

Formulæ for $\text{sn } 10u$, $\text{cn } 10u$, $\text{dn } 10u$ in terms of $\text{sn } u$.

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Formulæ for $\text{sn } 10u$, $\text{cn } 10u$, $\text{dn } 10u$ in terms of $x = \text{sn } u$ are given in the following four tables, in which the mode of arrangement will be apparent on inspection. They were calculated by two entirely different methods.

Firstly, they were deduced from the well-known equations :

$$\begin{aligned} \text{sn } 10u &= \frac{x \sqrt{1-x^2} \sqrt{1-k^2x^2} A(x^2)}{D(x^2)} = \frac{2 P Q R S}{S^4 - k^2 P^4}, \\ \text{cn } 10u &= \frac{B(x^2)}{D(x^2)} = \frac{S^4 - 2 S^2 P^2 + k^2 P^4}{S^4 - k^2 P^4}, \\ \text{dn } 10u &= \frac{C(x^2)}{D(x^2)} = \frac{S^4 - 2 k^2 S^2 P^2 + k^2 P^4}{S^4 - k^2 P^4}, \end{aligned}$$

where A, B, C, D denote algebraic rational integral functions of x^2 and

$$\begin{aligned} P = x \{ &5 - (20 + 20 k^2) x^2 + (16 + 94 k^2 + 16 k^4) x^4 - (80 k^2 + 80 k^4) x^6 - 105 k^4 x^8 \\ &+ (360 k^4 + 360 k^6) x^{10} - (240 k^4 + 780 k^6 + 240 k^8) x^{12} + (64 k^4 + 560 k^6 \\ &+ 560 k^8 + 64 k^{10}) x^{14} - (160 k^6 + 445 k^8 + 160 k^{10}) x^{16} + (140 k^8 + 140 k^{10}) x^{18} \\ &- 50 k^{10} x^{20} + k^{12} x^{24} \}, \end{aligned}$$

$$\begin{aligned} Q = \sqrt{1-x^2} \{ &1 - 12 x^2 + (16 + 50 k^2) x^4 - (80 k^2 + 140 k^4) x^6 + (335 k^4 + 160 k^6) x^8 \\ &- (264 k^4 + 464 k^6 + 64 k^8) x^{10} + (208 k^4 + 508 k^6 + 208 k^8) x^{12} - (64 k^4 \\ &+ 464 k^6 + 264 k^8) x^{14} + (160 k^6 + 335 k^8) x^{16} - (140 k^8 + 80 k^{10}) x^{18} + (50 k^{10} \\ &+ 16 k^{12}) x^{20} - 12 k^{12} x^{22} + k^{12} x^{24} \}, \end{aligned}$$

$$R = \sqrt{1 - k^2 x^2} \{ 1 - 12 k^2 x^2 + (50 k^2 + 16 k^4) x^4 - (140 k^2 + 80 k^4) x^6 + (160 k^2 + 335 k^4) x^8 - (64 k^2 + 464 k^4 + 264 k^6) x^{10} + (208 k^4 + 508 k^6 + 208 k^8) x^{12} - (264 k^6 + 464 k^8 + 64 k^{10}) x^{14} + (335 k^8 + 160 k^{10}) x^{16} - (80 k^8 + 140 k^{10}) x^{18} + (16 k^8 + 50 k^{10}) x^{20} - 12 k^{10} x^{22} + k^{12} x^{24} \},$$

$$S = 1 - 50 k^2 x^4 + (140 k^2 + 140 k^4) x^6 - (160 k^2 + 445 k^4 + 160 k^6) x^8 + (64 k^2 + 560 k^4 + 560 k^6 + 64 k^8) x^{10} - (240 k^4 + 780 k^6 + 240 k^8) x^{12} + (360 k^6 + 360 k^8) x^{14} - 105 k^8 x^{16} - (80 k^8 + 80 k^{10}) x^{18} + (16 k^8 + 94 k^{10} + 16 k^{12}) x^{20} - (20 k^{10} + 20 k^{12}) x^{22} + 5 k^{12} x^{24}.$$

To obtain PQR , PS was first formed and the result multiplied by Q and R successively. P^4 was got by multiplying P^2 by P twice in succession, S^4 was found by squaring S^2 , and, as a verification, they were substituted in the well-known relation—

$$S(x, k) = P\left(\frac{1}{kx}, k\right) k^{2p+1} x^{4p+1}$$

where $4p = n^2 - 1$, n being odd. $P^2 S^2$ was obtained by multiplying P^2 and S^2 and also by squaring PS . In every case, the result was further verified by putting $k^2 = 1$ and $k^2 = i$.

Secondly, A, B, C, D were calculated by a totally independent method given by Professor Fujisawa in his paper entitled "Researches on the Multiplication of Elliptic Functions" (this Journal, vol. VI, pp. 151-226).

In the course of calculation, these functions were considered to be arranged according to the powers of x^2 as well as according to the powers of k^2 . The formulæ to be used in the former case are given in the paper just alluded to, and the corresponding formulæ in the latter case are as follows :

$$A(x^2) = \sum_{r=0}^{r=p} \left\{ \sum_{m=r} A_{2m, 2r} x^{2m} \right\} k^{2r}, \quad n^2 - 4 = 4p,$$

where

$$A_{0,0} = n, \quad A_{2m,0} = (-1)^m \frac{n(n^2-4)(n^2-16)\dots(n^2-4m^2)}{(2m+1)!},$$

$$A_{2,2} = -\frac{n(n^2-4)}{3!}, \quad A_{2m,2} = (-1)^m \frac{n(n^2-4)(n^2-16)\dots\{n^2-4(m-1)^2\}}{(2m+1)!} \\ \times m\{m n^2 - (3m+1)\},$$

and, generally,

$$2m(2m+1)A_{2m,2r} + \{n^2(4r+1) - 4m^2\}A_{2m-2,2r} \\ = \{n^2(4r-3) - 2n^2(2m-1) + 4m^2\}A_{2m-2,2r-2} - (n^2 - 2m + 1)(n^2 - 2m)A_{2m-4,2r-2}.$$

$$D(x^2) = \sum_{r=0}^{r=p} \left\{ \sum_{m=r} D_{2m,2r} x^{2m} \right\} k^{2r}, \quad n^2 = 4p,$$

where

$$D_{0,0} = 1, \quad D_{2m,0} = 0,$$

$$D_{2,2} = 0, \quad D_{2m,2} = (-1)^{m-1} 2 \cdot 4^{m-2} \frac{n^2(n^2-1)(n^2-4)\dots\{n^2-(m-1)^2\}}{2m!},$$

$$D_{4,4} = 0, \quad D_{2m,4} = (-1)^{m-1} 4^{m-3} \frac{n^2(n^2-1)(n^2-4)\dots\{n^2-(m-2)^2\}}{2m!} \\ \times \{[2m^2 - 5(m-1)]n^2 - (m-1)(7m-5)\},$$

and, generally,

$$2m(2m-1)D_{2m,2r} + \{4rn^2 - 4(m-1)^2\}D_{2m-2,2r} \\ = \{n^2(4r-4) - 2n^2(2m-2) + 4(m-1)^2\}D_{2m-2,2r-2} \\ - (n^2 - 2m + 4)(n^2 - 2m + 3)D_{2m-4,2r-2}.$$

$$B(x^2) = \sum_{r=0}^{r=p} \left\{ \sum_{m=r} B_{2m,2r} x^{2m} \right\} k^{2r}, \quad n^2 = 4p,$$

where

$$B_{0,0} = 1, \quad B_{2m,0} = (-1)^m \frac{n^2(n^2-4)(n^2-16)\dots\{n^2-4(m-1)^2\}}{2m!},$$

$$B_{2,2} = 0, \quad B_{2m,2} = (-1)^m \frac{n^2(n^2-4)(n^2-16)\dots\{n^2-4(m-2)^2\}}{2m!} \\ \times m(m-1)(n^2-1),$$

and, generally,

$$2m(2m-1) B_{2m,2r} + \{n^2(4r+1) - 4(m-1)^2\} B_{2m-2,2r} \\ = \{n^2(4r-4) - 2n^2(2m-2) + 4(m-1)^2\} B_{2m-2,2r-2} \\ - (n^2-2m+4)(n^2-2m+3) B_{2m-4,2r-2}.$$
