Formulæ for sn $9u$.

By

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The following calculations of sn $9u$ were made to show the advantage of the method of finding the multiplication-formulæ of elliptic functions, given by Prof. Fujisawa in the second part of his paper *Researches on the Multiplication of Elliptic Functions* (this journal, vol. VI, pp. 151–226). For the notation adopted, as well as for a full account of the method of calculation, reference is to be made to that paper.

Considering the numerator and denominator of sn $9u$ as expressed in terms of $a\left(=k+\dfrac{1}{k}\right)$ and $\xi\,(=\sqrt{k}\ \mathrm{sn}\ u)$, and putting $n=9$, $q=10$ in the general formulæ for $H_q, H_{q-1}, H_{q-2}, H_{q-3}$, the values of H_{10}, H_9, H_8, H_7, were found without much difficulty, and, thence, by integrating differential equations (117) (*loc. cit.* p. 202), H_6, H_5, H_4, H_3, were successively obtained. On the other hand, H_0 which corresponds to H for $k^2=-1$, was derived from the expressions of sn $4\,(u, i)$ and sn $5\,(u, i)$ by addition, and, then, by again making use of the same differential equations, but this time in the reverse order, H_1, H_2, H_3, were got by successive differentiations. The agreement of the values of H_3 deduced in two different ways verified the results.

The value of E is at once obtained from that of H in virtue of equation (111) (*loc. cit.* p. 200).

Finally, the denominator of sn $9u$ was transformed into its usual form, where the variables are taken to be k and x ($=\text{sn}\,u$). The result was verified by comparing it with that derived from formulæ (146) and (150) (*loc. cit.* p. 215 and pp. 217–218).

In every case the calculation was performed in duplicate, by myself and Mr. Fujii, to whom I owe my best thanks. The results are given in the accompanying two tables, in which the mode of arrangement is obvious.

Table of H and E for sn $9u$.

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	
ξ^{0}	+ 1											ξ^{80}
ξ^{2}												ξ^{78}
ξ^{4}	− 540											ξ^{76}
ξ^{6}		+ 5544										ξ^{74}
ξ^{8}	− 7722		− 28512									ξ^{72}
ξ^{10}		+ 118800		+ 82368								ξ^{70}
ξ^{12}	− 275724		− 570960		− 139776							ξ^{68}
ξ^{14}		+ 2075976		+ 1322496		+ 138240						ξ^{66}
ξ^{16}	− 214731		− 6262272		− 1624320		− 73728					ξ^{64}
ξ^{18}		− 2747520		+ 9300480		+ 1019904		+ 16384				ξ^{62}
ξ^{20}	+ 7035984		+ 28003968		− 6773760		− 258048					ξ^{60}
ξ^{22}		− 73907424		− 94938624		+ 1935360						ξ^{58}
ξ^{24}	+ 40109256		+ 300769920		+ 173209344							ξ^{56}
ξ^{26}		− 295009344		− 673972992		− 202051584						ξ^{54}
ξ^{28}	+ 83514960		+ 957960000		+ 976181760		+ 165703680					ξ^{52}
ξ^{30}		− 562384608		− 1884824064		− 988793856		− 97517568				ξ^{50}
ξ^{32}	+ 135340722		+ 1807764480		+ 2501100288		+ 703217664		+ 38928384			ξ^{48}
ξ^{34}		− 999105408		− 3498582528		− 2240372736		− 331776000		− 9437184		ξ^{46}
ξ^{36}	+ 261189624		+ 3161982336		+ 4179875840		+ 1282842624		+ 93192192		+ 1048576	ξ^{44}
ξ^{38}		− 1679225040		− 5240563200		− 2992066560		− 424673280		− 11796480		ξ^{42}
ξ^{40}	+ 366014340		+ 4101762240		+ 4765582080		+ 1187020800		+ 61931520			ξ^{40}
ξ^{42}		− 1674487200		− 4811760000		− 2287411200		− 201523200				ξ^{38}
ξ^{44}	+ 256239000		+ 2694880800		+ 2716070400		+ 455270400					ξ^{36}
ξ^{46}		− 689919120		− 1653696000		− 503331840						ξ^{34}
ξ^{48}	+ 46878210		+ 276687360		− 56851200		− 90316800					ξ^{32}
ξ^{50}		+ 158433408		+ 721073664		+ 409522176		+ 22560768				ξ^{30}
ξ^{52}	− 57378672		− 721664640		− 766983168		− 116895744		− 3538944			ξ^{28}
ξ^{54}		+ 305542048		+ 758865408		+ 256198656		+ 20971520		+ 262144		ξ^{26}
ξ^{56}	− 49155768		− 417280896		− 306789120		− 53968896		− 1769472			ξ^{24}
ξ^{58}		+ 119517120		+ 215205120		+ 78852096		+ 5308416				ξ^{22}
ξ^{60}	− 13546224		− 87023808		− 71723520		− 9326592					ξ^{20}
ξ^{62}		+ 18343584		+ 41900544		+ 10644480						ξ^{18}
ξ^{64}	− 1537011		− 15759360		− 8273664							ξ^{16}
ξ^{66}		+ 3688320		+ 4465152								ξ^{14}
ξ^{68}	− 432828		− 1634688									ξ^{12}
ξ^{70}		+ 357480		− 4608								ξ^{10}
ξ^{72}	− 32106		+ 9120		+ 256							ξ^{8}
ξ^{74}		− 5616		− 576								ξ^{6}
ξ^{76}	+ 1044		+ 432									ξ^{4}
ξ^{78}		− 120										ξ^{2}
ξ^{80}	+ 9											ξ^{0}
	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	

The Common Denominator of sn $9u$, cn $9u$, dn $9u$ =

x^0	+		k^0																x^0
x^2																			x^2
x^4	−	540	k^2																x^4
x^6	+	5544	k^2+k^4																x^6
x^8	−	28512	k^2+k^6	−	64746	k^4													x^8
x^{10}	+	82368	k^2+k^8	+	365904	k^4+k^6													x^{10}
x^{12}	−	139776	k^2+k^{10}	−	1130064	k^4+k^8	−	2256300	k^6										x^{12}
x^{14}	+	138240	k^2+k^{12}	+	2013696	k^4+k^{10}	+	7425864	k^6+k^8										x^{14}
x^{16}	−	73728	k^2+k^{14}	−	2066688	k^4+k^{12}	−	13865472	k^6+k^{10}	−	23959755	k^8							x^{16}
x^{18}	+	16384	k^2+k^{16}	+	1134592	k^4+k^{14}	+	14744064	k^6+k^{12}	+	35926400	k^8+k^{10}							x^{18}
x^{20}	−	258048	k^4+k^{16}	−	8322048	k^6+k^{14}	−	2961792	k^8+k^{12}	+	17240400	k^{10}							x^{20}
x^{22}	+	1935360	k^6+k^{16}	−	85261824	k^8+k^{14}	−	339369696	$k^{10}+k^{12}$										x^{22}
x^{24}	+	173209344	k^8+k^{16}	+	993607296	$k^{10}+k^{14}$	+	1680905160	k^{12}										x^{24}
x^{26}	−	202051584	k^8+k^{18}	−	1684230912	$k^{10}+k^{16}$	−	4337444160	$k^{12}+k^{14}$										x^{26}
x^{28}	+	165703680	k^8+k^{20}	+	1970403840	$k^{10}+k^{18}$	+	7348242240	$k^{12}+k^{16}$	+	11170599120	k^{14}							x^{28}
x^{30}	−	97517568	k^8+k^{22}	−	1671416832	$k^{10}+k^{20}$	−	8876662272	$k^{12}+k^{18}$	−	19517910240	$k^{14}+k^{16}$							x^{30}
x^{32}	+	38928384	k^8+k^{24}	+	1014644736	$k^{10}+k^{22}$	+	7810401024	$k^{12}+k^{20}$	+	24540420096	$k^{14}+k^{18}$	+	35546811570	k^{16}				x^{32}
x^{34}	−	9437184	k^8+k^{26}	−	416710656	$k^{10}+k^{24}$	−	4902543360	$k^{12}+k^{22}$	−	22460465664	$k^{14}+k^{20}$	−	46699825536	$k^{16}+k^{18}$				x^{34}
x^{36}	+	1048576	k^8+k^{28}	+	103677952	$k^{10}+k^{26}$	+	2075566080	$k^{12}+k^{24}$	+	14612142080	$k^{14}+k^{22}$	+	44563088768	$k^{16}+k^{20}$	+	64108956408	k^{18}	x^{36}
x^{38}	−	11796480	$k^{10}+k^{28}$	−	530841600	$k^{12}+k^{26}$	−	6389452800	$k^{14}+k^{24}$	−	30109939200	$k^{16}+k^{22}$	−	63671501520	$k^{18}+k^{20}$				x^{38}
x^{40}	+	61931520	$k^{12}+k^{28}$	+	1682472960	$k^{14}+k^{26}$	+	13621789440	$k^{16}+k^{24}$	+	44437567680	$k^{18}+k^{22}$	+	65238653700	k^{20}				x^{40}
x^{42}	−	201523200	$k^{14}+k^{28}$	−	3698073600	$k^{16}+k^{26}$	−	20480803200	$k^{18}+k^{24}$	−	46037191200	$k^{20}+k^{22}$							x^{42}
x^{44}	+	455270400	$k^{16}+k^{28}$	+	5447692800	$k^{18}+k^{26}$	+	20388218400	$k^{20}+k^{24}$	+	31047831000	k^{22}							x^{44}
x^{46}	−	503331840	$k^{18}+k^{28}$	−	4170355200	$k^{20}+k^{26}$	−	10684325520	$k^{22}+k^{24}$										x^{46}
x^{48}	−	90316800	$k^{18}+k^{30}$	−	598752000	$k^{20}+k^{28}$	−	1305469440	$k^{22}+k^{26}$	−	1547190270	k^{24}							x^{48}
x^{50}	+	22560768	$k^{18}+k^{32}$	+	567447552	$k^{20}+k^{30}$	+	3242460672	$k^{22}+k^{28}$	+	7206503040	$k^{24}+k^{26}$							x^{50}
x^{52}	−	3538944	$k^{18}+k^{34}$	−	145207296	$k^{20}+k^{32}$	−	1567448064	$k^{22}+k^{30}$	−	5741214336	$k^{24}+k^{28}$	−	8688247920	k^{26}				x^{52}
x^{54}	+	262144	$k^{18}+k^{36}$	+	23330816	$k^{20}+k^{34}$	+	412436480	$k^{22}+k^{32}$	+	2502280704	$k^{24}+k^{30}$	+	5911158176	$k^{26}+k^{28}$				x^{54}
x^{56}	−	1769472	$k^{20}+k^{36}$	−	68124672	$k^{22}+k^{34}$	−	680147712	$k^{24}+k^{32}$	−	2553061248	$k^{26}+k^{30}$	−	3927693240	k^{28}				x^{56}
x^{58}	+	5308416	$k^{22}+k^{36}$	+	116011008	$k^{24}+k^{34}$	+	720942336	$k^{26}+k^{32}$	+	1739448000	$k^{28}+k^{30}$							x^{58}
x^{60}	−	9326592	$k^{24}+k^{36}$	−	127683072	$k^{26}+k^{34}$	−	513816768	$k^{28}+k^{32}$	−	80446680	k^{30}							x^{60}
x^{62}	+	10644480	$k^{26}+k^{36}$	+	95122944	$k^{28}+k^{34}$	+	250490016	$k^{30}+k^{32}$										x^{62}
x^{64}	−	8273664	$k^{28}+k^{36}$	−	48854016	$k^{30}+k^{34}$	−	82697715	k^{32}										x^{64}
x^{66}	+	4465152	$k^{30}+k^{36}$	+	17083776	$k^{32}+k^{34}$													x^{66}
x^{68}	−	1634688	$k^{32}+k^{36}$	−	3702204	k^{34}													x^{68}
x^{70}	−	4608	$k^{32}+k^{38}$	+	343656	$k^{34}+k^{36}$													x^{70}
x^{72}	+	256	$k^{32}+k^{40}$	+	10144	$k^{34}+k^{38}$	−	12330	k^{36}										x^{72}
x^{74}	−	576	$k^{34}+k^{40}$	−	7344	$k^{36}+k^{38}$													x^{74}
x^{76}	+	432	$k^{36}+k^{40}$	+	1908	k^{38}													x^{76}
x^{78}	−	120	$k^{38}+k^{40}$																x^{78}
x^{80}	+	9	k^{40}																x^{80}