Dilatational and Distorsional Waves generated from a Cylindrical or a Spherical Origin.

By

Katsutada SEZAWA.

Earthquake Research Institute.

圓壔形又は球形の源による縱横兩彈性波勢力分配の機構

所員 妹澤克惟

無限に擴がる彈性媒質中にある一點が種々の運動をする時に、それから生ずる彈性波の模樣は、ストークス、エル．ローレンツ、レーレー、ケルビン、ラブ等の諸氏による研究が既に發表されてゐるが、本問題は其延長とも見らるべきものであつて、源が或定つた大さの圓壔又は球形をなし、其が或適當な運動をする時に、横波と縱波とが或一定の割合及び形となつて傳播する機構を示した。

研究の方法は數學を用ひ、中心から出る縱波横波が前述の圓壔面又は球面に於て或適當な割合に結合して其面の境界條件を滿足する樣に調整する事が主な考となつてゐる。

論文は三章に分れ、第一章に於ては圓壔形の源が種々の節線を持ち、一定の境界彈性狀態にある振動をする時に出る縱横兩彈性波分配の機構を示し、第二章に於ては球形の源が斯る運動をなす場合、第三章には球形の源が單に横波のみ生ずる樣な場合を研究してある。

この研究によつて、縱波横波の分配は偶然でなく、源に於ける應力狀態によつて確定される事や、源を點として考へる時は容易に得られぬ複雜な運動及び有限な應力狀態、又は源からの方向によつて波の勢力が異なる事等が明かにせられる譯である。

圓壔形や球形の源を考へる事は、實際の地震運動とは勿論形式を異にするが、しかし、斯る計算の行ひ易ひ模型を考へて彈性體の性能を研究する事は、舊來からの力學的習慣であり、且つ又常に適當な處置であらうと思はれる。其れは、實驗的物理學者及び地震學者などが、斯る結果を複雜な現象の分解に於ける一成分と考へたり、又は特異現象を見出す爲の疑ひなき基準としたりする事によつても明かであらう。

I. Introduction.

Although the possibility of co-existence of two types of waves propagated with different velocities in an isotropic elastic solid body is well known, yet how the energy liberated from an origin with prescribed conditions of stress is partitioned into dilational and distorsional waves, is not yet completely studied. Stokes[1] showed, in connection with the diffraction of light, the distribution of energy of waves in various directions, when the waves were generated from a single point executing periodic vibrations. Also L. Lorenz[2] and Lord Rayleigh[3] dealt with the same subject in a similar manner. A more general problem was taken up by Lord Kelvin,[4] who obtained the effect due to an oscillating rigid sphere. Love[5] later considered cases under various conditions at the origin, when it is a single point.

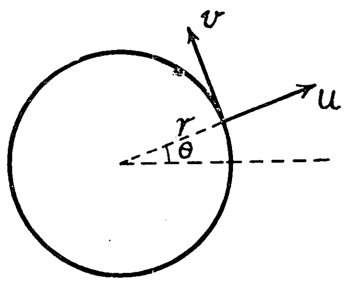
The present investigation has been undertaken to study the partition of strain energy into dilatational and distorsional waves, when the origin is of a cylindrical or spherical shape with prescribed conditions of stress.

II. Cylindrical Origin.

The equations of motion of elastic bodies in cylindrical coordinates, when the axial component of the motion is omitted, are expressed by

$$\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Delta}{\partial t^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Delta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2} \qquad (1)$$

$$\frac{\rho}{\mu}\frac{\partial^2\varpi}{\partial t^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varpi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varpi}{\partial\theta^2} \qquad (2)$$



where

$$\left.\begin{aligned}\Delta&=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\\ 2\varpi&=\frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\end{aligned}\right\} \qquad (3)$$

u, v=radial and transverse component of displacement respectively,
ρ=density of isotropic solid,
λ, μ=Lame's elastic constants,

1) Stokes's Math. and Phys. Paper, vol. 2 (1849)
2) L. Lorenz, J. f. Math. Bd. 58 (1861)
3) Lord Rayleigh, Theory of Sound, vol. 2, 276.
4) Phil. Mag. (1899)
5) London Math. Soc. Proc. (Ser. 2) vol. 1 (1904)

Writing $\Delta=\Delta_1 e^{i(n\theta+pt)}$ and $\varpi=\varpi_1 e^{i(n\theta+pt)}$ in (1) and (2) respectively and solving the resulting equations, we obtain,

$$\Delta_1=AJ_n(hr)+A'Y_n(hr) \tag{4}$$

$$\varpi_1=BJ_n(kr)+B'Y_n(kr) \tag{5}$$

in which $h^2=\dfrac{\rho p^2}{\lambda+2\mu_1},\ k^2=\dfrac{\rho p^2}{\mu_1},\ \dfrac{2\pi}{p}=$period of oscillations and n is a constant to denote the circumferential variation of amplitudes of waves.

Displacement (u_1, v_1) answering to Δ_1 in (4) and satisfying $\varpi=0$ is given by

$$u_1=-\frac{1}{h^2}\left\{A\frac{d}{dr}J_n(hr)+A'\frac{d}{dr}Y_n(hr)\right\} \tag{6}$$

$$v_1=-\frac{ni}{h^2}\left\{A\frac{J_n(hr)}{r}+A'\frac{Y_n(hr)}{r}\right\} \tag{7}$$

Displacement (u_2, v_2) derived from the value of ϖ_1 in (5) with the condition $\Delta=0$ is expressed by

$$u_2=\frac{ni}{k^2}\left\{B\frac{J_n(kr)}{r}+B'\frac{Y_n(kr)}{r}\right\} \tag{8}$$

$$v_2=-\frac{1}{k^2}\left\{B\frac{d}{dr}J_n(kr)+B'\frac{d}{dr}Y_n(kr)\right\} \tag{9}$$

The surface $r=a$ being assumed to be free from tangential traction and under a nomal pressure $p_1 e^{i(n\theta+pt)}$, the equations

$$\lambda\Delta+2\mu\frac{\partial u}{\partial r}=p_1 e^{i(n\theta+pt)} \tag{10}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)=0 \tag{11}$$

in which $u=u_1+u_2$ and $v=v_1+v_2$ must hold on that surface.

Putting the values of Δ_1, u_1+u_2 and v_1+v_2 from (4), (6), (7), (8) and (6) in (10) and (11), we get,

$$A\left[\lambda J_n(hr)-\frac{2\mu}{h^2}\frac{d^2}{dr^2}J_n(hr)\right]_{r=a}$$

$$+\frac{2iB\mu n}{k^2}\left[\frac{1}{r}\frac{J_n(kr)}{r}-\frac{J_n(kr)}{r^2}\right]_{r=a}=p_1 \tag{12}$$

$$\frac{2iAn}{k^2}\left[\frac{1}{r}J_n(hr)-\frac{1}{r^2}J_n(hr)\right]_{r=a}$$
$$+\frac{B}{k^2}\left[\frac{n^2J_n(kr)}{r^2}+\frac{d^2}{dr^2}J_n(kr)\right.$$
$$\left.-\frac{1}{r}\frac{d}{dr}J_n(kr)\right]_{r=a}=0 \qquad (13)$$

The above aquation (13) gives

$$\frac{B}{A}=-2i\frac{k^2}{h^2}n\frac{\left[\frac{1}{r}\frac{d}{dr}J_n(hr)-\frac{1}{r^2}J_n(hr)\right]_{r=a}}{\left[n^2\frac{J_n(kr)}{r^2}+\frac{d^2}{dr^2}J_n(kr)-\frac{1}{r}\frac{d}{dr}J_n(kr)\right]_{r=a}}\equiv -iR \qquad (14)$$

by which the relative values of the energy in dilatational and distorsional waves, when the origin executes the vibrations characterized by the equations (10) and (11), are obtained uniquely.

By the substitution of (14) in (12), we can determine the absolute magnitude of the enery of waves.

Each value of A and B being thus found, we can arrive at the general expressions for the dilatation and the distorsion, which can be written as follows :—

$$\Delta=A\cos n\theta\,\{J_n(hr)\cos pt+Y_n(hr)\sin pt\}$$
$$\varpi=AR\sin n\theta\,\{J_n(kr)\cos pt+Y_n(kr)\sin pt\}$$

where R is given in (14).

The velocities of propagation for Δ- and ϖ-waves are $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\sqrt{\frac{\mu}{\rho}}$ respectively.

III. Spherical Origin.

The equations of mortion of elastic bodies in spherical coordinates, when the azimuthal component of the motion is omitted, are expressed by

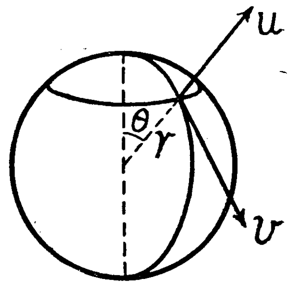
$$\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Delta}{\partial t^2}=\frac{\partial^2\Delta}{\partial r^2}+\frac{2}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\Delta}{\partial\theta}\cot\theta \qquad (1)$$

$$\frac{\rho}{\mu}\frac{\partial^2\varpi}{\partial t^2}=\frac{\partial^2\varpi}{\partial r^2}+\frac{2}{r}\frac{\partial\varpi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varpi}{\partial\theta^2}$$

$$+\frac{1}{r^2}\frac{\partial\varpi}{\partial\theta}\cot\theta-\frac{\varpi}{r^2}(1+\cot^2\theta) \qquad (2)$$

where

$$\left.\begin{aligned}\Delta&=\frac{\partial u}{\partial r}+\frac{2u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{v}{r}\cot\theta\\ 2\varpi&=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\end{aligned}\right\} \qquad (3)$$



u, v=radial and colatitudinal components of displacement respectively,

Writing $\Delta=\Delta_1 e^{ipt}$ and $\varpi=\varpi_1 e^{ipt}$ in (1) and (2) respectively and solving the resulting equations, we obtain,

$$\Delta_1=A\frac{P_n(\cos\theta)}{\sqrt{r}}\left\{J_{n+\frac{1}{2}}(hr)+A'\,Y_{n+\frac{1}{2}}(hr)\right\} \qquad (4)$$

$$\varpi_1=B\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{\sqrt{r}}\left\{J_{n+\frac{1}{2}}(kr)+B'\,Y_{n+\frac{1}{2}}(kr)\right\} \qquad (5)$$

in which $h^2=\dfrac{\rho p^2}{\lambda+2\mu}$ and $k^2=\dfrac{\rho p^2}{\mu}$.

Displacement (u_1, v_1) answering to Δ_1 in (4) and satisfying $\varpi=0$ is given by

$$u_1=-\frac{A}{h^2}P_n(\cos\theta)\left\{\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}+A'\frac{d}{dr}\frac{Y_{n+\frac{1}{2}}(hr)}{\sqrt{r}}\right\} \qquad (6)$$

$$v_1=-\frac{A}{h^2}\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{r^{\frac{3}{2}}}\left\{J_{n+\frac{1}{2}}(hr)+A'\,Y_{n+\frac{1}{2}}(hr)\right\} \qquad (7)$$

Displacement (u_2, v_2) derived from the value of ϖ_1 in (5) with the condition $\Delta=0$ is expressed by

$$u_2=-2B\frac{n(n+1)}{k^2}P_n(\cos\theta)\frac{1}{r^{\frac{3}{2}}}\left\{J_{n+\frac{1}{2}}(kr)+B'\,Y_{n+\frac{1}{2}}(kr)\right\} \qquad (8)$$

$$v_2=-\frac{2B}{k^2}\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{r}\left\{\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right)+B'\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(kr)\right)\right\} \tag{9}$$

The surface $r=a$ being assumed to be free from tangential traction and be under a normal pressure $p_1P_n(\cos\theta)\,e^{ipt}$, the equations

$$\lambda\Delta+2\mu\frac{\partial u}{\partial r}=p_1\,P_n(\cos\theta)\,e^{ipt} \tag{10}$$

$$\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}=0 \tag{11}$$

in which $u=u_1+u_2$ and $v=v_1+v_2$ must hold on that surface.

Putting the values of Δ, u_1+u_2 and v_1+v_2 from (4), (6), (7), (8) and (9) in (10) and (11), we have,

$$\frac{B}{A}=\frac{k^2}{h^2}\frac{\left[\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(hr)}{r^{\frac{3}{2}}}\right]_{r=a}}{\left[\frac{1}{\sqrt{r}}\frac{d^2}{dr^2}J_{n+\frac{1}{2}}(kr)-\frac{1}{r^{\frac{3}{2}}}\frac{d}{dr}J_{n+\frac{1}{2}}(kr)+\frac{1}{r^{\frac{5}{2}}}\left(n(n+1)-\frac{5}{4}\right)J_{n+\frac{1}{2}}(kr)\right]_{r=a}}\equiv R \tag{12}$$

$$A\left[\lambda\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}-\frac{2\mu}{h^2}\frac{d^2}{dr^2}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}-4\mu R\frac{n(n+1)}{k^2}\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}\right]_{r=a}=p_1 \tag{13}$$

The above equations (12) and (13) give us the ratio of the energy in dilatational and distorsional waves, together with the absolute magnitudes of of these waves.

Both dilatational and distorsional waves, which are thus partitioned to accord with the conditions on the surface of the spherical cavity, propagate outwards in the following manners :—

$$\left.\begin{aligned}\Delta&=AP_n(\cos\theta)\frac{1}{\sqrt{r}}\left\{J_{n+\frac{1}{2}}(hr)\cos pt+Y_{n+\frac{1}{2}}(hr)\sin pt\right\}\\ \varpi&=AR\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{\sqrt{r}}\left\{J_{n+\frac{1}{2}}(kr)\cos pt+Y_{n+\frac{1}{2}}(kr)\sin pt\right\}\end{aligned}\right\} \tag{14}$$

where A and AR can be obtained from (12) and (13).

The velocities of propagation of Δ- and ϖ-waves are $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\sqrt{\frac{\mu}{\rho}}$ respectively as in the preceding case.

The expressions of displacement for Δ-waves are in the following forms :—

$$u_1=-\frac{A}{h^2}P_n(\cos\theta)\left\{\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}\cos pt+\frac{d}{dr}\frac{Y_{n+\frac{1}{2}}(hr)}{\sqrt{r}}\sin pt\right\}$$

$$v_1=-\frac{A}{h^2}\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{r^{\frac{3}{2}}}\left\{J_{n+\frac{1}{2}}(hr)\cos pt+Y_{n+\frac{1}{2}}(hr)\sin pt\right\}$$

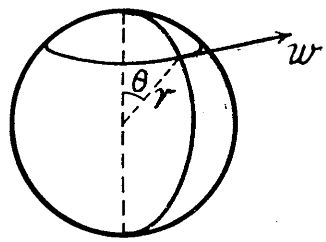
Similar expressions for ϖ-waves are

$$u_2=-\frac{2AR}{k^2}n(n+1)P_n(\cos\theta)\frac{1}{r^{\frac{3}{2}}}\left\{J_{n+\frac{1}{2}}(kr)\cos pt+Y_{n+\frac{1}{2}}(kr)\sin pt\right\}$$

$$v_2=-\frac{2AR}{k^2}\frac{dP_n(\cos\theta)}{d\theta}\frac{1}{r}\left\{\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right)\cos pt+\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(kr)\right)\sin pt\right\}$$

IV. Purely Distorsional Origin.

When stresses at the origin are entirely distorsional, the motion can be analysed more easily. As an example, let the displacement of a spherical origin be purely azimuthal. In such a case the equation of motion is written by



$$\frac{\rho}{\mu}\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial w}{\partial r}-\frac{2}{r^2}w+\frac{1}{r^2}\left\{\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial w}{\partial\theta}\cot\theta+w(1-\cot^2\theta)\right\}$$

n which w is the azimuthal component of displacement.

The appropriate solution of this differential equation is expressed by

$$w=\frac{A}{\sqrt{r}}\left\{J_{n+\frac{1}{2}}(kr)\cos pt+Y_{n+\frac{1}{2}}(kr)\sin pt\right\}\frac{d}{d\theta}P_n(\cos\theta)$$

where

$$\frac{2\pi}{p}=\text{period of oscillation,}$$

$$k^2=\frac{\rho p^2}{\mu}$$

It will be seen that the waves propagate towards infinity without change of type.

The motion of waves is purely equivoluminal and the velocity of propagation $\sqrt{\frac{\mu}{\rho}}$.

V. Conclusion.

The present investigation shows clearly that the partition of strain energy generated from an origin of a certain size into dilational and distorsional waves, is determined definitely by the conditions of stresses at the origin. It is also shown how the energy of waves is distributed in different directions. In concluding this paper the author wishes to express his indebtness to Professor Nagaoka and Professor Suyehiro of this Institute for valuable advices and suggestions.

December, 1926.