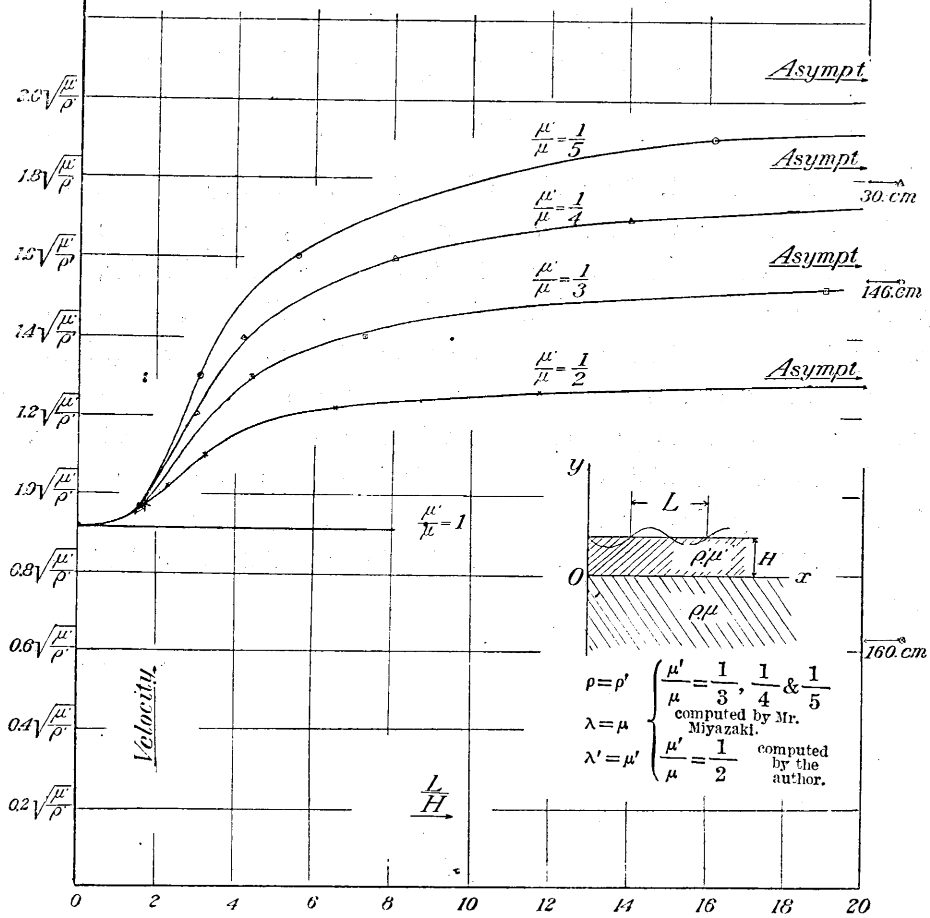


Fig. II. Propagation of Rayleigh-Waves over Curved Surface along Axial Directior.

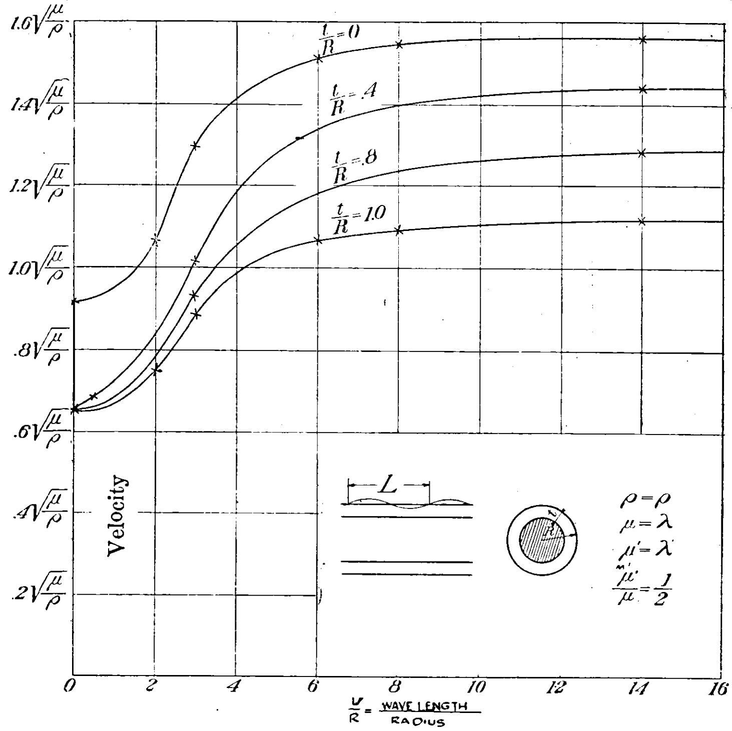


Fig. III. Propagation of Rayleigh-Waves along Hollow cylinder.

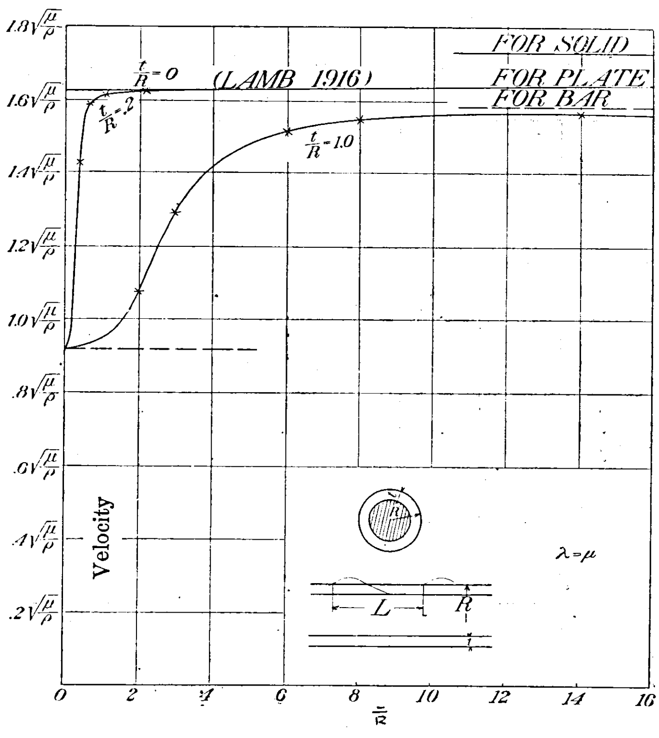


Fig. IV. Waves over Curved Surface along Circumferential Direction.

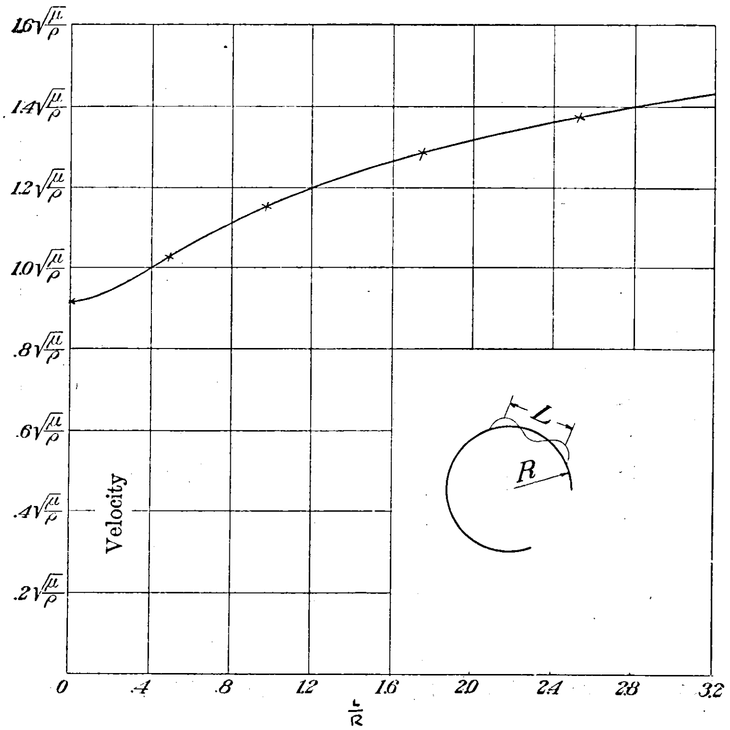


Fig. V. Propagation of Love-Waves over Curved Surface.