On the Propagation of Rayleigh-Waves on Plane and Spherical Surfaces.

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二次元又は球面上レーレー波の傳播

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彈性表面波が地表上を傳はる理論は、1887年レーレー卿によつて提出されたが、其は波が一方向にのみ傳播する場合であつた。其後1904年力學の大家ラム氏が應力函數を用ひて波が二次元に擴がる場合を研究し、而もレーレーとは異なる方法により、數學の困難さと優雅さとを示した。これより先、1882年同氏は直角座標を用ひる事によつて、球の自由振動を攻究したが、座標軸の性質上三節線以上の振動には極めて不適當な結果を殘した。最近1923年氣體力學の權威ジーンス氏が極座標を用ひ、重力迄も考慮に入れ、球上表面波の傳播に就て發表したが、計算は可なり近似的であり、重力の影響を除いても猶不正確を免れず、結局平面上の問題に歸して居る。

本問題は分つて二章より成り、第一章の研究は始めラム氏の文獻を知らず、ラム氏の方法を全く離れてレーレー卿に近き一方法を以てした。第二章中には、球面上に於ける勝手な波長のレーレー波につき極めて正確なる解を求めて傳播の狀況其他を示した。

此等の研究の結果によれば、二次元でも球面上でも、始めレーレー卿が考へたものと本質上何等異ならぬ傳播をするが、しかし其正確なる力學的機構や、又球面上では其が完全な彈性體である時、源から發した波が、源に對する赤道圓に到るまで次第に振幅を減じ、其から又次第に振幅を增し遂に對蹠點では、波動源に於けるものと同じ大さになる事などが示される。

地震時に當つてレーレー波が存在するか否かの議論はこの問題の關せぬ所であつて、要するに地球が完全な彈性體で、而も重力を持たぬ場合に存在する唯一の等週期性表面波である事には疑がなく、之が地球力學進歩への一階梯となるならば幸であらう。

著者は倘表面波の性質を表はすものとして現今最も信頼されつつある波即ち、彈性力學の泰斗ラブ氏によるラブ波を容易に球面上に擴張して見たが、其本質はやはり平面の時と類似をなすから論文には之を避けた。

For the study of the propagation of elastic surface waves the integration of the equations of motion is made to accord with prescribed boundary conditions of stresses or displacements. Lord Rayleigh[1] gave the integration of differential equations for the elastic surface waves propagating in one direction, and Lamb[2] extended the solution to the case in two dimensions. Lord Rayleigh obtained the solution by superposing two types of waves, one of them being dilational and the other distorsional. Lamb's method is based upon the application of stress functions. Regarding the elastic vibrations of a sphere, some problems in Cartesian coordinates were solved by Lamb[3], whose method is suited for one or two nodal vibrations. The same subject was attacked also by Jeans[4] by taking the effect of gravity into considerations ; the results obtained by him, however, are approximate even when the effect of gravity is neglected.

This investigation, which aims to obtain simple and correct methods of finding solutions of Rayleigh-waves in general, falls into two sections. First, a solution of the problem of the propagation of Rayleigh-waves in two dimensions has been found following Lord Rayleigh's method. Second, exact solutions concerning the propagation of surface waves having any wave length over an isotropic elastic sphere have been obtained by using spherical coordinates.

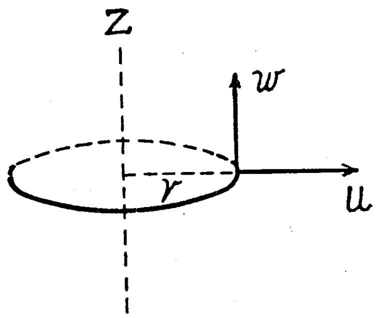
I. The Propagation of Rayleigh-Waves in Two Dimensions.

The equations of motion of elastic bodies in cylindrical coordinates, when the circumferential component of the motion is omitted, are expressed by

$$\frac{\partial^2\Delta}{\partial t^2}=\frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{\partial^2\Delta}{\partial z^2}\right) \qquad (1)$$

$$\frac{\partial^2\varpi}{\partial t^2}=\frac{\mu}{\rho}\left(\frac{\partial^2\varpi}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi}{\partial r}-\frac{\varpi}{r^2}+\frac{\partial^2\varpi}{\partial z^2}\right) \qquad (2)$$

$$\left.\begin{aligned}\Delta&=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}\\ 2\varpi&=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\end{aligned}\right\} \qquad (3)$$



1) London Math. Soc. Proc. vol. 17 (1887)
2) Phil. Trans. Roy. Soc. (Ser. A) vol. 203 (1904)
3) London Math. Soc. Proc. vol. 13 (1882)
4) Roy. Soc. Proc. (Ser. A) vol. 104 (1923)

u, v=radial and vertical components of displacement respectively at instant, t,

ρ=density of isotrophic solid,

λ, μ=Lame's elastic constants,

Writing $\Delta=\Delta_1 e^{-\alpha z+ipt}$ and $\varpi=\varpi_1 e^{-\beta z+ipt}$ in (1) and (2) respectively and solving the resulting equations, we have

$$\Delta_1=AJ_0(kr)+A'\,Y_0(kr) \qquad (4)$$

$$\varpi_1=BJ_1(kr)+B'\,Y_1(kr) \qquad (5)$$

in which $k^2=\alpha^2+\dfrac{\rho p^2}{\lambda+2\mu}=\beta^2+\dfrac{\rho p}{\mu}$

Displacement (u_1, w_1) answering to Δ_1 in (4) and satisfying $\omega=0$ is given by the forms,

$$u_1=\frac{k}{k^2-\alpha^2}\,e^{-\alpha z+ipt}\left\{AJ_1(kr)+A'Y_1(kr)\right\} \qquad (6)$$

$$w_1=\frac{\alpha}{k^2-\alpha^2}\,e^{-\alpha z+ipt}\left\{AJ_0(kr)+A'Y_0(kr)\right\} \qquad (7)$$

Displacement (u_2, w_2) derived from the value of ω_1 in (5) under the condition, $\Delta=0$, is expressed by

$$u_2=\frac{2\beta}{k^2-\beta^2}\,e^{-\beta z+ipt}\left\{BJ_1(kr)+B'Y_1(kr)\right\} \qquad (8)$$

$$w_2=\frac{2k}{k^2-\beta^2}\,e^{-\beta z+ipt}\left\{BJ_0(kr)+B'Y_0(kr)\right\} \qquad (9)$$

The surface $z=0$ being free from traction, the equations

$$\lambda\Delta+2\mu\frac{\partial w}{\partial z}=0 \qquad (10)$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}=0 \qquad (11)$$

in which $u=u_1+u_2$ and $w=w_1+w_2$ must hold on that surface.

These equations give

$$\left.\begin{aligned}\frac{B}{A}&=\frac{\lambda}{2\mu}\frac{k^2-a^2}{ak}-\frac{a}{k}\\ \frac{B}{A}&=-\frac{2\beta k}{\beta^2+k^2}\end{aligned}\right\}\tag{12}$$

Eliminating $\frac{B}{A}$, we obtain,

$$j^8-8j^6+24j^4-16(1+h^2)j^2+16h^2=0\tag{13}$$

where $h^2=\frac{\rho p^2}{(\lambda+2\mu)k^2}$ and $j^2=\frac{\rho p^2}{\mu k^2}$.

The equations (13) is of the same form as the equation derived by Lord Rayleigh to determine the velocity of propagation of plane surface waves.

The velocity of propagation at infinity is given by

$$\frac{2\pi/k}{2\pi/p}=C\sqrt{\frac{\mu}{\rho}}$$

where $C=0.9553$ for incompressible materials and 0.9194 for materials whose Poisson's ratio is $\frac{1}{4}$.

Near the origin the velocity is greater than that at infinity because of the large distances between zero points (or wave lengths) of Bessel's function for small arguments. Thus the velocity near the origin should be written.

$$\frac{2\pi/k\,(1-\epsilon)}{2\pi/p}=\frac{C}{1-\epsilon}\sqrt{\frac{\mu}{\rho}}$$

The ratio of $\frac{B}{A}$ being determined from (5), we get $B/A=-0.543$ for incompressible case and -0.682 for materials which fulfils Poisson's condition. The moduli of decay $(\alpha,\ \beta)$ in semi-infinite solid body are the same as those for ordinary Rayleigh-waves. For examples, $\alpha^2=k^2$, $\beta^2=0.08737\,k^2$ for incompressible body and $\alpha^2=0.7182\,k^2$, $\beta^2=0.1546\,k^2$ for a body whose Poisson's ratio is $\frac{1}{4}$.

The general expressions for displacement (taking $\lambda=\mu$) are

$$u=u_1+u_2=\frac{A}{k}\left\{\frac{e^{-.847kz}}{.847}-.682e^{-.394kz}\right\}\left\{J_1(kr)\sin pt-Y_1(kr)\cos pt\right\}$$

$$w=w_1+w_2=\frac{A}{k}\left\{ e^{-.847kz}-\frac{.682}{.394}\,e^{-.394kz}\right\}\left\{J_0(kr)\sin pt-Y_0'(kr)\cos pt\right\}$$

The displacement at infinity is of the forms,

$$u=\frac{A}{\sqrt{kr}}\,f(z)\sin\left(pt-kr+\frac{3}{4}\pi\right)$$

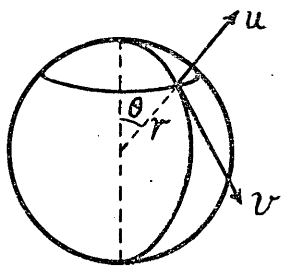
$$w=\frac{A}{\sqrt{kr}}\,F(z)\cos\left(pt-kr+\frac{3}{4}\pi\right)$$

II. The Propagation of Rayleigh-Waves on a Spherical Surface.

The equations of motion of elastic bodies in spherical coordinates, when the azimuthal component of the motion is omitted, are expressed by

$$\frac{\partial^2\Delta}{\partial t^2}=\frac{\lambda+2\mu}{\rho}\left\{\frac{\partial^2\Delta}{\partial r^2}+\frac{2}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\Delta}{\partial\theta}\cot\theta\right\}\qquad(1)$$

$$\frac{\partial^2\varpi}{\partial t^2}=\frac{\mu}{\rho}\left\{\frac{\partial^2\varpi}{\partial r^2}+\frac{2}{r}\frac{\partial\varpi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varpi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial\varpi}{\partial\theta}\cot\theta-\frac{\varpi}{r^2}(1+\cot^2\theta)\right\}\qquad(2)$$



where

$$\left.\begin{aligned}\Delta&=\frac{\partial u}{\partial r}+\frac{2u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{v}{r}\cot\theta\\ 2\varpi&=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\end{aligned}\right\}\qquad(3)$$

u, v.=radial and co-latitudinal components of displacement respectively at instant, t,

Writing $\Delta=\Delta_1 e^{ipt}$ and $\varpi=\varpi_1 e^{ipt}$ in (1) and (2) respectively and solving the resulting equations, we obtain,

$$\Delta_1=A\,\frac{J_m(hr)}{\sqrt{r}}\left\{P_n(\cos\theta)+A'\,Q_n(\cos\theta)\right\}\qquad(4)$$

$$\varpi_1 = B\,\frac{J_m(kr)}{\sqrt{r}}\left\{\frac{dP_n(\cos\theta)}{d\theta} + B'\,\frac{dQ_n(\cos\theta)}{d\theta}\right\} \qquad (5)$$

in which $h^2 = \dfrac{\rho p^2}{\lambda+2\mu}$, $k^2 = \dfrac{\rho p^2}{\mu}$, $m = n + \dfrac{1}{2}$ and n is a constant to denote the number of waves on the whole spherical surface.

Displacement $(u_1,\ v_1)$ answering to Δ_1 in (4) and satisfying $\varpi=0$ is given by

$$u_1 = -\frac{A}{h^2}\,\frac{d}{dr}\,\frac{J_m(hr)}{\sqrt{r}}\left\{P_n(\cos\theta) + A'\,Q_n(\cos\theta)\right\}$$

$$v_1 = -\frac{A}{h^2}\,\frac{J_m(hr)}{r^{\frac{3}{2}}}\left\{\frac{dP_n(\cos\theta)}{d\theta} + A'\,\frac{dQ_n(\cos\theta)}{d\theta}\right\}$$

Displacement $(u_2,\ v_2)$ derived from the value of ϖ_1 in (5) under the condition, $\Delta=0$, is expressed by

$$u_2 = -2B\,\frac{n(n+1)}{k^2}\,\frac{J_m(kr)}{r^{\frac{3}{2}}}\left\{P_n(\cos\theta) + B'\,Q_n(\cos\theta)\right\}$$

$$v_2 = -2B\,\frac{1}{k^2}\,\frac{1}{r}\,\frac{d}{dr}\left\{\sqrt{r}\,J_m(kr)\right\}\left\{\frac{dP_n(\text{ocs}\,\theta)}{d\theta} + B'\,\frac{dQ_n(\cos\theta)}{d\theta}\right\}$$

The surface $r=a$ being free from traction, the equations

$$\lambda\Delta + 2\mu\,\frac{\partial u}{\partial r} = 0 \quad \text{and} \quad \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial u}{\partial \theta} = 0 \qquad (6)$$

in which $u = u_1 + u_2$ and $v = v_1 + v_2$ must hold on that surface.

Eliminating A and B in applying the above boundary conditions. at $r=a$, we obtain the following equation to determine the velocity of propagation of surface waves.

$$\begin{aligned}
(\lambda+2\mu)\,h^2\,\frac{J_m(ha)}{\sqrt{a}}\Bigg[\left(-k^2 + \frac{2(n^2+n-1)}{a^2}\right)\frac{J_m(ka)}{\sqrt{a}} - \frac{2}{a}\left(\frac{d}{dr}\,\frac{J_m(kr)}{\sqrt{r}}\right)_{r=a}\Bigg]& \\
-2\mu\Bigg[\frac{n(n+1)}{a^2}\left(-k^2 + \frac{2(n-1)(n+2)}{a^2}\right)\frac{J_m(ha)\,J_m(ka)}{a}& \\
-\frac{2}{a}\left(-k^2 + \frac{(n-1)(n+2)}{a^2}\right)\frac{J_m(ka)}{\sqrt{a}}\left(\frac{d}{dr}\,\frac{J_m(hr)}{\sqrt{r}}\right)_{r=a}& \\
-\frac{2(n-1)(n+2)}{a^2}\left(\frac{d}{dr}\,\frac{J_m(kr)}{\sqrt{r}}\,\frac{d}{dr}\,\frac{J_m(kr)}{\sqrt{r}}\right)_{r=a}\Bigg] = 0& \qquad (7)
\end{aligned}$$

This equation is too complicated to study the nature of the motion, so that we must be contented with the following simplified form :—

$$\left(\frac{k^2}{f^2}-2\right)^4=16\left(1-\frac{h^2}{f^2}\right)\left(1-\frac{k^2}{f^2}\right) \tag{7'}$$

where $\dfrac{h^2}{f^2}=\dfrac{ha}{n}$ and $\dfrac{k}{f}=\dfrac{ka}{n}$.

In this equation the largeness of n is taken into consideration ; even when $n=100$, the wave length on the surface of the Earth is so large as 360 kilometers. Also, for simplicitie's sake, for the values of Bessel's functions in (7), the first term of the asymptotic expansion due to P. Debye, namely

$$J_n(x)=\frac{1}{\pi}\,e^{-x(\tau\cosh\tau-\sinh\tau)}\sum_{q=l}^{q=l}B_q(\tau)\frac{\Gamma\left(q+\frac{1}{2}\right)}{\left(\frac{x}{2}\sinh\tau\right)^{q+\frac{1}{2}}},$$

in which $\cosh\tau=\dfrac{n}{x}>1$, is taken.

The equation (7′) is of the same form as that derived by Lord Rayleigh to determine the velocity of propagation of plane surface waves.

The velocity of propagation excepting near the origin, when n is large, is given by

$$\frac{2\pi/f}{2\pi/p}=C\sqrt{\frac{\mu}{\rho}}$$

where $C=0.9553$ for incompressible materials and 0.9194 for materials whose Poisson's ratio is $\frac{1}{4}$.

The value of B/A is obtained from one of the relations in (6). If the second relation is used,

$$\frac{B}{A}=\left\{-\frac{k^2}{h^2}\,\frac{\left[\dfrac{1}{r}\dfrac{d}{dr}\dfrac{J_n(hr)}{\sqrt{r}}-\dfrac{1}{r^2}\dfrac{J_n(hr)}{\sqrt{r}}\right]}{\left(-k^2+\dfrac{2(n^2+n-1)}{r^2}\right)\dfrac{J_n(kr)}{\sqrt{r}}-\dfrac{2}{r}\dfrac{d}{dr}\dfrac{J_n(kr)}{\sqrt{r}}}\right\}_{r=a} \tag{8}$$

Under a similar process by which we deduced (7′) from (7), we obtain, from (8)

$$\frac{B}{A}=\frac{\dfrac{-\sqrt{1-\left(\dfrac{h}{f}\right)^2}}{n\left[2\left(\dfrac{h}{k}\right)^2-\left(\dfrac{h}{f}\right)^2\right]}\left\{\dfrac{1-\dfrac{k^2}{f^2}}{1-\dfrac{h^2}{f^2}}\right\}^{\frac{1}{4}}}{\exp.\quad n\left[\log\left\{\dfrac{1+\sqrt{1-\left(\dfrac{h}{f}\right)^2}}{1+\sqrt{1-\left(\dfrac{k}{f}\right)^2}}\,\dfrac{k}{h}\right\}-\sqrt{1-\dfrac{h^2}{f^2}}-\sqrt{1-\dfrac{k^2}{f^2}}\right]}$$

The value of B/A being thus found, we can arrive at the general expressions for displacement, which can be written as follows :—

$$\left.\begin{aligned}
u&=A\left[\frac{1}{h^2}\frac{d}{dr}\frac{J_m(hr)}{\sqrt{r}}+2\left(\frac{B}{A}\right)\frac{n(n+1)}{k^2}\frac{J_m(kr)}{r^{\frac{3}{2}}}\right]\times\\
&\qquad\left[P_n(\cos\theta)\sin pt+D_n\,Q_n(\cos\theta)\cos pt\right]\\
v&=A\left[\frac{1}{h^2}\frac{J_m(kr)}{r^{\frac{3}{2}}}+\frac{2}{k^2}\left(\frac{B}{A}\right)\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,J_m(kr)\right\}\right]\times\\
&\qquad\left[\frac{dP_n(\cos\theta)}{d\theta}\sin pt+D_n\frac{dQ_n(\cos\theta)}{d\theta}\cos pt\right]
\end{aligned}\right\}\quad(9)$$

where D_n is a positive number, whose value is constant, for a given n and which is to be so adjusted that $P_n(\cos\theta)$ and $D_n\,Q_n(\cos\theta)$ may have enveloping curves in common, excepting near the origin and the antipode within a zenith distance corresponding to, say, $\frac{1}{4}$ wave-length. Evidently motions in these regions must be excluded from the consideration.

The expressions (9) clearly tell the fact that, while the displacements gradually decrease as the waves approach the equatorial circle, the displacements again increase as they proceed towards the antipode.

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