Scattering of Elastic Waves and Some Allied Problems

By

Katsutada SEZAWA

Earthquake Research Institute.

彈性波の散逸と之に關聯する一二の問題

所員　妹澤克惟

音波が障碍によつて散逸する問題は多少試みられて居るが、彈性波の場合は殆んど研究されて居らぬ。唯末廣博士が本研究に必要なる或部分に既に手を着けられて居る位なものである。本研究は八部に分れ、第一部第二部は中空及び固定された圓壔形及び球形の障碍による彈波性の散逸、第三部は障碍が楕圓形である場合、第四部第五部は媒體と共に位置を變へる障碍による散逸、第六、第七、第八部は散逸體又は波動源に復原力がある場合之等が次第に老衰する機構を示した。

本研究は數學を用ひ、散逸の問題は入射波を簡單の爲に一方向の粗密波とし、其波の障碍面に垂直及び平行なる種々の分波が反射されるといふ考のもとに計算を推し進めてある。波源の老衰に就ては入射波がないから比較的簡單に取扱はれた。

理論的に面白い結果を摘錄すれば

1. 入射波は單に粗密波でも散逸波は粗密波及び非變容波の二種が伴ふ。
2. 中空障碍の前面では多少の靜止波が現はれ、後面では波の蔭が出來、又其邊緣には波動の廻折が想像される。之等は波長の大さにあつて左右され、長波に就ては以上の事實が存在せぬ。
3. 媒體の中へ障碍を押込んだものは波長によつて見掛の質量が存在する。
4. 波動源に復原力がある時は波動を周圍に與へつゝ次第に自身の老衰を示し、而して其老衰性は媒質に對する自身の比較密度、波源の形狀、大さ及び周圍に傳はる波動速度に關係し、復原の力には無關係となる。

The effect of obstacles upon the propagation of elastic waves is an important problem, for modern seismology as well as for the theory of elasticities related with the scattering and diffraction of elastic waves. Yet, such problems have not been much studied on account of the reason that the present problem involves mathematical difficulty and complication.

The present investigation consists of eight sections; the first and the second deal with the scattering and diffraction of waves by cylindrical and spherical cavities and also by solid cylindrical and spherical obstacles, the third gives mathematical formulæ for that due to an elliptical obstacle, the fourth and the fifth treats of the effects of imbedded particles upon the transmission of waves and in the remaining three sections the decay of the restitutive origins due to the emission of the elastic waves is studied.

I. Cylindrical Obstacle of a Circular Form.

The expressions for primary waves, when they are dilatational, are given by

$$\left.\begin{aligned} \Delta &= A \sin(pt-hx) \\ u_0 &= \frac{A\cos\theta}{h}\cos(pt-hx) \\ v_0 &= -\frac{A\sin\theta}{h}\cos(pt-hx) \end{aligned}\right\} \qquad (1)$$

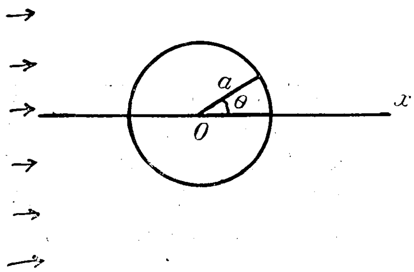

Fig. 1.

where

$$\frac{2\pi}{p} = \text{period of waves}$$

$$h^2 = \frac{\rho p^2}{\lambda+2\mu}$$

$$\frac{2\pi}{h} = \text{wave length}$$

Δ, u_0, v_0 = dilatation, radial and circumferential displacements.

In the above equations, $\cos(hx)$ and $\sin(hx)$ can be expanded in the following forms:—

$$\left.\begin{aligned} \cos(hx) &= J_0(hr) - 2J_2(hr)\cos 2\theta + 2J_4(hr)\cos 4\theta - \ldots\ldots \\ \sin(hx) &= 2J_1(hr)\cos\theta - 2J_3(hr)\cos 3\theta + 2J_5(hr)\cos 5\theta - \ldots\ldots \end{aligned}\right\} \qquad (2)$$

The expressions of the secondary waves are given by

$$\left.\begin{aligned} \Delta' &= \sum_{n=0}^{\infty} B_n\Big\{J_n(hr)\sin(pt+\varepsilon_n) - Y_n(hr)\cos(pt+\varepsilon_n)\Big\}\cos n\theta \\ \varpi' &= \sum_{n=1}^{\infty} C_n\Big\{J_n(kr)\sin(pt+\sigma_n) - Y_n(kr)\cos(pt+\sigma_n)\Big\}\sin n\theta \\ u_1 &= -\sum_{n=0}^{\infty}\frac{B_n}{h^2}\Big\{\frac{dJ_n(hr)}{dr}\sin(pt+\varepsilon_n) - \frac{dY_n(hr)}{dr}\cos(pt+\varepsilon_n)\Big\}\cos n\theta \\ v_1 &= \sum_{n=1}^{\infty}\frac{B_n}{h^2}\frac{n}{r}\Big\{J_n(hr)\sin(pt+\varepsilon_n) - Y_n(hr)\cos(pt+\varepsilon_n)\Big\}\sin n\theta \end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} u_2&=\sum_{n=1}^{\infty}\frac{C_n}{k^2}\frac{n}{r}\left\{J_n(kr)\sin(pt+\sigma_n)-Y_n(kr)\cos(pt+\sigma_n)\right\}\cos n\theta \\ v_2&=-\sum_{n=1}^{\infty}\frac{C_n}{k^2}\left\{\frac{dJ_n(kr)}{dr}\sin(pt+\sigma_n)-\frac{dY_n(kr)}{dr}\cos(pt+\sigma_n)\right\}\sin n\theta \end{aligned}\right\}$$

where

$$k^2=\frac{\rho p^2}{\mu}$$

Δ', ϖ' = secondary dilatational and distorsional waves.

u_1, v_1 = radial and circumferential components of displacement corresponding to Δ'

u_2, v_2 = radial and circumferential components of displacement corresponding to ϖ'

If there is a hollow cylinderical cavity, $r=a$ at the origin, the boundary conditions are given by

$$\left.\begin{aligned} &\lambda(\Delta+\Delta')+2\mu\frac{\partial}{\partial a}(u_0+u_1+u_2)=0 \\ &\frac{\partial}{\partial a}(v_0+v_1+v_2)-\frac{1}{a}(v_0+v_1+v_2)+\frac{1}{a}\frac{\partial}{\partial\theta}(u_0+u_1+u_2)=0 \end{aligned}\right\} \tag{4}$$

Substituting from (1), (2) and (3) in (4), and equating the cœfficients of $\cos pt\cos n\theta$, $\cos pt\sin n\theta$, $\sin pt\cos n\theta$ and $\sin pt\sin n\theta$ respectively to zero, we obtain the following relations to determine the constants, B_n', B_n'', C_n' and C_n'', in which $B_n'=B_n\cos\varepsilon_n$, $B_n''=B_n\sin\varepsilon_n$, $C_n'=C_n\cos\sigma_n$ and $C_n''=C_n\sin\sigma_n$

When n is an even number,

$$\left.\begin{aligned} &A(-1)^{\frac{n}{2}}\left[2\lambda J_n(ha)-\frac{2\mu}{h}\left(\frac{dJ_{n-1}(ha)}{da}-\frac{dJ_{n+1}(ha)}{da}\right)\right] \\ &\quad+B_n'\left[\lambda J_n(ha)-\frac{2\mu}{h^2}\frac{d^2J_n(ha)}{da^2}\right]_{①} \\ &\quad+B_n''\left[\lambda Y_n(ha)-\frac{2\mu}{h^2}\frac{d^2Y_n(ha)}{da^2}\right]_{②} \\ &\quad+C_n'\left[\frac{2\mu n}{k^2}\frac{d}{da}\frac{J_n(ka)}{a}\right]_{③}+C_n''\left[\frac{2\mu n}{k^2}\frac{d}{da}\frac{Y_n(ka)}{a}\right]_{④}=0 \\ &A(-1)^{\frac{n}{2}}\left[\frac{1}{h}\left(\frac{dJ_{n-1}(ha)}{da}+\frac{dJ_{n+1}(ha)}{da}+\frac{n-1}{ha}\right)\left(J_{n-1}(ha)+J_{n+1}(ha)\right)\right] \\ &\quad+B_n'\left[\frac{2n}{h^2}\frac{d}{da}\frac{J_n(ha)}{a}\right]_{⑤}+B_n''\left[\frac{2n}{h^2}\frac{d}{da}\frac{Y_n(ha)}{a}\right]_{⑥} \end{aligned}\right\} \tag{5}$$

$$
\left.\begin{aligned}
&+C_n'\left[\frac{2}{k^2a}\frac{dJ_n(ka)}{da}+\left(1-\frac{n^2+n}{k^2a^2}\right)J_n(ka)\right]_{⑦}\\
&+C_n''\left[\frac{2}{k^2a}\frac{dY_n(ka)}{da}+\left(1-\frac{n^2+n}{k^2a^2}\right)Y_n(ka)\right]_{⑧}=0\\
&B_n''[\ ①\ ]-B_n'[\ ②\ ]+C_n''[\ ③\ ]-C_n'[\ ④\ ]=0\\
&B_n''[\ ⑤\ ]-B_n'[\ ⑥\ ]+C_n''[\ ⑦\ ]-C_n'[\ ⑧\ ]=0
\end{aligned}\right\}
$$

and when n is odd,

$$
\left.\begin{aligned}
&A(-1)^{\frac{n+1}{2}}\left[2\lambda J_n(ha)-\frac{2\mu}{h}\left(\frac{dJ_{n-1}(ha)}{da}-\frac{dJ_{n+1}(ha)}{da}\right)\right]\\
&\qquad+B_n''[\ ①\ ]-B_n'[\ ②\ ]+C_n''[\ ③\ ]-C_n'[\ ④\ ]=0\\
&A(-1)^{\frac{n+1}{2}}\left[\frac{1}{h}\left(\frac{dJ_{n-1}(ha)}{da}+\frac{dJ_{n+1}(ha)}{da}\right)\right.\\
&\qquad\qquad\left.+\frac{(n-1)}{ha}\left(J_{n-1}(ha)+J_{n+1}(ha)\right)\right]\\
&\qquad+B_n''[\ ⑤\ ]-B_n'[\ ⑥\ ]+C_n''[\ ⑦\ ]-C_n'[\ ⑧\ ]=0\\
&B_n'[\ ①\ ]+B_n''[\ ②\ ]+C_n'[\ ③\ ]+C_n''[\ ④\ ]=0\\
&B_n'[\ ⑤\ ]+B_n''[\ ⑥\ ]+C_n'[\ ⑦\ ]+C_n''[\ ⑧\ ]=0
\end{aligned}\right\} \quad (6)
$$

When $n=0$

$$
\left.\begin{aligned}
&A\left[\lambda J_0(ha)+\frac{2\mu}{h}\frac{dJ_1(ha)}{da}\right]+B_0'\left[\lambda J_0(ha)-\frac{2\mu}{h^2}\frac{d^2J_0(ha)}{da^2}\right]\\
&\qquad\qquad+B_0''\left[\lambda Y_0(ha)-\frac{2\mu}{h^2}\frac{d^2Y_0(ha)}{da^2}\right]=0\\
&B_0''\left[\lambda J_0(ha)-\frac{2\mu}{h^2}\frac{d^2J_0(ha)}{da^2}\right]-B_0'\left[\lambda Y_0(ha)-\frac{2\mu}{h^2}\frac{d^2Y_0(ha)}{da^2}\right]=0
\end{aligned}\right\} \quad (7)
$$

As a simplest example, take $\lambda=\mu$ and $ha=0.5$, then the principal terms of the scattered waves are given by

$$
\Delta'=-.393A\left\{J_0\left(0.5\frac{r}{a}\right)\sin(pt+67°)-Y_0\left(0.5\frac{r}{a}\right)\cos(pt+67°)\right\}
$$

$$
+.173A\left\{J_1\left(0.5\frac{r}{a}\right)\sin(pt+45°)-Y_1\left(0.5\frac{r}{a}\right)\cos(pt+45°)\right\}
$$

$$
\varpi'=-.927A\left\{J_1\left(0.86\frac{r}{a}\right)\sin(pt+37°)-Y_1\left(0.86\frac{r}{a}\right)\cos(pt+37°)\right\}
$$

For large values of ha, more terms than these should be taken as principal ones. The calculated maximum displacements on the front and rear surfaces for various ratio of a/L are illustrated in Fig. I, L being the wave length. This diagram shows that the increase of the displacement at the front is more remarkable than the decease of the displacement at the back. We can see from this diagram that, while the front is in a stationary vibration, there appears a shadowy space on the rear side; these extensions depend upon the ratio of a/L. The fact that stationary vibration at the front and also the shadowy space at the back gradually disappear as the radial distance is increased, may be seen from the well-known nature of Bessel's functions, with which the solutions are composed.

We may now add the effect of a rigid cylinder fixed in space.

$$u_0+u_1+u_2=0 \qquad v_0+v_1+v_2=0 \qquad \text{at} \qquad r=a \tag{8}$$

Substituting from (1), (2) and (3) in (8), we obtain the relation between the constants as follows:—

For even n,

$$\left.\begin{aligned}
&-\frac{1}{h^2}\left(B_n'\frac{dJ_n(ha)}{da}+B_n''\frac{dY_n(ha)}{da}\right)_{①}\\
&\qquad+\frac{n}{k^2a}\left(C_n'J_n(ka)+C_n''Y_n(ka)\right)_{②}\\
&\qquad+\frac{A}{h}(-1)^{\frac{n}{2}}\left(J_{n+1}(ha)-J_{n-1}(ha)\right)=0\\
&-\frac{1}{h^2}\left(B_n''\frac{dJ_n(ha)}{da}-B_n'\frac{dY_n(ha)}{da}\right)_{③}\\
&\qquad+\frac{n}{k^2a}\left(C_n''J_n(ka)-C_n'Y_n(ka)\right)_{④}=0\\
&\frac{n}{h^2a}\left(B_n'J_n(ha)+B_n''Y_n(ha)\right)_{⑤}-\frac{1}{k^2}\left(C_n'\frac{dJ_n(ka)}{da}+C_n''\frac{dY_n(ka)}{da}\right)_{⑥}\\
&\qquad+\frac{A}{h}(-1)^{\frac{n}{2}}\left(J_{n-1}(ha)+J_{n+1}(ha)\right)=0\\
&\frac{n}{h^2a}\left(B_n''J_n(ha)-B_n'Y_n(ha)\right)_{⑦}\\
&\qquad-\frac{1}{k^2}\left(C_n''\frac{dJ_n(ka)}{da}-C_n'\frac{dY_n(ka)}{da}\right)_{⑧}=0
\end{aligned}\right\} \tag{9}$$

For odd n,

$$
\left.\begin{aligned}
&[\ ①\ ]+[\ ②\ ]=0\\
&[\ ③\ ]+[\ ④\ ]+\frac{A}{h}(-1)^{\frac{n-1}{2}}\Big(J_{n-1}(ha)-J_{n+1}(ha)\Big)=0\\
&[\ ⑤\ ]+[\ ⑥\ ]=0\\
&[\ ⑦\ ]+[\ ⑧\ ]+\frac{A}{h}(-1)^{\frac{n+1}{2}}\Big(J_{n-1}(ha)+J_{n+1}(ha)\Big)=0
\end{aligned}\right\} \qquad (10)
$$

For $n=0$

$$
\left.\begin{aligned}
&-\frac{1}{h^2}\left(B_0'\frac{dJ_0(ha)}{da}+B_0''\frac{dY_0(ha)}{da}\right)\frac{A}{h}J_1(ha)=0\\
&B_0''\frac{dJ_0(ha)}{da}-B_0'\frac{dY_0(ha)}{da}=0
\end{aligned}\right\} \qquad (11)
$$

II. Spherical Obstacle.

The expressions for primary waves, when they are dilational, may be the same as for the cylindrical obstacle, but the expansion of $\cos(hx)$ and $\sin(hx)$ in polar coordinates must be expressed in the following forms:—

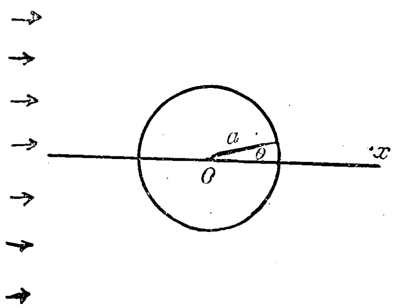

Fig. 2.

$$
\left.\begin{aligned}
\cos(hx)&=J_{\frac{1}{2}}\frac{(hr)}{\sqrt{hr}}-5J_{\frac{5}{2}}(hr)\frac{1}{\sqrt{hr}}P_2(\cos\theta)\\
&\qquad+9J_{\frac{9}{2}}(hr)\frac{1}{\sqrt{hr}}P_4(\cos\theta)-\ldots\ldots\\
\sin(hx)&=3J_{\frac{3}{2}}(hr)\frac{1}{\sqrt{hr}}P_1(\cos\theta)-7J_{\frac{7}{2}}(hr)\frac{1}{\sqrt{hr}}P_3(\cos\theta)\\
&\qquad+11J_{\frac{11}{2}}(hr)\frac{1}{\sqrt{hr}}P_5(\cos\theta)-\ldots\ldots
\end{aligned}\right\} \qquad (12)
$$

The expression of the secondary waves are given by

$$
\left.\begin{aligned}
\varDelta' &= \sum_{n=0}^{\infty} \frac{B_n}{\sqrt{hr}} \Big\{ J_{n+\frac{1}{2}}(hr) \sin(pt+\varepsilon_n) \\
&\qquad - Y_{n+\frac{1}{2}}(hr) \cos(pt+\varepsilon_n) \Big\} P_n(\cos\theta) \\
\varpi' &= \sum_{n=1}^{\infty} \frac{C_n}{2\sqrt{hr}} \Big\{ J_{n+\frac{1}{2}}(kr) \sin(pt+\sigma_n) \\
&\qquad - Y_{n+\frac{1}{2}}(kr) \cos(pt+\sigma_n) \Big\} \frac{dP_n(\cos\theta)}{d\theta} \\
u_1 &= -\sum_{n=0}^{\infty} \frac{B_n}{h^2} \Big\{ \frac{d}{dr} \frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{hr}} \sin(pt+\varepsilon_n) \\
&\qquad - \frac{d}{dr} \frac{Y_{n+\frac{1}{2}}(hr)}{\sqrt{hr}} \cos(pt+\varepsilon_n) \Big\} P_n(\cos\theta) \\
v_1 &= -\sum_{n=1}^{\infty} \frac{B_n}{h^{2+\frac{1}{2}} r^{\frac{1}{2}}} \Big\{ J_{n+\frac{1}{2}}(hr) \sin(pt+\varepsilon_n) \\
&\qquad - Y_{n+\frac{1}{2}}(hr) \cos(pt+\varepsilon_n) \Big\} \frac{dP_n(\cos\theta)}{d\theta} \\
u_2 &= -\sum_{n=1}^{\infty} \frac{C_n n(n+1)}{k^{2+\frac{1}{2}} r^{\frac{3}{2}}} \Big\{ J_{n+\frac{1}{2}}(kr) \sin(pt+\sigma_n) \\
&\qquad - Y_{n+\frac{1}{2}}(kr) \cos(pt+\sigma_n) \Big\} P_n(\cos\theta) \\
v_2 &= -\sum_{n=1}^{\infty} \frac{C_n}{k^{2+\frac{1}{2}} r} \Big\{ \frac{d}{dr}\Big(\sqrt{r}\, J_{n+\frac{1}{2}}(kr)\Big) \sin(pt+\sigma_n) \\
&\qquad - \frac{d}{dr}\Big(\sqrt{r}\, Y_{n+\frac{1}{2}}(kr)\Big) \cos(pt+\sigma_n) \Big\} \frac{dP_n(\cos\theta)}{d\theta}
\end{aligned}\right\} \tag{13}
$$

where

u_1, v_1 = radial and colatitudinal components of displacement respectively corresponding to $\varDelta'$

u_2, v_2 = radial and colatitudinal components of displacement respectively corresponding to ϖ'

If the obstacle is a hollow spherical cavity, $r=a$, the boundary conditions are given by

$$
\left.\begin{aligned}
&\lambda(\varDelta+\varDelta') + 2\mu \frac{\partial}{\partial a}(u_0+u_1+u_2) = 0 \\
&\frac{\partial}{\partial a}(v_0+v_1+v_2) - \frac{1}{a}(v_0+v_1+v_2) + \frac{1}{a}\frac{\partial}{\partial\theta}(u_0+u_1+u_2) = 0
\end{aligned}\right\} \tag{14}
$$

Substituting from (1), (12) and (13) in (14), and equating the cœfficients of $\cos pt \cos n\theta$, $\cos pt \sin n\theta$, $\sin pt \cos n\theta$ and $\sin pt \sin n\theta$ respectively to zero, we obtain the following equations to determine the constants, B_n', B_n'', C_n' and C_n'', in which $B_n'=B_n\cos\varepsilon_n$, $B_n''=B_n\sin\varepsilon_n$, $C_n'=C_n\cos\sigma_n$ and $C_n''=C_n\sin\sigma_n$. For even n

$$\left.\begin{aligned}
&A(-1)^{\frac{n}{2}}\Big[\lambda(2n+1)\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{ha}}-\frac{2\mu}{h}\Big(n\frac{d}{da}\frac{J_{n-\frac{1}{2}}(ha)}{\sqrt{ha}}\\
&\quad -\overline{n+1}\frac{d}{da}\frac{J_{n+\frac{3}{2}}(ha)}{\sqrt{ha}}\Big)\Big]+B_n'\Big[\frac{\lambda}{\sqrt{ha}}J_{n+\frac{1}{2}}(ha)\\
&\quad -\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{a}}\Big]_{①}-C_n'\Big[\frac{2\mu n(n+1)}{k^{2+\frac{1}{2}}}\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ka)}{a^{\frac{3}{2}}}\Big]_{③}\\
&\quad +B_n''\Big[\frac{\lambda}{\sqrt{ha}}Y_{n+\frac{1}{2}}(ha)-\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{Y_{n+\frac{1}{2}}(ha)}{\sqrt{a}}\Big]_{②}\\
&\quad -C_n''\Big[\frac{2\mu n(n+1)}{k^{2+\frac{1}{2}}}\frac{d}{da}\frac{Y_{n+\frac{1}{2}}(ka)}{a^{\frac{3}{2}}}\Big]_{④}=0\\
&\frac{A}{h}(-1)^{\frac{n}{2}}\Big[\Big(\frac{d}{da}\frac{J_{n-\frac{1}{2}}(ha)}{\sqrt{ha}}+\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{ha}}\Big)\\
&\quad +\frac{1}{a}\Big(\overline{n-1}\frac{1}{\sqrt{ha}}J_{n-\frac{1}{2}}(ha)-n\frac{1}{\sqrt{ha}}J_{n+\frac{3}{2}}(ha)\Big)\Big]\\
&\quad +B_n'\Big[\frac{2}{h^{2+\frac{1}{2}}}\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ha)}{a^{\frac{3}{2}}}\Big]_{⑤}-C_n'\Big[\frac{2}{k^{2+\frac{1}{2}}a^{\frac{3}{2}}}\frac{dJ_{n+\frac{1}{2}}(ka)}{da}\\
&\quad +\Big(\frac{1}{(ka)^{\frac{1}{2}}}-\frac{2(n^2+n)-1}{(ka)^{2+\frac{1}{2}}}\Big)J_{n+\frac{1}{2}}(ka)\Big]_{⑦}\\
&\quad +B_n''\Big[\frac{2}{h^{2+\frac{1}{2}}}\frac{d}{da}\frac{Y_{n+\frac{1}{2}}(ha)}{a^{\frac{3}{2}}}\Big]_{⑥}-C_n''\Big[\frac{2}{k^{2+\frac{1}{2}}a^{\frac{3}{2}}}\frac{dY_{n+\frac{1}{2}}(ka)}{da}\\
&\quad +\Big(\frac{1}{(ka)^{\frac{1}{2}}}-\frac{2(n^2+n)-1}{(ka)^{2+\frac{1}{2}}}\Big)Y_{n+\frac{1}{2}}(ka)\Big]_{⑧}=0\\
&B_n''[\ ①\ ]-B_n'[\ ②\ ]-C_n''[\ ③\ ]+C_n'[\ ④\ ]=0\\
&B_n''[\ ⑤\ ]-B_n'[\ ⑥\ ]-C_n''[\ ⑦\ ]+C_n'[\ ⑧\ ]=0
\end{aligned}\right\}\quad(15)$$

For odd n

$$\begin{aligned}
&A(-1)^{\frac{n+1}{2}}\Big[\lambda(2n+1)\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{ha}}-\frac{2\mu}{h}\Big(n\frac{d}{da}\frac{J_{n-\frac{1}{2}}(ha)}{\sqrt{ha}}\\
&\quad -\overline{n+1}\frac{d}{da}\frac{J_{n+\frac{3}{2}}(ha)}{\sqrt{ha}}\Big)\Big]\\
&\quad +B_n''[\ ①\ ]-B_n'[\ ②\ ]-C_n''[\ ③\ ]+C_n'[\ ④\ ]=0
\end{aligned}$$

$$\left.\begin{aligned}&\frac{A}{h}(-1)^{\frac{n+1}{2}}\Big[\frac{d}{da}\frac{J_{n-\frac{1}{2}}(ha)}{\sqrt{ha}}+\frac{d}{da}\frac{J_{n+\frac{1}{2}}(ha)}{\sqrt{ha}}\\&\quad+\frac{1}{a}\Big(\overline{n-1}\frac{1}{\sqrt{ha}}J_{n-\frac{1}{2}}(ha)-n\frac{1}{\sqrt{ha}}J_{n+\frac{3}{2}}(ha)\Big)\Big]\\&\quad+B_n''[\ ⑤\ ]-B_n'[\ ⑥\ ]-C_n''[\ ⑦\ ]+C_n'[\ ⑧\ ]=0\\&\quad B_n'[\ ①\ ]+B_n''[\ ②\ ]-C_n'[\ ③\ ]-C_n''[\ ④\ ]=0\\&\quad B_n'[\ ⑤\ ]+B_n''[\ ⑥\ ]-C_n'[\ ⑦\ ]-C_n''[\ ⑧\ ]=0\end{aligned}\right\}\quad(16)$$

When $n=0$

$$\left.\begin{aligned}&A\Big[\frac{\lambda J_{\frac{1}{2}}(ha)}{\sqrt{ha}}+\frac{2\mu}{h}\frac{d}{da}\frac{J_{\frac{3}{2}}(ha)}{\sqrt{ha}}\Big]\\&\quad+B_0'\Big[\frac{\lambda}{\sqrt{ha}}J_{\frac{1}{2}}(ha)-\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{J_{\frac{1}{2}}(ha)}{\sqrt{a}}\Big]\\&\quad+B_0''\Big[\frac{\lambda}{\sqrt{ha}}Y_{\frac{1}{2}}(ha)-\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{Y_{\frac{1}{2}}(ha)}{\sqrt{a}}\Big]=0\\&B_0''\Big[\frac{\lambda}{\sqrt{ha}}J_{\frac{1}{2}}(ha)-\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{J_{\frac{1}{2}}(ha)}{\sqrt{a}}\Big]\\&\quad-B_0'\Big[\frac{\lambda}{\sqrt{ha}}Y_{\frac{1}{2}}(ha)-\frac{2\mu}{h^{2+\frac{1}{2}}}\frac{d^2}{da^2}\frac{Y_{\frac{1}{2}}(ha)}{\sqrt{a}}\Big]=0\,.\end{aligned}\right\}\quad(17)$$

The calculated maximum displacements at the front and rear surfaces are illustrated in Fig. II, L being the wave length. In this case, too, the increase of the displacement at the front according to the value of a/L is more rapid than the decrease of the displacement at the rear.

III. Cylindrical Obstacle of an Elliptic Form.

The expressions of the primary waves, when they are dilatational, are given by

$$\left.\begin{aligned}\Delta&=A\sin(pt-hy)\\&=A[\sin pt\cos(hc\sinh\xi\sin\eta)-\cos pt\sin(hc\sin h\xi\sin\eta)\end{aligned}\right\}\quad(18)$$

where

$$\cos(hc\sinh\xi\sin\eta)=\sum_{n=0}^{\infty}B_n\,\mathrm{ce}_n\Big(\frac{\xi}{i},\ q\Big)\,\mathrm{ce}_n(\eta,\ q)$$

$$\sin(hc\sinh\xi\sin\eta)=\sum_{n=1}^{\infty}C_n\,\mathrm{se}_n\Big(\frac{\xi}{i},\ q\Big)\,(\mathrm{se}_n\,\eta,\ q)$$

$$B_n=1\Big/\int_{\pi}^{\pi}\Big[\mathrm{ce}_n(\eta,\ q)\Big]^2 d\eta$$

$$C_n=1\Big/\int_{\pi}^{\pi}\Big[\mathrm{se}_n(\eta,\ q)\Big]^2 d\eta$$

$$q=-\frac{h^2c^2}{32},\qquad h^2=\frac{\rho p^2}{\lambda+2\mu},$$

$2C$=length between foci.

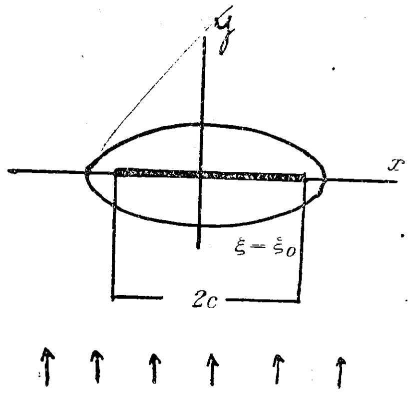

Fig. 3.

The displacement corresponding to Δ in (18) is given by

$$\left.\begin{aligned}
u_0=&-\sum_{n=0}^{\infty}\frac{AB_n}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\frac{\partial\,\mathrm{ce}_n\left(\frac{\xi}{i},\ q\right)}{\partial\xi}\mathrm{ce}_n(\eta,\ q)\sin pt\\
&+\sum_{n=1}^{\infty}\frac{AC_n}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\frac{\partial\,\mathrm{se}_n\left(\frac{\xi}{i},\ q\right)}{\partial\xi},\ \mathrm{se}_n(\eta,\ q)\cos pt\\
v_0=&-\sum_{n=0}^{\infty}\frac{AB_n}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\mathrm{ce}_n\left(\frac{\xi}{i},\ q\right)\frac{\partial\mathrm{ce}_n(\eta,\ q)}{\partial\eta}\sin pt\\
&+\sum_{n=1}^{\infty}\frac{AC_n}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\mathrm{se}_n\left(\frac{\xi}{i},\ q\right)\frac{\partial\,\mathrm{se}_n(\eta,\ q)}{\partial\eta}\cos pt
\end{aligned}\right\}\qquad(19)$$

The expressions of the secondary waves are given by

$$\begin{aligned}
\Delta'=&\sum_{n=0}^{\infty}B_n'\Big\{\mathrm{ce}_n\left(\frac{\xi}{i},\ q\right).\sin(pt+\varepsilon_n)\\
&-\mathrm{se}_n\left(\frac{\xi}{i},\ q\right)\cos(pt+\varepsilon_n)\Big\}\ \mathrm{ce}_n(\eta,\ q)\\
\varpi'=&\sum_{n=1}^{\infty}C_n'\Big\{\mathrm{ce}_n\left(\frac{\xi}{i},\ q'\right)\sin(pt+\sigma_n)\\
&-\mathrm{se}_n\left(\frac{\xi}{i},\ q'\right)\cos(pt+\sigma_n)\Big\}\ \mathrm{se}_n(\eta,\ q)\\
u_1=&-\sum_{n=0}^{\infty}\frac{B_n'}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\Bigg\{\frac{\partial\,\mathrm{ce}_n\left(\frac{\xi}{i},\ q\right)}{\partial\xi}\sin(pt+\varepsilon_n)\\
&-\frac{\partial\,\mathrm{se}_n\left(\frac{\xi}{i},\ q\right)}{\partial\xi}\cos(pt+\varepsilon^n)\Bigg\}\ \mathrm{ce}_n(\eta,\ q).
\end{aligned}$$

$$
\left.\begin{aligned}
v_1=&-\sum_{n=1}^{\infty}\frac{B_n'}{k^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\left\{\mathrm{ce}_n\left(\frac{\xi}{i},\ q\right)\sin(pt+\varepsilon_n)\right.\\
&\left.-\mathrm{se}_n\left(\frac{\xi}{i},\ q\right)\cos(pt+\varepsilon_n)\right\}\frac{\partial\,\mathrm{ce}_n(\eta,\ q)}{\partial\eta}\\
u_2=&\sum_{n=1}^{\infty}\frac{2C_n'}{k^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\left\{\mathrm{ce}_n\left(\frac{\xi}{i},\ q'\right)\sin(pt+\sigma_n)\right.\\
&\left.-\mathrm{se}_n\left(\frac{\xi}{i},\ q'\right)\cos(pt+\sigma_n)\right\}\frac{\partial\,\mathrm{se}_n(\eta,\ q')}{\partial\eta}\\
v_2=&-\sum_{n=1}^{\infty}\frac{2C_n'}{k^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\left\{\frac{\partial\,\mathrm{ce}_n\left(\frac{\xi}{i},\ q'\right)}{\partial\xi}\sin(pt+\sigma_n)\right.\\
&\left.-\frac{\partial\,\mathrm{se}_n\left(\frac{\xi}{i},\ q'\right)}{\partial\eta}\cos(pt+\sigma_n)\right\}\mathrm{se}_n(\eta,\ q')
\end{aligned}\right\}\qquad(20)
$$

where $\qquad k^2=\dfrac{\rho p^2}{\mu},\qquad q'=-\dfrac{k^2c^2}{32}$

As the boundary conditions, we have, when the boundary is free,

$$
\left.\begin{aligned}
&\lambda(\Delta+\Delta')+\frac{2\mu}{c(\cosh^2\xi_0-\cos^2\eta)}\frac{\partial}{\partial\xi_0}\\
&\qquad\left[(u_0+u_1+u_2)\sqrt{\cosh^2\xi_0-\cos^2\eta}\right]=0\\
&\frac{\partial}{\partial\xi_0}\left[(v_0+v_1+v_2)\sqrt{\cosh^4\xi_0-\cos^2\eta}\right]\\
&\qquad+\frac{\partial}{\partial\eta}\left[(u_0+u_1+u_2)\sqrt{\cosh^2\xi_\theta-\cos^2\eta}\right]=0
\end{aligned}\right\}\qquad(21)
$$

and when the boundary is rigid and fixed,

$$
\left.\begin{aligned}
u_0+u_1+u_2=0\\
v_0+v_1+v_2=0
\end{aligned}\right\}\qquad(22)
$$

Substituting from (18), (19) and (20) in (21) or (22), we can determine the various constants in the expressions for the scattered waves.

Though the derivatives, $\dfrac{\partial\,\mathrm{ce}_n(\eta,\ q)}{\partial\eta}$ and $\dfrac{\partial\,\mathrm{se}_n(\eta,\ q)}{\partial\eta}$ in (19) seem at first sight to be easily obtainable from their expanded forms in $\sin x$ or $\cos x$, yet in reality their values cannot be given in a simple form, so that we cannot find the numerical values of the constants from (21) or (22). We can see, however, from the nature of the solutions (20) that, at a great distance from the origin relative to the size of an obstacle, the scattered waves and the shadowy space become insignificant, the latter being the effect of the diffraction.

IV. Imbedded Cylindrical Obstacle.

The effect of an imbedded cylindrical obstacle upon the trausmission of elastic waves may not be unimportant. The expressions of the primary waves are given by

$$\left.\begin{aligned}
\Delta &= Ae^{i(fx-pt)} = A[J_0(hr)+2iJ_1(hr)\cos\theta+\dots\\
&\qquad +2i^nJ_n(hr)\cos n\theta+\dots]e^{-ipt}\\
u_0 &= -\frac{A}{h^2}\left[\frac{\partial J_0(hr)}{\partial r}+2i\frac{\partial J_1(hr)}{\partial r}\cos\theta+\dots\right.\\
&\qquad \left.+2i^n\frac{\partial J_n(hr)}{\partial r}\cos n\theta+\dots\right]e^{-ipt}\\
v_0 &= \frac{A}{h^2r}\left[2iJ_1(hr)\sin\theta+\dots+2i^n nJ_n(hr)\sin n\theta+\dots\right]e^{-ipt}
\end{aligned}\right\} \quad (23)$$

The expressions of the secondary waves are given by

$$\left.\begin{aligned}
\Delta' &= \sum_{n=0}^{\infty} B_nH_n^{(1)}(hr)\cos n\theta e^{-ipt}\\
2\varpi' &= \sum_{n=1}^{\infty} C_nH_n^{(1)}(kr)\sin n\theta e^{-ipt}\\
u_1 &= -\sum_{n=0}^{\infty}\frac{B_n}{h^2}\frac{dH_n^{(1)}(hr)}{dr}\cos n\theta\, e^{-ipt}\\
v_1 &= \sum_{n=1}^{\infty}\frac{B_n}{h^2}\frac{n}{r}H_n^{(1)}(hr)\sin n\theta\, e^{-ipt}\\
u_2 &= \sum_{n=1}^{\infty}\frac{C_n}{k^2}\frac{n}{r}H_n^{(1)}(kr)\cos n\theta\, e^{-ipt}\\
v_2 &= -\sum_{n=1}^{\infty}\frac{C_n}{k^2}\frac{dH_n^{(1)}(kr)}{dr}\sin n\theta\, e^{-ipt}
\end{aligned}\right\} \quad (24)$$

where $h^2=\frac{\rho p^2}{\lambda+2\mu}$, $k^2=\frac{\rho p^2}{\mu}$, $H_n^{(1)}(x)=J_n(x)+iY_n(x)$ and $\frac{2\pi}{p}$ = period of waves.

The equation of motion of the rigid cylinder is expressed by

$$M\left(\frac{d^2U}{dt^2}+\sigma_0^2U\right)=\int_{-\pi}^{\pi}\left[\left(\lambda\Delta+2\mu\frac{\partial u}{dr}\right)\cos\theta\right.$$

$$\left.-\mu\left(\frac{\partial v}{dr}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\sin\theta\right]ad\theta \qquad (25)$$

where

M=mass of cylinder per unit length,

U=displacement of cylinder in direction of x,

$\frac{2\pi}{\sigma_0}$=period of free vibration of cylinder, when it is subjected to a restitutive force,

a=radius of the cylinder

$u=u_0+u_1+u_2$

and $v=v_0+v_1+v_2$

Substituting from (23) and (24) in (25), and writing $U=U'e^{-ipt}$, we get

$$(\sigma_0^2-p^2)\frac{MU'}{\pi a}=2iA(\lambda+2\mu)J_1(ha)+B_1(\lambda+2\mu)H_1^{(1)}(ha)-C_1\mu H_1^{(1)}(ka); \qquad (26)$$

the remaining terms vanishes by the well known nature of integration of the products of harmonics of different orders.

The boundary conditions are given by

$$u=U\cos\theta \qquad v=-U\sin\theta \qquad \text{at} \qquad r=a$$

from which we get by means of harmonic analysis

$$\left.\begin{aligned}U'&=-\frac{2iA}{h^2}\frac{dJ_1(ha)}{da}-\frac{B_1}{h^2}\frac{dH_1^{(1)}(ha)}{da}+\frac{C_1}{k^2}\frac{H_1^{(1)}(ka)}{a}\\ U'&=-\frac{2iA}{h^2}\frac{J_1(ha)}{a}-\frac{B_1}{h^2}\frac{H_1^{(1)}(ha)}{a}+\frac{C_1}{k^2}\frac{dH_1^{(1)}(ka)}{da}\end{aligned}\right\} \qquad (27)$$

Eliminating U' between (26) and (27) and solving the resulting equations for relatively large values of ha and ka, we obtain

$$\left.\begin{aligned}B_1&=\sin\left(2ha-\frac{3\pi}{2}\right)A\\ C_1&=\frac{2k}{h^2a}\sqrt{\frac{k}{h}}\left(\frac{\rho}{\rho_0}\frac{1}{1-\frac{T^2}{T_0^2}}-1\right)\cos\left(ha+ka-\frac{3\pi}{2}\right)A\end{aligned}\right\} \qquad (28)$$

where

ρ, ρ_0=densities of medium and cylinder respectively,

T, T_0=periods of waves and free oscillation of the cylinder respectively.

Equations (28) together with (24) give the expressions of the principal scattered waves.

The motion of the cylinder can be determined from (27) and (28) and expressed by

$$U=2\sqrt{\frac{2}{\pi}}\frac{A}{h^2a}\left[\sqrt{\frac{k}{h}}\left(\frac{\rho}{\rho_0}\frac{1}{1-\frac{T^2}{T_0^2}}-1\right)-\frac{1}{\sqrt{ha}}\right]\sin\left(ha-\frac{3\pi}{4}+pt\right) \quad (29)$$

When the restitutive force is existing and $T \doteqdot T_0$, the resonanse of waves occurs.

The other harmonics of the scattered waves can be determined by the formulæ (5), (6) and (7); the results, however, are not shown here, as they have not much importance.

V. Imbedded Spherical Particle.

The case of an imbedded spherical obstacle has many pnysical applications. The expressions of the primary waves are given by

$$\left.\begin{aligned}
\Delta &= Ae^{i(fx-pt)}=A\sum_{n=0}^{\infty}(2n+1)i^n\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{hr}}P_n(\cos\theta)e^{-ipt}\\
u_0 &= -\frac{A}{h^2}\sum_{n=0}^{\infty}(2n+1)i^n\frac{\partial}{\partial r}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{hr}}P_n(\cos\theta)\,e^{-ipt}\\
v_0 &= -\frac{A}{h^2}\sum_{n=1}^{\infty}(2n+1)i^n\frac{J_{n+\frac{1}{2}}(hr)}{r\sqrt{hr}}\frac{dP_n(\cos\theta)}{d\theta}e^{-ipt}
\end{aligned}\right\} \quad (30)$$

Tne expressions of the secondary waves are given by

$$\left.\begin{aligned}
\Delta' &= \sum_{n=0}^{\infty}\frac{B_n}{\sqrt{hr}}H^{(1)}_{n+\frac{1}{2}}(hr)P_n(\cos\theta)\,e^{-ipt}\\
2\bar{\omega}' &= \sum_{n=1}^{\infty}\frac{C_n}{\sqrt{hr}}H^{(1)}_{n+\frac{1}{2}}(kr)\frac{dP_n(\cos\theta)}{d\theta}e^{-ipt}\\
u_1 &= -\sum_{n=0}^{\infty}\frac{B_n}{h^2}\frac{d}{dr}\frac{H^{(1)}_{n+\frac{1}{2}}(hr)}{\sqrt{hr}}P_n(\cos\theta)\,e^{-ipt}\\
v_1 &= -\sum_{n=1}^{\infty}\frac{B_n}{h^{\frac{5}{2}}r^{\frac{3}{2}}}H^{(1)}_{n+\frac{1}{2}}(hr)\frac{dP_n(\cos\theta)}{d\theta}e^{-ipt}
\end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned} u_2 &= -\sum_{n=1}^{\infty} \frac{C_n n(n+1)}{k^{\frac{5}{2}} r} H^{(1)}_{n+\frac{1}{2}}(kr) P_n(\cos\theta) e^{-ipt} \\ v_2 &= -\sum_{n=1}^{\infty} \frac{C_n}{k^{\frac{5}{2}}} \frac{1}{r} \frac{d}{dr} \sqrt{r}\, H^{(1)}_{n+\frac{1}{2}}(kr) \frac{dP_n(\cos\theta)}{d\theta} e^{-ipt} \end{aligned}\right\}$$

where $h^2=\dfrac{\rho p^2}{\lambda+2\mu}$, $k^2=\dfrac{\rho p^2}{\mu}$, $H^{(2)}_{n+\frac{1}{2}}(x)=J_{n+\frac{1}{2}}(x)+iY_{n+\frac{1}{2}}(x)$, $\dfrac{2\pi}{p}=\text{period}.$

The equation of motion of the rigid sphere is expressed by

$$M\left(\frac{d^2U}{dt^2}+\sigma_0^2 U\right)=\iint\left[\left(\lambda\Delta+2\mu\frac{\partial u}{\partial r}\right)\cos\theta\right.$$

$$\left.-\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\sin\theta\right]2\pi a^2\sin\theta\, d\theta \qquad (32)$$

in which

M=mass of sphere,

U=displacement of cylinder in direction of x

$u=u_0+u_1+u_2$

and $v=v_0+v_1+v_2$

Substituting from (30) and (31) in (32) and writing $U=U'e^{-ipt}$, we obtain

$$(\sigma_0^2-p^2)\frac{MU'}{4\pi a^2}=iA(\lambda+2\mu)\frac{J_{\frac{3}{2}}(ha)}{\sqrt{ha}}$$

$$+\frac{B_1}{3}(\lambda+2\mu)\frac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}}+\frac{2\mu}{3}C_1\frac{H^{(1)}_{\frac{3}{2}}(ka)}{\sqrt{ka}}; \qquad (33)$$

the other terms vanish as described previously under (26).

The boundary conditions are given by

$$u=U\cos\theta \qquad v=-U\sin\theta \qquad \text{at} \qquad r=a$$

from which we have

$$\left.\begin{aligned} U' &= -\frac{3iA}{h^2}\frac{d}{da}\frac{J_{\frac{3}{2}}(ha)}{\sqrt{ha}}-\frac{B_1}{h^2}\frac{d}{da}\frac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}}-\frac{2C_1}{k^{\frac{5}{2}}a^{\frac{3}{2}}}H^{(1)}_{\frac{3}{2}}(ka) \\ U' &= -\frac{3iA}{h^2}\frac{J_{\frac{3}{2}}(ha)}{\sqrt{ha}}-\frac{B_1}{h^{\frac{5}{2}}a^{\frac{3}{2}}}H_{\frac{3}{2}}(ha)-\frac{C_1}{k^3 a}\frac{d}{da}\sqrt{ka}\,H^{(1)}_{\frac{3}{2}}(ka) \end{aligned}\right\} \qquad (34)$$

When ha and ka are very small, the values of B_1 and C_1 are found by the elimination of U' between (33) and (34), as follows:—

$$\left.\begin{aligned} B_1&=\frac{237\left(\frac{a}{L}\right)^3\left[1-\frac{\rho_0}{\rho}\left(1-\frac{T^2}{T_0^2}\right)\right]}{2.54\frac{\rho_0}{\rho}\left(1-\frac{T^2}{T_0^2}\right)-1}A \\ C_1&=\frac{3\times 237\left(\frac{a}{L}\right)^3\left[1-\frac{\rho_0}{\rho}\left(1-\frac{T^2}{T_0^2}\right)\right]}{2.54\frac{\rho_0}{\rho}\left(1-\frac{T^2}{T_0^2}\right)-1}A \end{aligned}\right\} \quad (35)$$

in which

$\lambda=\mu$

a=radius if sphere

L=wave length.

The displacement U of the sphere is approximately the same as that of the primary waves.

When $T \doteqdot T_0\left(1-\frac{1}{2.54}\frac{\rho}{\rho_0}\right)=T_1$, the resonance will take place. T_1 may be taken to be the effective period of free vibration for a small rigid particle, and the expression shows that the free oscillation is affected by the surrounding elastic medium.

When ha and ka are relatively large, the elimination of U' between (33) and (34) gives

$$\left.\begin{aligned} B_1&=\frac{3}{2}\sin(2ha)\,A \\ C_1&=-3\frac{k^2}{h^3a}\left(\frac{\rho}{\rho_0}\frac{1}{1-\frac{T^2}{T_0^2}}-1\right)\cos(ha+ka)\,A \end{aligned}\right\} \quad (36)$$

Equation (36) together with (31) gives the expressions of the principal scattered waves.

The motion of the cylinder can be determined from (29) and (30) and expressed by

$$U=-3\sqrt{\frac{2}{\pi}}\frac{A}{h^3a^2}\left[\frac{k}{h}\left(\frac{\rho}{\rho_0}\frac{1}{1-\frac{T^2}{T_0^2}}-1\right)-1\right]\sin(ha+pt) \quad (37)$$

When the restitutive force is existing and $T \doteqdot T_0$, the resonance will occur. In this case the motion of the sphere is free from the effect of the inertia mass.

VI. Decay of Plane Restitutive Origin emitting Elastic Waves.

The equations of motion and the solutions, appeared in IV and V, which treats of the problems of scattering due to imbedded particles, may be extended with a slight modification, to the problem of the decay of the vibratory motion of the scattering particle and of the restitutive origin emitting elastic waves. In the first place we shall treat the simplest case where the origin is of a plane rigid body and the generated waves are purely dilatational.

Taking the plane $x=0$ coincident with undisturbed middle surface of the plate with thickness 2a, the equation of motion of the elastic medium is expressed by

$$\nabla^2 \Delta = \frac{\rho}{\lambda+2\mu} \frac{\partial^2 \Delta}{\partial t^2}$$

The solution of the above may be expressed by the form:—

$$\Delta = A_1 e^{\alpha(v_1 t - x + a)} \qquad x > a$$

$$\Delta = A_2 e^{\alpha(v_1 t + x + a)} \qquad x < -a$$

$$U = B e^{\alpha v_1 t}$$

where

U : displacement of plate

$$v_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

The equation of motion of the plate is expressed by

$$M\left(\frac{d^2U}{dt^2} + \sigma_0^2 U\right) = \left(\lambda\Delta + 2\mu\frac{\partial u}{\partial x}\right)_{x=a} + \left(\lambda\Delta + 2\mu\frac{\partial u}{\partial x}\right)_{x=-a} \qquad (38)$$

in which

$\frac{2\pi}{\sigma_0}$ = period of free oscillation of origin,

M = mass of origin per unit length.

The boundary condition are given by

$$U = u_{x=a} = u_{x=-a} \qquad (39)$$

From (37), (38) and (39), we get

$$\alpha = -\frac{\rho}{M} \pm i\sqrt{\frac{\sigma_0^2}{v_1^2} - \left(\frac{\rho}{M}\right)^2} \doteqdot -\frac{\rho}{M} \pm i\frac{\sigma_0}{v_1}$$

Thus the motion of the propagated waves and of the vibratory origin are expressed by

$$\left.\begin{aligned} u_1&=Be^{-\frac{\rho}{M}(v_1t-x+a)}\cos\frac{\sigma_0}{v_1}(v_1t-x+a)\\ u_2&=Be^{-\frac{\rho}{M}(v_1t+x+a)}\cos\frac{\sigma_0}{v_1}(v_1t+x+a)\\ U&=Be^{-\frac{\rho}{M}v_1t}\cos\sigma_0t \end{aligned}\right\}\tag{40}$$

The origin decays gradually by the emission of the wave energy in the surrounding medium, the decay factor being expressed by

$$e^{-\frac{\rho}{\rho_0}\frac{v_1}{2a}t}$$

where ρ and ρ_0 are the densities of the medium and the rigid origin respectively.

VII. Decay of Cylindrical Origin.

The expressions for the radiating waves are the same as shown in (24) of section IV.

The motion of the cylinder is given by

$$M\left(\frac{d^2U}{dt^2}+\sigma_0^2U\right)=\int_{-\pi}^{\pi}\left[\left(\lambda\Delta+2\mu\frac{\partial u}{\partial r}\right)\cos\theta -\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{d\theta}\right)\sin\theta\right]ad\theta \tag{41}$$

where

M : mass of cylinder per unit length,

a : radius of cylinder.

Substituting from (24) in (41) and writing $U=U'e^{-ipt}$, we have

$$(\sigma_0^2-p^2)\frac{MU'}{\pi a}=B_1(\lambda+2\mu)H_1^{(1)}(ha)-C_1\mu H_1^{(1)}(ka) \tag{42}$$

At the boundary $r=a$, we shall have

$$u=U\cos\theta \qquad v=-U\sin\theta$$

from which

$$\left.\begin{aligned} U&=-\frac{B_1}{h^2}\frac{dH_1^{(1)}(ha)}{da}+\frac{C_1}{k^2}\frac{H_1^{(1)}(ka)}{a}\\ U&=-\frac{B_1}{h^2}\frac{H_1^{(1)}(ha)}{a}+\frac{C_1}{k^2}\frac{dH_1^{(1)}(ka)}{da} \end{aligned}\right\}\tag{43}$$

Between (42) and (43) eliminate U, B_1 and C_1, then we have

$$\begin{vmatrix} (p^2-\sigma_0^2)a\frac{\rho_0}{\rho} & (\lambda+2\mu)H_1^{(1)}(ha) & \mu H_1^{(1)}(ha) \\ 1 & \frac{1}{h^2}\frac{dH_1^{(1)}(ha)}{da} & \frac{1}{k^2}\frac{H_1^{(1)}(ka)}{a} \\ 1 & \frac{1}{h^2}\frac{H_1^{(1)}(ha)}{a} & \frac{1}{k^2}\frac{dH_1^{(1)}(ka)}{da} \end{vmatrix}=0 \qquad (44)$$

in which

ρ_0, ρ=densities of cylinder and medium respectively.

Calculating the determinant* for relatively large values of ha and ka, we obtain,

$$p^4+\frac{\rho}{\rho_0}\frac{v_1+v_2}{a}ip^3-\left[\sigma_0^2-\frac{v_1v_2}{a^2}+\frac{2\rho}{\rho_0}\frac{v_1v_2}{a^2}\right]p^2-\frac{v_1v_2}{a^2}\sigma_0^2=0 \qquad (45)$$

Neglecting terms which are very small, we get

$$p^2+\frac{\rho}{\rho_0}\frac{v_1+v_2}{a}ip-\sigma_0^2=0$$

$$p \doteqdot \sigma_0-\frac{\rho}{\rho_0}\frac{v_1+v_2}{2a}i \qquad (46)$$

where $v_1=\sqrt{\frac{\lambda+2\mu}{\rho}}$, $v_2=\sqrt{\frac{\mu}{\rho}}$

and $h=\frac{\sigma_0}{v_1}-\frac{\rho}{\rho_0}\frac{v_1+v_2}{2av_1}i \qquad (47)$

The motion of the propagated waves are expressed by

$$\left.\begin{aligned} \Delta &\doteqdot -\frac{\sigma_0}{v_1}U'\sqrt{\frac{a}{r}}e^{\frac{\rho}{\rho_0}\frac{v_1+v_2}{2av_1}(r-a-v_1t)}\cos\theta\sin\frac{\sigma_0}{v_1}(r-a-v_1t) \\ 2\varpi &\doteqdot \frac{\sigma_0}{v_1}U'\sqrt{\frac{a}{r}}e^{\frac{\rho}{\rho_0}\frac{v_1+v_2}{2av_1}(r-a-v_2t)}\sin\theta\sin\frac{\sigma_0}{v_2}(r-a-v_2t) \end{aligned}\right\} \qquad (48)$$

and the decay factor of the origin is given by

$$e^{-\frac{\rho}{\rho_0}\frac{v_1+v_2}{2a}t} \qquad (49)$$

VIII. Decay of Spherical Origin.

The expressions for the propagated waves are the same as shown in (31) of the section V.

The equation of motion is expressed by

$$M\left(\frac{d^2U}{dt^2}+\sigma_0^2U\right)=\iint\left[\left(\lambda\varDelta+2\mu\frac{u\partial}{\partial r}\right)\cos\theta\right.$$
$$\left.-\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)\sin\theta\right]2\pi a^2\sin\theta\, d\theta \tag{50}$$

Substituting from (26) in (50) and writing $U=U'e^{-ipt}$, we have

$$(\sigma_0^2-p^2)\frac{MU'}{\pi a}=B_1(\lambda+2\mu)H^{(1)}_{\frac{3}{2}}(ha)-C_1\mu H^{(1)}_{\frac{3}{2}}(ka) \tag{51}$$

At the boundary $r=a$, we have

$$u=U\cos\theta \qquad \text{and} \qquad v=-U\sin\theta$$

so that,

$$\left.\begin{aligned} U&=-\frac{B_1}{h^2}\frac{d}{da}\frac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}}-\frac{2C}{k^2a}H^{(1)}_{\frac{3}{2}}(ka)\\ U&=-\frac{B_1}{h^{\frac{5}{2}}a^{\frac{3}{2}}}H^{(1)}_{\frac{3}{2}}(ha)-\frac{C}{k^3a}\frac{d}{da}\sqrt{ka}H^{(1)}_{\frac{3}{2}}(ka)\end{aligned}\right\} \tag{52}$$

Between (51) and (52) eliminate U, B_1 and C_1, then

$$\begin{vmatrix} (p^2-\sigma_0^2)\dfrac{a\rho_0}{\rho} & (\lambda+2\mu)\dfrac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}} & 2\mu\dfrac{H^{(1)}_{\frac{3}{2}}(ka)}{\sqrt{ka}} \\ 1 & \dfrac{1}{h^2}\dfrac{d}{da}\dfrac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}} & \dfrac{2}{k^{\frac{5}{2}}}\dfrac{H^{(1)}(ka)}{a^{\frac{3}{2}}} \\ 1 & \dfrac{1}{h^2a}\dfrac{H^{(1)}_{\frac{3}{2}}(ha)}{\sqrt{ha}} & \dfrac{1}{k^3a}\dfrac{d}{da}\sqrt{ka}H^{(1)}_{\frac{3}{2}}(ka)\end{vmatrix}=0 \tag{53}$$

ρ_0, ρ=densities of sphere and medium respectively.

For relatively large values of ha and ka, we have

$$p^4-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{a}ip^3-\left[\sigma_0^2-\frac{2v_1v_2}{a^2}+\frac{4\rho}{\rho_0}\frac{v_1v_2}{a^2}\right]p^2-\frac{v_1v_2}{a^2}\sigma_0^2=0 \tag{54}$$

Neglecting terms which are very small, we get

$$p^2-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{a}ip-\sigma_0^2=0$$

$$p\doteqdot\sigma_0-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2a}i$$

$$h=\frac{\sigma_0}{v_1}-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2av_1}i$$

The propagated waves are expressed by

$$\left.\begin{aligned}\Delta &\doteqdot -\frac{\sigma_0}{v_1}U'\frac{a}{r}e^{\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2av_1}(r-a-v_1t)}\cos\theta\sin\frac{\sigma_0}{v_1}(r-a-v_1t)\\ 2\varpi &\doteqdot \frac{\sigma_0}{v_2}U'\frac{a}{r}e^{\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2av_2}(r-a-v_2t)}\sin\theta\sin\frac{\sigma_0}{v_2}(r-a-v_2t)\end{aligned}\right\} \quad (55)$$

with decay factor $e^{-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2a}t}$

The decay factors of three cases are tabulated as follows:—

Dimensions	1-dimensional	2-dimensional	3-dimensional
Decay factor	$e^{-\frac{\rho}{\rho_0}\frac{v_1}{2a}t}$	$e^{-\frac{\rho}{\rho_0}\frac{v_1+v_2}{2a}t}$	$e^{-\frac{\rho}{\rho_0}\frac{v_1+2v_2}{2a}t}$

where $v_1=\sqrt{\frac{\lambda+2\mu}{\rho}}$ $\qquad v_2=\sqrt{\frac{\mu}{\rho}}$

It will be seen that the retitutive origins emitting elastic waves decay with a rate depending upon the size, the shape, the relative density of the origin and velocity of elastic waves in the surrounding medium; also that it is independent of the intensity the resititutive forces. The effect of dimensional extension of obstacles on the decay factor is evidently due to the difference of the amount of the wave energy generated.

In seismic records of near earthquakes, the small tails to the principal tremors may partly be accounted for by such decaying nature of the origin.

IX. Résumé.

We have thus obtained some results having practical importance on the seismology by the use of harmonic functions, such as circular functions, Bessel's functions, Legendre's functions and Mathieu's functions; the principal results may be enumerated as follows:—

1. The scattering waves are composed of both dilatational and distorsional waves even when the incident ones are purely dilatational.

2. In the front of a hollow cavity the vibrations are nearly of standing type and in its rear part appears a space which is practically quiescent. These

phenomena do not happen if the obstacle is small compared with wave length.

3. Motions of the imbedded particles influence, among other things, their inertia masses for a certain range of wave lengths.

4. The vibration of the restitutive origin itself decays by emitting elastic waves. The decay factor depends on its density relative to the surrounding medium, the size, the shape and the velocities of waves in the surrounding medium.

In conclusion the author's sincere thanks are due to Professors Suyehiro and Terada for their kind advices and suggestions.

May, 1927.

Fig. I. (Cylinder) Maximum Displacement at F. & R. ($\lambda=\mu$)

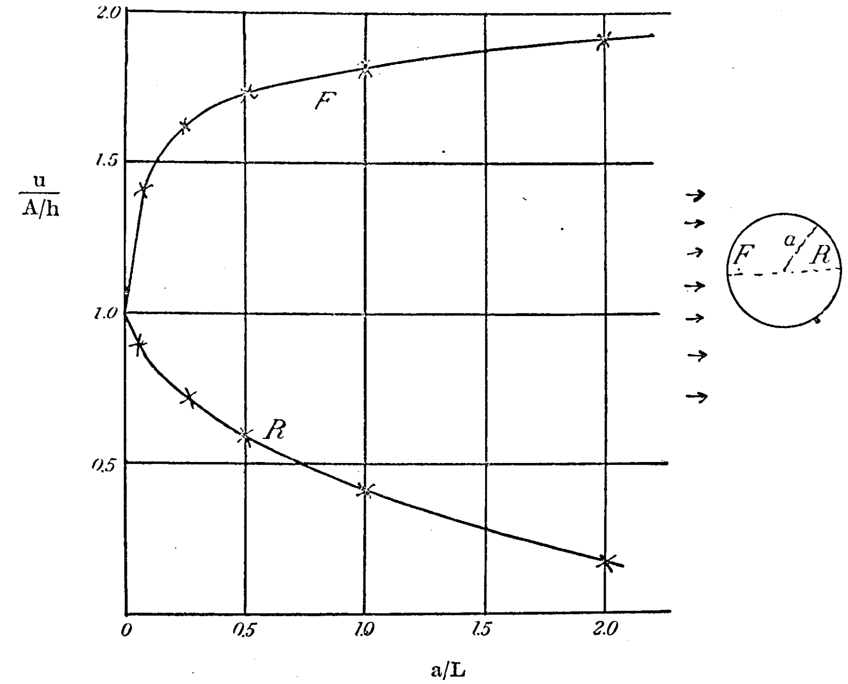

Fig. II. (Sphere) Maximum Displacement at F. & R. ($\lambda=\mu$)