Propagation of Elastic Waves from an Elliptic or a Spheroidal Origin.

By

Katsutada SEZAWA.

Earthquake Research Institute.

楕圓形又は廻轉楕圓體形の源による彈性波の傳播

所員　妹澤克惟

楕圓形又は廻轉楕圓體形の源から彈性波が無窮遠の方向へ傳播する場合、如何なる特性が現はれるかは相當興味のある問題であるが、數學が困難な爲か、或は單に注意されなかつた爲か、研究が餘り見當らぬ様である。唯靜力學的研究はワンゲリン氏によつて幾等か試みられて居る。

本論文は三章より成り、第一章は楕圓形の源によるもの、第二章は長廻轉楕圓體形、第三章は短廻轉楕圓體形の源による彈性波の傳播の研究を夫々包含するが、其方法に就ての重要點は、曲座標の應用と、マテユー氏微分方程式の解答とに歸する事が出來る。其源の周圍の方向の勢力分布に就ては殊に數學的困難を伴ひ、第一章の部分は幸にマテユー氏積分方程式を利用し得るが、第二章第三章の縦波に關する部分は、僅に著者の創意に成る積分方程式によつて解答を求める事が出來、其横波の部分に到つては未だ正確なる解を得るに到らぬ。然しながら、波動の大體の數學的機構及び諸性質は相當詳細の點まで知るに充分である。

例へば源に於ては極的不對稱の振動が、源の長さの數倍の遠方では、源が恰も一點であるかの如き波動をなす事、又源に於ては種々の傳播速度可能なるに拘らず、遠方では一定の速度となるが如きは著しい事實であらう。前者は地震の觀測に多少暗示を與へる事にならうし、又又後者は彈性波の傳播と構造體の振動理論とに對して共に好都合を與へる事にならう。其他方向による勢力の大小、又波動距離による振幅の減少等に一定の數量的結果を與へる事なども可なり意味がある筈である。

The propagation of elastic waves, started from an origin of an elliptical or a spheroidal form, under prescribed periodic tractions, is an interesting problem, because the propagation of the waves is affected by the shape of the origin. Yet such a problem has not been much studied for the reason that it involves mathematical difficulty and more probably has escaped the notice of investigators. So far as the author is aware, successful attempts hitherto made are for statical problems by Wangerin and a few others. Although the existing formulae in mathematics are not sufficient for solving the present subject, yet the author has succeeded to carry out a mathematical investigation more or less satisfactorily.

The present investigation consists of three sections : the first is on two-dimensional propagation of waves from an elliptical origin, the second on the transmission of waves from an origin of prolate spheroid and the third on that from an origin of a oblate spheroid. In the following analysis, special attention should be given to the vibratory motion at the origin, wave fronts at infinity and the distribution of energy of waves in all directions.

I. Elliptic Origin.

The equations of motion of elastic bodies in ellip'ic coordinates are expressed by

$$\left.\begin{aligned}(\lambda+2\mu)h_1\frac{\partial\Delta}{\partial\xi}-2\mu h_2\frac{\partial\varpi}{\partial\eta}&=\rho\frac{\partial^2u}{\partial t^2}\\(\lambda+2\mu)h_2\frac{\partial\Delta}{\partial\eta}+2\mu h_1\frac{\partial\varpi}{\partial\xi}&=\rho\frac{\partial^2v}{\partial t^2}\end{aligned}\right\}\qquad(1)$$

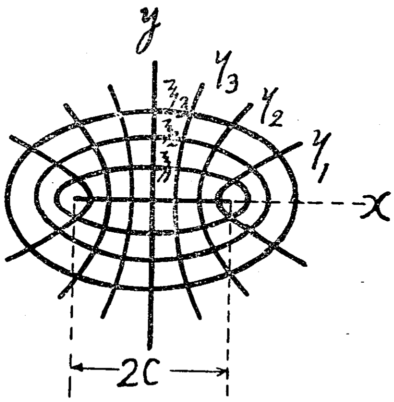
where

ξ, η=curvilinear coordinates related to Cartesian coordinates (x, y) in the form,

$$x+iy=c\cosh(\xi+i\eta)$$

or

$$\left.\begin{aligned}\frac{x^2}{c^2\cosh^2\xi}+\frac{y^2}{c^2\sinh^2\xi}&=1\\\frac{x^2}{c^2\cos^2\eta}-\frac{y^2}{c^2\sin^2\eta}&=1\end{aligned}\right\}$$



u, v=components of displacement referred to curvilinear coordinates, ξ and η, respectively at instant, t,

$$\Delta=h_1h_2\left[\frac{\partial}{\partial\xi}\left(\frac{u}{h_2}\right)+\frac{\partial}{\partial\eta}\left(\frac{v}{h_1}\right)\right]\qquad(2)$$

$$2\varpi = h_1 h_2 \left[\frac{\partial}{\partial \xi}\left(\frac{v}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u}{h_1}\right) \right] \tag{3}$$

$$\left. \begin{aligned} \frac{1}{h_1^2} &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = c^2 (\cosh^2 \xi - \cos^2 \eta) \\ \frac{1}{h_2^2} &= \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 = c^2 (\cosh^2 \xi - \cos^2 \eta) \end{aligned} \right\} \tag{4}$$

ρ=density of isotropic solid,

λ, μ=Lame's elastic constants.

From (1), (2), (3) and (4), we get

$$\left. \begin{aligned} \rho \frac{\partial^2 \Delta}{\partial t^2} &= \frac{\lambda + 2\mu}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2 \Delta}{\partial \xi^2} + \frac{\partial^2 \Delta}{\partial \eta^2} \right) \\ \rho \frac{\partial^2 \varpi}{\partial t^2} &= \frac{\mu}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2 \varpi}{\partial \xi^2} + \frac{\partial^2 \varpi}{\partial \eta^2} \right) \end{aligned} \right\} \tag{5}$$

Writing $\Delta = \Delta_1 e^{ipt}$, $\varpi = \varpi_1 e^{ipt}$, $\frac{\rho p^2}{\lambda + 2\mu} = h^2$ and $\frac{\rho p^2}{\mu} = k^2$, we obtain

$$\left. \begin{aligned} \frac{\partial^2 \Delta_1}{\partial \xi^2} + \frac{\partial^2 \Delta_1}{\partial \eta^2} + h^2 c^2 (\cosh^2 \xi - \cos^2 \eta) \Delta_1 &= 0 \\ \frac{\partial^2 \varpi_1}{\partial \xi^2} + \frac{\partial^2 \varpi_1}{\partial \eta^2} + k^2 c^2 (\cosh^2 \xi - \cos^2 \eta) \varpi_1 &= 0 \end{aligned} \right\} \tag{6}$$

From the symmetrical nature of the equations, it will be sufficient to study only one of the above equations.

Putting $\Delta_1 = \Xi(\xi)\,\mathrm{H}(\eta)$, the first of the equations in (6) will be transformed into Mathieu's differential equations, namely

$$\left. \begin{aligned} \frac{d^2 \Xi}{d\xi^2} + (h^2 c^2 \cosh^2 \xi - A_n)\, \Xi &= 0 \\ \frac{d^2 \mathrm{H}}{d\eta^2} + (h^2 c^2 \cos^2 \eta - A_n)\, \mathrm{H} &= 0 \end{aligned} \right\} \tag{7}$$

in which A_n is an arbitrary constant.

As the lower equation of (7) satisfies the integral equation of the type,

$$\mathrm{H}(\eta) = a \int_{-\pi}^{\pi} e^{ihc \cos \eta \cos \theta}\, \mathrm{H}(\theta)\, d\theta \tag{8}$$

in which α is any arbitrary constant, we can derive the value of $H(\eta)$ in an expanded form, by substituting some series in (8) and adjusting the accompanying constants.

When $A_n=n^2=0$, a convenient form for $H(\eta)$ is expressed by

$$H(\eta)=ce_0(\eta,\ q)=1+\sum_{r=1}^{\infty}\left\{\frac{2^{r+1}q^r}{r!\,r!}-\frac{2^{r+3}r(3r+4)}{(r+1)!\,(r+1)!}q^{r+2}+O(q^{r+4})\right\}\cos 2r\eta \tag{9}$$

where $q=-\dfrac{h^2c^2}{32}$

The domains of convergence of the series have not yet been determined The series converge for sufficiently small value of q.

For a prescribed distribution of $H(\eta)$, we may put,

$$H(\eta)=\sum_{n=0}^{\infty}\alpha_n ce_n(\eta,\ q)+\sum_{n=0}^{\infty}\beta_n se_n(\eta,\ q) \tag{10}$$

in which

$$n^2=A_n$$

$$\alpha_n=\int_{-\pi}^{\pi}H(\eta)\,ce_n(\eta,\ q)\,d\eta\Big/\int_{-\pi}^{\pi}\{ce_n(\eta,\ q)\}^2\,d\eta$$

$$\beta_n=\int_{-\pi}^{\pi}H(\eta)\,se_n(\eta,\ q)\,d\eta\Big/\int_{-\pi}^{\pi}\{se_n(\eta,\ q)\}^2\,d\eta$$

Another way of solving the lower equation of (7) is the application of the expanded form of $H(\eta)$, namely,

$$H(\eta)=\frac{1}{2}a_0+a_1\cos 2\eta+a_2\cos 4\eta+\ldots\ldots \tag{11}$$

where $a_0,\ a_1,\ a_2\ldots.$ are given, be means of (7), in the forms,

$$a_1=\frac{16}{h^2c^2}\left(\frac{n^2}{4}-\frac{h^2c^2}{8}\right)a_0$$

$$a_2=-\frac{16}{h^2c^2}\left(1-\frac{n^2}{4}+\frac{h^2c^2}{8}\right)a_1-a_0$$

$$\left.\begin{aligned} a_3 &= -\frac{16}{h^2 c^2}\left(4-\frac{n^2}{4}+\frac{h^2 c^2}{8}\right)a_2 - a_1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m+1} &= -\frac{16}{h^2 c^2}\left(m^2-\frac{n^2}{4}+\frac{h^2 c^2}{8}\right)a_m - a_{m-1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (12)$$

The important part of this investigation is to make clear the distribution of Ξ for all points of ξ. The difficulties in solving the upper equation of (7), oblige us to be contented with the solutions only at the origin and at infinity. Needless to say, the upper equations of (7) can be integrated by the same method as that characterised by the expansion in (11) ; but in this case, the series diverge. The infinity above described, however, may not be very far from the origin ; even a distance of say 2 c or 3 c from the centre of the origin may be regarded as infinity.

The upper equation of (7) for a small value of ξ can be written in the form,

$$\frac{d^2\Xi}{d\xi^2}+\left[(h^2 c^2-n^2)+h^2 c^2\left(\xi^2+\frac{\xi^4}{3}+\ldots\right)\right]\Xi=0 \qquad (13)$$

Putting Ξ in the expanded form,

$$\Xi=a_0+a_1\xi+a_1\xi^2+\ldots\ldots\ldots$$

and neglecting the small quantities of the second order, we obtain,

$$\Xi=e^{\pm i\sqrt{h^2c^2-n^2}\,\xi} \qquad (14)$$

Thus the solution for Δ is

$$\Delta=\sum_n \gamma_n e^{i(pt-\sqrt{h^2c^2-n^2}\,\xi)}\,\mathrm{H}_n(\eta) \qquad (15)$$

The velocity at **O** is given by

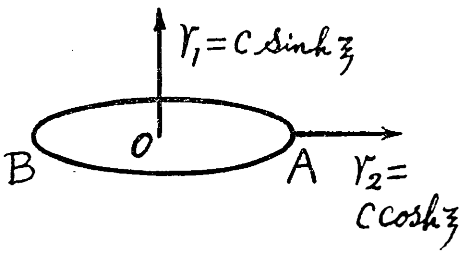
$$\frac{dr_1}{dt}=c\frac{d\sinh\xi}{dt}\doteqdot c\frac{d\xi}{dt}=\frac{1}{\sqrt{1-\frac{n^2}{h^2c^2}}}\sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$=\sqrt{\frac{\lambda+2\mu}{\rho}} \qquad \text{for} \quad n=0$$

The velocity at A is given by

$$\frac{dr_2}{dt}=c\frac{d\cosh\xi}{dt}$$

$$=c\sinh\xi\frac{d\xi}{dt}\doteqdot 0$$



Proceeding to the solution of Ξ for distant points, the upper equation of (7) can be transformed into,

$$(j^2+\xi_1^2)\frac{d^2\Xi}{d\xi_1^2}+\xi_1\frac{d\Xi}{d\xi_1}(\xi_1^2+M^2)\Xi=0 \tag{16}$$

in which

$$hc\sinh\xi=\xi_1 \qquad \frac{h^2c^2}{2}-n^2=a$$

$$M^2=a+\frac{j^2}{2} \qquad hc=j$$

Putting $\Xi=\frac{e^{i\xi_1}u}{\sqrt{\xi_1}}$ in (16), we get,

$$(\xi_1^2+j^2)\frac{d^2u}{d\xi_1^2}+\left[2i\,\overline{\xi_1^2+j^2}-\xi_1-\frac{2j^2}{\xi_1}\right]\frac{du}{d\xi_1}$$

$$-\left[\frac{2ij^2}{\xi_1}+\overline{j^2-M^2+1}-i\xi_1-\frac{2j^2}{\xi_1^2}\right]u=0 \tag{17}$$

Substitute

$$u=1+\frac{a_1}{\xi_1}+\frac{a_2}{\xi_1^2}+\ldots\ldots+\frac{a_n}{\xi_1^n}+\ldots\ldots$$

in (17) ; we then find that

$$a_1=\frac{i}{2}(j^2-M^2+1)$$

$$a_2=-\frac{j^2}{2}+\frac{1}{8}(j^2-M^2+1)(2-j^2+M^2)$$

and we have the recurrence formula for a_n s as follows :—

$$i(2n+1)\,a_{n+1}=\Big[-j^2+M^2-1+n(n+2)\Big]\,a_n-2inj^2\,a_{n-1}+n(n-1)\,j^2\,a_{n-2}$$

Thus the general solution for Δ is written in the form,

$$\Delta=CH_1(\eta)\frac{1}{\sqrt{\xi_1}}\Big[\cos(\xi_1-pt)\Big(1+\frac{a_2}{\xi_1^2}+\frac{a_4}{\xi_1^4}+\dots\Big)$$

$$+\sin(\xi_1-pt)\Big(\frac{a_1^1}{\xi_1}+\frac{a_3^1}{\xi_1^3}+\dots\Big)\Big] \qquad (18)$$

where $a_n^1=\frac{a_n}{i}$ and $H_1(\eta)$ is the modified form of $H(\eta)$ at infinity.

When ξ_1 is sufficiently large, Δ is given by

$$\Delta=CH_1(\eta)\frac{\cos\xi_1-pt)}{\sqrt{\xi_1}}$$

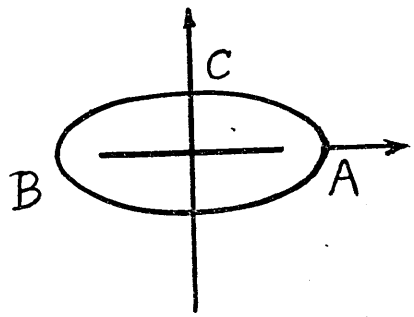
The velocities of propagation are given by

$$\frac{dr_1}{dt}=\frac{d(c\sinh\xi)}{dt}$$

$$=\frac{1}{h}\frac{d\xi_1}{dt}=\sqrt{\frac{\lambda+2\mu}{\rho}} \qquad \text{at } C$$

$$\frac{dr_2}{dt}=\frac{\sinh\xi}{\cosh\xi}\frac{d\xi_1}{dt}=\sqrt{\frac{\lambda+2\mu}{\rho}}$$

at A



both of which are of constant magnitude and independent of the value of n.

Proportions of energy in Δ-and ω-waves can be determined uniquely from the conditions of stress at the origin as follows :—

Displacement (u_1, v_1) answering to Δ in (6) and (2) and satisfying $\omega=0$ is given by

$$\left.\begin{aligned} u_1&=-\frac{1}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\frac{\partial\Xi}{\partial\xi}H \\ v_1&=-\frac{1}{h^2c\sqrt{\cosh^2\xi-\cos^2\eta}}\Xi\frac{\partial H}{\partial\eta} \end{aligned}\right\} \qquad (19)$$

Displacement (u_2, v_2) derived from the value of ω in (6) and (3) with the condition $\Delta=0$ is expressed by

$$\left.\begin{aligned} u_2 &= -\frac{2}{k^2 c\sqrt{\cosh^2\xi-\cos^2\eta}}\,\Xi'\,\frac{\partial \mathrm{H}'}{\partial\eta} \\ v_2 &= -\frac{2}{k^2 c\sqrt{\cosh^2\xi-\cos^2\eta}}\,\frac{\partial \Xi'}{\partial\xi}\,\mathrm{H}' \end{aligned}\right\} \tag{20}$$

On the surface $\xi=\xi_0$, two equations

$$\lambda\Delta+2\mu\, h_1 h_2\frac{\partial}{\partial\xi_0}\left(\frac{u}{h_2}\right)=\text{normal stress.} \tag{21}$$

$$\mu\, h_1 h_2\left[\frac{\partial}{\partial\xi_0}\left(\frac{u}{h_2}\right)+\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right]=\text{tangential stress.} \tag{22}$$

in which $u=u_1+u_2$ and $v=v_1+v_2$ must hold.

Putting the values of Δ, u_1+u_2 and v_1+v_2 from (15), (19) and (20) in (21) and (22), we get the absolute magnitudes of the enegy of both kinds of waves.

II. Origin of Prolate Spheroid.

The equations of motion of elastic bodies in spheroidal coordinates are expressed by

$$\left.\begin{aligned} (\lambda+2\mu)\, h_1\frac{\partial\Delta}{\partial\xi}-2\mu\, h_2 h_3\frac{\partial}{\partial\eta}\left(\frac{\varpi}{h_3}\right) &= \rho\frac{\partial^2 u}{\partial t^2} \\ (\lambda+2\mu)\, h_2\frac{\partial\Delta}{\partial\eta}+2\mu\, h_1 h_3\frac{\partial}{\partial\xi}\left(\frac{\varpi}{h_3}\right) &= \rho\frac{\partial^2 v}{\partial t^2} \end{aligned}\right\} \tag{1}$$

where

ξ, η = curvilinear coordinates related with Cartesian coordinates (x, y, z) in the form,

$$z+i\sqrt{x^2+y^2}=c\cosh(\xi+i\eta)$$

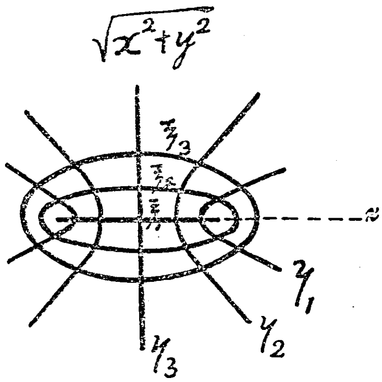
or

$$\frac{x^2+y^2}{c^2\sinh^2\xi}+\frac{z^2}{c^2\cosh^2\xi}=1,\qquad \frac{z^2}{c^2\cos^2\eta}-\frac{x^2+y^2}{c^2\sin^2\eta}=1$$

u, v = components of displacement referred to curvilinear coordinates, ξ and η, respectively at instant, t,

$$\Delta=h_1 h_2 h_3\left[\frac{\partial}{\partial\xi}\left(\frac{u}{h_2 h_3}\right)+\frac{\partial}{\partial\eta}\left(\frac{v}{h_1 h_3}\right)\right] \tag{2}$$

$$2\omega = h_1 h_2 \left[\frac{\partial}{\partial \xi}\left(\frac{v}{h_2}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{h_1}\right) \right] \qquad (3)$$



$$\left.\begin{aligned}
\frac{1}{h_1^2} &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \\
&= c^2 (\cosh^2 \xi - \cos^2 \eta) \\
\frac{1}{h_2^2} &= \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 \\
&= c^2 (\cosh^2 \xi - \cos^2 \eta) \\
\frac{1}{h_3^2} &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \\
&= c^2 \sinh^2 \xi \ \sin^2 \eta
\end{aligned}\right\} \qquad (4)$$

From (1), (2), (3) and (4), we get,

$$\left.\begin{aligned}
\rho \frac{\partial^2 \Delta}{\partial t^2} &= \frac{\lambda + 2\mu}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2 \Delta}{\partial \xi^2} + \coth \xi \frac{\partial \Delta}{\partial \xi} \right. \\
&\quad \left. + \frac{\partial^2 \Delta}{\partial \eta^2} + \cot \eta \frac{\partial \Delta}{\partial \eta} \right) \\
\rho \frac{\partial^2 \varpi}{\partial t^2} &= \frac{\mu}{c^2 (\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2 \varpi}{\partial \xi^2} + \coth \xi \frac{\partial \varpi}{\partial \xi} \right. \\
&\quad \left. - \frac{\varpi}{\sinh^2 \xi} + \frac{\partial^2 \varpi}{\partial \eta^2} + \cot \eta \frac{\partial \varpi}{\partial \eta} - \frac{\varpi}{\sin^2 \eta} \right)
\end{aligned}\right\} \qquad (5)$$

Writing $\Delta = \Delta_1 e^{ipt}$, $\varpi = \varpi_1 e^{ipt}$, $\dfrac{\rho p^2}{\lambda + 2\mu} = h^2$ and $\dfrac{\rho p^2}{\mu} = k^2$ in (5), we ob

$$\left.\begin{aligned}
&\frac{\partial^2 \Delta_1}{\partial \xi_1^2} + \coth \xi \frac{\partial \Delta_1}{\partial \xi} + \frac{\partial^2 \Delta_1}{\partial \eta^2} + \cot \eta \frac{\partial \Delta_1}{\partial \eta} \\
&\qquad\qquad + h^2 c^2 (\cosh^2 \xi - \cos^2 \eta) \Delta_1 = 0 \\
&\frac{\partial^2 \varpi_1}{\partial \xi^2} + \coth \xi \frac{\partial \varpi_1}{\partial \xi} - \frac{\varpi_1}{\sinh^2 \xi} + \frac{\partial^2 \varpi_1}{\partial \eta^2} + \cot \eta \frac{\partial \varpi_1}{\partial \eta} \\
&\qquad - \frac{\varpi_1}{\sin^2 \eta} + k^2 c^2 (\cosh^2 \xi - \cos^2 \eta) \varpi_1 = 0
\end{aligned}\right\} \qquad (6)$$

Putting $\Delta_1 = \Xi(\xi)\, \mathrm{H}(\eta)$ and $\omega_1 = \Xi'(\xi)\, \mathrm{H}'(\eta)$, the equations (6) are decomposed into,

$$\left.\begin{aligned}&\frac{d^2\Xi}{d\xi^2}+\coth\xi\frac{d\Xi}{d\xi}+(h^2c^2\cosh^2\xi-A_n)\,\Xi=0\\&\frac{d^2\mathrm{H}}{d\eta^2}+\cot\eta\frac{d\mathrm{H}}{d\eta}-(h^2c^2\cos^2\eta-A_n)\,\mathrm{H}=0\end{aligned}\right\}\qquad(7)$$

$$\left.\begin{aligned}&\frac{d^2\Xi'}{d\xi^2}+\coth\xi\frac{d\Xi'}{d\xi}+\Big(k^2c^2\cosh^2\xi\\&\qquad\qquad-\frac{1}{\sinh^2\xi}-A'_n\Big)\,\Xi'=0\\&\frac{d^2\mathrm{H}'}{d\eta^2}+\cot\eta\frac{d\mathrm{H}'}{d\eta}-\Big(k^2c^2\cos^2\eta\\&\qquad\qquad+\frac{1}{\sin^2\eta}-A'_n\Big)\,\mathrm{H}'=0\end{aligned}\right\}\qquad(8)$$

in which A_n and A'_n are arbitrary constants.

The author found that the solution of H was obtainable by using the integral equation of the following form with an arbitrary constant α :—

$$\mathrm{H}(\eta)=\alpha\int_{-\pi}^{\pi}e^{ihc\cos\eta\cos\theta}\sin\theta\,\mathrm{H}(\theta)\,d\theta \qquad (9)$$

which satisfies the differential equation,

$$\frac{d^2\mathrm{H}}{d\eta^2}+\cot\eta\frac{d\mathrm{H}}{d\eta}-(h^2c^2\cos^2\eta-A_n)\,\mathrm{H}=0 \qquad (10)$$

The solution of H′, however, has not yet been found.

The solutions of Ξ and Ξ′ are derived in the subsequent manners.

Taking the special case, $c=0$, the equations relating to H and H′ become

$$\frac{d^2\mathrm{H}}{d\eta^2}+\cot\eta\frac{d\mathrm{H}}{d\eta}+A_n\mathrm{H}=0$$

$$\frac{d^2\mathrm{H}'}{d\eta^2}+\cot\eta\frac{d\mathrm{H}'}{d\eta}+\left[A'_n-\frac{1}{\sin^2\eta}\right]\mathrm{H}'=0$$

so that, by putting $A_n=A'_n=n(n+1)$, we can write

$$\frac{d^2\Xi}{d\xi^2}+\coth\xi\frac{d\Xi}{d\xi}+\left[h^2c^2\cosh^2\xi-n(n+1)\right]\Xi=0$$

$$\left.\begin{aligned}\frac{d^2\Xi'}{d\xi^2}+\coth\xi\frac{d\Xi'}{d\xi}+\Big[h^2c^2\cosh^2\xi-n(n+1)\\ -\frac{1}{\sinh^2\xi}\Big]\Xi'=0\end{aligned}\right\}\qquad(11)$$

The motion at the origin is obtained by solving the equations

$$\left.\begin{aligned}&\frac{d^2\Xi}{d\xi^2}+\frac{1}{\xi}\frac{d\Xi}{d\xi}+\Big[h^2c^2-n(n+1)\Big]\Xi=0\\ &\frac{d^2\Xi'}{d\xi^2}+\frac{1}{\xi}\frac{d^2\Xi'}{d\xi}+\Big[h^2c^2-n(n+1)-\frac{1}{\xi^2}\Big]\Xi'=0\end{aligned}\right\}\qquad(12)$$

the integrals of which are

$$\left.\begin{aligned}\Xi=&AJ_0\left(hc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\\ &+A'Y_0\left(hc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\\ \Xi'=&BJ_1\left(kc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\\ &+B'Y_1\left(kc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right).\end{aligned}\right\}\qquad(13)$$

The general expressions for Δ and ϖ are

$$\left.\begin{aligned}\Delta=&A\Big\{J_0\left(hc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\sin pt\\ &-Y_0\left(hc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\cos pt\Big\}\mathrm{H}(\eta)\\ \varpi=&B\Big\{J_1\left(kc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\,\xi\right)\sin pt\\ &-Y_1\left(kc\sqrt{1-\frac{n(n+1)}{k^2c^2}}\,\xi\right)\cos pt\Big\}\mathrm{H}'(\eta).\end{aligned}\right\}\qquad(14)$$

The velocities at the middle and the end of a line-origin are

$$\left.\begin{aligned} \frac{dr_1}{dt} &\doteqdot \sqrt{\frac{\lambda+2\mu}{\rho}}\Big/\sqrt{1-\frac{n(n+1)}{h^2c^2}} \\ \frac{dr_2}{dt} &\doteqdot 0 \end{aligned}\right\} \text{ for } \Delta\text{-waves}$$

Going to the state of Ξ at infinity, the upper equation of (7) can be transformed into

$$(j^2+\xi_1^2)\frac{d^2\Xi}{d\xi_1^2}+\frac{2\xi_1^2+j^2}{\xi_1}\frac{d\Xi}{d\xi_1}+(\xi_1^2+M^2)\,\Xi=0 \tag{15}$$

in which

$$hc\sinh\xi=\xi_1 \qquad \frac{h^2c^2}{2}-n(n+1)\doteqdot a$$

$$hc=j \qquad M^2=a+\frac{j^2}{2}$$

Putting $\Xi=\frac{e^{i\xi_1}}{\xi_1}u$ in (15), we get,

$$(\xi_1^2+j^2)\frac{d^2u}{d\xi_1^2}+\left[2i(\xi_1^2+j^2)-\frac{j^2}{\xi_1}\right]\frac{du}{d\xi_1}$$
$$+\left[-\frac{ij^2}{\xi_1}-j^2+M^2+\frac{j^2}{\xi_1^2}\right]u=0 \tag{16}$$

Substitute

$$u=1+\frac{a_1}{\xi_1}+\frac{a_2}{\xi_1^2}+\dots+\frac{a_n}{\xi_1^n}+\dots$$

in (16); we then find that

$$a_1=\frac{i}{2}(j^2-M^2)$$

$$a_2=-\frac{i}{4}+\frac{1}{8}(j^2-M^2)(2+M^2-j^2)$$

and we have the recurrence formula for a_n s as follows :—

$$2i(n+1)a_{n+1}=[-j^2+M^2+n(n+1)]\,a_n$$
$$-(2n-1)ij^2a_{n-1}+(n-1)^2j^2a_{n-2}$$

Thus the general solution for Δ is written by

$$\Delta=CH_1(\eta)\frac{1}{\xi_1}\Big[\cos(\xi_1-pt)\Big(1+\frac{a_2}{\xi_1^2}+\frac{a_4}{\xi_1^4}+\ldots\Big)$$
$$+\sin(\xi_1-pt)\Big(\frac{a_1'}{\xi_1}+\frac{a_3'}{\xi_1^3}+\ldots\ldots\Big)\Big] \qquad (17)$$

wher $a_n'=\frac{a_n}{i}$ and $H_1(\eta)$ is the modified form of $H(\eta)$ at infinity.

In this case too the velocity of propagation at infinity becomes constant, namely $\sqrt{\frac{\lambda+2\mu}{\rho}}$, in spite of various velocities at the origin.

The asymptotic expansion for Ξ' is obtained by the same process. Rewriting the upper equation of (8),

$$\frac{d^2\Xi'}{d\xi^2}+\coth\xi\,\frac{d\Xi'}{d\xi}+\Big\{\Big(\frac{k^2c^2}{2}-n(n+1)\Big)$$
$$-\frac{h^2c^2}{2}\cosh 2\xi-\frac{1}{\sinh^2\xi}\Big\}\,\Xi'=0 \qquad (18)$$

and, by like manner as in Ξ, we have,

$$(\xi_1^2+j^2)\frac{d^2u}{d\xi_1^2}+\Big[2i(\xi_1^2+j^2)-\frac{j^2}{\xi_1}\Big]\frac{du}{d\xi_1}+\Big[-\frac{ij^2}{\xi_1}-j^2+M^2\Big]u=0$$

$$kc\sinh\xi=\xi_1 \qquad \frac{k^2c^2}{2}-n(n+1)=a$$

$$kc=j \qquad M^2=a+\frac{j^2}{2}$$

$$a_1=\frac{i}{2}(j^2-M^2)$$

$$a_2=-\frac{j^2}{4}+\frac{1}{8}(j^2-M^2)(2-j^2+M^2)$$

and

$$2i(n+1)a_{n+1}=[-j^2+M^2+n(n+1)]a_n$$
$$-(2n-1)ij^2a_{n-1}+n(n-2)j^2a_{n-2}$$

The general solution for ϖ is written in the form,

$$\varpi=DH_1'(\eta)\frac{1}{\xi_1}\Big[\cos(\xi_1-pt)\Big(1+\frac{a_2}{\xi_1^2}+\frac{a_4}{\xi_1^4}+\ldots\Big)$$
$$+\sin(\xi_1-pt)\Big(\frac{a_1'}{\xi_1}+\frac{a_3'}{\xi_1^3}+\ldots\ldots\Big)\Big] \qquad (19)$$

where $a_n' = \frac{a_n}{i}$ and $H_1'(\eta)$ is the modified form of $H'(\eta)$ at infinity.

The velocity of propagation at a great distance is constant being $\sqrt{\frac{\mu}{\rho}}$.

Proportions of energy in Δ- and ϖ-waves can be determined uniquely from the conditions of stress at the origin as follows :—

Displacement (u_1, v_1) answering to Δ in (2) and (6) and satisfying $\varpi=0$ is given by

$$\left.\begin{aligned} u_1 &= -\frac{1}{h^2 c\sqrt{\cosh^2\xi-\cos^2\eta}}\frac{\partial \Xi}{\partial \xi}\mathrm{H} \\ v_1 &= -\frac{1}{h^6 c\sqrt{\cosh^2\xi-\cos^2\eta}}\,\Xi\frac{\partial \mathrm{H}}{\partial \eta} \end{aligned}\right\} \tag{20}$$

Displacement (u_2, v_2) derived from the value of ϖ in (3) and (6) with the condition $\Delta=0$ is expressed by

$$\left.\begin{aligned} u_2 &= \frac{2}{k^2 c\sin\eta\sqrt{\cosh^2\xi-\cos^2\eta}}\,\Xi'\frac{\partial(\mathrm{H}'\sin\eta)}{\partial \eta} \\ v_2 &= -\frac{2}{k^2 c\sinh\xi\sqrt{\cosh^2\xi-\cos^2\eta}}\frac{\partial(\Xi'\sinh\xi)}{\partial \xi}\mathrm{H}' \end{aligned}\right\} \tag{21}$$

On the surface $\xi=\xi_0$, two equations

$$\lambda\Delta+2\mu\, h_1 h_2 h_3\frac{\partial}{\partial \xi_0}\left(\frac{u}{h_2 h_3}\right)=\text{normal stress} \tag{22}$$

$$\mu\, h_1 h_2\left[\frac{\partial}{\partial \xi_0}\left(\frac{v}{h^2}\right)+\frac{\partial}{\partial \eta}\left(\frac{u}{h_1}\right)\right]=\text{tangential stress} \tag{23}$$

in which $u=u_1+u_2$ and $v=v_1+v_2$ must hold.

Putting the values of Δ, u_1+u_2 and v_1+v_2 from (14), (20) and (21) in (22) and (23), we obtain the ratio of the energy in dilatational and distorsional waves, together with the absolute magnitudes of these waves.

III. Origin of Oblate Spheroid.

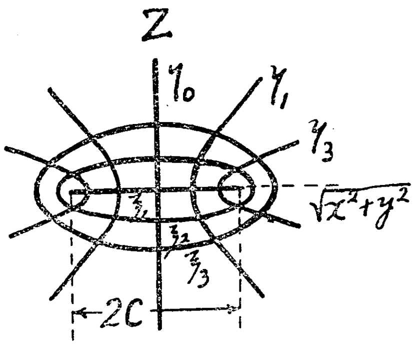
The equations of motion of elastic bodies in spheroidal coordinates are, as before, expressed by

$$\left.\begin{aligned}(\lambda+2\mu)h_1\frac{\partial\Delta}{\partial\xi}-2\mu h_2 h_3\frac{\partial}{\partial\eta}\left(\frac{\varpi}{h_3}\right)=\rho\frac{\partial^2 u}{dt^2}\\(\lambda+2\mu)h_2\frac{\partial\Delta}{\partial\eta}+2\mu h_1 h_3\frac{\partial}{\partial\xi}\left(\frac{\varpi}{h_3}\right)=\rho\frac{\partial^2 v}{dt^2}\end{aligned}\right\}\quad(1)$$

ξ, η=curvilinear coordinates related with Cartesian coordinates in the form,

$$z+i\sqrt{x^2+y^2}=c\sinh(\xi+i\eta)$$

or



$$\left.\begin{aligned}\frac{z^2}{c^2\sinh^2\xi}+\frac{x^2+y^2}{c^2\cosh^2\xi}=1\\\frac{x^2+y^2}{c^2\sin^2\eta}+\frac{z^2}{c^2\cos^2\eta}=1\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{1}{h_1^2}=\frac{1}{h_2^2}=c^2(\cosh^2\xi-\sin^2\eta)\\\frac{1}{h_3^2}=c^2\cosh^2\xi\ \sin^2\eta\end{aligned}\right\}\quad(2)$$

From (1), (2) and expressions of Δ and ϖ in terms of u, and v, we get,

$$\left.\begin{aligned}\rho\frac{\partial^2\Delta}{\partial t^2}=\frac{\lambda+2\mu}{c^2(\cosh^2\xi-\text{sih}^2\eta)}\left(\frac{\partial^2\Delta}{\partial\xi^2}+\tanh\xi\frac{\partial\Delta}{\partial\xi}\right.\\\left.+\frac{\partial^2\Delta}{\partial\eta^2}+\cot\eta\frac{\partial\Delta}{\partial\eta}\right)\\\rho\frac{\partial^2\varpi}{\partial t^2}=\frac{\mu}{c^2(\cosh^2\xi-\sin^2\eta)}\left(\frac{\partial^2\varpi}{\partial\xi^2}+\tanh\xi\frac{\partial\varpi}{\partial\xi}\right.\\\left.+\frac{\varpi}{\cosh^2\xi}+\frac{\partial^2\varpi}{\partial\eta^2}+\cot\eta\frac{\partial\varpi}{\partial\eta}-\frac{\varpi}{\sin^2\eta}\right)\end{aligned}\right\}\quad(3)$$

Writing $\Delta=\Delta_1 e^{ipt}$, $\varpi=\varpi_1 e^{ipt}$, $\frac{\rho p^2}{\lambda+2\mu}=h^2$ and $\frac{\rho p^2}{\mu}=k^2$ in (3), we obtain,

$$\begin{aligned}\frac{\partial^2\Delta_1}{\partial\xi^2}+\tanh\xi\frac{\partial\Delta_1}{\partial\xi}+\frac{\partial^2\Delta_1}{\partial\eta^2}+\cot\eta\frac{\partial\Delta_1}{\partial\eta}\\+h^2c^2(\cosh^2\xi-\sin^2\eta)\Delta_1=0\end{aligned}$$

$$\left.\begin{aligned}&\frac{\partial^2\varpi_1}{\partial\xi^2}+\tanh\xi\,\frac{\partial\varpi_1}{\partial\xi}+\frac{\varpi_1}{\cosh^2\xi}+\frac{\partial^2\varpi_1}{\partial\eta^2}+\cot\eta\,\frac{\partial\varpi_1}{\partial\eta}\\&\qquad-\frac{\varpi_1}{\sin^2\eta}+k^2c^2(\cosh^2\xi-\sin^2\eta)\,\varpi_1=0\end{aligned}\right\}\tag{4}$$

Putting $\Delta_1=\Xi(\xi)\,\mathrm{H}(\eta)$ and $\omega_1=\Xi'(\xi)\,\mathrm{H}'(\eta)$, the above equations are decomposed into

$$\left.\begin{aligned}&\frac{d^2\Xi}{d\xi^2}+\tanh\xi\,\frac{d\Xi}{d\xi}+(h^2c^2\cosh^2\xi-A_n)\,\Xi=0\\&\frac{d^2\mathrm{H}}{d\eta^2}+\cot\eta\frac{d\mathrm{H}}{d\eta}+(k^2c^2\sin^2\eta-A_n)\,\mathrm{H}=0\end{aligned}\right\}\tag{5}$$

$$\left.\begin{aligned}&\frac{d^2\Xi'}{d\xi^2}+\tanh\xi\,\frac{d\Xi'}{d\xi}+\Big(k^2c^2\cosh^2\xi\\&\qquad\qquad+\frac{1}{\cosh^2\xi}-A'_n\Big)\,\Xi'=0\\&\frac{d^2\mathrm{H}'}{d\eta^2}+\cot\eta\,\frac{d\mathrm{H}'}{d\eta}-\Big(k^2c^2\sin^2\eta\\&\qquad\qquad+\frac{1}{\sin^2\eta}-A'_n\Big)\,\mathrm{H}'=0\end{aligned}\right\}\tag{6}$$

in which A_n and A'_n are arbitrary constants.

H and H′ can be treated of in the same manner as described in the previous chapter for the integral equation (9); we shall here leave them out for simplicity's sake.

The solutions of Ξ and Ξ' are obtained also by similar processes as that given in II.

Thus the motion at the origin is given by

$$\left.\begin{aligned}&\Delta=\alpha\,{\sin\atop\cos}\left(pt-hc\sqrt{1-\frac{n(n+1)}{h^2c^2}}\;\xi\right)\mathrm{H}(\eta)\\&\omega=\beta\,{\cos\atop\sin}\left(pt-kc\sqrt{1-\frac{n(n+1)-1}{k^2c^2}}\;\xi\right)\mathrm{H}'(\eta)\end{aligned}\right\}\tag{7}$$

The velocities at the centre and the rim of a plate-origin are

$$\left.\begin{aligned} &\frac{dr_1}{dt}\sqrt{\frac{\lambda+2\mu}{\rho}}\Big/\sqrt{1-\frac{n(n+1)}{h^2c^2}} \\ &\frac{dr_2}{dt}=0 \end{aligned}\right\} \text{ for } \Delta\text{-waves}$$

$$\left.\begin{aligned} &\frac{dr_1}{dt}=\sqrt{\frac{\mu}{\rho}}\Big/\sqrt{1-\frac{n(n+1)-1}{h^2c^2}} \\ &\frac{dr_2}{dt}=0 \end{aligned}\right\} \text{ for } \varpi\text{-waves}$$

Going to the state of Ξ at infinity, the upper equation of (6) can be transformed into

$$(\xi_1^2-j^2)\frac{d^2\Xi}{d\xi_1^2}+\left(2\,\xi_1-\frac{j^2}{\xi_1}\right)\frac{d\Xi}{d\xi_1}+\left(a+\xi_1^2-\frac{j^2}{2}\right)\Xi=0 \qquad (8)$$

where

$$hc\cosh\xi=\xi_1 \qquad \frac{h^2c^2}{2}-n(n+1)=a$$

$$hc=j \qquad M'^2=a-\frac{j^2}{2}$$

Putting $\Xi=\frac{e^{i\xi_1}}{\xi_1}u$ in (8), we get,

$$(\xi_1^2-j^2)\frac{d^2u}{d\xi_1^2}+\left[2i(\xi_1^2-j^2)+\frac{j^2}{\xi_1}\right]\frac{du}{d\xi_1}$$

$$+\left[\frac{ij^2}{\xi_1}+j^2+M'^2-\frac{j^2}{\xi_1^2}\right]u=0 \qquad (9)$$

Substitute

$$u=1+\frac{a_1}{\xi_1}+\frac{a_2}{\xi_1^2}+\ldots+\frac{a_n}{\xi_1^n}+\ldots\ldots$$

in (9); we then find that

$$a_1=-\frac{i}{2}(j^2+M'^2)$$

$$a_2=\frac{j^2}{4}-\frac{1}{8}(j^2+M'^2+2)(j^2+M'^2)$$

and for the recurrence formula for a_n s, we have

$$2i\,(n+1)\,a_{n+1}=[j^2+M'^2+n\,(n+1)]\,a_n+(2n-1)\,ij^2\,a_{n-1}-(n-1)^2 j^2\,a_{n-2}$$

The general expression for Δ is given by

$$\Delta=c\mathrm{H}_1(\eta)\frac{1}{\xi_1}\Big[\cos(\xi_1-pt)\Big(1+\frac{a_2}{\xi_1^2}+\frac{a_4}{\xi_1^4}+\dots\Big)$$

$$+\sin(\xi_1-pt)\Big(\frac{a_1'}{\xi_1}+\frac{a_3'}{\xi_1^3}+\dots\Big) \qquad (10)$$

where $a_n'=\frac{a_n}{i}$ and $\mathrm{H}_1(\eta)$ is the modified from of $\mathrm{H}(\eta)$ at infinity.

The velocitv at infinity is constant, namely $\sqrt{\frac{\lambda+2\mu}{\rho}}$, in spite of the variable velocities at the origin.

The asymptotic expansion for Ξ' is obtained similarly. In the equation

$$\frac{d^2\Xi'}{d\eta^2}+\tanh\xi\,\frac{d\Xi'}{d\xi}+\Big\{\Big(\frac{k^2c^2}{2}-n(n+1)\Big)+\frac{h^2c^2}{2}\cosh 2\xi$$

$$+\frac{1}{\cosh^2\xi}\Big\}\,\Xi'=0 \qquad (11)$$

where

$$kc\cosh\xi=\xi_1 \qquad \frac{k^2c^2}{2}-n(n+1)=a$$

$$kc=j \qquad M'^2=a-\frac{j^2}{2}$$

Operate as in (9); we then find that

$$(\xi_1^2-j^2)\frac{d^2u}{d\xi_1^2}+\Big[2i\,(\xi_1^2-j^2)+\frac{j^2}{\xi_1}\Big]\frac{du}{d\xi_1}$$

$$+\Big[\frac{ij^2}{\xi_1}+j^2+M'^2\Big]u=0\,, \qquad (12)$$

$$u=1+\frac{a_1}{\xi_1}+\frac{a_2}{\xi_1^2}+\dots$$

$$a_1=-\frac{i}{2}(j^2+M'^2)$$

$$a_2=\frac{j^2}{4}-\frac{1}{8}(j^3+M'^2)(2+j^2+M'^2)$$

and for the recurrence formula for a_n s, we have

$$2i(n+1)a_{n+1}=[j^2+M'^2+n(n+1)]a_n+(2n-1)ij^2a_{n-1}-n(n-2)j^2a_{n-2}$$

The general expression for ϖ is given by

$$\varpi=\mathrm{DH}'_1(\eta)\frac{1}{\xi_1}\Big[\cos(\xi_1-pt)\Big(1+\frac{a_2}{\xi_1^2}+\frac{a_4}{\xi_1^4}+\dots\Big)$$
$$+\sin(\xi_1-pt)\Big(\frac{a_1'}{\xi_1}+\frac{a_3'}{\xi_1^3}+\dots\Big)\Big]$$

The velocity at infinity is constant, namely $\sqrt{\frac{\mu}{\rho}}$, in spite of variable velocities near the origin.

Proportions of energy in Δ- and ϖ-waves can be determined uniquely from the conditions of stress at the origin as follows :—

Displacement (u_1, v_1) answering to Δ in (4) and (2) of II and satisfying $\varpi=0$ is given by

$$\left.\begin{aligned} u_1&=-\frac{1}{h^2c^2\sqrt{\cosh^2\xi-\sin^2\eta}}\frac{\partial\Xi}{\partial\xi}\mathrm{H}\\ v_1&=-\frac{1}{h^2c^2\sqrt{\cosh^2\xi-\sin^2\eta}}\Xi\frac{\partial\mathrm{H}}{\partial\eta}\end{aligned}\right\}\qquad(13)$$

Displacement (u_2, v_2) derived from the value of ϖ in (4) and (3) of II with the condition $\Delta=0$ is expressed by

$$\left.\begin{aligned} u_2&=-\frac{2}{k^2c^2\sin\eta\sqrt{\cosh^2\xi-\sin^2\eta}}\Xi'\frac{\partial(\mathrm{H}'\sin\eta)}{\partial\eta}\\ v_2&=-\frac{2}{k^2c^2\cosh\xi\sqrt{\cosh^2\xi-\sin^2\eta}}\frac{\partial(\Xi'\cosh\xi)}{\partial\xi}\mathrm{H}'\end{aligned}\right\}\qquad(14)$$

On the surface $\xi=\xi_0$, two equations

$$\lambda\Delta+2\mu h_1h_2h_3\frac{\partial}{\partial\xi_0}\Big(\frac{u}{h_2h_3}\Big)=\text{normal stress.}\qquad(15)$$

$$\mu h_1h_2\Big[\frac{\partial}{\partial\xi_0}\Big(\frac{v}{h_1}\Big)+\frac{\partial}{\partial\eta}\Big(\frac{u}{h_1}\Big)\Big]=\text{tangential stress.}\qquad(16)$$

in which $u=u_1+u_2$ and $v=v_1+v_2$ must hold.

Putting the values of Δ, u_1+u_2 and v_1+v_2 from (7), (13) and (14) in (15) and (16), we obtain the ratio of the energy in dilatational and distorsional waves, together with the absolute magnitudes of these waves.

Concluding Remarks.

The proceeding results obtained by mathematical investigations shows an important fact, that the fronts of waves generated from an elliptical or spheroidal origin tend to become circular or spherical in progressing towards infinity, in spite of its polar unsymmetry in the neighbourhood of the origin. It also worths noticing that the velocity of propagation at infinity is uniform, notwithstanding of its various velocities at the origin. The fact that the energy of waves is transmitted along the systems of hyperbolas or hyperboloids of revolution, and thereby the amplitudes of wave fronts are modified, may be remarkable. Not less important is the fact that the decay of amplitudes progressing towards distant points is of the same nature as that of waves started from a single point.

In conclusion the author wishes to express his indebtness to Professor Nagaoka and Professor Suyehiro for valuable advices and suggestions.

December, 1926.