学位論文

Accurate Numerical Computation for Floquet Multipliers of Periodic Orbits of Ordinary Differential Equations

常微分方程式の周期軌道と Floquet 乗数の高精度数値計算

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2009年3月

Abstract

This thesis describes numerical stability analysis of periodic orbits of ordinary differential equations. Stability of the periodic orbits is determined by eigenvalues of matrix solutions of variational equations corresponding to the orbits. These eigenvalues are called Floquet multipliers. There are some conventional methods to compute periodic orbits and Floquet multipliers. However, it has been reported that these methods may produce inaccurate Floquet multipliers even if periodic orbits are accurate enough. It is because variational equations are solved as an initial value problem for which it is difficult to control numerical errors. Then this thesis proposes an iterative method to solve ordinary differential equations and the corresponding variational equations.

The basic ideas of the proposed method are to construct an iterative method for variational equations using a property of dynamical systems, and to compute Floquet multipliers using eigenvectors of the fundamental matrix of variational equations. The iterative method enables us to control errors of computed results of differential equations, and utilization of eigenvectors reduces round-off errors in the computation of Floquet multipliers.

Numerical examples for the nonlinear Mathieu equations and the FitzHugh-Nagumo equations showed that the proposed method can produce highly accurate results with practical computational costs. The errors of the computed results of differential equations were bounded by 10^{-6}, and Floquet multipliers were much more accurate than those computed by the conventional method.

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Chapter 1

Introduction

This thesis considers numerical stability analysis of periodic orbits of ordinary differential equations. We can examine stability of periodic orbits using Floquet multipliers which are defined by solutions of the corresponding variational equations. There are some conventional numerical methods to compute periodic orbits and its Floquet multipliers. However, those methods may generate erroneous results of Floquet multipliers, even if periodic orbits are computed with enough accuracy. In order to overcome this trouble, we propose an iterative method for the variational equations.

This section introduces background of stability analysis of dynamical systems, and shows the objectives of this work and the basic ideas of the proposed method.

Background of stability analysis of dynamical systems

In the wide range of fields of science and engineering, time variation of phenomena is modeled by ordinary differential equations such as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{x}) \quad \text{with} \quad \boldsymbol{x}(t, \boldsymbol{x}) = \boldsymbol{f}(t+T, \boldsymbol{x}) \quad \text{and} \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,, \quad (1.1)$$

where $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. For example, nerve axon [15, 32], population dynamics [42], influence of virus [27, 46], electrical

circuits [26, 44], plasma dynamics [34], astro dynamics [9, 10], etc [17, 23]. Generally, solutions of the systems (1.1) with nonlinear vector fields $\boldsymbol{f}$ cannot be solved analytically. Then we try to numerically catch approximate solutions of initial value problems. However, it is well known that nonlinear systems (1.1) can produce quite complex behaviours, namely bifurcation phenomena, with variation of parameters [7, 18, 25, 45]. In order to systematically understand complicated nonlinear phenomena, we consider qualitative changes of invariant sets such as equilibria, periodic orbits, and strange attractors in the state space. Invariant sets can be characterized using stability, stationary distribution, fractal dimensions, Lyapunov exponents, and topological entropy [36, 40]. Various numerical methods for computation of invariant sets [30, 43, 2, 3, 24, 11, 41] and numerical stability analysis of them [12, 14, 19, 22, 31, 37] have been developed.

This thesis considers numerical stability analysis of periodic orbits. Let $\boldsymbol{x}(t)$ denote a periodic orbit with the period T satisfying

$$\boldsymbol{x}(t+T) = \boldsymbol{x}(t) \quad \text{for} \quad \forall t\,. \tag{1.2}$$

It follows from linear stability analysis that a perturbation $\boldsymbol{v}(t)$ of the periodic orbit $\boldsymbol{x}(t)$ can be expressed as

$$\boldsymbol{v}(t) = \boldsymbol{X}_0(\tau)\,\boldsymbol{v}_0\,, \tag{1.3}$$

with the matrix solution $\boldsymbol{X}_0(\tau) \in \mathbb{R}^{N\times N}$ $(\tau = t - t_0)$ of the corresponding variational equations given by

$$\frac{\mathrm{d}\boldsymbol{X}_0}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_0+\tau,\ \boldsymbol{x}=\varphi^{\tau}_{t_0}(\boldsymbol{x}_0)} \boldsymbol{X}_0(\tau) \quad \text{with} \quad \boldsymbol{X}_0(0) = I\,, \tag{1.4}$$

where I denotes the identity matrix. Eigenvalues λ_j $(j = 1, 2, \cdots, N)$ of the matrix solution $\boldsymbol{X}_0(T)$ determine stability of the periodic orbit. These eigenvalues are called Floquet multipliers.

There are various commonly used numerical methods to compute periodic orbits of (1.1) and Floquet multipliers such as the finite difference method, the shooting method, and the collocation method [25, 8]. Also, some standard bifurcation analysis tools such as AUTO [12] are available. However,

it has been reported that these methods can produce incorrect Floquet multipliers under some conditions, even if the periodic orbits are approximated accurately [14,29,31].

Figures 1.1(a) and (b) show examples of computed results of the periodic orbits for the nonlinear Mathieu equations [18] defined by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -(1+p\cos t)\,\sin x, \end{aligned} \tag{1.5}$$

using one of the conventional methods, the multiple shooting method summarized in Section 2.2. Figure 1.1(a) shows the orbits in the state space and Figure 1.1(b) shows the time series for the parameters $p = 2$, 20, and 40, respectively. Figure 1.2(a) shows the Floquet multipliers λ_1 and λ_2 computed with the conventional method. We can find that the smaller Floquet multiplier λ_2 is computed incorrectly for the parameter $p > 20$, even if the periodic orbits are computed accurately as shown in Figures 1.1(a) and (b). Note that the ratio of the Floquet multipliers $\frac{|\lambda_1|}{|\lambda_2|}$ ($|\lambda_1| > |\lambda_2|$) drastically increases with p for $p < 20$. Such systems with large ratio of the Floquet multipliers are called 'stiff'. It is known that it is not straightforward to numerically obtain solutions of 'stiff' equations with enough accuracy, even if systems are theoretically stable. In order to overcome this difficulty for 'stiff' equations, implicit methods such as BDF (Backward Differentiation Formulae) and implicit Runge-Kutta methods [6,16,20] are used in general. However, as shown later in Chapter 4, the implicit methods do not improve accuracy of the Floquet multipliers.

Also, algebraic computation of eigenvalues of such $\boldsymbol{X}_0(T)$ causes large round-off errors. Lust developed an improved algorithm to avoid the round-off errors [29], as shown in Section 2.3. Lust's method divides a periodic orbit into some sub-orbits, and computes the expanding or contraction rates of perturbations on each sub-orbit using the periodic Schur decomposition [4] of $\boldsymbol{X}_0(T)$. Then the Floquet multipliers are obtained as products of those rates. Lust's method produces accurate Floquet multipliers. However, his method cannot control errors of variational equations (1.4). It is because his method solves the variational equations (1.4) as an initial value problem,

similarly to the other conventional methods.

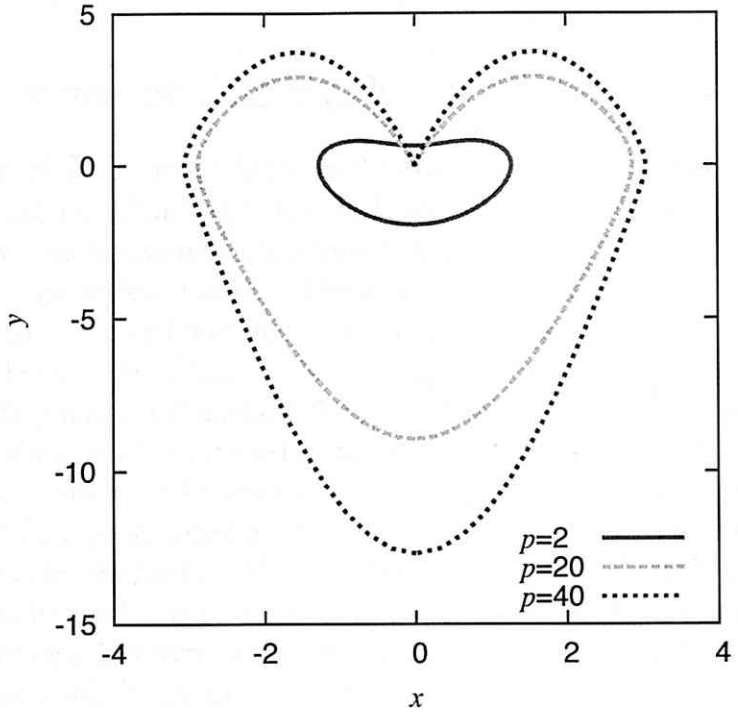
Objectives of this work

The objectives of this work are to develop a new method to compute solutions of the variational equations (1.4) and Floquet multipliers with enough accuracy, and to examine effectiveness of the proposed method using some numerical examples. For that, the basic ideas are to construct an iterative method for the variational equations (1.4) and to use eigenvectors of $\boldsymbol{X}_0(T)$ as initial condition of (1.4).

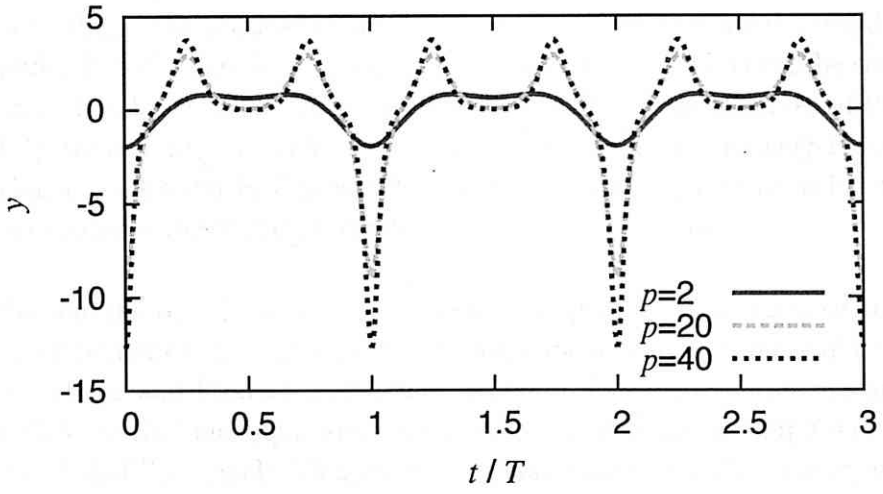
Existing numerical methods for initial value problems can produce inaccurate solutions of variational equations (1.4) and Floquet multipliers [33]. Then we propose an iterative method to solve (1.4) using a general property of solutions of ordinary differential equations, as shown in Chapter 3. This iterative method enables us to control errors of the solutions $\boldsymbol{X}_0(T)$ of the variational equations. On the other hand, when computed Floquet multipliers are incorrect as shown in Figure 1.2(a), some elements of $\boldsymbol{X}_0(T)$ are considerably large. In such a case, a standard computational method of eigenvalues using a transformation of $\boldsymbol{X}_0(T)$ gives incorrect results due to round-off errors, even if the computed $\boldsymbol{X}_0(T)$ is accurate. This work proposes an alternative method to obtain eigenvalues without using the standard methods. It is the key that the initial condition $\boldsymbol{X}_0(0)$ of (1.4) can be set to any non-singular matrix. Using eigenvectors of $\boldsymbol{X}_0(T)$ as initial conditions of (1.4), we can obtain a relation for eigenvalues of $\boldsymbol{X}_0(T)$, namely Floquet multipliers, as shown in Chapter 3. Computation of eigenvalues using this relation reduces the round-off errors.

The outline of this thesis is as follows: Chapter 2 summarizes the conventional numerical methods to compute periodic orbits of ordinary differential equations and Floquet multipliers. Chapter 3 describes a new iterative method for variational equations and an effective method to compute eigenvalues of $\boldsymbol{X}_0(T)$, namely Floquet multipliers. Chapter 4 shows computed results of some numerical examples using the proposed method, and compares those with the conventional methods. Finally, we discuss computed

results in Chapter 5, and conclude this thesis in Chapter 6.

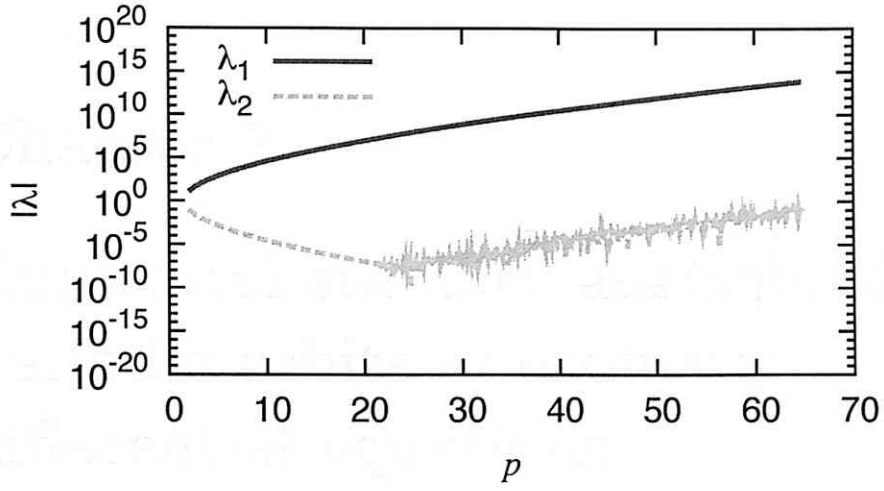


(a) Periodic orbits in the state space

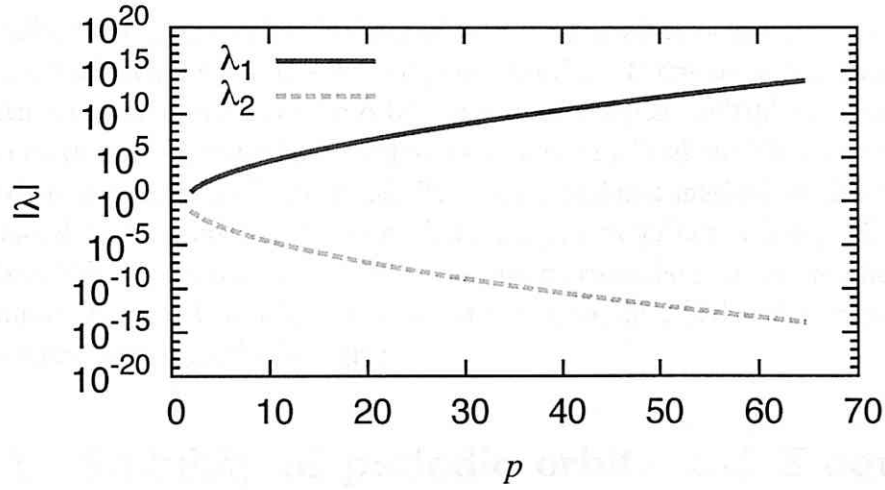


(b) Time series of y

Figure 1.1: Computed results of the periodic orbits of the Mathieu equations (1.5) with the multiple shooting method. T : the period. p : the parameter of the equations.



(a) conventional method



(b) proposed method

Figure 1.2: Computed results of the Floquet multipliers λ_j $(j = 1, 2)$ for the nonlinear Mathieu equations (1.5). p : the parameter of the equations.

Chapter 2

Numerical stability analysis of periodic orbits of ordinary differential equations

Stability of periodic orbits is determined by eigenvalues of matrix solutions of the variational equations for the periodic orbit. There are some standard methods to compute periodic orbits and their Floquet multipliers. For example, the shooting method, the finite difference method and the collocation method are often used [39, 25, 8]. The computational method in this work is based on the shooting method. This chapter describes stability of periodic orbits and Floquet multipliers, and summarizes the shooting method to compute them. Also a brief review on Lust's method [29] which improves the conventional methods is given.

2.1 Stability of periodic orbits and Floquet multipliers

This thesis considers nonlinear dynamical systems defined by ordinary differential equations such as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \text{with} \quad \boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{f}(t + T, \boldsymbol{x}), \tag{2.1}$$

where $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. Periodic orbits of the ordinary differential equations (2.1) are defined by solutions $\boldsymbol{x}(t)$ which satisfy

$$\boldsymbol{x}(t+T) = \boldsymbol{x}(t)\,. \tag{2.2}$$

The solution of the non-autonomous system (2.1) with an initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written in the form

$$\boldsymbol{x}(t) = \varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0)\,, \tag{2.3}$$

and the periodic solution $\boldsymbol{x}_0$ with the period T satisfies

$$\boldsymbol{x}(t_0+T) = \varphi_{t_0}^{T}(\boldsymbol{x}_0) = \boldsymbol{x}_0\,. \tag{2.4}$$

Substituting a perturbed solution of the periodic orbit, namely,

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t) + \boldsymbol{v}(t) = \varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0) + \boldsymbol{v}(t) \quad \text{with} \quad \boldsymbol{v}(t_0) = \boldsymbol{v}_0\,, \tag{2.5}$$

into $\boldsymbol{x}$ in (2.1), we get

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{f}(t,\boldsymbol{x}) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t,\,\boldsymbol{x}=\varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0)} \boldsymbol{v}(t) + O(\|\boldsymbol{v}\|^2)\,. \tag{2.6}$$

Assuming that $\|\boldsymbol{v}\|$ is sufficiently small, we can linearize (2.6) with respect to $\boldsymbol{v}(t)$ and get differential equations for time variation of the perturbation $\boldsymbol{v}(t)$ as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t,\,\boldsymbol{x}=\varphi_{t_0}^{t-t_0}(\boldsymbol{x}_0)} \boldsymbol{v}(t) \quad \text{with} \quad \boldsymbol{v}(t_0) = \boldsymbol{v}_0\,. \tag{2.7}$$

These equations are called the variational equations, and the solution $\boldsymbol{v}(t)$ of (2.7) can be expressed as

$$\boldsymbol{v}(t) = \boldsymbol{X}_0(\tau)\,\boldsymbol{v}_0\,, \tag{2.8}$$

with the fundamental matrix solution $\boldsymbol{X}_0(\tau) \in \mathbb{R}^{N\times N}$ ($\tau = t - t_0$) of (2.7). The matrix $\boldsymbol{X}_0(\tau)$ is a solution of the variational equations given by

$$\frac{\mathrm{d}\boldsymbol{X}_0}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_0+\tau,\ \boldsymbol{x}=\varphi_{t_0}^{\tau}(\boldsymbol{x}_0)} \boldsymbol{X}_0(\tau) \quad \text{with} \quad \boldsymbol{X}_0(0) = I\,, \tag{2.9}$$

where I denotes the identity matrix. Expansion or contraction of perturbations $\boldsymbol{v}(t_0 + \tau)$ is governed by $\boldsymbol{X}_0(\tau)$. Thus, we can examine stability of the periodic orbit using eigenvalues of $\boldsymbol{X}_0(T)$. These eigenvalues are called Floquet multipliers.

2.2 The shooting method for periodic orbits

Simple shooting method

The simple shooting method is one of standard methods to compute periodic orbits. Let $\boldsymbol{x}(t_0+\tau) = \varphi_{t_0}^{\tau}(\boldsymbol{x}_0)$ denote a periodic orbit of (2.1) with the period T, which satisfies (2.4), namely,

$$\boldsymbol{g}_0(\boldsymbol{x}_0) := \varphi_{t_0}^{T}(\boldsymbol{x}_0) - \boldsymbol{x}_0 = \boldsymbol{0}\,. \tag{2.10}$$

For non-autonomous systems (2.1), the period T is given. We can obtain $\varphi_{t_0}^{T}(\boldsymbol{x}_0)$ by computing (2.1) with the initial condition $\boldsymbol{x}_0$ using the Runge-Kutta method. The unknown variable is $\boldsymbol{x}_0$ at a certain time t_0. The simple shooting method solves $\boldsymbol{x}_0$ using an iterative method for $\boldsymbol{g}_0$ in (2.10) such as Newton's method.

Although this method is relatively simple, it is difficult to compute periodic orbits with large period T or large Floquet multipliers. It is because that errors for computation of $\varphi_{t_0}^{T}(\boldsymbol{x}_0)$ accumulate with time t, and convergence region for $\boldsymbol{x}_0$ becomes small. For such periodic orbits, the multiple shooting method below is used.

Multiple shooting method

The basic idea of the multiple shooting method [39,25,8] is the same as the simple shooting method, but this method divides a periodic orbit into $M+1$ sub-orbits such as

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}(t_{k+1}) = \varphi_{t_k}^{t_{k+1}-t_k}(\boldsymbol{x}_k) \quad \text{for} \quad k = 0, 1, \cdots, M\,, \tag{2.11}$$

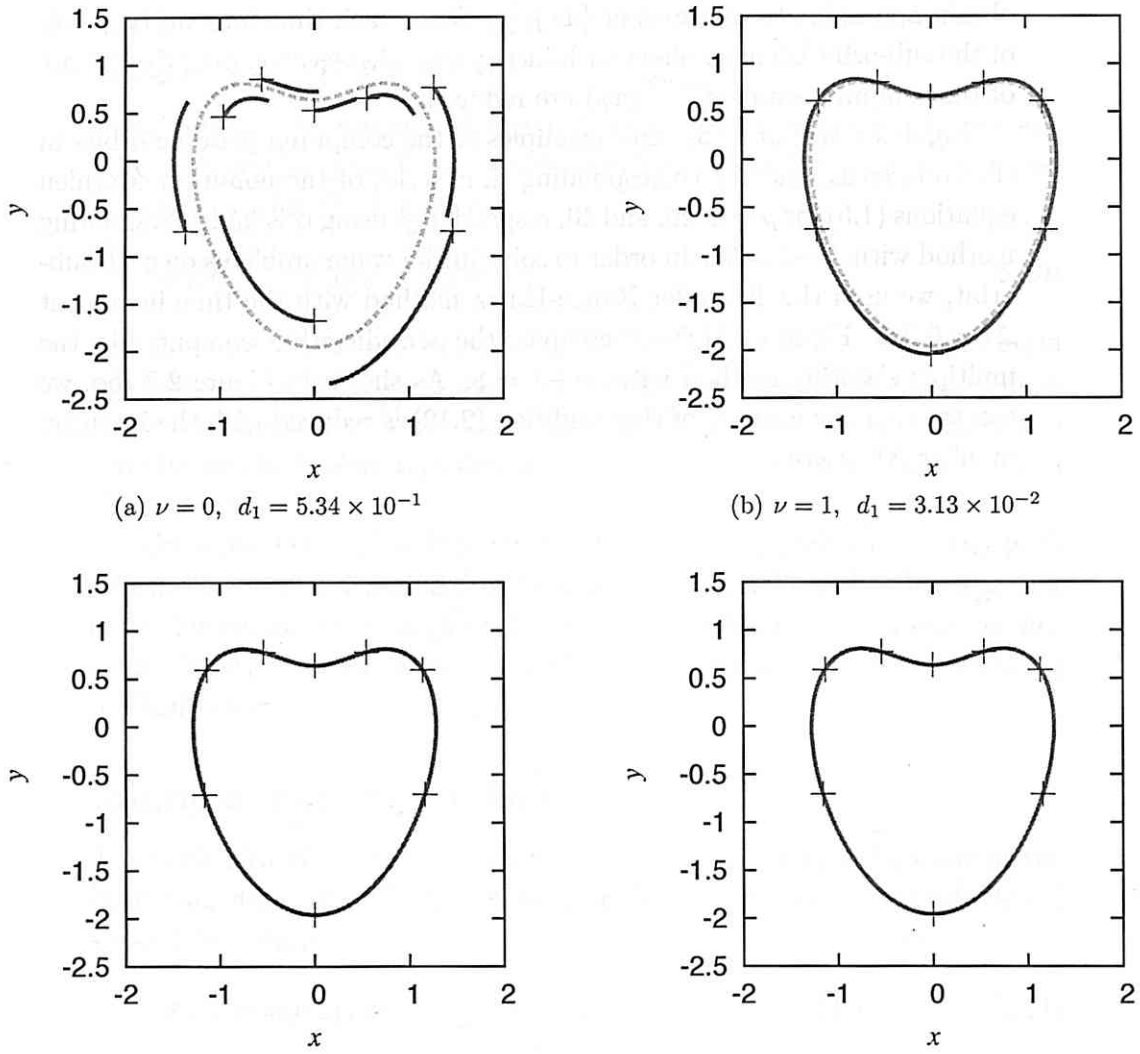
where $t_0 < t_1 < \cdots < t_M < t_{M+1} = t_0 + T$ and $\boldsymbol{x}_{M+1} = \boldsymbol{x}_0$. Since the period T is given for non-autonomous systems (2.1), each time t_k can be fixed. Equation (2.11) can be rewritten as

$$\boldsymbol{g}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}) := \varphi_{t_k}^{t_{k+1}-t_k}(\boldsymbol{x}_k) - \boldsymbol{x}_{k+1} = \boldsymbol{0}\,. \tag{2.12}$$

The solutions $\varphi_{t_k}^{t_{k_1}-t_k}(\boldsymbol{x}_k)$ in (2.12) are computed on each sub-orbit between the time sections t_k and t_{k+1} with the initial conditions $\boldsymbol{x}_k$ at t_k using the

Runge-Kutta method. Using an iterative method for $\boldsymbol{g}_k$ in (2.12), we can obtain approximate solutions of $\{\boldsymbol{x}_k\}_{k=0}^{M}$. Since each time interval $t_{k+1} - t_k$ of the sub-orbit becomes short with increase of M, we can expect that errors of the computation of $\varphi_{t_k}^{t_{k_1}-t_k}(\boldsymbol{x}_k)$ are reduced.

Figures 1.1(a) and (b) show examples of the computed periodic orbits in the state space and the corresponding time series of the nonlinear Mathieu equations (1.5) for $p = 2, 20$, and 40, respectively, using this multiple shooting method with $M+1 = 50$. In order to solve initial value problems on each sub-orbit, we used the 4th order Runge-Kutta method with the time increment $\Delta t = 0.025$. Figures 2.1(a)–(d) compare the periodic orbits computed by the multiple shooting method with $M+1 = 8$. As shown in Figure 2.2 too, we can see that the error d_1 of the condition (2.12) is reduced with the iteration number ν increases.



(a) $\nu = 0,\ d_1 = 5.34 \times 10^{-1}$

(b) $\nu = 1,\ d_1 = 3.13 \times 10^{-2}$

(c) $\nu = 2,\ d_1 = 1.92 \times 10^{-3}$

(d) $\nu = 3,\ d_1 = 5.43 \times 10^{-6}$

Figure 2.1: Examples of computed periodic orbits for the nonlinear Mathieu equations (1.5) using the multiple shooting method with $M + 1 = 8$. ν : the iteration number. d_1 : the error of the condition $\boldsymbol{g}_k$ in (2.12). See (3.10) for details. $+$: approximate solutions $\boldsymbol{x}_k^{(\nu)}$ of $\boldsymbol{x}_k$ $(k = 0, 1, \cdots, 7)$ after the ν-th iteration. Dotted line : the best approximate solution with $d_1 = 5.03 \times 10^{-16}$.

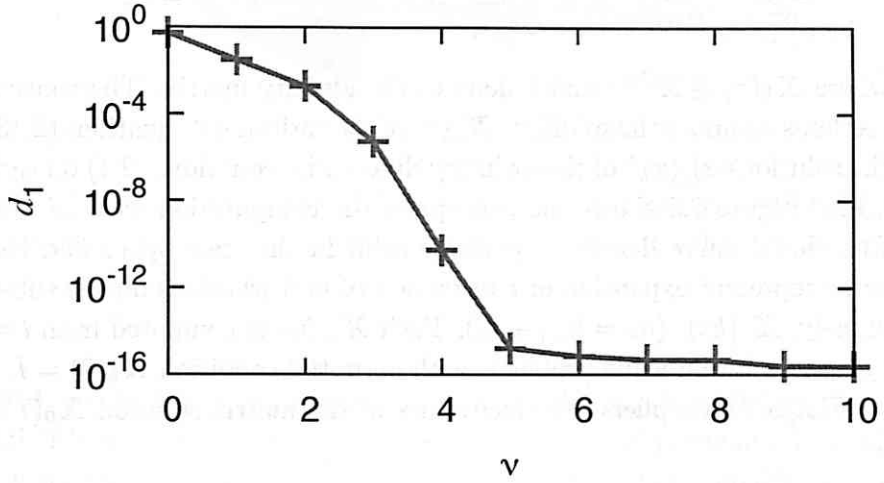


Figure 2.2: Errors d_1 of the condition $\boldsymbol{g}_k$ (2.12) for periodic orbits of the nonlinear Mathieu equations (1.5) with $p = 2$. ν : the iteration number. d_1 : See (3.10) for details.

2.3 Computational methods of Floquet multipliers

In order to examine stability of the periodic orbit with Floquet multipliers, the standard methods compute the solution $\boldsymbol{X}_0(T)$ of the variational equations (2.9) and its eigenvalues. Writing $\tau = t - t_k$ for $k = 0, 1, \cdots, M$, we can express the variational equation (2.9) for each sub-orbit as

$$\frac{\mathrm{d}\boldsymbol{X}_k}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_k+\tau,\ \boldsymbol{x}=\varphi^{\tau}_{t_k}(\boldsymbol{x}_k)} \boldsymbol{X}_k(\tau) \quad \text{with } \boldsymbol{X}_k(0) = I\,, \tag{2.13}$$

where $\boldsymbol{X}_k(\tau) \in \mathbb{R}^{N\times N}$ and I denotes the identity matrix. The conventional methods compute the solutions $\boldsymbol{X}_k(\tau)$ of the variational equations (2.13) with the solutions $\varphi^{\tau}_{t_k}(\boldsymbol{x}_k)$ of the ordinary differential equations (2.1) on each sub-orbit. Figure 2.3 shows the concept of the computation with $M + 1 = 3$. The closed curve denotes a periodic orbit in the state space and the gray areas represent expansion or contraction of perturbations on the sub-orbits, namely, $\boldsymbol{X}_k(h_k)$ $(h_k = t_{k+1} - t_k)$. Each $\boldsymbol{X}_k(h_k)$ is computed from $t = t_k$ to t_{k+1} as an initial value problems with an initial condition $\boldsymbol{X}_k(0) = I$.

Floquet multipliers are eigenvalues of the matrix solution $\boldsymbol{X}_0(T)$ given by

$$\boldsymbol{X}_0(T) = \boldsymbol{X}_M(h_M)\,\boldsymbol{X}_{M-1}(h_{M-1}) \cdots \boldsymbol{X}_0(h_0)\,. \tag{2.14}$$

We can numerically get $\boldsymbol{X}_k(h_k)$ using the Runge-Kutta method, and the Floquet multipliers using standard tools for linear algebra such as LAPACK [1]. This is one of the commonly used methods for stability analysis of periodic orbits using the multiple shooting method. Hereafter we call this method "Method 1".

Figure 2.4 shows computed results of the Floquet multipliers for the nonlinear Mathieu equations (1.5) by "Method 1". It is found that "Method 1" inaccurately computes the smaller Floquet multiplier λ_2 for the parameter $p > 20$. Also, we can find the ratio of the Floquet multipliers $\frac{|\lambda_1|}{|\lambda_2|}$ drastically increases with p for $p < 20$. It should be noted that the periodic orbit gets closer to the equilibrium $\boldsymbol{x}_*$ at the origin with increase of p, as shown in Figure 1.1(a). The point $\boldsymbol{x}(t)$ on the periodic orbit moves very slowly with time

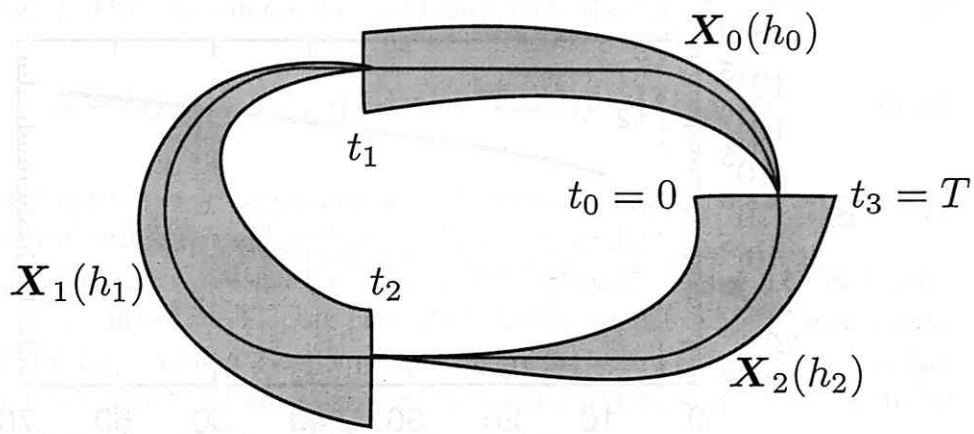


Figure 2.3: Closed curve : a periodic orbit in the state space. Gray areas : expansion or contraction of perturbations on each sub-orbit between t_k and t_{k+1}, namely, $\boldsymbol{X}_k(h_k)$.

near the equilibrium $\boldsymbol{x}_*$ and very fast away from $\boldsymbol{x}_*$. In such cases, it is not straightforward to numerically solve the initial value problems. In addition, we can find that some elements of the matrix $\boldsymbol{X}_0(T)$ are considerably large. Then round-off errors can produce erroneous results of eigenvalues of $\boldsymbol{X}_0(T)$.

As the example in Figure 2.4 illustrates, the conventional method may produce inaccurate numerical results of Floquet multipliers when periodic orbits get closer to equilibria, and ratios of Floquet multipliers $|\lambda_i|/|\lambda_j|$ ($|\lambda_i| > |\lambda_j|$) increase. We consider that the followings cause this problem:

(i) In general, it is difficult to control numerical errors of initial value problems of differential equations.

(ii) When some elements of $\boldsymbol{X}_0(T)$ are large, round-off errors of transformation of $\boldsymbol{X}_0(T)$ in computation of the eigenvalues can lead to erroneous results of Floquet multipliers.

For (ii), Lust proposed the following improved method [29].

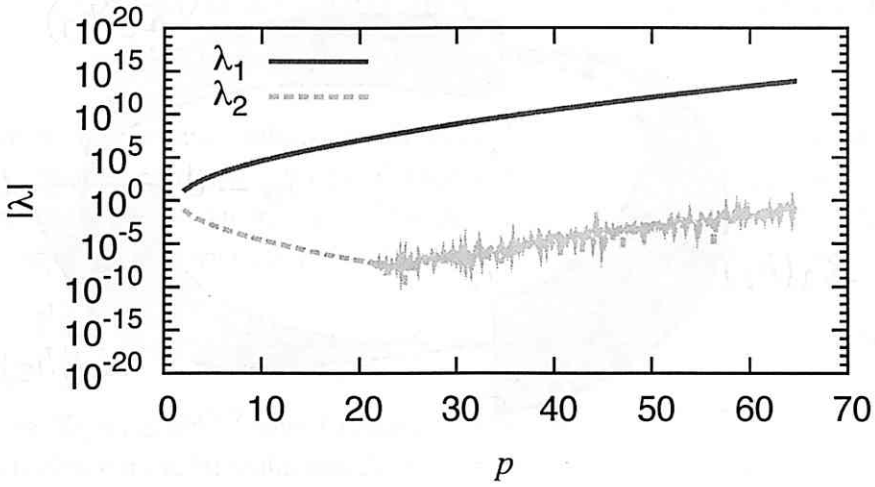


Figure 2.4: The Floquet multipliers λ_j $(j = 1, 2)$ computed with "Method 1" for the nonlinear Mathieu equations (1.5). p : the parameter of the equations.

Lust's method [29]

Lust developed an improved method [29] to obtain accurate Floquet multipliers. This method computes expanding or contracting rates of perturbations on each sub-orbit using transformation of $\boldsymbol{X}_k(h_k)$ by the periodic Schur decomposition [4] of $\boldsymbol{X}_0(T)$ such as

$$\begin{aligned}\Lambda &= Q_0^{\mathrm{T}} \boldsymbol{X}_0(T) Q_0 \\ &= Q_0^{\mathrm{T}} \boldsymbol{X}_M(h_M) \cdots \boldsymbol{X}_1(h_1) \boldsymbol{X}_0(h_0) Q_0 \\ &= Q_0^{\mathrm{T}} \boldsymbol{X}_M(h_M) Q_M \cdots Q_2^{\mathrm{T}} \boldsymbol{X}_1(h_1) Q_1 Q_1^{\mathrm{T}} \boldsymbol{X}_0(h_0) Q_0 \\ &= A_M \cdots A_1 A_0 , \end{aligned} \tag{2.15}$$

where $Q_k \in \mathbb{R}^{N \times N}$ are orthogonal matrices , and Λ and $A_k = Q_{k+1}^{\mathrm{T}} \boldsymbol{X}_k(h_k) Q_k$ are upper triangular matrices, respectively. Since the triangular matrix Λ is linearly conjugate to $\boldsymbol{X}_0(T)$, the eigenvalues of Λ are the same as those of

$\boldsymbol{X}_0(T)$. Thus, we can obtain the Floquet multipliers λ_j $(j = 1, 2, \cdots, N)$ as

$$\lambda_j = \Lambda(j,j) = A_M(j,j) \cdots A_1(j,j) A_0(j,j) = \prod_{k=0}^{M} A_k(j,j) \,. \tag{2.16}$$

Existence of the orthogonal matrices Q_k to transform $\boldsymbol{X}_k(h_k)$ is proved in [4] and construction of Q_k is described in Appendix A.

In this thesis, we call Lust's method "Method 2". Figure 2.5 shows computed results of the Floquet multipliers for the nonlinear Mathieu equations (1.5) using "Method 2". Comparing Figure 2.4 with Figure 2.5, we can find that "Method 2" produces accurate Floquet multipliers even for $p > 20$, for which the results of "Method 1" are erroneous.

In Lust's method, the variational equations are computed as an initial value problem, similarly to some conventional methods. If the solution $\boldsymbol{X}_0(T)$ of the initial value problem is accurately obtained, then Lust's method is very effective. However, it is not straightforward to control errors of computed results of initial value problems of differential equations. Then, in Chapter 3, we will propose a new iterative method for the variational equations, and an improved method to obtain Floquet multipliers with small round-off errors.

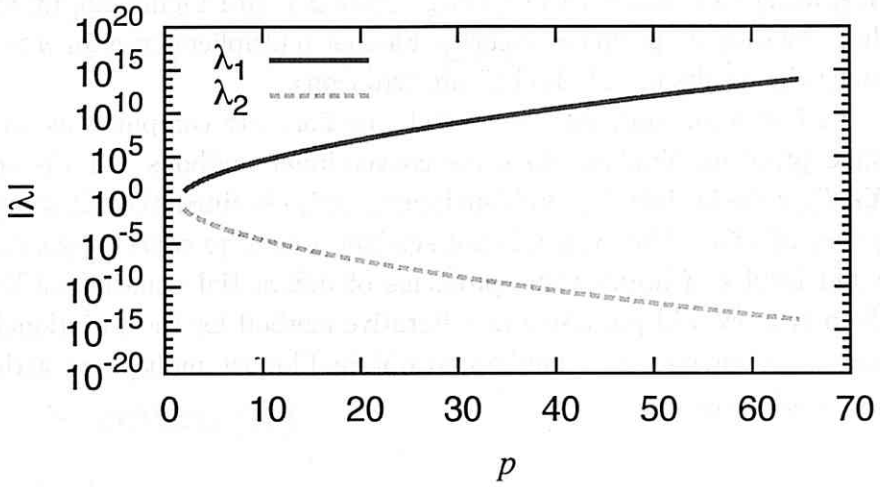


Figure 2.5: The Floquet multipliers λ_j $(j = 1, 2)$ computed with "Method 2" for the nonlinear Mathieu equations (1.5). p : the parameter of the equations.

Chapter 3

A new iterative method for periodic orbits and Floquet multipliers

In order to overcome troubles in the conventional method shown in Chapter 2, this thesis proposes an improved method based on the following two ideas:

(I) Construction of an iterative method to solve variational equations using a property of dynamical systems

(II) Reduction of round-off errors in the computation of Floquet multipliers using eigenvectors of $\boldsymbol{X}_0(T)$ as the initial condition for variational equations

Sections 3.1 and 3.2 describe the ideas (I) and (II), respectively. Section 3.4 shows some indices for accuracy of computed results.

3.1 An iterative method using a property of dynamical systems

The multiple shooting method in Section 2.2 computes periodic solutions $\{\boldsymbol{x}_k\}_{k=0}^{M}$ such that

$$\boldsymbol{x}_{k+1} = \varphi_{t_k}^{t_{k+1}-t_k}(\boldsymbol{x}_k) \quad \text{for} \quad k = 0, 1, \cdots, M\,, \tag{3.1}$$

and

$$\boldsymbol{x}_{M+1} = \boldsymbol{x}_0\,, \tag{3.2}$$

using an iterative method for these conditions. On the other hand, solutions $\boldsymbol{X}_k(h_k)$ $(h_k = t_{k+1} - t_k)$ of the variational equations (2.13) are solved as initial value problems on each sub-orbit between t_k and t_{k+1}. In general, it is difficult to control errors of computed results using standard numerical methods for initial value problems of ordinary differential equations, such as the Runge-Kutta method. This section derives an iterative method for variational equations (2.13) using a property of solutions of differential equations.

First, consider ordinary differential equations (2.1) with an initial condition $\boldsymbol{x}(t_k) = \boldsymbol{x}_k$. The map $\varphi_{t_k}^{\tau}$ of a solution $\boldsymbol{x}(t) = \varphi_{t_k}^{\tau}(\boldsymbol{x}_k)$ $(\tau = t - t_k)$ has the following property:

$$\varphi_{t_k}^{\tau+\sigma} = \varphi_{t_k+\tau}^{\sigma} \circ \varphi_{t_k}^{\tau} \quad \text{for} \quad \tau, \sigma \in \mathbb{R}\,. \tag{3.3}$$

Note that this property is also used as definition of dynamical systems. The property (3.3) gives the following condition:

$$\varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k) = \varphi_{t_k+h_k}^{-s_k} \circ \varphi_{t_k}^{h_k}(\boldsymbol{x}_k) = \varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1})\,, \tag{3.4}$$

with

$$s_k = \rho_k\, h_k\,, \tag{3.5}$$

where $\rho_k \in [0, 1]$ for $k = 0, 1, \cdots, M$. Then we get

$$\boldsymbol{g}_k(\boldsymbol{x}_k,\, \boldsymbol{x}_{k+1}) := \varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k) - \varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1}) = \boldsymbol{0}\,. \tag{3.6}$$

Figure 3.1 shows the meaning of this condition. The closed curve denotes a periodic orbit in the state space, and the solid curve represents a sub-orbit between t_k and t_{k+1}. Consider an intermediate point on the sub-orbit, and let s_k denote the time interval from the end at t_{k+1} to the point. Using the Runge-Kutta method, we can compute the intermediate point at $t = t_k + h_k - s_k$ as $\varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k)$ or $\varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1})$ with the initial conditions $\boldsymbol{x}_k$ or $\boldsymbol{x}_{k+1}$, respectively. Since these two solutions express the same point, the solutions must be equal to each other (3.4).

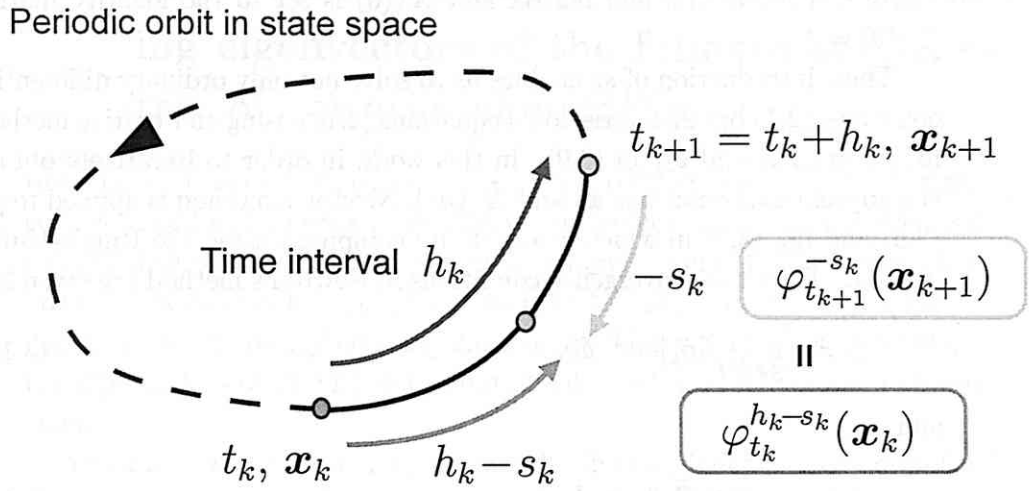


Figure 3.1: Image of the meaning of the condition (3.4) for solutions of ordinary differential equations. Closed curve : a periodic orbit in the state space. Solid curve : a sub-orbit between t_k and t_{k+1}. s_k : the time interval from the end at t_{k+1} to an intermediate point on the sub-orbit. $\varphi_{t_k}^{h_k - s_k}(\boldsymbol{x}_k)$, $\varphi_{t_{k+1}}^{-s_k}(\boldsymbol{x}_{k+1})$: representations of the intermediate point computed with the initial conditions $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$, respectively.

Next, consider variational equations (2.13). Solutions $\boldsymbol{X}_k(\tau) = \Psi_{t_k, \boldsymbol{x}_k}^{\tau}(\boldsymbol{X}_k(0))$ of (2.13) also have the same property as (3.3), namely

$$\Psi_{t_k, \boldsymbol{x}_k}^{\tau+\sigma} = \Psi_{t_k+\tau, \boldsymbol{x}(t_k+\tau)}^{\sigma} \circ \Psi_{t_k, \boldsymbol{x}_k}^{\tau} \quad \text{for} \quad \tau, \sigma \in \mathbb{R}\,. \tag{3.7}$$

Similarly to (3.4), this property yields

$$\Psi_{t_k, \boldsymbol{x}_k}^{h_k - s_k}(\boldsymbol{X}_k(0)) = \Psi_{t_k+h_k, \boldsymbol{x}(t_k+h_k)}^{-s_k} \circ \Psi_{t_k, \boldsymbol{x}_k}^{h_k}(\boldsymbol{X}_k(0)) = \Psi_{t_{k+1}, \boldsymbol{x}_{k+1}}^{-s_k}(\boldsymbol{X}_k(h_k))\,, \tag{3.8}$$

and it follows that

$$\boldsymbol{Q}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \boldsymbol{X}_k(h_k)) = \Psi_{\boldsymbol{x}_k}^{h_k - s_k}(\boldsymbol{X}_k(0)) - \Psi_{t_{k+1}, \boldsymbol{x}_{k+1}}^{-s_k}(\boldsymbol{X}_k(h_k)) = O\,, \tag{3.9}$$

where O denotes the null matrix and $\boldsymbol{X}_k(0)$ is set to the identity matrix $\boldsymbol{X}_k(0) = I$.

Thus, introduction of s_k enables us to solve not only ordinary differential equations (2.1) but also variational equations (2.13) using an iterative method for $\boldsymbol{g}_k$ in (3.6) and $\boldsymbol{Q}_k$ in (3.9). In this work, in order to iteratively obtain the approximate solutions $\tilde{\boldsymbol{x}}_k$ and $\tilde{\boldsymbol{X}}_k(h_k)$, Newton's method is applied to $\boldsymbol{g}_k$ (3.6) and $\boldsymbol{Q}_k$ (3.9) in which φ and Ψ are computed using the Runge-Kutta method. Then the convergence conditions in Newton's method are given by

$$d_1 := \max_{0 \le k \le M} \|\tilde{\boldsymbol{g}}_k\| \ < \ \delta_1 \,, \tag{3.10}$$

and

$$d_2 := \max_{0 \le k \le M} \|\tilde{\boldsymbol{Q}}_k\| \ < \ \delta_2 \,, \tag{3.11}$$

with sufficiently small $\delta_1, \delta_2 > 0$ and

$$\|\tilde{\boldsymbol{Q}}_k\| = \|(\tilde{\boldsymbol{q}}_1 \, \tilde{\boldsymbol{q}}_2 \cdots \tilde{\boldsymbol{q}}_N)\| \ := \ \max_{1 \le j \le N} \|\tilde{\boldsymbol{q}}_j\| \,, \tag{3.12}$$

where $\tilde{\boldsymbol{q}}_j$ denotes the j-th column vector of the matrix $\tilde{\boldsymbol{Q}}_k$. Using the threshold δ, this iterative method enables us to control accuracy for computed results of the ordinary differential equations and the variational equations.

3.2 Improvement of the proposed method using eigenvectors of the fundamental matrix of variational equations

In Section 3.1, in order to control errors of the computed results of variational equations (2.13), we derived an iterative method for (2.13). This section considers reduction of round-off errors in the computation of Floquet multipliers. For that, we can try to suitably choose initial values of variational equations, which can be set to any non-singular matrix. This section shows that choosing eigenvectors of $\boldsymbol{X}_0(T)$ as the initial value helps us reduce the round-off errors.

Let $\boldsymbol{v}_{j,0}$ $(j = 1, 2, \cdots, N)$ denote the linearly independent normalized eigenvectors corresponding to the j-th Floquet multipliers $\lambda_j \in \mathbb{C}$, namely,

$$\lambda_j \boldsymbol{v}_{j,0} = \boldsymbol{X}_0(T)\, \boldsymbol{v}_{j,0}\,. \tag{3.13}$$

Consider a solution $\boldsymbol{v}_j(t) = \psi^{t-t_0}_{t_0,\boldsymbol{x}_0}(\boldsymbol{v}_{j,0})$ of (2.7) with the initial value $\boldsymbol{v}_{j,0}$. Write solutions $\boldsymbol{v}_j(t)$ at $t = t_k$ as $\boldsymbol{v}_{j,k} := \boldsymbol{v}_j(t_k) = \psi^{t_k-t_0}_{t_0,\boldsymbol{x}_0}(\boldsymbol{v}_{j,0})$ for $k = 0, 1, \cdots, M+1$. Then $\boldsymbol{v}_{j,k-1}$ and $\boldsymbol{v}_{j,k}$ are related by

$$\boldsymbol{v}_{j,k} = \psi^{h_{k-1}}_{t_{k-1},\boldsymbol{x}_{k-1}}(\boldsymbol{v}_{j,k-1}) = \boldsymbol{X}_{k-1}(h_{k-1})\boldsymbol{v}_{j,k-1}\,, \tag{3.14}$$

where $h_k = t_{k+1} - t_k$ and $\boldsymbol{X}_k(h_k)$ is a solution of (2.13). It follows that

$$\boldsymbol{v}_{j,k} = \boldsymbol{X}_{k-1}(h_{k-1})\boldsymbol{X}_{k-2}(h_{k-2})\cdots\boldsymbol{X}_0(h_0). \tag{3.15}$$

Also, equations (2.14), (3.13), and (3.14) yield

$$\begin{aligned}
&\lambda_j \boldsymbol{v}_{j,k} \\
&= \lambda_j \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\,\boldsymbol{v}_{j,0} \\
&= \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\,(\lambda_j \boldsymbol{v}_{j,0}) \\
&= \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\boldsymbol{X}_0(T)\,\boldsymbol{v}_{j,0} \\
&= \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\boldsymbol{X}_M(h_M)\cdots\boldsymbol{X}_{k+1}(h_{k+1})\boldsymbol{X}_k(h_k) \\
&\quad \boldsymbol{X}_{k-1}(h_{k-1})\cdots\boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\,\boldsymbol{v}_{j,0} \\
&= \boldsymbol{X}_k(T)\,\boldsymbol{v}_{j,k}\,.
\end{aligned} \tag{3.16}$$

Here the following relation is used:

$$\boldsymbol{X}_k(T) = \boldsymbol{X}_{k-1}(h_{k-1}) \cdots \boldsymbol{X}_1(h_1)\boldsymbol{X}_0(h_0)\boldsymbol{X}_M(h_M) \cdots \boldsymbol{X}_{k+1}(h_{k+1})\boldsymbol{X}_k(h_k)\,. \tag{3.17}$$

Equation (3.16) means that the j-th Floquet multiplier λ_j is equal to the j-th eigenvalue of each $\boldsymbol{X}_k(T)$, and $\boldsymbol{v}_{j,k}$ is the eigenvector corresponding to λ_j.

Note that $\boldsymbol{v}_j(t_0+T) = \boldsymbol{X}_0(T)\boldsymbol{v}_{j,0}$. From equation (3.13), we get

$$\lambda_j\, \boldsymbol{v}_{j,0} = \boldsymbol{X}_0(T)\boldsymbol{v}_{j,0} = \boldsymbol{v}_j(t_0+T) = \boldsymbol{v}_{j,\,M+1} = \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{v}_{j,M})\,, \tag{3.18}$$

and

$$\lambda_j = \langle\, \boldsymbol{v}_{j,0},\ \boldsymbol{v}_j(t_0+T)\,\rangle = \langle\, \boldsymbol{v}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{v}_{j,M})\,\rangle\,, \tag{3.19}$$

where $\langle\cdot\,,\ \cdot\rangle$ denotes the inner product. This representation (3.19) of the Floquet multiplier enables us to compute the Floquet multiplier λ_j with relatively small round-off errors, even if some elements of the matrix $\boldsymbol{X}_0(T)$ are considerably large. It is because that (3.19) does not require transformation of $\boldsymbol{X}_0(T)$ which is commonly used in computation of eigenvalues of a matrix and may cause round-off errors.

In addition, in order to avoid overflow of computed results, we normalize $\boldsymbol{v}_j(t)$ at every $t = t_k$, similarly to the computation of Lyapunov exponents [40], as follows :

$$\boldsymbol{u}_{j,k} = \frac{\boldsymbol{v}_{j,k}}{\|\boldsymbol{v}_{j,k}\|}\,. \tag{3.20}$$

Then, $\boldsymbol{u}_{j,k+1}$ can be expressed as

$$\boldsymbol{u}_{j,k+1} = \frac{\boldsymbol{v}_{j,k+1}}{\|\boldsymbol{v}_{j,k+1}\|} = \frac{\boldsymbol{v}_{j,k+1}}{\|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{v}_{j,k})\|} = \frac{\boldsymbol{v}_{j,k+1}}{\|\boldsymbol{v}_{j,k}\|\|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|}\,. \tag{3.21}$$

From (3.21), we get

$$\|\boldsymbol{v}_{j,k+1}\| = \|\boldsymbol{v}_{j,k}\|\|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|\,. \tag{3.22}$$

Using (3.20)–(3.22), we can rewrite (3.19) in the form

$$\begin{aligned}
\lambda_j &= \left\langle \boldsymbol{u}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\|\boldsymbol{v}_{j,M}\|\boldsymbol{u}_{j,M}) \right\rangle \\
&= \left\langle \boldsymbol{u}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M}) \right\rangle \| \boldsymbol{v}_{j,M} \| \\
&= \left\langle \boldsymbol{u}_{j,0},\ \frac{\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})}{\|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\|} \right\rangle \|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\| \\
&\quad \| \psi^{h_{M-1}}_{t_{M-1},\boldsymbol{x}_{M-1}}(\boldsymbol{u}_{j,M-1}) \| \| \boldsymbol{v}_{j,M-1} \| \\
&= \left\langle \boldsymbol{u}_{j,0},\ \frac{\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})}{\|\psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\|} \right\rangle \prod_{k=0}^{M} \|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\| \,. \qquad (3.23)
\end{aligned}$$

In order to obtain accurate Floquet multipliers using (3.23), we utilize the idea of the method in Section 3.1, and iteratively compute the normalized eigenvectors $\boldsymbol{u}_{j,k}$ of $\boldsymbol{X}_k(T)$.

Assume that eigenvectors $\boldsymbol{u}_{1,k}, \boldsymbol{u}_{2,k}, \cdots$, and $\boldsymbol{u}_{N,k}$ are linearly independent. Then, any perturbations $\boldsymbol{v}(t_k) = (v_1, v_2, \cdots, v_N)^{\mathrm{T}} \in \mathbb{R}^N$ can be expressed as

$$\boldsymbol{v}(t_k) = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_{1,k} & \boldsymbol{u}_{2,k} & \cdots & \boldsymbol{u}_{N,k} \end{pmatrix} \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_N \end{pmatrix} = \hat{\boldsymbol{X}}_k(0)\, \hat{\boldsymbol{v}}(t_k)\,, \quad (3.24)$$

where

$$\hat{\boldsymbol{X}}_k(0) = \begin{pmatrix} \boldsymbol{u}_{1,k} & \boldsymbol{u}_{2,k} & \cdots & \boldsymbol{u}_{N,k} \end{pmatrix}\,, \qquad (3.25)$$

and $\hat{\boldsymbol{v}}(t_k) = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_N)^{\mathrm{T}} \in \mathbb{R}^N$. Write the solution at $t = t_k + \tau$ of (2.13) with the initial condition $\hat{\boldsymbol{X}}_k(0)$ as $\hat{\boldsymbol{X}}_k(\tau)$. Then, we get

$$\begin{aligned}
\boldsymbol{v}(t) &= \boldsymbol{X}_k(\tau)\, \boldsymbol{v}(t_k) \\
&= \Psi^{\tau}_{t_k,\boldsymbol{x}_k}(I)\hat{\boldsymbol{X}}_k(0)\, \hat{\boldsymbol{v}}(t_k) \\
&= \Psi^{\tau}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0))\, \hat{\boldsymbol{v}}(t_k) \\
&= \hat{\boldsymbol{X}}_k(\tau)\, \hat{\boldsymbol{v}}(t_k)\,. \qquad (3.26)
\end{aligned}$$

Thus, time variation of perturbations can be expressed as linear combination of column vectors of the non-singular matrix $\hat{\boldsymbol{X}}_k(\tau)$. Furthermore, equations (3.14), (3.18)–(3.21) yield

$$\begin{aligned}\hat{\boldsymbol{X}}_k(h_k) &= \Psi^{h_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0)) \\ &= \left(\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{1,k}) \quad \psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{2,k}) \quad \cdots \quad \psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{N,k})\right) \\ &= \left(\boldsymbol{u}_{1,k+1} \quad \boldsymbol{u}_{2,k+1} \quad \cdots \quad \boldsymbol{u}_{N,k+1}\right) D_k \\ &= \hat{\boldsymbol{X}}_{k+1}(0)\, D_k \quad \text{for} \quad k=0,1,\cdots,M\,, \end{aligned} \tag{3.27}$$

where $\hat{\boldsymbol{X}}_{M+1}(0) = \hat{\boldsymbol{X}}_0(0)$ and $D_k \in \mathbb{R}^{N\times N}$ are the diagonal matrices such that

$$D_k = \operatorname*{diag}_{1\le j\le N} \|\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})\|\,, \tag{3.28}$$

for $k = 0, 1, \cdots, M-1$ and

$$D_M = \operatorname*{diag}_{1\le j\le N} \langle \boldsymbol{u}_{j,0},\ \psi^{h_M}_{t_M,\boldsymbol{x}_M}(\boldsymbol{u}_{j,M})\rangle\,. \tag{3.29}$$

Since $\boldsymbol{u}_{j,k}$ and $\psi^{h_k}_{t_k,\boldsymbol{x}_k}(\boldsymbol{u}_{j,k})$ are the j-th columns of $\hat{\boldsymbol{X}}_k(0)$ and $\Psi^{h_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0))$, respectively, D_k depends on $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$. Then, equation (3.9) can be rewritten as

$$\begin{aligned}&\boldsymbol{Q}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \hat{\boldsymbol{X}}_k(0), \hat{\boldsymbol{X}}_{k+1}(0)) \\ &\quad = \Psi^{h_k-s_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0)) - \Psi^{-s_k}_{t_{k+1},\boldsymbol{x}_{k+1}}(\hat{\boldsymbol{X}}_k(h_k)) \\ &\quad = \Psi^{h_k-s_k}_{t_k,\boldsymbol{x}_k}(\hat{\boldsymbol{X}}_k(0)) - \Psi^{-s_k}_{t_{k+1},\boldsymbol{x}_{k+1}}(\hat{\boldsymbol{X}}_{k+1}(0)\, D_k) \\ &\quad = O\,. \end{aligned} \tag{3.30}$$

Using an iterative method for (3.6) and (3.30), we can control errors of approximate solutions $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$, namely, the corresponding eigenvectors $\boldsymbol{u}_{j,k}$ with a threshold δ in (3.10) and (3.11). Also, Floquet multipliers are accurately computed with the approximate solutions of $\boldsymbol{u}_{j,k}$ and equation (3.23). We call this method "Method 3".

We should emphasize that equation (3.30) can be separated to each column $\boldsymbol{q}_{j,k}$ of $\boldsymbol{Q}_k$. That is, for each j, we can iteratively compute $\boldsymbol{u}_{j,k}$ and

$\boldsymbol{u}_{j,k+1}$ using

$$\begin{aligned}
&\boldsymbol{q}_{j,k}(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \boldsymbol{u}_{j,k}, \boldsymbol{u}_{j,k+1}) \\
&\quad = \psi^{h_k - s_k}_{t_k, \boldsymbol{x}_k}(\boldsymbol{u}_{j,k}) - \psi^{-s_k}_{t_{k+1}, \boldsymbol{x}_{k+1}}(D_k(j,j)\,\boldsymbol{u}_{j,k+1}) \\
&\quad = \psi^{h_k - s_k}_{t_k, \boldsymbol{x}_k}(\boldsymbol{u}_{j,k}) - D_k(j,j)\,\psi^{-s_k}_{t_{k+1}, \boldsymbol{x}_{k+1}}(\boldsymbol{u}_{j,k+1}) \\
&\quad = \boldsymbol{0}\,. \qquad (3.31)
\end{aligned}$$

Therefore, we can obtain each Floquet multiplier independently for each j. This leads to reduction of round-off errors in the computation of Floquet multipliers. In addition, even if eigenvectors $\{\boldsymbol{u}_{j,k}\}$ are not linearly independent, (3.31) enables us to obtain Floquet multipliers of which the total number is less than N.

Finally, we refer to computational costs of this proposed method for simplicity using $M+1 = 1$, namely, the simple shooting method. Since unknown variables are $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $\hat{\boldsymbol{X}}_0(0) \in \mathbb{R}^{N \times N}$, the number of the unknown scalar values is $(N + N^2)$. In order to reduce computational costs, in this thesis, we solve the unknown variables $\boldsymbol{x}_0$ and $\hat{\boldsymbol{X}}_0(0)$ separately using Newton's method for $\boldsymbol{g}_0$ (3.6) and for $\boldsymbol{Q}_0$ (3.9), respectively. Otherwise, the derivative $\frac{\partial \boldsymbol{Q}_0}{\partial \boldsymbol{x}_0}$ is required in Newton's method, then we must additionally compute N^3 dimensional solutions of differential equations as an initial value problem. Therefore, we separate the proposed method into two steps. First, we compute the periodic solutions $\boldsymbol{x}_k$ using Newton's method for $\boldsymbol{g}_k$(3.6). Next, we can get Floquet multipliers λ_j and $\hat{\boldsymbol{X}}_k(0)$ using Newton's method for $\boldsymbol{Q}_k$(3.30). Figure 3.2 shows the computation time for periodic orbits and Floquet multipliers of the Mathieu equations (4.1) using the "Method 3". We can see the computation time of the method is reduced using $\boldsymbol{g}_k$ and $\boldsymbol{Q}_k$ separately.

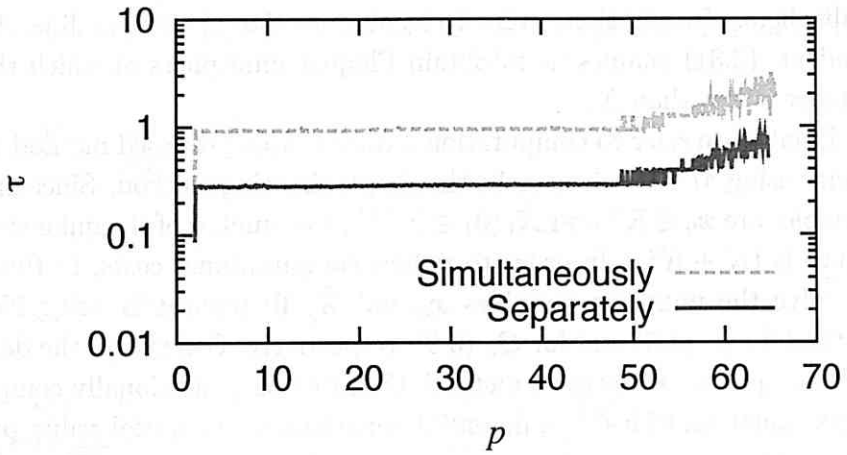


Figure 3.2: Computation time τ [seconds] of "Method 3 " for periodic orbits and Floquet multipliers for the nonlinear Mathieu equations (4.1) using both (3.6) and (3.30) simultaneously and separately, respectively. p : the parameter of the equations.

3.3 Application of the proposed method to autonomous systems

The proposed method "Method 3" can be applied to autonomous systems defined by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) \tag{3.32}$$

where $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. For autonomous systems, the time intervals $h_k = t_{k+1} - t_k \ \ (k = 0, 1, \cdots, M)$ are unknown, and the period T is expressed as

$$T = \sum_{k=0}^{M} h_k \,. \tag{3.33}$$

These intervals h_k must be computed simultaneously with $\boldsymbol{x}_k$ and $\hat{\boldsymbol{X}}_k(0)$ so that (3.6) and (3.30) are satisfied. Since the initial time t_0 can be set arbitrarily, we set $t_0 = 0$. Unknown variables are $\{h_k, \boldsymbol{x}_k, \hat{\boldsymbol{X}}_k(0)\}_{k=0}^{M}$. The number of the unknown scalar values is $(1+N+N\times N)\times(M+1)$, and is more than the number of equations (3.6) and (3.30). In order to close this problem and determine the unknown variables, we use an additional condition under which each $\boldsymbol{x}_k$ is located on Poincaré sections H_k transverse to the periodic orbit as shown in Figure 3.3. Consider a given approximate solution $\tilde{\boldsymbol{x}}_k$ of $\boldsymbol{x}_k$. A section orthogonal to the vector field $\boldsymbol{f}(\tilde{\boldsymbol{x}}_k)$ at $\tilde{\boldsymbol{x}}_k$ is transverse to the periodic orbit when $\tilde{\boldsymbol{x}}_k$ has enough accuracy. Thus, we can express the additional condition $\boldsymbol{x}_k \in H_k$ as

$$b_k(\boldsymbol{x}_k) = \langle \boldsymbol{x}_k - \tilde{\boldsymbol{x}}_k \,,\ \boldsymbol{f}(\tilde{\boldsymbol{x}}_k) \rangle = 0 \,. \tag{3.34}$$

Using Newton's method for (3.6) and (3.30) with (3.34), we can iteratively obtain approximate solutions of $\{h_k, \boldsymbol{x}_k, \hat{\boldsymbol{X}}_k(0)\}_{k=0}^{M}$, and from (3.33) we can determine t_k and the period T.

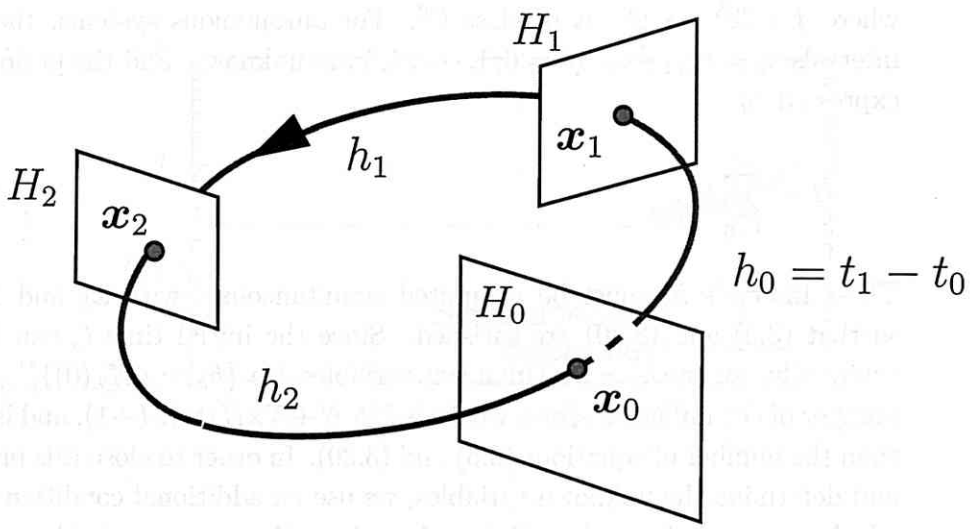


Figure 3.3: The additional condition (3.34) of the multiple shooting method for autonomous systems. Closed curve : a periodic orbit in the state space. H_k $(k = 1, 2, 3)$: Poincaré sections transverse to the periodic orbit.

3.4 Accuracy of computed results

As stated in Section 3.2, the j-th Floquet multiplier λ_j is equal to the j-th eigenvalue $\lambda_{j,k}$ $(k = 0, 1, \cdots, M)$ of (3.17). However, numerical results of $\lambda_{j,k}$ may be different from each other. Thus we can use

$$R(\lambda_j) = \frac{1}{\max_{0 \le k \le M} |\lambda_{j,k}|} \left| \max_{0 \le k \le M} \lambda_{j,k} - \min_{0 \le k \le M} \lambda_{j,k} \right| , \tag{3.35}$$

as an index of accuracy of the computed Floquet multipliers.

Additionally, for autonomous systems, one of the Floquet multipliers λ_{u} must be equal to unity [36,31]. Thus we can use

$$E(\lambda_{\mathrm{u}}) = \max_{0 \le k \le M} |\lambda_{\mathrm{u},k} - 1| , \tag{3.36}$$

as an index of error of numerical results of λ_{u}.

Using these indices $R(\lambda_j)$ and $E(\lambda_{\mathrm{u}})$, we will compare accuracy of Floquet multipliers computed by "Method 1", "Method 2", and "Method 3".

Chapter 4

Numerical examples

As shown in Chapter 2, the conventional method "Method 1" can compute inaccurate Floquet multipliers when periodic orbits get closer to equilibria. This chapter compares numerical examples computed with "Method 1", "Method 2", and "Method 3". The first example is a two dimensional non-autonomous system defined by the nonlinear Mathieu equations [18, 13, 38]. Next, we show the results of a three dimensional autonomous system defined by the FitzHugh-Nagumo equations [15, 32]. In the computations with "Method 3", s_k (3.5) in (3.6) and (3.30) is fixed to $\frac{h_k}{2}$. We may expect that suitable choice of s_k can improve computed results, but this is left for future works. Eigenvalues and eigenvectors of $\boldsymbol{X}_k(T)$ (3.17) were computed with "geev" of LAPACK [1]. The eigenvalues are the results of Floquet multipliers with "Method 1", and the eigenvectors are used as initial approximate solutions of "Method 3".

Using indices $R(\lambda_j)$ (3.35) and $E(\lambda_u)$ (3.36) in Section 3.4, we compare accuracy of the computed Floquet multipliers. Also, we check errors of the conditions (3.10) and (3.11) for the periodic orbits and the solutions of the corresponding variational equations, respectively.

4.1 The Mathieu equations

The nonlinear Mathieu equations [18, 13, 38] are given by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -(q + p\cos t)\,\sin x\,, \end{aligned} \tag{4.1}$$

where p is the free parameter and q is set to 1 in this thesis. These equations are a mathematical model for prametrically excited nonlinear oscillation, for example, the forced motion of a swing [5], vibration of elastic cables [35], the seesaw oscillator [28], oscillatory motion of a ship [38], etc. Note that this system has an equilibrium $\boldsymbol{x}_*$ at the origin in the state space (x, y).

Figures 4.1(a) and (b) show the computed periodic orbits with the period $T = 2\pi$ in the state space and the corresponding time series for $p = 2, 20$, and 40, respectively. We can find that the periodic orbit gets closer to the equilibrium $\boldsymbol{x}_*$ with increase of p, and that the point $\boldsymbol{x}(t)$ on the periodic orbit moves very slowly with time near $\boldsymbol{x}_*$ and quickly away from $\boldsymbol{x}_*$.

In these computations, the proposed method "Method 3" with $M+1 = 100$, $t_k = \frac{T}{M+1}k$, and $h_k = t_{k+1} - t_k = \frac{T}{M+1}$ $(k = 0, 1, \cdots, M)$ are used. The thresholds δ_1 and δ_2 of the convergence conditions (3.10) and (3.11) of "Method 3" are set to $\delta_1 = \delta_2 = 10^{-6}$. Initial value problems of the ordinary differential equations (2.1) and the corresponding variational equations (2.13) are numerically solved using the 4-th order explicit Runge-Kutta method with the time increment $\Delta t = 0.025$. The conventional multiple shooting method "Method 1" also produced accurate enough results of the periodic orbits.

Figure 4.2 shows the computed Floquet multipliers using "Method 1". The ratio of the Floquet multipliers $\frac{|\lambda_1|}{|\lambda_2|}$ $(|\lambda_1| > |\lambda_2|)$ becomes considerably large as the periodic orbit gets closer to the equilibrium $\boldsymbol{x}_*$ with increase of p. Then, "Method 1" gives inaccurate Floquet multiplier λ_2, even if the corresponding periodic orbit is accurately computed.

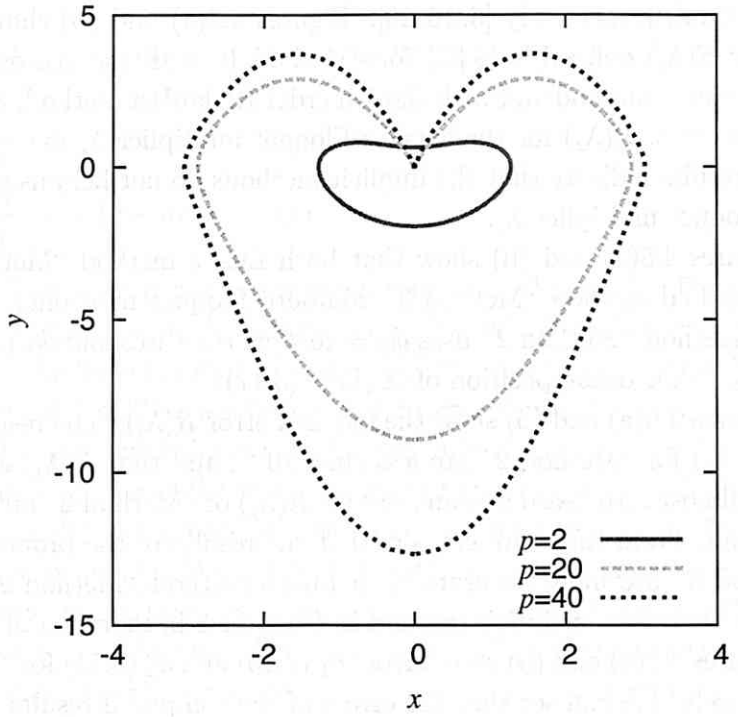
We may expect that some commonly used implicit methods for initial value problems of ordinary differential equations improve accuracy of computed results of Floquet multipliers. Figures 4.3(a) and (b) show the computed Floquet multipliers using "Method 1" with the implicit Runge-Kutta

methods called the 3rd order RadauIIA method and the 4th order LobattoIIIA method, respectively [6, 16, 20]. Figures 4.4(a) and (b) show the index of error $R(\lambda_j)$ defined by(3.35) for "Method 1" with the 4th order explicit Runge-Kutta method and with the 4th order RadauIIA method, respectively. In both cases, $R(\lambda_2)$ for the smaller Floquet multiplier λ_2 increases with p. These results indicate that the implicit methods do not help us obtain accurate Floquet multiplier λ_2.

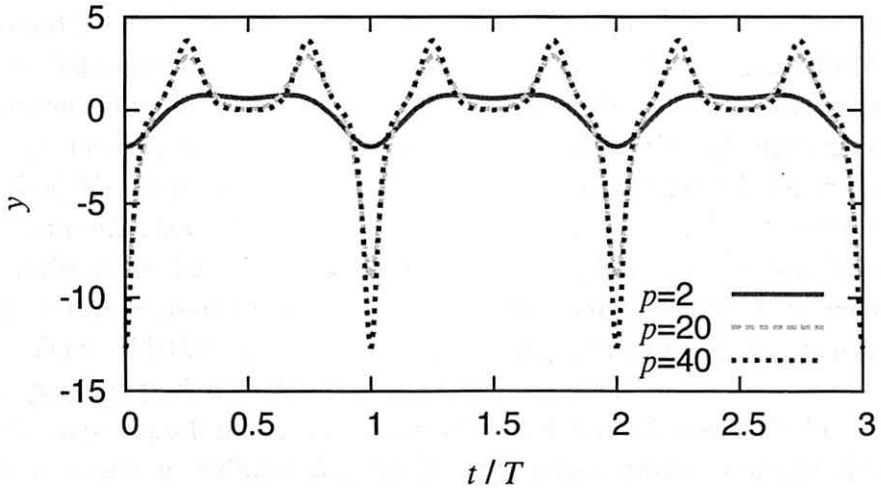
Figures 4.5(a) and (b) show that both Lust's method "Method 2" and the proposed method "Method 3" compute Floquet multipliers accurately. Lust's method "Method 2" uses $\delta_Q = 10^{-6}$ as the threshold δ_Q (A.5) for the periodic Schur decomposition of $\boldsymbol{X}_k(T)$ (3.17).

Figures 4.6(a) and (b) show the index of error $R(\lambda_j)$. The results indicate that $R(\lambda_j)$ for "Method 2" are less than 10^{-5}, and that $R(\lambda_j)$ for "Method 3" are almost zero. Also we can see that $R(\lambda_j)$ of "Method 2" are large when p is small. From these, we can say that the results of the proposed method "Method 3" are more accurate than Lust's method "Method 2". We will discuss the results of Lust's method in Chapter 5 in more detail.

Figures 4.7(a) and (b) show errors d_1 (3.10) and d_2 (3.11) for "Method 3", respectively. We can see that the errors of the computed results of "Method 3" are bounded by the thresholds $\delta_1 = \delta_2 = 10^{-6}$.



(a) Periodic orbits in the state space



(b) Time series of y

Figure 4.1: Computed results of the periodic orbits of the nonlinear Mathieu equations (4.1) with "Method 3". T : the period. p : the parameter of the equations.

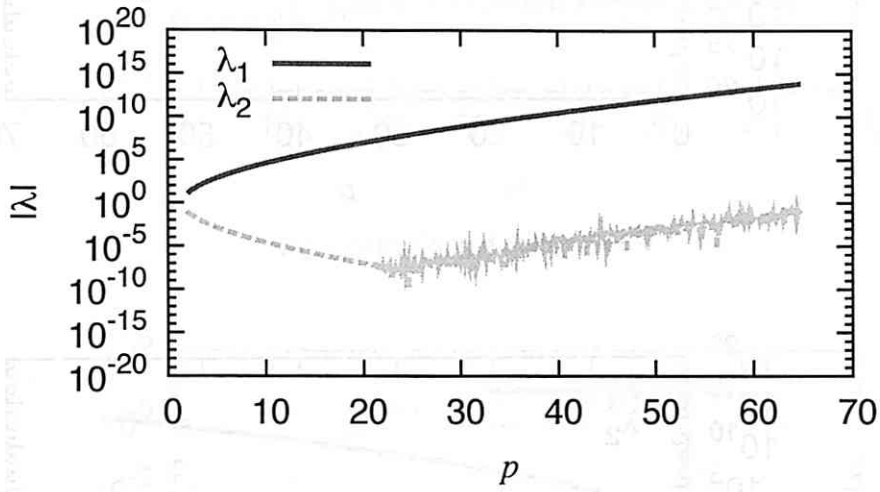
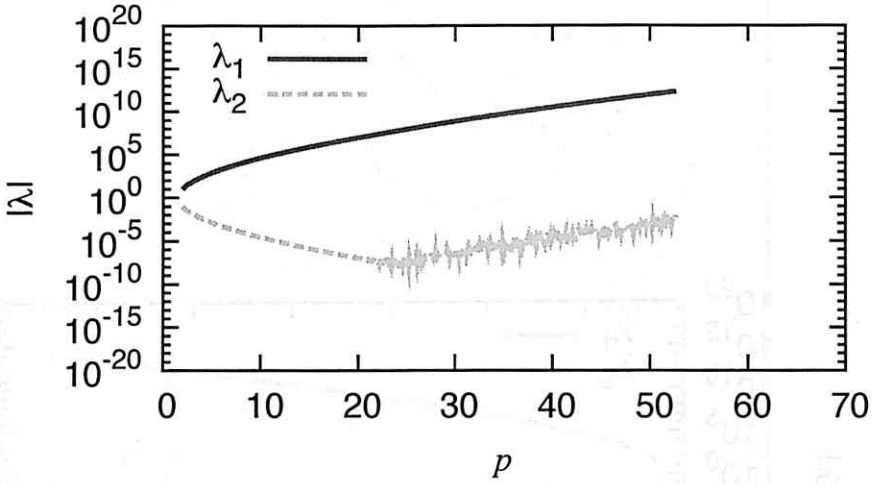
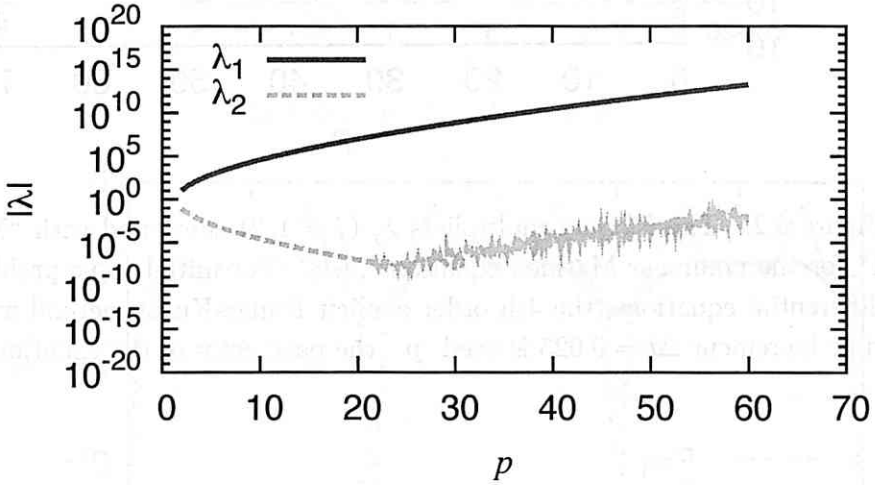


Figure 4.2: The Floquet multipliers λ_j $(j = 1, 2)$ computed with "Method 1" for the nonlinear Mathieu equations (4.1). For initial value problems of differential equations, the 4th order explicit Runge-Kutta method with the time increment $\Delta t = 0.025$ is used. p : the parameter of the equations.

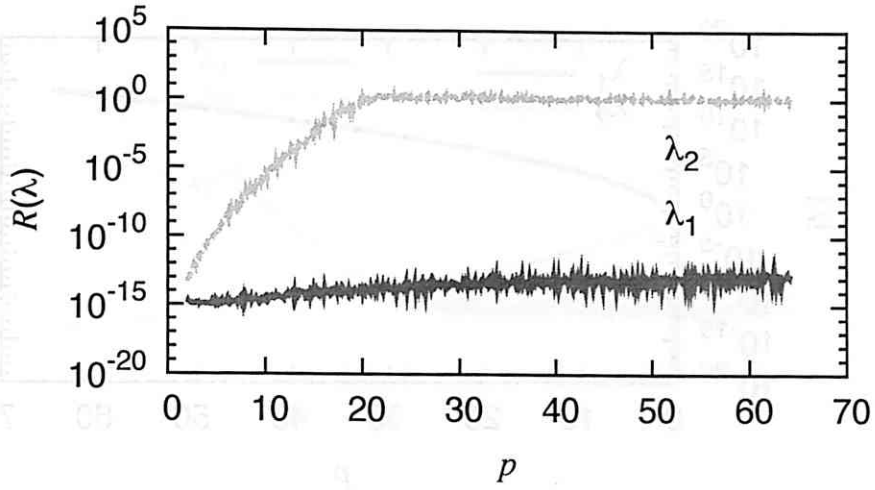


(a) 3rd order RadauIIA

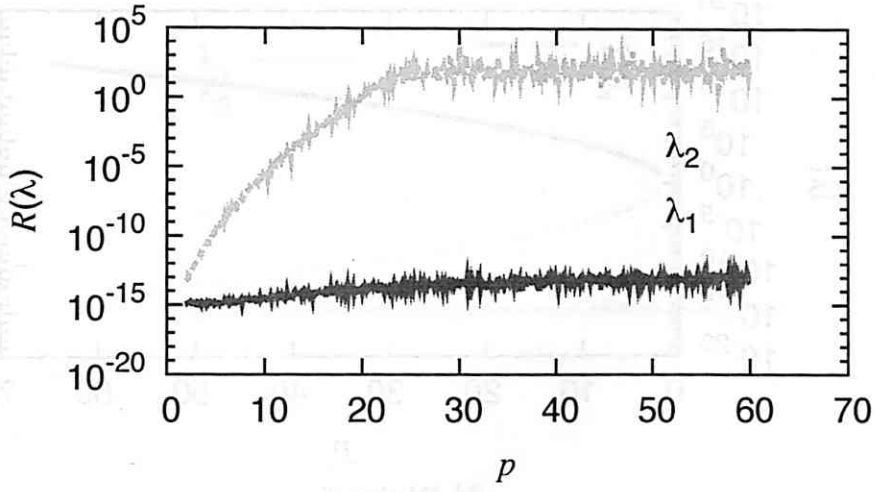


(b) 4th order LobattoIIIA

Figure 4.3: Comparison of the computed Floquet multipliers λ_j $(j = 1, 2)$ for the nonlinear Mathieu equations (4.1) with the implicit Runge-Kutta methods called the 3rd order RadauIIA and the 4th order LobattoIIIa, respectively. The time increment Δt of the methods is set to $\Delta t = 0.025$. p : the parameter of the equations.

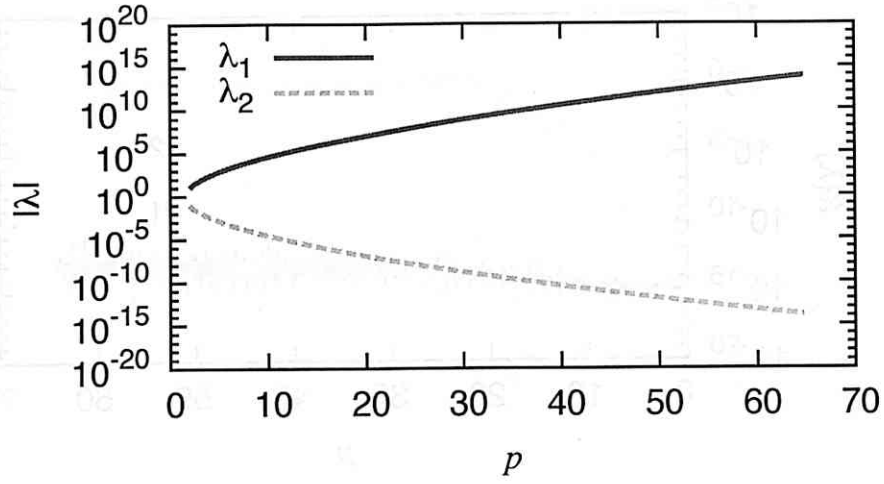


(a) 4th order Explicit Runge-Kutta

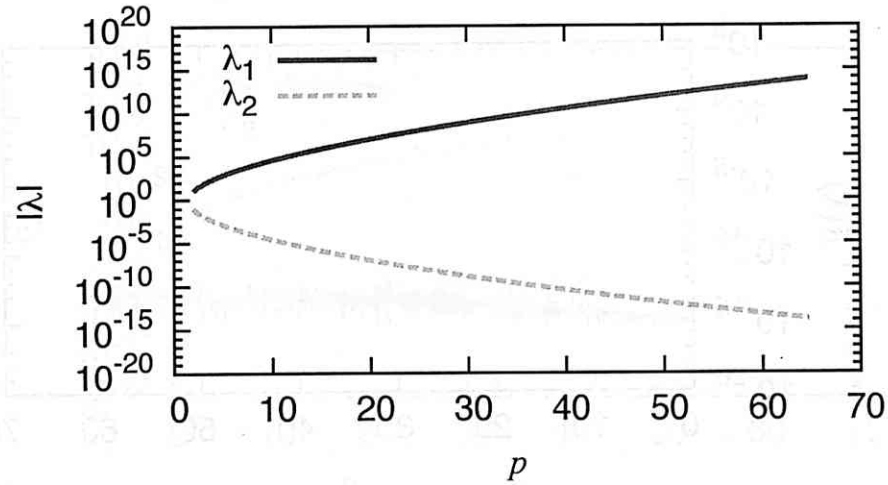


(b) 4th order Implicit Runge-Kutta (LobattoIIIA)

Figure 4.4: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2)$ by "Method 1" with the implicit methods for the nonlinear Mathieu equations (4.1). $R(\lambda_j)$: See (3.35). p : the parameter of the equations.

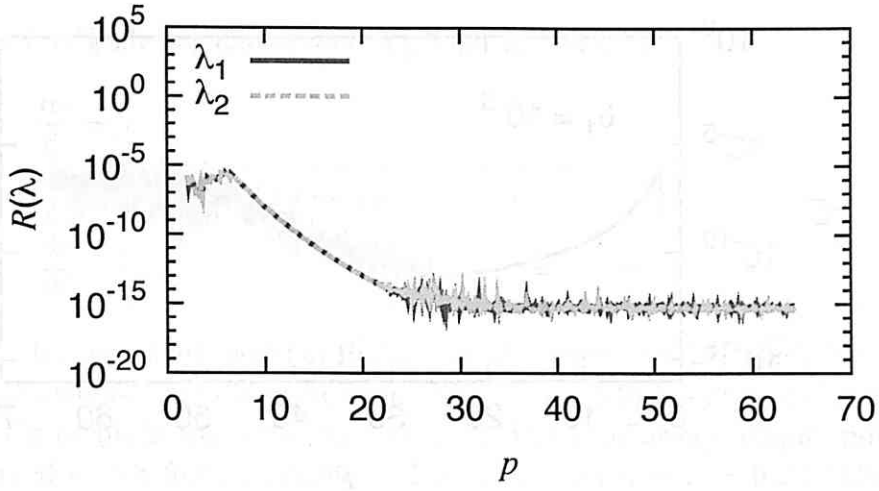


(a) "Method 2"

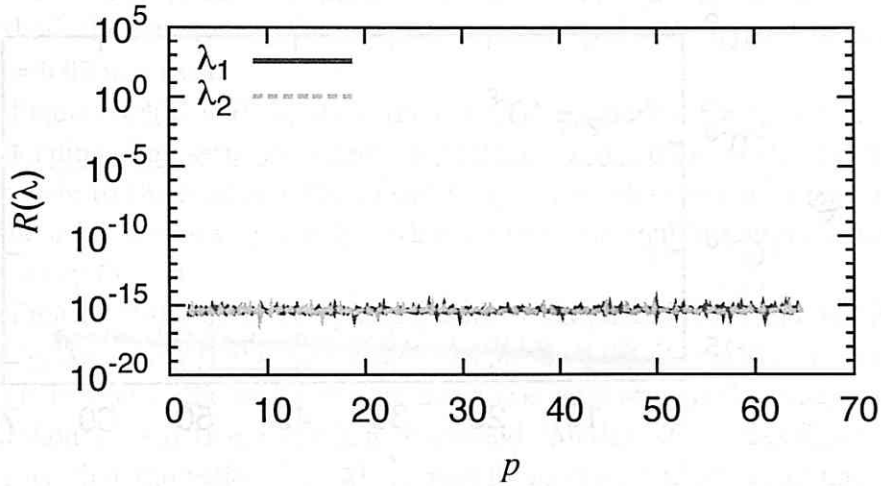


(b) "Method 3"

Figure 4.5: Comparison of the computed Floquet multipliers λ_j $(j = 1, 2)$ by "Method 2" and "Method 3" for the nonlinear Mathieu equations (4.1). For initial value problems of differential equations, the 4th order explicit Runge-Kutta method with $\Delta t = 0.025$ is used. p : the parameter of the equations.

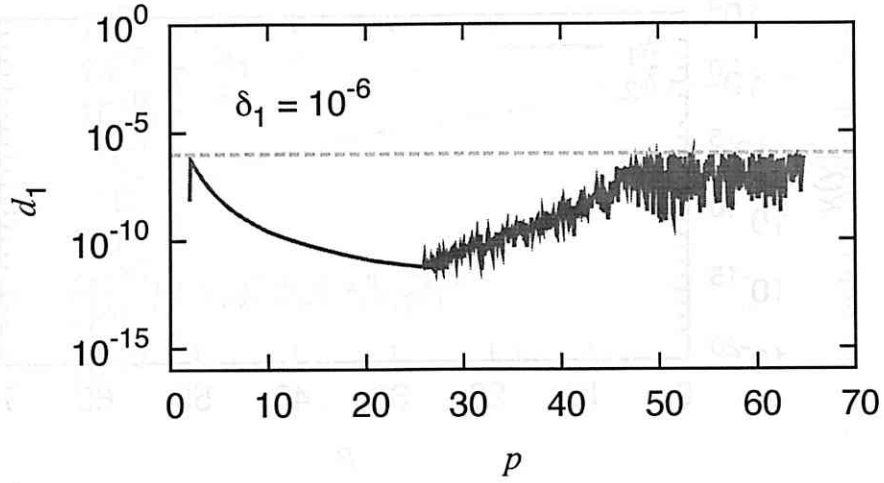


(a) "Method 2"

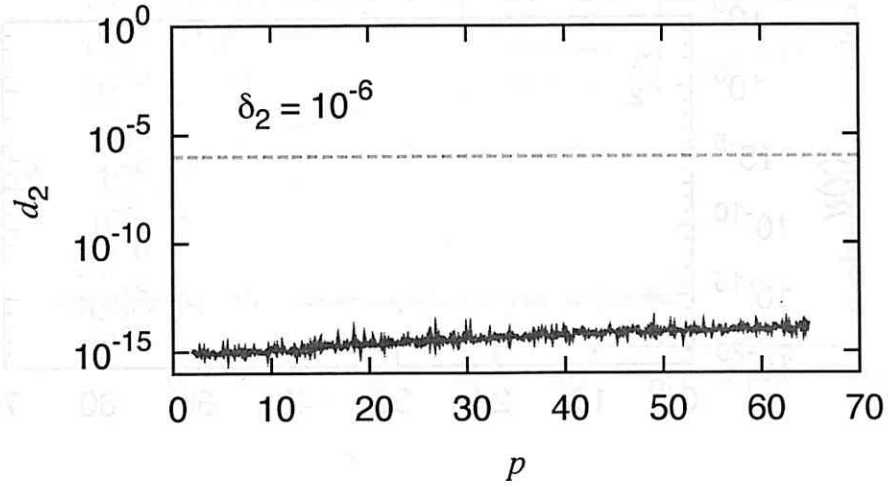


(b) "Method 3"

Figure 4.6: Comparison of accuracy of the computed results of the Floquet multipliers λ_j $(j = 1, 2)$ by "Method 2" and "Method 3" for the nonlinear Mathieu equations (4.1). $R(\lambda_j)$: See (3.35). p : the parameter of the equations.



(a) Errors d_1 for the periodic orbits



(b) Errors d_2 for the variational equations

Figure 4.7: Errors d_1 (3.10) and d_2 (3.11) for the convergence conditions of Newton's method in "Method 3" for the nonlinear Mathieu equations (4.1). p : the parameter of the equations. δ_1, δ_2 : threhsolds for the errors d_1 and d_2, respectively.

4.2 The FitzHugh-Nagumo equations

The FitzHugh-Nagumo equations [15,32] are given by

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y\,,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= qy + x\,(x-1)(x-p) + z\,,\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{r}{q}x\,.\end{aligned} \tag{4.2}$$

This is a simplified model of Hodgkin-Huxley equations [21] which describes excitation dynamics of nerve axon. This model (4.2) has an equilibrium $\boldsymbol{x}_*$ at the origin in the state space (x, y, z). In the following computations, p is used as the free parameter, and q and r are set to $q = 0.28288600873$ and $r = 0.0025$, respectively. We computed periodic orbits near $\boldsymbol{x}_*$ using the proposed method "Method 3" with $M+1 = 100$ and thresholds $\delta_1 = \delta_2 = 10^{-6}$. For initial value problems of differential equations in the iterative method, the 4th order explicit Runge-Kutta method with the time increment $\Delta t = 0.05$ was used.

Figures 4.8(a) and (b) show the computed periodic orbits, and the corresponding time series of equations (4.2) for $p = -0.015, 0.01$, and 0.015. Similarly to the results of the Mathieu equations, the point $\boldsymbol{x}(t)$ on the periodic orbit moves very slowly with time near the equilibrium $\boldsymbol{x}_*$ and very fast away from $\boldsymbol{x}_*$.

Figure 4.9 shows the computed Floquet multipliers λ_1, λ_2, and λ_3 ($|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$) using the proposed method "Method 3". It was found that this result is almost the same as that using the conventional shooting method "Method 1" and that using Lust's method "Method 2" in this figure. We can see that the ratios $\frac{|\lambda_1|}{|\lambda_2|}$ and $\frac{|\lambda_1|}{|\lambda_3|}$ rapidly increase with p. Here note that one of the Floquet multipliers λ_{u} of this autonomous system must be equal to unity [36,31]. In this case, the corresponding eigenvalue λ_{u} is λ_2.

Figures 4.10(a) and (b) compare the index of error $R(\lambda_j)$ (3.35) of the "Method 1" using the 4th order explicit Runge-Kutta method and the 4th order implicit Runge-Kutta method, respectively. The time increment Δt is $\Delta t = 0.05$. Similarly to the computed results of the Mathieu equations, the

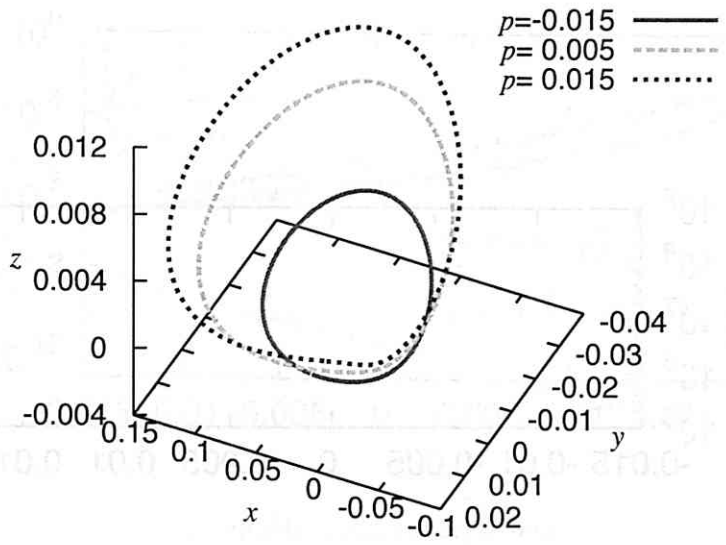
implicit method for initial value problems with a sufficiently small threshold $\delta_1 = 10^{-12}$, does not help us obtain accurate Floquet multipliers.

Figures 4.11(a) and 4.12(a) show the computed Floquet multipliers using Lust's method "Method 2" and the proposed method "Method 3", respectively. It is found that the index of error $R(\lambda_j)$ of "Method 2" and "Method 3" are less than 10^{-5}. Furthermore, Figures 4.11(b) and 4.12(b) show that, with smaller thresholds $\delta_1 = \delta_2 = \delta_Q = 10^{-12}$, both methods reduce $R(\lambda_j)$ less than 10^{-11}. Since the periodic Schur decomposition for autonomous systems is constructed such that $\lambda_u = \lambda_2 = 1$ as shown in Appendix A, $R(\lambda_2)$ of "Method 2" is not displayed in the figures.

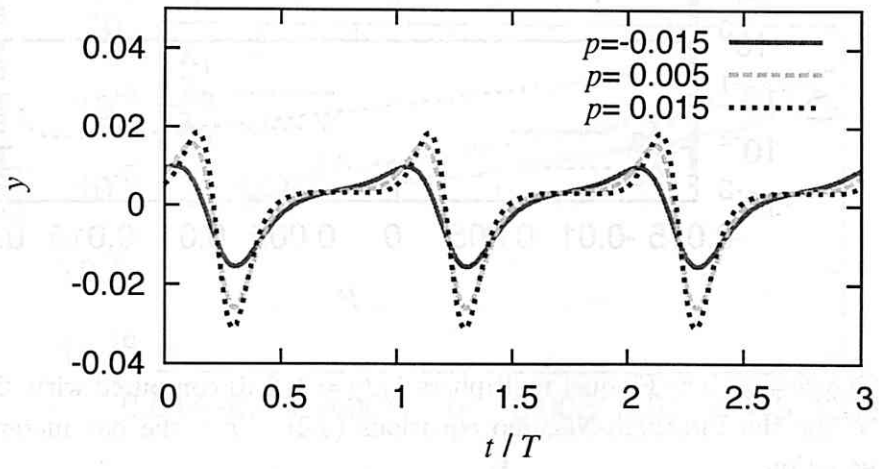
Figures 4.13(a) and (b) compare the error $E(\lambda_u)$ of "Method 1" with that of "Method 3". We can see that $E(\lambda_u)$ can be reduced using "Method 3" with smaller time increment Δt of the Runge-Kutta method, but not using "Method 1".

Figures 4.14(a) and (b) show the errors d_1 (3.10) and d_2 (3.11) for the convergence conditions of "Method 3", respectively. Figures 4.15(a) and (b) show the iteration number ν. The thresholds δ_1 and δ_2 are set to $\delta_1 = \delta_2 = 10^{-12}$. It is found that the proposed method "Method 3" enables us to control accuracy of not only the periodic orbits but also the solutions of the variational equations with a few iterations, even for considerably small thresholds.

Finally, Figure 4.16 compares computation time for the periodic orbits and Floquet multipliers. In this figure, $\tau_A+\tau_1$, $\tau_A+\tau_2$, and $\tau_B+\tau_3$ represent the total computation time of the conventional shooting method "Method 1", Lust's method "Method 2", and the proposed method "Method 3", respectively. These results show that the proposed method "Method 3" can produce accurate Floquet multipliers in practical computation time.



(a) Periodic orbits in the state space



(b) Time series of y

Figure 4.8: Computed results of the periodic orbits of the FitzHugh-Nagumo equations (4.2) with "Method 3". p : the parameter of the equations. T : the period.

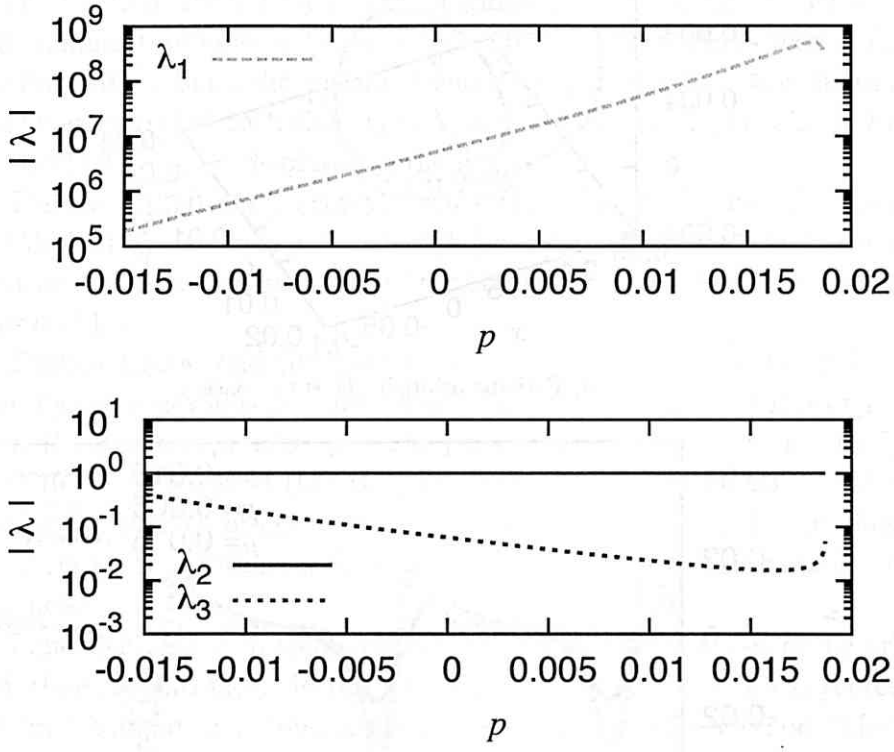
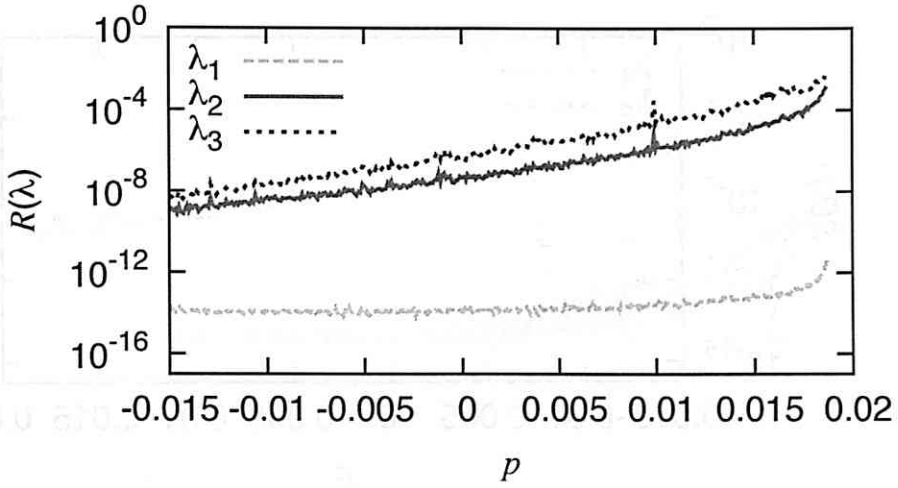
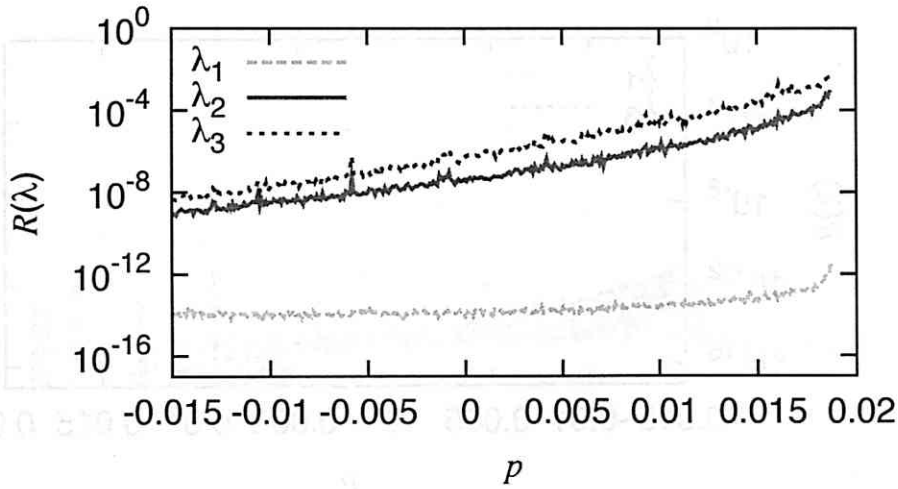


Figure 4.9: The Floquet multipliers λ_j $(j = 1, 2, 3)$ computed with "Method 3" for the FitzHugh-Nagumo equations (4.2). p : the parameter of the equations.

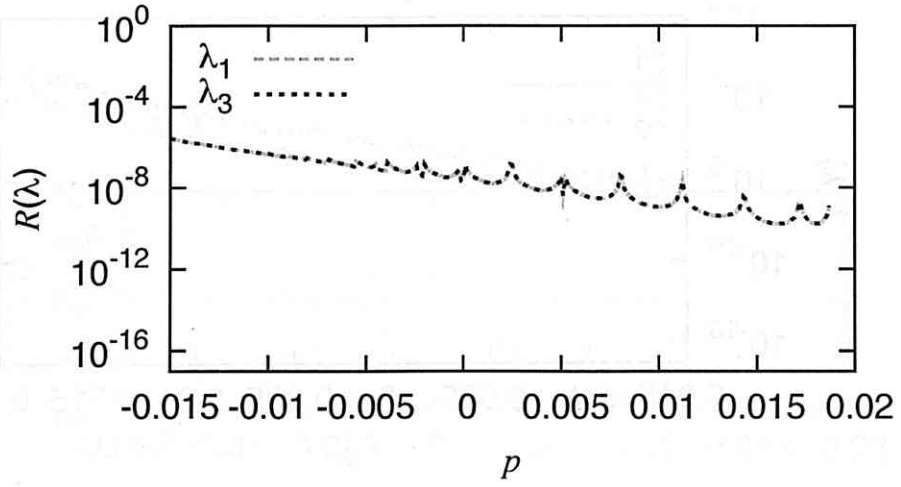


(a) 4th order explicit Runge-Kutta

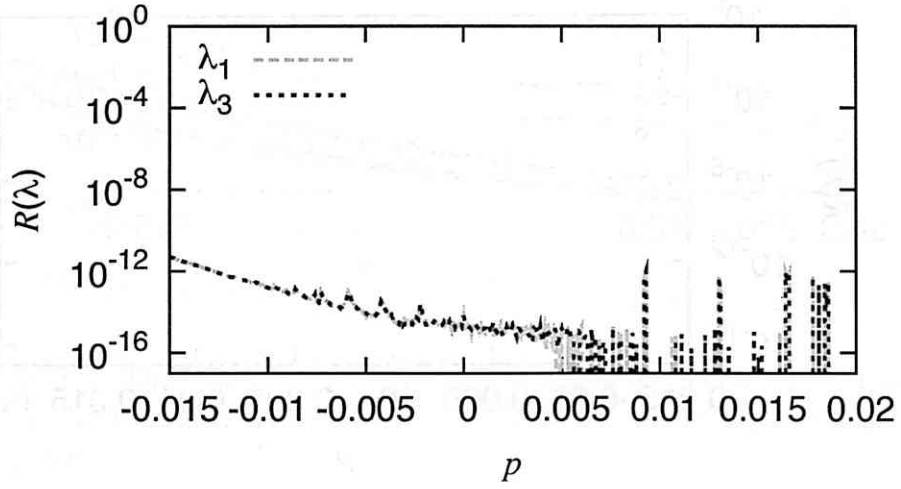


(b) 4th order implicit Runge-Kutta (LobattoIIIA)

Figure 4.10: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2, 3)$ by "Method 1" for the FitzHugh-Nagumo equations (4.2). For initial value problems of differential equations, (a) the 4th order explicit Runge-Kutta method and (b) the 4th order implicit Runge-Kutta method called LobattoIIIA are used. $R(\lambda_j)$: see (3.35). p : the parameter of the equations.

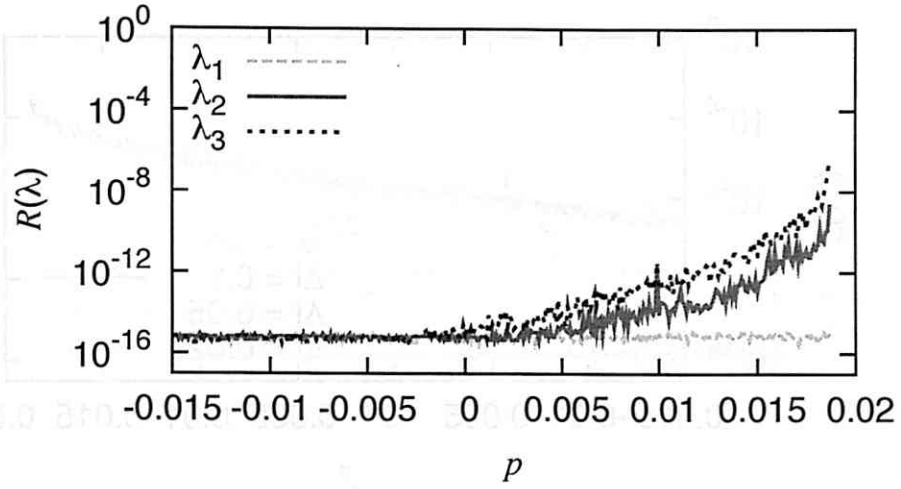


(a) $\delta_1 = \delta_Q = 10^{-6}$

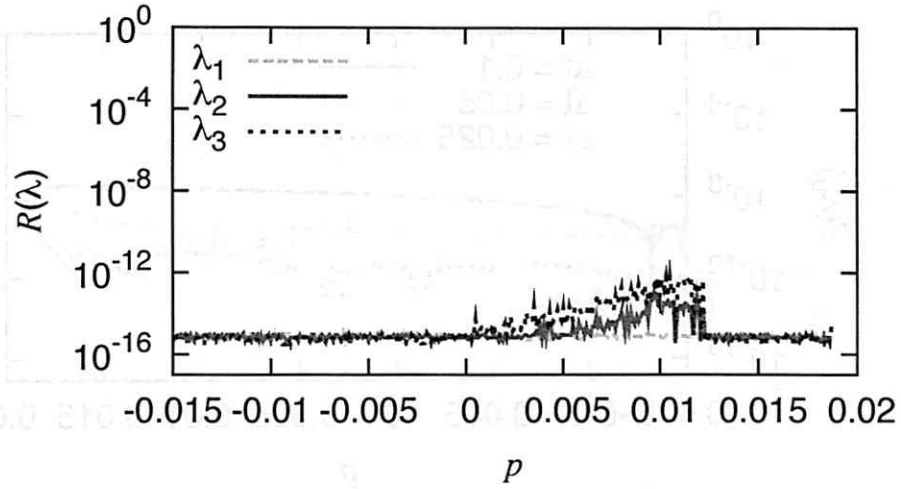


(b) $\delta = \delta_Q = 10^{-12}$

Figure 4.11: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2, 3)$ by "Method 2" for the FitzHugh-Nagumo equations (4.2). δ_1 : see (3.10). δ_Q : see (A.5). $R(\lambda_j)$: see (3.35). p : the parameter of the equations.

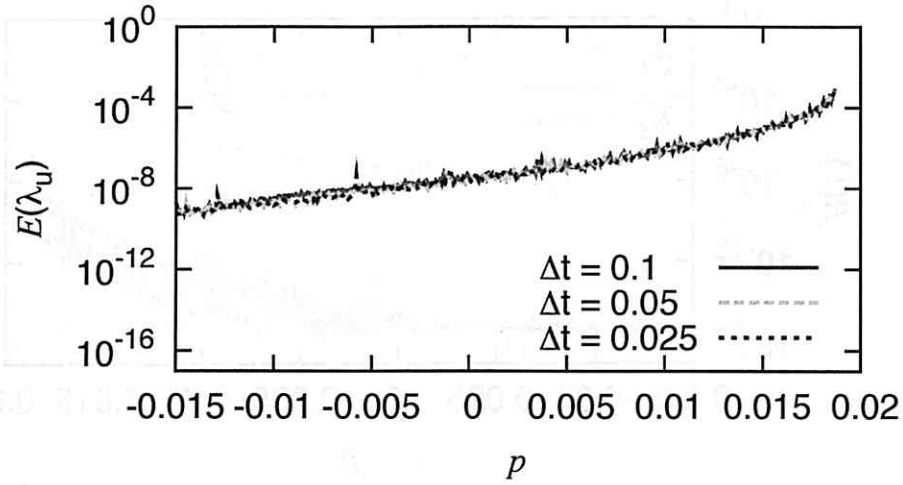


(a) $\delta_1 = \delta_2 = 10^{-6}$

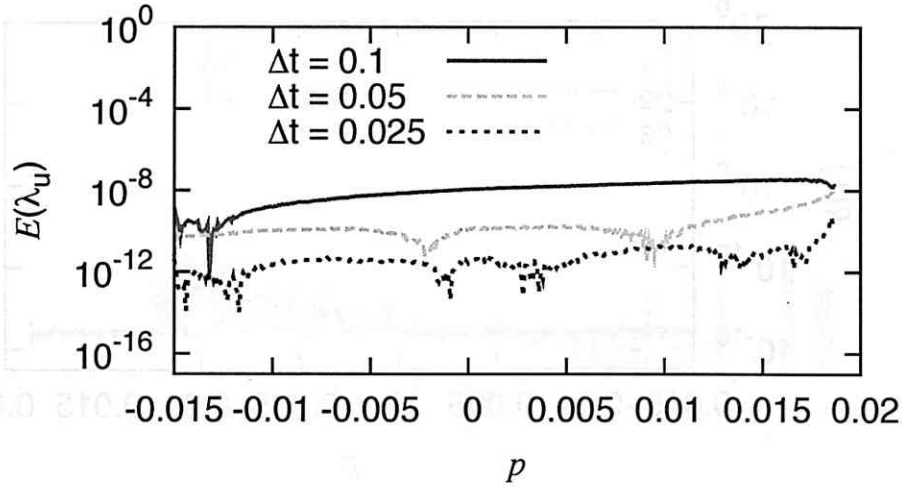


(b) $\delta_1 = \delta_2 = 10^{-12}$

Figure 4.12: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2, 3)$ by "Method 3" for the FitzHugh-Nagumo equations (4.2). δ_1, δ_2 : see (3.10) and (3.11), respectively. $R(\lambda_j)$: see (3.35). p : the parameter of the equations.

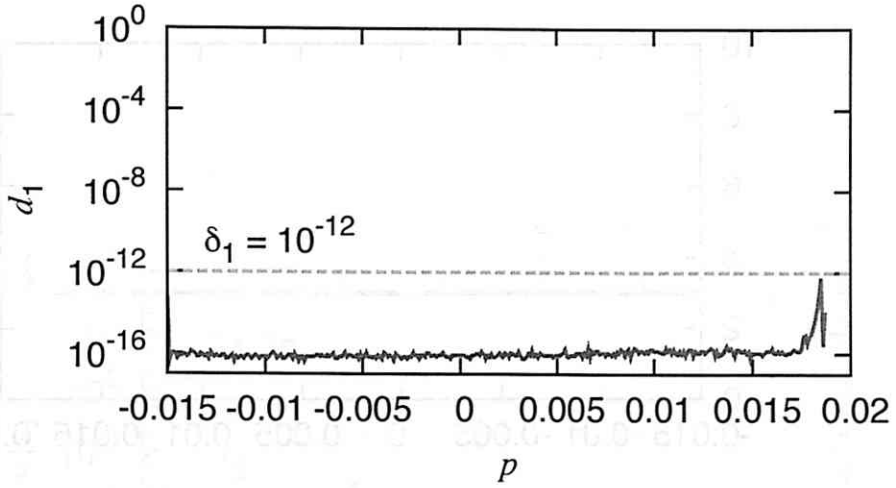


(a) "Method 1"

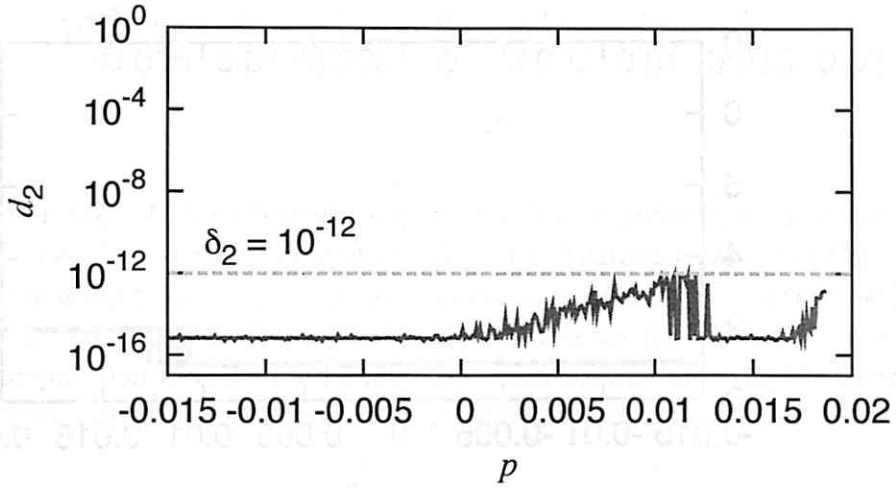


(b) "Method 3"

Figure 4.13: Comparison of errors of the unit Floquet multiplier $\lambda_u(=\lambda_2)$ for the FitzHugh-Nagumo equations (4.2) by "Method 1" and by "Method 3". Δt : the time increment of the 4th order explicit Runge-Kutta method. $E(\lambda_u)$: the error defined by (3.36). p : the parameter of the equations.

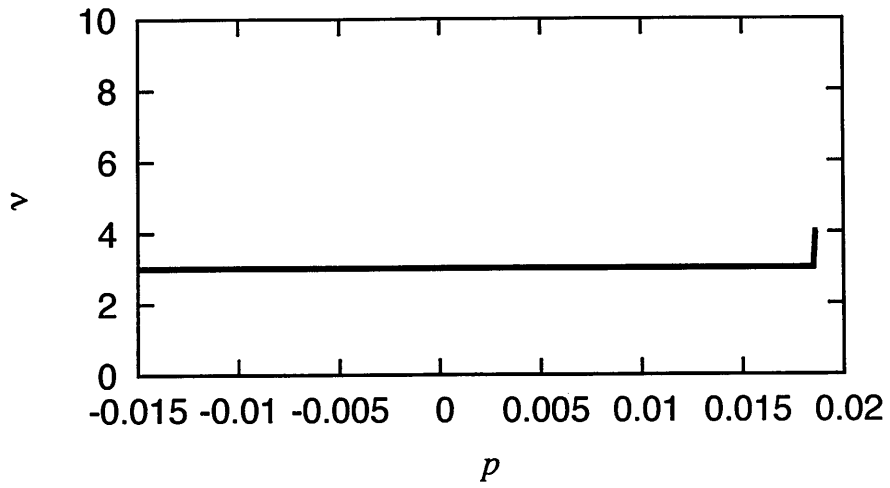


(a) Error d_1 of the condition for periodic orbits

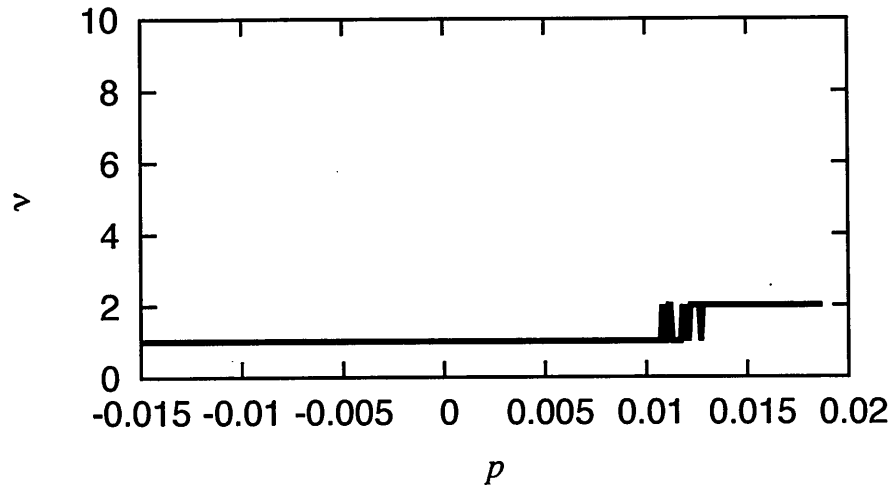


(b) Error d_2 of the condition for variational equations

Figure 4.14: Errors of the convergence conditions of Newton's method in "Method 3" for the FitzHugh-Nagumo equations (4.2). d_1, d_2 : see (3.10) and (3.11), respectively. p : the parameter of the equations.



(a) Periodic orbits



(b) Variational equations

Figure 4.15: The iteration number ν of "Method 3" with $\delta_1 = \delta_2 = 10^{-12}$ for the FitzHugh-Nagumo equations (4.2). p : the parameter of the equations.

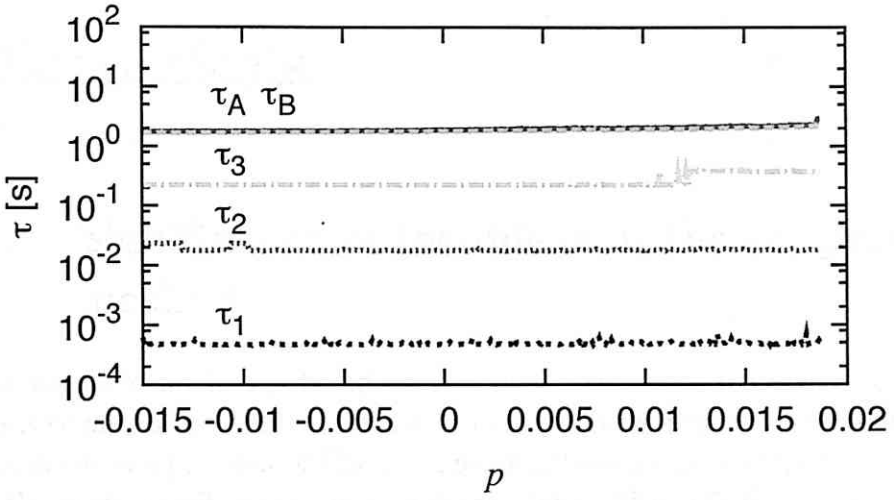


Figure 4.16: Comparison of computation time τ [seconds] for periodic orbits and Floquet multipliers of the FitzHugh-Nagumo equations (4.2). τ_A : computation time for the periodic orbits using "Method 1" and "Method 2". τ_B : computation time for the periodic orbits using "Method 3". τ_1, τ_2, τ_3 : computation time for the Floquet multipliers using "Method 1", "Method 2", and "Method 3" , respectively. p : the parameter of the equations.

Chapter 5

Discussions

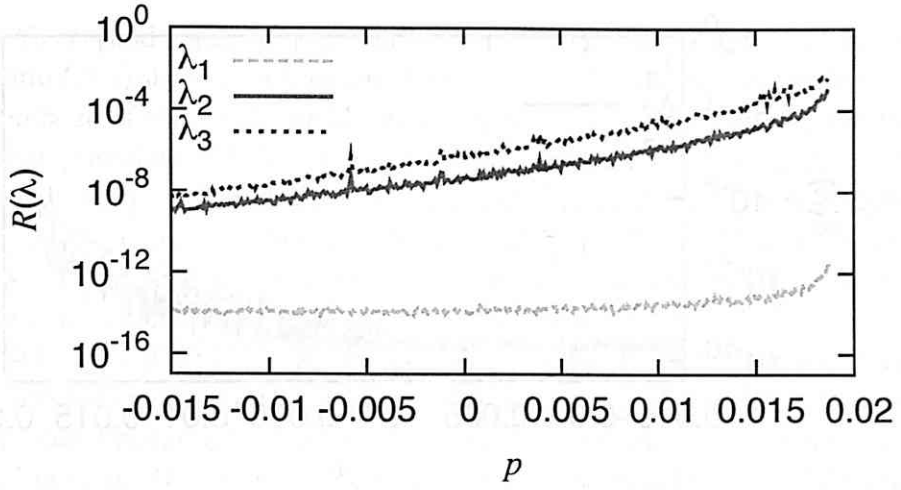
5.1 Significance of two ideas in the proposed method

The proposed method is based on two main ideas, (I) construction of the iterative method for variational equations and (II) reduction of round-off errors in the computation of Floquet multipliers using eigenvectors of $\boldsymbol{X}_0(T)$ as the initial condition of variational equations. Although these two ideas (I) and (II) are separately described in Sections 3.1 and 3.2, it should be emphasized that both (I) and (II) are required for accurate computation of Floquet multipliers. Actually, in Section 3.2, eigenvectors of $\boldsymbol{X}_0(T)$ are computed using the iterative algorithm for variational equations. The followings indicate significance of both ideas (I) and (II) in the proposed method.

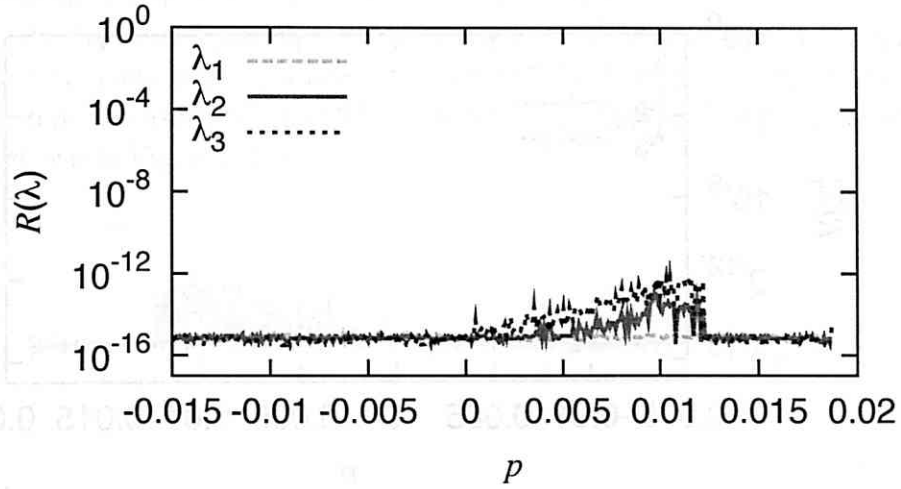
We can compute Floquet multipliers without using eigenvectors of $\boldsymbol{X}_0(T)$, namely "Method 3 without eigenvectors". Figure 5.1(a) shows the computed results of Floquet multipliers for the FitzHugh-Nagumo equations (4.2) using "Method 3 without eigenvectors" in which variational equations are iteratively solved. Although $\boldsymbol{X}_0(T)$ is accurately computed, the results using "Method 3 with eigenvectors" shown in Figure 5.1(b) are more accurate than "Method 3 without eigenvectors". Thus reduction of the round-off errors using eigenvectors is critical in the proposed method.

On the other hand we can compute Floquet multipliers without using

the iterative method for variational equations, namely "Method 3 without iteration". Figure 5.2(a) shows the computed results of Floquet multipliers for the FitzHugh-Nagumo equations (4.2) using "Method 3 without iteration" in which eigenvectors of $\boldsymbol{X}_0(T)$ are used the initial condition of variational equations. Although utilization of eigenvectors of $\boldsymbol{X}_0(T)$ reduces the round-off errors, the results using "Method 3 with iteration" in Figure 5.2(b) are more accurate than "Method 3 without iteration". It is because we cannot control the errors of solutions of the variational equations without iteration. From these, we can see that the iterative computation of variational equations is essentially important in the proposed method.

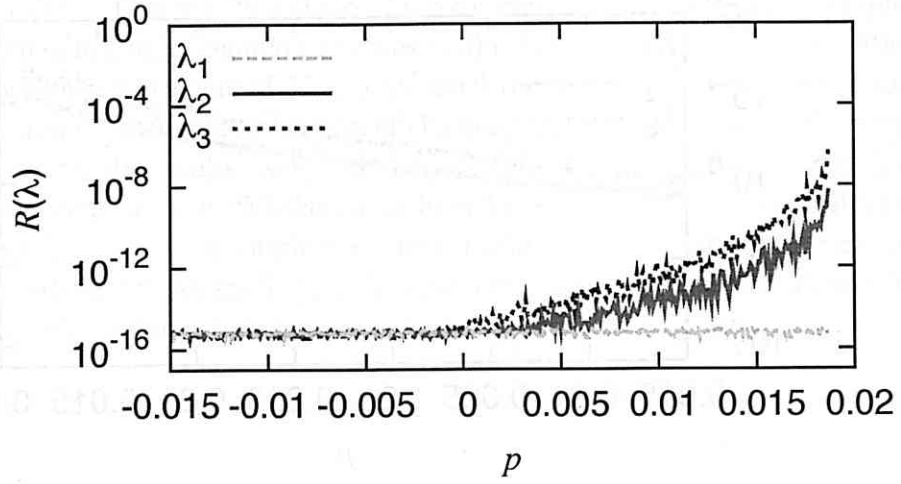


(a) "Method 3 without eigenvectors"

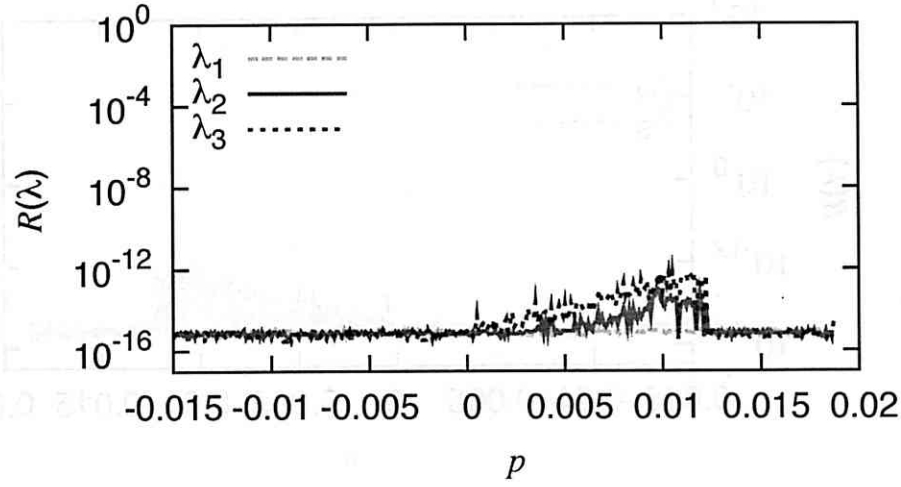


(b) "Method 3 with eigenvectors"

Figure 5.1: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2, 3)$ for the FitzHugh-Nagumo equations (4.2) . p : the parameter of the equations. $R(\lambda_j)$: see (3.35).



(a) "Method 3 without iteration"



(b) "Method 3 with iteration"

Figure 5.2: Comparison of accuracy of the numerical results of the Floquet multipliers λ_j $(j = 1, 2, 3)$ for the FitzHugh-Nagumo equations (4.2) . p : the parameter of the equations. $R(\lambda_j)$: see (3.35).

5.2 Comparison with Lust's method

Lust's method improves some other conventional methods. For example, Figure 4.5(a) shows that Lust's method "Method 2" produces accurate results for $p > 20$ of the nonlinear Mathieu equations (4.1) which the conventional shooting method "Method 1" cannot solve correctly. However, Figure 4.6(a) shows that the index $R(\lambda_j)$ (3.35) of error of Floquet multipliers computed by Lust's method "Method 2" is large for small p near the bifurcation point. That is, the computed results using Lust's method can be erroneous near the bifurcation point. We can consider the reason as follows:

The idea of Lust's method is to use the periodic Schur decomposition of matrix solutions of variational equations as shown in Appendix A. Then the variational equations are solved as an initial value problem until the orthogonal matrices Q_k converge. Since the transient state is long near bifurcation points in general, it takes time for Q_k to converge and the errors of the computed results accumulate due to long time calculation of the initial value problem. This results in the errors in Figure 4.6(a).

On the other hand, the proposed method "Method 3" iteratively solves the variational equations using Newton's method. Consequently, the computed results using "Method 3" are accurate enough even near the bifurcation as shown in Figure 4.6(b).

Chapter 6

Conclusions

This thesis has considered numerical stability analysis of periodic orbits of ordinary differential equations, and proposed a new method to compute Floquet multipliers with enough accuracy. The basic ideas of the proposed method are to construct an iterative method to compute the variational equations, and to utilize eigenvectors of the matrix solution $\boldsymbol{X}_0(T)$ of the variational equations. Numerical examples showed effectiveness of the proposed method.

The followings summarize each chapter:

Chapter 2 introduced the conventional method to numerically investigate stability of periodic orbits of non-autonomous ordinary differential equations given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{x}) \quad \text{with} \quad \boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{f}(t+T, \boldsymbol{x})\,. \tag{6.1}$$

We can examine stability of periodic orbits $\boldsymbol{x}(t) = \varphi_{t_0}^{\tau}(\boldsymbol{x}_0)$ with the period T using the variational equations

$$\frac{\mathrm{d}\boldsymbol{X}_0}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t_0+\tau,\, \boldsymbol{x}=\varphi_{t_0}^{\tau}(\boldsymbol{x}_0)} \boldsymbol{X}_0(\tau) \quad \text{with} \quad \boldsymbol{X}_0(0) = I\,. \tag{6.2}$$

Eigenvalues of $\boldsymbol{X}_0(T)$, namely Floquet multipliers, determine stability of the periodic orbit. The proposed method is based on the shooting method which is one of the commonly used methods. It was shown using numerical examples

that the shooting method and the standard method to obtain eigenvalues of $\boldsymbol{X}_0(T)$ can produce inaccurate results when the periodic orbit passes near the equilibrium point or the ratio of the maximum Floquet multiplier to the minimum one is very large. These computational errors are due to the numerical method for the variational equations and round-off errors in the computation of eigenvalues of $\boldsymbol{X}_0(T)$.

Chapter 3 proposed a new method to numerically obtain Floquet multipliers. The basic ideas are to construct an iterative method for the variational equations, and to utilize eigenvalues of $\boldsymbol{X}_0(T)$ as the initial condition of the variational equations. The iterative method enables us to control errors of the computed results of the variational equations. Utilization of eigenvalues of $\boldsymbol{X}_0(T)$ reduces the round-off errors in the computation of Floquet multipliers.

Chapter 4 showed some computed results for the nonlinear Mathieu equations and the FitzHugh-Nagumo equations. Comparison with the conventional methods indicated that the proposed method can yield Floquet multipliers with enough accuracy and practical computational cost.

Chapter 5 discusses significance of the two main ideas of the proposed method and compares the proposed method with Lust's method. Numerical examples illustrate the importance of the ideas to accurately compute Floquet multipliers.

In this thesis, accuracy of computed Floquet multipliers is examined using some indices given by necessary conditions of them. However, the indices are not sufficient to verify accuracy of numerical results. We may further investigate accuracy of numerical solutions using the proposed method, for example, following the idea of numerical verification methods [41,39,8]. This remains as future works. Also, the eigenvectors of $\boldsymbol{X}_0(T)$ can be used in approximation of (un)stable manifolds of the periodic orbits. We expect to apply the proposed method for computation of (un)stable manifolds.

Appendix A

The periodic Schur decomposition

An upper triangular matrix $\Lambda = Q_0^{\mathrm{T}} \boldsymbol{X}_0(T) Q_0$ with an orthogonal matrix Q_0 is linearly conjugate to a matrix $\boldsymbol{X}_0(T) \in \mathbb{R}^{N\times N}$ (2.14) which is the matrix product of $\boldsymbol{X}_k(h_k)$ $(k = 0, 1, \cdots, M)$. Using orthogonal matrices Q_k $(k = 0, 1, \cdots, M)$, the periodic Schur decomposition [4] of $\boldsymbol{X}_0(T)$ represents the conjugate matrix Λ as the matrix product of upper triangular matrices $A_k \in \mathbb{R}^{N\times N}$ such as

$$\begin{aligned}\Lambda &= Q_0^{\mathrm{T}} \boldsymbol{X}_0(T) Q_0 \\ &= Q_0^{\mathrm{T}} \boldsymbol{X}_M(h_M) \cdots \boldsymbol{X}_1(h_1) \boldsymbol{X}_0(h_0) Q_0 \\ &= Q_0^{\mathrm{T}} \boldsymbol{X}_M(h_M) Q_M \cdots Q_2^{\mathrm{T}} \boldsymbol{X}_1(h_1) Q_1 Q_1^{\mathrm{T}} \boldsymbol{X}_0(h_0) Q_0 \\ &= A_M \cdots A_1 A_0 \,, \end{aligned} \tag{A.1}$$

where

$$A_k = Q_{k+1}^{\mathrm{T}} \boldsymbol{X}_k(h_k)\, Q_k \quad \text{for} \quad k = 0, 1, \cdots, M-1 \,, \tag{A.2}$$

and

$$A_M = Q_0^{\mathrm{T}} \boldsymbol{X}_M(h_M)\, Q_M \,. \tag{A.3}$$

The orthogonal matrices Q_k are constructed by the following steps.

(i) Construct an arbitrarily orthogonal matrix $Q_0 \in \mathbb{R}^{N\times N}$.

(ii) Using Gram-Schmidt orthonormalization, decompose $\boldsymbol{X}_k(h_k)\,Q_k$ such as

$$\boldsymbol{X}_k(h_k)\,Q_k = Q_{k+1}\,A_k \tag{A.4}$$

for $k = 0, 1, \cdots, M$.

(iii) If the following convergence condition between Q_{M+1} and Q_0 is satisfied, finish the iteration and recompute A_M as (A.3), or else, substitute Q_{M+1} into Q_0 and return step (ii).

The convergence condition in step (iii) is given as follows. Let $\boldsymbol{w}_{j,M+1}$ and $\boldsymbol{w}_{j,0}$ denote the j-th columns $(j = 1, 2, \cdots, N)$ of Q_{M+1} and Q_0, respectively. This iteration is continued until $\boldsymbol{w}_{j,M+1}$ becomes oriented to the same direction of $\boldsymbol{w}_{j,0}$ except for the sign, namely, the convergence condition is given as

$$d_Q := \max_{1\le j\le N} \|\boldsymbol{w}_{j,M+1} - \mathrm{sign}(\langle \boldsymbol{w}_{j,0},\, \boldsymbol{w}_{j,M+1}\rangle)\; \boldsymbol{w}_{j,0}\| < \delta_Q\,, \tag{A.5}$$

for sufficiently small $\delta_Q > 0$ where $\langle\cdot,\,\cdot\rangle$ denotes the inner product.

It should be noted that $\boldsymbol{X}_k(h_k)\,Q_k = \Psi^{h_k}_{t_k,\boldsymbol{x}_k}(Q_k)$, namely solutions of the variational equations (2.13) with the initial condition Q_k. The computation of $\Psi^{h_k}_{t_k,\boldsymbol{x}_k}(Q_k)$ is iterated in (i), (ii), and (iii) until Q_k are converged so that the condition (A.5) is satisfied. Then, the 1st vector $\boldsymbol{w}_{1,k}$ of Q_k is most expanded to the direction of the eigenvector for the largest Floquet multiplier, namely, the largest eigenvalue of $\boldsymbol{X}_k(T)$.

The method above is available for both of non-autonomous systems and autonomous systems, but for autonomous systems, we can set more accurate 1st vectors $\boldsymbol{w}_{1,k}$ using a theoretical property. The property is that perturbations directed to the vector field $\boldsymbol{f}$ (3.32) is not expanded or contracted on the trajectory. Write a solution $\boldsymbol{x}(t)$ with an initial condition $\boldsymbol{x}(t_k) = \boldsymbol{x}_k$ as $\boldsymbol{x}(t) = \varphi^{\tau}_{t_k}(\boldsymbol{x}_k)$ $(\tau = t - t_k)$. Then, it follows that

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\,\varphi^{\tau}_{t_k}(\boldsymbol{x}_k)}{\mathrm{d}t} \\ \Leftrightarrow\; &\boldsymbol{f}(\boldsymbol{x}(t)) = \left.\frac{\partial\varphi^{\tau}_{t_k}(\boldsymbol{x})}{\partial\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_k} \boldsymbol{f}(\boldsymbol{x}_k) = \boldsymbol{X}_k(\tau)\boldsymbol{f}(\boldsymbol{x}_k)\,, \end{aligned} \tag{A.6}$$

and this property gives

$$\boldsymbol{f}_{k+1} = \boldsymbol{X}_k(h_k)\boldsymbol{f}_k\,, \tag{A.7}$$

where $\boldsymbol{f}_k$ denotes $\boldsymbol{f}(\boldsymbol{x}_k)$. Therefore, we can select the orthogonal matrices Q_k as

$$Q_k = \left(\frac{\boldsymbol{f}_k}{\|\boldsymbol{f}_k\|} \quad S_k\right)\,, \tag{A.8}$$

where $S_k \in \mathbb{R}^{N\times(N-1)}$ are orthonormal matrices. With minor modifications of steps **(i)**, **(ii)**, **(iii)**, we can iteratively obtain appropriate S_k as follows.

(i′) Construct an arbitrarily S_0 orthonormal to $\boldsymbol{f}_0$ so that Q_0 is orthogonal.

(ii′) Using Gram-Schmidt orthonormalization, decompose $\boldsymbol{X}_k(h_k)Q_k$ as

$$\begin{aligned}
\boldsymbol{X}_k(h_k)\,Q_k &= \left(\tfrac{1}{\|\boldsymbol{f}_k\|}\boldsymbol{X}_k(h_k)\boldsymbol{f}_k \quad \boldsymbol{X}_k(h_k)S_k\right)\\
&= \left(\tfrac{1}{\|\boldsymbol{f}_k\|}\boldsymbol{f}_{k+1} \quad \boldsymbol{X}_k(h_k)S_k\right)\\
&= \left(\tfrac{\|\boldsymbol{f}_{k+1}\|}{\|\boldsymbol{f}_k\|}\tfrac{\boldsymbol{f}_{k+1}}{\|\boldsymbol{f}_{k+1}\|} \quad \boldsymbol{X}_k(h_k)S_k\right)\\
&= \left(\tfrac{\boldsymbol{f}_{k+1}}{\|\boldsymbol{f}_{k+1}\|} \quad S_{k+1}\right)\begin{pmatrix}\tfrac{\|\boldsymbol{f}_{k+1}\|}{\|\boldsymbol{f}_k\|} & \boldsymbol{b}_k^{\mathrm{T}}\\ 0 & B_k\end{pmatrix}\\
&= Q_{k+1}A_k\,,
\end{aligned} \tag{A.9}$$

where $B_k \in \mathbb{R}^{(N-1)\times(N-1)}$ are upper triangular matrices and $\boldsymbol{b}_k \in \mathbb{R}^{(N-1)}$ for $k = 0, 1, \cdots, M$.

(iii′) If the convergence condition (A.5) is satisfied, finish the iteration and recompute A_M as (A.3), or else substitute S_{M+1} into S_0 and return step **(ii′)**.

As stated in Section 2.3, we can compute each Floquet multiplier λ_j as j-th eigenvalue of the conjugate matrix Λ, namely, the product of elements $A_k(j, j)$ of the upper triangular matrices A_k. It should be noted that this decomposition of $\boldsymbol{X}_0(T)$ for autonomous systems gives the unit Floquet multiplier λ_{u} as

$$\lambda_{\mathrm{u}} = \prod_{k=0}^{M} A_k(1,1) = \prod_{k=0}^{M}\frac{\|\boldsymbol{f}_{k+1}\|}{\|\boldsymbol{f}_k\|} = \frac{\|\boldsymbol{f}_{M+1}\|}{\|\boldsymbol{f}_0\|} = 1\,. \tag{A.10}$$

Acknowledgements

The author would especially like to thank Professor Sunao Murashige at Future University-Hakodate for his frequent encouragements and suggestions over five long years. During this time, the author received continuous guidance to explore the numerical analysis of nonlinear dynamical systems.

The author would also like to express his gratitude to Professor Hirosuke Yamamoto at the University of Tokyo for his full support during the author's doctoral research. This work could not have been accomplished without his support.

In addition, the author is grateful to the other University of Tokyo faculty members who served on the author's doctoral committee – Professor Mitsuhiro Toriumi, Professor Masato Okada, and Associate Professor Takayasu Matsuo. Their valuable suggestions are deeply appreciated and contributed to the writing of this thesis.

The author appreciates the comments and discussions from the many members of the laboratory. In particular, to fellow doctoral students Mr. Elliot T. Taniguchi and Mr. Kunihiko Harada.

Finally, the author thanks his parents for supporting for his education.

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