On the Decay of Waves in Visco-Elastic Solid Bodies

By

Katsutada SEZAWA

Earthquake Research Institute

粘彈性體に於ける波動の老衰に就て

所員 妹澤克惟

彈性體が粘性を有する場合、其に傳はる波動が如何なる影響を受けるかは、考へただけでは極めて平凡な樣であるが、其を研究して見ると種々面白い結果が得られる。

本論文は五章に分れ、第一章では一般の運動方程式や其彈性論の關係を示し、第二章第四章には表面に無關係なる波動が粘質彈性體を傳はる場合、第三章は表面を傳はる波動、第五章には彈性體の中で最も特別な性質を持つ梁を波動が傳播する力學を示した。

此等の計算の結果を列記すれば、

1) 波源の働きが如何に鋭い性質のものでも、遠方へ行くに從ひ、漸次鈍くなり、波動形の長さは次第に長くなる。
2) 地震學に於ける志田博士等の「押しと引き」の法則は可なり複雜な場合にも成立する。
3) 構造物の强制振動の分析は甚だ厄介なものであるが自由波の問題を解く事によつて、容易に研究する事が出來る。
4) 粘質彈性體を傳播する波動を研究する事によつて、工學上重要なる衝撃の問題を數學的に研究する事が出來る。

Although the possibilities of the modification of the form and the decay of waves in visco-elastic solid bodies are well known, yet the effect of solid viscosity on the elastic waves caused by arbitrary disturbance has not been completely studied. So long ago as 1906, Professor Nagaoka[1] investigated the propagation of the seismic waves in an elastic body having viscosity. It seems that at that time very little was known about the nature of the solid viscosity. Galitzin[2] and Hosali[3] dealt with the same subject for very simple cases. Hosali's investigation based upon the assumption which well conforms with recent view.

1) Tokyo Math. and Phys. Soc. Proc., vol. 3 (1906).
2) Bull. de l'Acad. des Sc. St. Pets. (1912).
3) Roy. Soc. Proc. (1923).

In the present investigation the author attempts to study the propagation of arbitrary waves in visco-elastic solid bodies, and also with a slight extension to the same propagated along a beam, which may have important bearings on constructive materials and problems of shocks on them.

I. General Equations of Motion.

The equations of motion in rectangular coordinates are expressed by

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}&=\rho\frac{\partial^2 u}{\partial t^2} \\ \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}&=\rho\frac{\partial^2 v}{\partial t^2} \\ \frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z}&=\rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \quad (1)$$

The stress-strain relations for visco-elastic solid body are given by

$$\left.\begin{aligned} X_x&=\lambda\Delta+2\mu\frac{\partial u}{\partial x}+\lambda'\frac{\partial\Delta}{\partial t}+2\mu\frac{\partial^2 u}{\partial t\,\partial x} \\ Y_y&=\lambda\Delta+2\mu\frac{\partial v}{\partial y}+\lambda'\frac{\partial\Delta}{\partial t}+2\mu\frac{\partial^2 v}{\partial t\,\partial y} \\ Z_z&=\lambda\Delta+2\mu\frac{\partial w}{\partial z}+\lambda'\frac{\partial\Delta}{\partial t}+2\mu\frac{\partial^2 w}{\partial t\,\partial z} \\ Y_z&=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\ Z_x&=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ X_y&=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad (2)$$

where

λ, μ =Lame's elastic constants.

λ', μ'=voluminal and equivoluminal viscosities.

Substantial from (2) in (1), we get

$$\left.\begin{aligned}
&\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\frac{\partial \varDelta}{\partial x}-2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_3}{\partial y}+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_2}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2}\\
&\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\frac{\partial \varDelta}{\partial y}-2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_1}{\partial z}+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_3}{\partial x}=\rho\frac{\partial^2 v}{\partial t^2}\\
&\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\frac{\partial \varDelta}{\partial z}-2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_2}{\partial x}+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial \varpi_1}{\partial y}=\rho\frac{\partial^2 w}{\partial t^2}
\end{aligned}\right\} \quad (3)$$

where

$$\varDelta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \qquad 2\varpi_1=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),$$

$$2\varpi_2=\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right), \qquad 2\varpi_3=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

ρ=density

u, v, w=components of displacement in x, y, z–directions.

In symmetrical cases, we have

$$\left.\begin{aligned}
&(\lambda+2\mu)\nabla^2\varDelta+(\lambda'+2\mu')\frac{\partial}{\partial t}\nabla^2\varDelta=\rho\frac{\partial^2\varDelta}{\partial t^2}\\
&\mu\nabla^2\varpi+\mu'.\frac{\partial}{\partial t}\nabla^2\varpi=\rho\frac{\partial^2\varpi}{\partial t^2}
\end{aligned}\right\} \quad (3')$$

II. Purely Dilatational Plane Waves.

The equation of motion is expressed by

$$\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\nabla^2 u=\rho\frac{\partial^2 u}{\partial t^2} \quad (1)$$

Putting $u=Ae^{i(pt-fx)}$ we have

$$\frac{p}{f}=-\frac{\lambda'+2\mu'}{2\rho}fi\pm\sqrt{\frac{\lambda+2\mu}{\rho}}$$

Writing $\sqrt{\dfrac{\lambda+2\mu}{\rho}}=v_1$ and $\dfrac{\lambda'+2\mu'}{2\rho}=w_1$, we get

$$u=Ae^{if(x-v_1t)}e^{-w_1f_2t} \quad (2)$$

To generalise this, we introduce Fourier's double integral:

$$f(x)=\frac{1}{\pi}\int_0^\infty d\xi\int_{-\infty}^\infty f(\lambda)\cos\xi(x-\lambda)\,d\lambda \quad (3)$$

so that

$$u=\frac{1}{\pi}\int_0^\infty df\cos fv_1t\, e^{-w_1f^2t}\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-\lambda)\,d\lambda \qquad (4)$$

for a given initial disturbance $u=\varphi(x)$ and

$$u=\frac{1}{\pi}\int_0^\infty df\, e^{-w_1f^2t}\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-v_1t-\lambda)\,d\lambda \qquad (5)$$

for a given initial wave profile $u=\varphi(x)$.

Now we take a few examples corresponding to (4).

1) Put $\varphi(\lambda)=\frac{B}{b}e^{-\frac{\lambda^2}{b^2}}$, then

$$u=\frac{B}{\sqrt{\pi}}\int_0^\infty e^{-\left(w_1t+\frac{b^2}{4}\right)f^2}\cos fx\cos fv_1t\,df$$

Calculating this, we have

$$u=\frac{B}{2\sqrt{4w_1t+b^2}}\left[e^{-\frac{(x+v_1t)^2}{4w_1t+b^2}}+e^{-\frac{(x-v_1t)^2}{4w_1t+b^2}}\right]$$

This solution gives us a pair of pulses propagated in both positive and negative directions of x axis. The above solutions also shows that, however sharp the initial form of disturbance may be, the forms of tremors become gradually flat and long; this indicates-the nature of decay.

2) Put $\varphi(\lambda)=\frac{B\lambda}{b^2}e^{-\frac{\lambda^2}{b^2}}$, then

$$u=\frac{B}{2\sqrt{\pi}}\int_0^\infty e^{-\left(w_1t+\frac{b^2}{4}\right)f^2}bf\sin fx\cos fv_1t\,df$$

Calculating this, we get

$$u=\frac{Bb}{2}\frac{1}{(4w_1t+b^2)^{\frac{3}{2}}}\left[(x+v_1t)\,e^{-\frac{(x+v_1t)}{4w_1t+b^2}}+(x-v_1t)\,e^{-\frac{(x-v_1t)^2}{4w_1t+b^2}}\right]$$

which is due to an unsymmetrical disturbance. This solution shows that, besides the nature as in the former case, "the law of pull and push" in seismology due to Professor Shida is applicable even to very complicated disturbances.

3) Putting $\varphi(\lambda)=\dfrac{Ba^2}{b^2+\lambda^2}$, we get

$$u=\frac{Ba^2}{b}\int_0^\infty e^{-w_1tf^2+bf}\cos fx\cos fv_1t\,df$$

For large values of t, we have

$$u\doteqdot\frac{Ba^2}{4b}\frac{\sqrt{\pi}}{\sqrt{w_1t}}\left[e^{-\frac{(x+v_1t)^2}{4w_1t}}\cos\frac{b(x+v_1t)}{w_1t}+e^{-\frac{(x-v_1t)^2}{4w_1t}}\cos\frac{b(x-v_1t)}{w_1t}\right]e^{\frac{b^2}{4w_1t}}$$

approximately.

4) Put $\varphi(\lambda)=\dfrac{Ba\lambda}{b^2+\lambda^2}$, then

$$u=Ba\int_0^\infty e^{-(w_1tf^2+bf)}\sin fx\cos fv_1t\,df$$

For large values of t, we have

$$u=\frac{Ba}{4}\frac{\sqrt{\pi}}{\sqrt{w_1t}}\left[e^{-\frac{(x+v_1t)^2}{4w_1t}}\sin\frac{b(x+v_1t)}{w_1t}+e^{-\frac{(x-v_1t)^2}{4w_1t}}\sin\frac{b(x-v_1t)}{w_1t}\right]e^{\frac{b^2}{4w_1t}}$$

approximately.

Next, taking the cases corresponding to (5), we get the following results.

5) Put $\varphi(\lambda)=\dfrac{B}{b}e^{-\frac{\lambda^2}{b^2}}$, then

$$u=\frac{B}{\sqrt{\pi}}\int_0^\infty e^{-\left(w_1t+\frac{b^2}{4}\right)f^2}\cos f(x-v_1t)\,df$$

or

$$u=\frac{B}{\sqrt{4w_1t+b^2}}e^{-\frac{(x-v_1t)^2}{4w_1t+b^2}}$$

6) Put $\varphi(\lambda)=\dfrac{B\lambda}{b^2}e^{-\frac{\lambda^2}{b^2}}$, then

$$u=\frac{Bb(x-v_1t)}{(4w_1t+b^2)^{\frac{3}{2}}}e^{-\frac{(x-v_1t)^2}{4w_1t+b^2}}$$

III. Surface Waves.

The elementary solutions of displacements, when $\lambda=\mu$, are expressed by

$$\left.\begin{aligned} u_1 &= i\,3{\cdot}55\,Ae^{ry+i(pt-fx)-w_1h^2t} \\ v_1 &= -3{\cdot}01\,Ae^{ry+i(pt-fx)-w_1h^2t} \\ u_2 &= -i\,2{\cdot}04\,Ae^{sy+i(pt-fx)-w_2k^2t} \\ v_2 &= 5{\cdot}21\,Ae^{sy+i(pt-fx)-w_2k^2t} \end{aligned}\right\}\quad \left\{\begin{aligned} w_1 &= \frac{\lambda'+2\mu'}{2\rho} \\ w_2 &= \frac{\mu'}{2\rho} \\ h^2 &= \frac{\rho p^2}{\lambda+2\mu} \\ k^2 &= \frac{\rho p^2}{\mu} \\ r^2 &= f^2-h^2 \\ s^2 &= f^2-k^2 \end{aligned}\right.$$

Here the boundary conditions are given by

$$\left.\begin{aligned} \lambda\Delta+2\mu\frac{\partial v}{\partial y}+\frac{\partial}{\partial t}\left(\lambda'\Delta+2\mu'\frac{\partial v}{\partial y}\right)&=0 \\ \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)&=0 \end{aligned}\right\}\ \text{on } y=0$$

If $\frac{\lambda}{\mu}=\frac{\lambda'}{\mu'}$, the above boundary conditions are satisfied.

We put now $r=\cdot 847f$, $s=\cdot 393f$, $w_1h^2=\cdot 845f^2w_2$ and $w_2k^2=\cdot 845f^2w_2$. The elementary solutions are thus given by

$$\left.\begin{aligned} v &= v_1+v_2=\left(-3{\cdot}01\,e^{\cdot 874fy-\cdot 845\,w_2f^2t}+5{\cdot}21\,e^{\cdot 393fy-\cdot 845\,w_2f^2t}\right)Ae^{i(pt-fx)} \\ u &= u_1+u_2=i\left(3{\cdot}55\,e^{\cdot 847fy-\cdot 845\,w_2f^2t}-2{\cdot}04\,e^{\cdot 393fy-\cdot 845\,w_2f^2t}\right)Ae^{i(pt-fx)} \end{aligned}\right\}$$

Generalising this, we obtain for the initial disturbance $v=\rho(x)$ ou $y=0$,

$$\left.\begin{aligned} v &= \frac{1}{2{\cdot}20\pi}\int_0^\infty df\cos fvt\,e^{-\cdot 845\,w_2tf^2}\left(-3{\cdot}01e^{\cdot 847fy}+5{\cdot}21e^{\cdot 393fy}\right)\int_{-\infty}^{\infty}\varphi(\lambda)\cos f(x-\lambda)d\lambda \\ u &= \frac{1}{2{\cdot}20\pi}\int_0^\infty df\cos fvt\,e^{-\cdot 845\,w_2tf^2}\left(3{\cdot}55e^{\cdot 847fy}-2{\cdot}04e^{\cdot 393fy}\right)\int_{-\infty}^{\infty}\varphi(\lambda)\sin f\,x-\lambda)\,d\lambda \end{aligned}\right\}$$

1) If $\varphi(\lambda)=\frac{Ba^2}{b^2+\lambda^2}$, we have the integrals

$$\left.\begin{aligned} v&=\frac{Ba}{2{\cdot}20\,b}\int_0^\infty e^{-{\cdot}845\,w_2tf^2}\left(-3{\cdot}01\,e^{({\cdot}847\,yf-bf)}+5{\cdot}21\,e^{({\cdot}393\,yf-bf)}\right)\cos fx\cos fvt\,df \\ u&=\frac{Ba^2}{2{\cdot}20\,b}\int_0^\infty e^{-{\cdot}845\,w_2tf^2}\left(3{\cdot}55\,e^{({\cdot}847\,yf-bf)}-2{\cdot}04\,e^{({\cdot}393\,yf-bf)}\right)\sin fx\cos fvt\,df \end{aligned}\right\}$$

2) If $\varphi(\lambda)=\dfrac{Ba\lambda}{b^2+\lambda^2}$, we get the integrals

$$\left.\begin{aligned} v&=\frac{Ba}{2{\cdot}20}\int_0^\infty e^{-{\cdot}845\,w_2tf^2}\left(-3{\cdot}01\,e^{({\cdot}847\,yf-bf)}+5{\cdot}21\,e^{({\cdot}393\,yf-bf)}\right)\sin fx\cos fvt\,df \\ u&=\frac{-Ba}{2{\cdot}20}\int_0^\infty e^{-{\cdot}845\,w_2tf^2}\left(3{\cdot}55\,e^{({\cdot}847\,yf-bf)}-2{\cdot}04\,e^{({\cdot}394\,yf-bf)}\right)\cos fx\cos fvt\,df \end{aligned}\right\}$$

When $w_2=0$, both integrals, 1) and 2), can be easily evaluated. The exact integrals, when $w_2\neq 0$, are left for furthur study.

IV. Propagation of Dilatational Waves in Two Dimensions.

The equation of motion is expressed by

$$\left[(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right]\nabla^2 u=\rho\frac{\partial^2 u}{\partial t^2}$$

Putting $u=AJ_0(fr)\cos pt$, we get

$$\frac{p}{f}=-\frac{\lambda'+2\mu'}{2\rho}f\,i\pm\sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$\therefore\quad u=AJ_0(fr)\,e^{-w_1f^2t}\cos fv_1t$$

Now $\quad \varphi(r)=\displaystyle\int_0^\infty J_0(fr)f\,df\int_0^\infty \varphi(\lambda)\,J_0(f\lambda)\,\lambda\,d\lambda$

Thus

$$u=\int_0^\infty J_0(fr)\,e^{-w_1f^2t}\cos fv_1t\cdot f\,df\int_0^\infty \varphi(\lambda)\,J_0(f\lambda)\,\lambda\,d\lambda$$

Put $\varphi(\lambda)=Be^{-\frac{\lambda^2}{b^2}}$, then

$$B\int_0^\infty e^{-\frac{\lambda^2}{b^2}}J_0(f\lambda)\,\lambda\,d\lambda=\frac{Bb^2}{2}e^{-\frac{f^2b^2}{4}}$$

Thus

$$u=\frac{Bb^2}{2}\int_0^{\infty}J_0(fr)\,e^{-\left(w_1t+\frac{b^2}{4}\right)f^2}\cos fv_1t.fdf$$

For a small value of t, we may expand the integrand and find the approximately value of this as follows:—

$$u=\frac{Bb^2}{2}\int_0^{\infty}\left\{f-\frac{f^3v_1{}^2t^2}{2!}+\frac{f^5v^4t^4}{4!}-\ldots\right\}J_0(fr)\,e^{-\left(w_1t+\frac{b}{4}\right)f^2}\,df$$

Now

$$\int_0^{\infty}J_0(fr)\,e^{-\left(w_1t+\frac{b}{4}\right)f^2}f^n\,df=\frac{\Gamma\left(\frac{n+1}{2}\right)\;e^{-\frac{r^2}{w_1t+b^2}}}{2\left(w_1t+\frac{b^2}{4}\right)^{\frac{n+1}{2}}}\;{}_1F_1\left(\frac{1-n}{2};1;\frac{r^2}{4w_1t+b^2}\right)$$

and
$$_1F_1\left(\frac{1-n}{2};1;\frac{r^2}{4w_1t+b^2}\right)=1+\frac{\frac{1-n}{2}}{2!\,1.}\frac{r^2}{4w_1t+b^2}+\ldots.$$

Thus for small value of t, we find,

$$u=\frac{Bb^2}{4\left(w_1t+\frac{b^2}{4}\right)}e^{-\frac{r^2}{4w_1t+b^2}}\left(1-\frac{v_1{}^2t^2}{w_1t+\frac{b^2}{4}}+\ldots\right)$$

V. Waves propagated in Beam from Fixed End.

The equation of motion of a beam is given by

$$\rho\frac{\partial^2y}{\partial t^2}+Ek^2\frac{\partial^4y}{\partial x^4}+wk^2\frac{\partial^5y}{\partial t\,\partial x^4}=0$$

where

y=lateral deflection
ρ=density
E=Young's modulus
k=radius of gyration of section of beam
w=solid viscosity corresponding to bending.

Putting $y=Ae^{-mx+ipt}+Be^{i(pt-mx)}$, we have

$$(A+B)\left(p^2-ipm^4\frac{wk^2}{\rho}-m^4\frac{Ek^2}{\rho}\right)=0$$

from which we have

$$m^4=\frac{p^2(c_1-ipc_2)}{c_1^2+p^2c_2^2}$$

or
$$m\doteqdot\frac{\sqrt{p}}{c_1^{1/4}}-i\frac{c_2}{4c_1^{5/4}}p^{\frac{3}{2}}=m_1+im_2\ (m_1=\frac{\sqrt{p}}{c_1^{1/4}},\ m_2=\frac{c_2}{4c_1^{5/4}}p^{\frac{3}{2}}\doteqdot 0)$$

where
$$c_1=\frac{Ek^2}{\rho}\ \text{and}\ c_2=\frac{wk^2}{\rho}$$

The solution satisfying the boundary conditions $y=a\cos pt$, $\frac{dy}{dx}=0$ at $x=0$ is of the following form, which expresses the waves propagating along the beam from the fixed and oscillating end:—

$$y=\frac{a}{\sqrt{2}}e^{-m_1x}\cos\left(pt+m_2x-\frac{\pi}{4}\right)+\frac{a}{\sqrt{2}}e^{-m_2x}\cos\left(m_1x-pt-\frac{\pi}{4}\right)$$

in which the first term of the right-hand side is the term of decay of the end condition and the second gives the propagated waves.

The solution thus found can be directly applied to the forced vibrations of the constructive materials. Because the nature of forced oscillations is naturally equivalent to that of free waves propagated from a disturbed origin.

Generalising the solution, we have

$$y=\frac{1}{\sqrt{2\pi}}\int_0^\infty dp\int_{-\infty}^\infty f(\lambda)\Big\{e^{-m_1x}\cos\left(p\,\overline{t-\lambda}+m_2x-\frac{\pi}{4}\right)$$
$$+e^{-m_2x}\cos\left(m_1x-p\,\overline{t-\lambda}-\frac{\pi}{4}\right)\Big\}d\lambda$$

1) Putting $f(\lambda)=\frac{Ba^2}{b^2+\lambda^2}$, we get

$$y=\frac{Ba^2}{\sqrt{2\pi}}\int_0^\infty dp\int_{-\infty}^\infty\frac{1}{b^2+\lambda^2}\Big\{e^{-m_1x}\cos\left(p\,\overline{t-\lambda}+m_2x-\frac{\pi}{4}\right)$$
$$+e^{-m_2x}\cos\left(m_1x-p\,\overline{t-\lambda}-\frac{\pi}{4}\right)d\lambda$$

which is integrated by the definite integral, $\int_{-\infty}^\infty\frac{\cos rx}{b^2+x^2}dx=\frac{\pi}{b}e^{-br}$, in the form

$$y=\frac{Ba^2}{\sqrt{2}\,b}\int_0^\infty\Big\{e^{-(m_1x+pb)}\cos\left(pt+m_2x-\frac{\pi}{4}\right)+e^{-(m_2x+pb)}\cos\left(m_1x-pt-\frac{\pi}{4}\right)dp$$

For a small value of t, we may put the integral, as before, in the form,

$$y=\frac{Ba^2}{2b}\int_0^\infty\Big[\Big(1-m_1x+\frac{m_1^2x^2}{2!}-..\Big)\Big(1+m_2x-\frac{m_2^2x^2}{2!}-..\Big)e^{-pb}\cos pt$$
$$+\Big(1-m_1x+\frac{m_1^2x^2}{2!}-..\Big)\Big(1-m_2x-\frac{m_2^2x^2}{2!}+..\Big)e^{-pb}\sin pt$$
$$+\Big(1-m_2x+\frac{m_2^2x^2}{2!}-..\Big)\Big(1+m_1x-\frac{m_1^2x^2}{2!}-..\Big)e^{-pb}\cos pt$$
$$-\Big(1-m_2x+\frac{m_2^2x^2}{2!}-..\Big)\Big(1-m_1x-\frac{m_1^2x^2}{2!}+..\Big)e^{-pb}\sin pt\Big]dp$$

Taking to the second order of x, we obtain

$$y=\frac{Ba^2}{2b}\int_0^\infty\Big[\Big(2+\frac{c_2}{2c_1^{3/2}}p^2x^2\Big)e^{-pb}\cos pt+\Big(\frac{p}{\sqrt{c_1}}-\frac{c_2p^3}{16c_1^{5/2}}\Big)x^2e^{-pb}\sin pt\Big]dp$$

Integrating we have

$$y=\frac{Ba^2}{b^2+t^2}\Big[1+\frac{x^2}{\sqrt{c_1}\,(b^2+t^2)}\Big\{t+\frac{c_2}{2c_1}\frac{b^2-3t^2}{b^2+t^2}-\frac{3}{4}\Big(\frac{c_2}{c_1}\Big)^2\frac{t(b^2-t^2)}{(b^2+t^2)^2}\Big\}\Big]$$

2) Writing $f(\lambda)=\dfrac{Ba\lambda}{b^2+\lambda^2}$, we have

$$y=\frac{Ba}{\sqrt{2}\,\pi}\int_0^\infty dp\int_{-\infty}^\infty\frac{\lambda}{b^2+\lambda^2}\Big\{e^{-m_1x}\cos\Big(p\,\overline{t-\lambda}+m_2x-\frac{\pi}{4}\Big)$$
$$+e^{-m_2x}\cos\Big(m_1x-p\,\overline{t-\lambda}-\frac{\pi}{4}\Big)\Big\}d\lambda$$

As $\displaystyle\int_{-\infty}^\infty\frac{x\sin rx}{b^2+\lambda^2}dx=\pi e^{-br}$, we get

$$y=\frac{Ba}{\sqrt{2}\,\pi}\int_0^\infty\Big[e^{-(m_1x+pb)}\sin\Big(pt+m_2x-\frac{\pi}{4}\Big)$$
$$-e^{-(m_2x+pb)}\sin\Big(m_1x-pt-\frac{\pi}{4}\Big)\Big]dp$$

Similarly as before, we get

$$y=\frac{Ba}{2}\int_0^\infty\Big[(m_2^2-m_1^2)x^2e^{-pb}\cos pt+(2+2m_1m_2x^2)e^{-pb}\sin pt\Big]dp$$

and finally we obtain

$$y=\frac{Ba}{b^2+t^2}\Big[t+\frac{x^2}{\sqrt{c_1}\,(b^2+t^2)}\Big\{\frac{b^2-t^2}{2}+\frac{c_2}{2c_1}\frac{t(3b^2-t^2)}{b^2+t^2}-\frac{3}{4}\Big(\frac{c_2}{c_1}\Big)^2\frac{(b^4-6b^2t^2+t^4)}{(b^2+t^2)^2}\Big\}\Big]$$

We have seen that arbitrary tremors transmitted along a beam from a fixed end shows dispersive nature and that, by the above method of investigation of propagated waves, the problem of the forced oscillation of constructive materials can be solved without difficulty. Another important problem is that of a beam subjected to a suddenly applied load; empirical methods of attacking "shock" can thus be taken place by this purely analytical method.

VI. Résumé.

We have now obtained some results having theoretical interest, by a purely mathematical analysis. The essential parts can be summed up as follows:—

1) However sharp the initial form of disturbance may be, the pulses in visco-elastic solid bodies take gradually flat forms, their apparent wave length being prolonged.

2) The law of "pull and push" in seismology is applicable even to very complicated cases of the disturbances.

3) By the use of the equations of free waves in elastic bodies the problem of forced vibrations of constructive materials can be solved without difficulty.

4) Using the equation of pulses in visco-elastic solid bodies, the problem of "shock" can be easily analysed by a purely mathematical method.

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