Researches on the Multiplication of Elliptic Functions.

By

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Jacobi, in one of the *Suites des notices sur les fonctions elliptiques*, gives, without demonstration, remarkable expressions of sn $2u$, sn $3u$, sn $4u$ and sn $5u$ in terms of the differential coefficients of $\sqrt{x^2(1-x^2)(1-k^2x^2)}$ and $\sqrt{\dfrac{(1-x^2)(1-k^2x^2)}{x^2}}$ taken with respect to x^2, whereby x stands for sn u.* Namely, writing

$$\sqrt{x^2(1-x^2)(1-k^2x^2)} = A, \quad \sqrt{\frac{(1-x^2)(1-k^2x^2)}{x^2}} = B,$$

we have

$$\mathrm{sn}\, 2u = -\frac{1}{x^2}\,\frac{1}{\dfrac{dB}{d(x^2)}},$$

$$\mathrm{sn}\, 3u = -\,x^3\,\frac{\dfrac{d^2B}{d(x^2)^2}}{\dfrac{d^2A}{d(x^2)^2}},$$

$$\mathrm{sn}\, 4u = -\frac{1}{x^4}\,\frac{\dfrac{1}{2.3}\dfrac{d^3A}{d(x^2)^3}}{\dfrac{1}{2}\dfrac{d^2B}{d(x^2)^2}\,\dfrac{1}{2}\dfrac{d^2B}{d(x^2)^2}-\dfrac{dB}{d(x^2)}\,\dfrac{1}{2.3}\dfrac{d^3B}{d(x^2)^3}},$$

$$\mathrm{sn}\, 5u = x^5\,\frac{\dfrac{1}{2.3}\dfrac{d^3B}{d(x^2)^3}\,\dfrac{1}{2.3}\dfrac{d^3B}{d(x^2)^3}-\dfrac{1}{2}\dfrac{d^2B}{d(x^2)^2}\,\dfrac{1}{2.3.4}\dfrac{d^4B}{d(x^2)^4}}{\dfrac{1}{2.3}\dfrac{d^3A}{d(x^2)^3}\,\dfrac{1}{2.3}\dfrac{d^3A}{d(x^2)^3}-\dfrac{1}{2}\dfrac{d^2A}{d(x^2)^2}\,\dfrac{1}{2.3.4}\dfrac{d^4A}{d(x^2)^4}}.$$

* Crelle's Journal, Bd. IV, pp. 185-193, and Jacobi's gesammelte Werke, Bd. I, pp. 266-275.

Jacobi adds "*la loi général de la composition de ces expressions est aisée à saisir*," and further remarks that we shall have analogous formulæ by using, instead of the differential coefficients of A and B, those of

$$\frac{1}{\sqrt{x^2(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad \frac{1}{\sqrt{\frac{(1-x^2)(1-k^2x^2)}{x^2}}}.$$

The general form of these expressions is, however, by no means easy to infer from the particular cases just given, and I have tried to trace among the writings of Jacobi, the steps which might have led him to these expressions, but without success.

The present memoir is divided into two parts. In the first part, the multiplication-formulæ of elliptic functions are derived from Abel's theorem for the elliptic integral of the first kind. It will be seen that one of the results arrived at is the general formula in question. It appears, however, highly improbable that Jacobi obtained his formulæ in this way.

In the paper just alluded to, Jacobi gives, also without demonstration, the partial differential equation satisfied by the numerators and denominator of the multiplication-formulæ. This partial differential equation has since been obtained by Betti,* Cayley,** Briot et Bouquet,† and others; but the final results to be obtained by applying it to the actual evaluation of the numerical constants involved in the multiplication-formulæ, has not, to my knowledge, hitherto been developed with much completeness or success.

In the second part, Jacobi's partial differential equation is derived in a manner which is most probably the one followed by Jacobi

* Betti, Annali di Matematica, Vol. IV, p. 32.

** Cayley, Cambridge and Dublin Mathematical Journal, Vol. II, pp. 256-266.

† Briot et Bouquet, Theorie des Fonctions Elliptiques, p. 529.

himself, and then applied to the investigation of the multiplication-formulæ. The present paper is thus composed of two parts, each of them having reference to the theory of multiplication but otherwise unconnected.

Throughout the paper, I have adopted the notation of *Fund. Nova*, the only exception being that I write θ_1, θ_2, θ_3, θ instead of θ and H, whereby I follow Jacobi in his lectures.*

Part First.

§. 1.

Denote the fundamental elliptic irrationality by s and the corresponding Riemann's surface by T. Let S stand for an algebraic function of z, one-valued on the surface T, and of the order q, having for its zero-points

$$\delta_\mu \{z_\mu \mid s_\mu\}, \quad \mu = 1, 2, 3, \ldots\ldots q,$$

and for its infinity-points

$$\varepsilon_\nu \{\zeta_\nu \mid \sigma_\nu\}, \quad \nu = 1, 2, 3, \ldots\ldots q,$$

then Abel's theorem is expressed by

$$(1.) \qquad \sum_{\mu=1}^{\mu=q} \frac{dz_\mu}{s_\mu} - \sum_{\nu=1}^{\nu=q} \frac{d\zeta_\nu}{\sigma_\nu} = 0.$$

Now S is necessarily of the form

$$\frac{P + Q\,s}{R},$$

where P, Q, R are integral functions of z of the degree q, $q-2$ and q

* Jacobi, Gesammelte Werke, Bd. I, p. 501.

respectively. The numerator $P+Qs$ must be so determined that it vanishes in q points δ_μ and, besides, in another set of q points

$$\varepsilon'_\nu \;\{\zeta_\nu \mid -\sigma_\nu\}, \quad \nu = 1, 2, 3, \ldots\ldots q.$$

It will be convenient to write $\delta_{\mu+\nu}$ instead of ε'_ν, and then $P+Qs$ vanishes in $2q$ points

$$\delta_\mu \;\{z_\mu \mid s_\mu\}, \qquad \mu = 1, 2, 3, \ldots q, q+1, \ldots 2q.$$

Equation (1.) then takes the form

$$\text{(2.)} \qquad \sum_{\mu=1}^{\mu=2q} \frac{dz_\mu}{s_\mu} = 0.$$

We now take for the fundamental irrationality Riemann's form

$$\sqrt{z.\overline{1-z}.\overline{1-k^2z}}\,,$$

and write

$$2u = \int_0^z \frac{dz}{\sqrt{z.\overline{1-z}.\overline{1-k^2z}}},$$

so that

$$\sqrt{z} = \operatorname{sn} u\,.$$

Two cases are to be distinguished according as n is odd or even.

§. 2.

When n is odd, put $n=2m+1=2q-1$, and let one of the $2q$ points in which $P+Qs$ vanishes, coincide with the point $\{\zeta \mid \sigma\}$ and the remaining $(2q-1)$ points with the point $\{z \mid s\}$. Equation (2.) then becomes

$$(3) \qquad \frac{d\zeta}{\sigma} + n\frac{dz}{s} = 0, \quad n \text{ odd};$$

and, writing $P+Qs$ in full,

$$P+Qs = (a_0 + a_1 z + a_2 z^2 + \ldots + a_{m+1} z^{m+1}) + (b_0 + b_1 z + b_2 z^2 + \ldots + b_{m-1} z^{m-1})s,$$

we must have, denoting differentiation with respect to z by D,

$$(4)\left\{\begin{array}{r}
a_0 + a_1\zeta + a_2\zeta^2 + \ldots + a_{m+1}\zeta_{m+1} + b_0\sigma + b_1(\sigma\zeta) + \ldots + b_{m-1}(\sigma\zeta^{m-1}) = 0, \\
a_0 + a_1 z + a_2 z^2 + \ldots + a_{m+1} z^{m+1} + b_0 s + b_1(sz) + \ldots + b_{m-1}(sz^{m-1}) = 0, \\
a_1 + a_2 2z + \ldots + a_{m+1}(m+1)z^m + b_0 Ds + b_1 D(sz) + \ldots + b_{m-1} D(sz^{m-1}) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
b_0 D^{m+2}s + b_1 D^{m+2}(sz) + \ldots \\
+ b_{m-1} D^{m+2}(sz^{m-1}) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
b_0 D^{2m}s + b_1 D^{2m}(sz) + \ldots \\
+ b_{m-1} D^{2m}(sz^{m-1}) = 0.
\end{array}\right.$$

Since $a_0, a_1, \ldots\ldots, b_0, b_1 \ldots$ do not all vanish, the determinant obtained by eliminating $a_0, a_1, \ldots\ldots, b_0, b_1 \ldots$ must vanish, that is,

$$(5) \qquad \left|\begin{array}{lllll}
1, \zeta, \zeta^2, \ldots\ldots \zeta^{m+1}, & \sigma, & \sigma\zeta, & \sigma\zeta^2, & \ldots\ldots \sigma\zeta^{m-1} \\
1, z, z^2, \ldots\ldots z^{m+1}, & s, & sz, & sz^2, & \ldots\ldots sz^{m-1} \\
\quad 1, 2z, \ldots\ldots (m+1)z^m, & Ds, & D(sz), & D(sz^2), & \ldots\ldots D(sz^{m-1}) \\
\dots\dots\dots\dots & & & & \\
\quad (m+1)\,!\,, & D^{m+1}s, & D^{m+1}(sz), & D^{m+1}(sz^2), & \ldots\ldots D^{m+1}(sz^{m-1}) \\
 & D^{m+2}s, & D^{m+2}(sz), & D^{m+2}(sz^2), & \ldots\ldots D^{m+2}(sz^{m-1}) \\
 & D^{m+3}s, & D^{m+3}(sz), & D^{m+3}(sz^2), & \ldots\ldots D^{m+3}(sz^{m-1}) \\
 & \dots\dots & & & \\
 & D^{2m}s, & D^{2m}(sz), & D^{2m}(sz^2), & \ldots\ldots D^{2m}(sz^{m-1})
\end{array}\right| = 0.$$

Let the expansion of this determinant according to the elements of the first row be written,

$$(6)\quad \{P_0+P_1\zeta+\ldots+P_{m+1}\zeta^{m+1}\}+\{Q_0+Q_1\zeta+\ldots+Q_{m-1}\zeta^{m-1}\}\sigma=0.$$

Then $(2m+2)$ roots of the equation

$$\{P_0+P_1Z+\ldots+P_{m+1}Z^{m+1}\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2.Z(1-Z)(1-k^2Z)=0$$

are ζ, and z repeated $(2m+1)$ times. Thus we obtain the following identity :

$$(7)\quad \{P_0+P_1Z+\ldots+P_{m+1}Z^{m+1}\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2.Z(1-Z)(1-k^2Z)$$
$$\equiv P_{m+1}^2(Z-\zeta)\,(Z-z)^{2m+1}.$$

Herein putting $Z=0$, $Z=1$, $Z=\frac{1}{k^2}$ successively and reducing, we obtain

$$(8)\quad \begin{cases} \sqrt{\zeta}\;\; z^{\frac{2m+1}{2}}=\dfrac{P_0}{P_{m+1}}, \\[2ex] \sqrt{1-\zeta}\;\;(1-z)^{\frac{2m+1}{2}}=\dfrac{P_0+P_1+\ldots+P_{m+1}}{P_{m+1}}, \\[2ex] \sqrt{1-k^2\zeta}\;\;(1-k^2z)^{\frac{2m+1}{2}}=\dfrac{P_0k^{2m+2}+P_1k^{2m}+\ldots+P_{m+1}}{P_{m+1}}. \end{cases}$$

In extracting square root, strictly speaking, we have to prefix the double sign $\pm$; but by taking some particular value of n or by putting $k=0$ in the final results to be hereafter obtained, it comes out that we have to take the $+$ sign.

§ 3.

Let us now investigate the expressions which occur on the right-hand side of equation (8). For this purpose, put

$$(9.)\qquad \Delta \equiv \begin{vmatrix} D^{m+1}s, & D^{m+1}sz, & \ldots & D^{m+1}sz^{m-1} \\ D^{m+2}s, & D^{m+2}sz, & \ldots\ldots & D^{m+2}sz^{m-1} \\ \ldots & \ldots & \ldots & \ldots \\ D^{2m}s, & D^{2m}sz, & \ldots\ldots & D^{2m}sz^{m-1} \end{vmatrix};$$

further let the expansion of Δ according to the elements of the first row be written :

$$(10.)\qquad \Delta \equiv D^{m+1}s\,.\,\Delta_1 + D^{m+1}sz\,.\,\Delta_2 + \ldots + D^{m+1}sz^{m-1}\,.\,\Delta_m\,.$$

Now

$$(11.)\qquad P_0 = \begin{vmatrix} z, z^2, \ldots & z^{m+1}, & s, & sz, \ldots\ldots\ldots & sz^{m-1} \\ 1, 2z, \ldots (m+1)z^m, & Ds, & & Dsz, \ldots\ldots\ldots & Dsz^{m-1} \\ \ldots & \ldots & & \ldots & \ldots \\ & (m+1)\,!\,, & D^{m+1}s, & D^{m+1}sz, \ldots\ldots & D^{m+1}sz^{m-1} \\ & & D^{m+2}s, & D^{m+2}sz, \ldots\ldots & D^{m+2}sz^{m-1} \\ & & \ldots & \ldots & \ldots \\ & & D^{2m}s, & D^{2m}sz, \ldots\ldots & D^{2m}sz^{m-1} \end{vmatrix}$$

$$= \Delta \begin{vmatrix} z, z^2, \ldots\ldots & z^m, & z^{m+1} \\ 1, 2z, \ldots\ldots & mz^{m-1}, & (m+1)z^m \\ \ldots & \ldots & \ldots \\ & m\,!\,, & (m+1)!\,z \end{vmatrix} - (m+1)!\Delta_1 \begin{vmatrix} z, z^2, \ldots\ldots & z^m, & s \\ 1, 2z, \ldots\ldots & mz^{m-1}, & Ds \\ \ldots & \ldots & \ldots \\ & m\,!\,, & D^m s \end{vmatrix}$$

$$- (m+1)!\Delta_2 \begin{vmatrix} z, z^2, \ldots\ldots & z^m, & sz \\ 1, 2z, \ldots\ldots & mz^{m-1}, & Dsz \\ \ldots & \ldots & \ldots \\ & m\,!\,, & D^m sz \end{vmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (m+1)!\Delta_m \begin{vmatrix} z, z^2, \ldots\ldots & z^m, & sz^{m-1} \\ 1, 2z, \ldots\ldots & mz^{m-1}, & Dsz^{m-1} \\ \ldots & \ldots & \ldots \\ & m\,!\,, & D^m sz^{m-1} \end{vmatrix}.$$

Writing, for shortness,

$$1!\,2!\,\ldots\,m! \equiv m!!,$$

we have

$$\begin{vmatrix} z, z^2, \ldots\ldots & z^m, & z^{m+1} \\ 1, 2z, \ldots\ldots mz^{m-1}, & (m+1)z^m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & m!, & (m+1)!\,z \end{vmatrix} = m!!\,z^{m+1};$$

also

$$\begin{vmatrix} z, z^2, \ldots & z^m, & s \\ 1, 2z, \ldots mz^{m-1}, & Ds \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & m!, & D^m s \end{vmatrix} = \begin{vmatrix} 1, 1, \ldots & 1, & s \\ 1, 2, \ldots & m, & zDs \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & m!, & z^m D^m s \end{vmatrix}$$

$$= (m-1)!!\,\{z^m D^m s - mz^{m-1}D^{m-1}s + \ldots + (-1)^m m!\,s\}$$

$$= (m-1)!!\,z^{m+1}D^m\left(\frac{s}{z}\right);$$

similarly

$$\begin{vmatrix} z, z^2, \ldots\ldots & z^m, & sz \\ 1, 2z, \ldots mz^{m-1}, & Dsz \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & m!, & D^m sz \end{vmatrix} = (m-1)!!\,z^{m+1}D^m s,$$

$$\begin{vmatrix} z, z^2, \ldots\ldots & z^m, & sz^2 \\ 1, 2z, \ldots mz^{m-1}, & Dsz^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & m!, & D^m sz^2 \end{vmatrix} = (m-1)!!\,z^{m+1}D^m sz,$$

....

$$\begin{vmatrix} z, z^2, \ldots & z^m, & sz^{m-1} \\ 1, 2z, \ldots mz^{m-1}, & & D(sz^{m-1}) \\ \ldots\ldots & & \ldots\ldots \\ & m\,!\,, & D^m(sz^{m-1}) \end{vmatrix} = (m-1)\,!!\; z^{m+1} D^m(sz^{m-2})\,.$$

Hence

$$(12.)\quad P_0 = m\,!!\; z^{m+1}\left\{\Delta - (m+1)\left[D^m\left(\frac{s}{z}\right).\Delta_1 + D^m s.\Delta_2 + \ldots + D^m sz^{m-2}.\Delta_m\right]\right\}.$$

Again

$$(13.)\quad P_{m+1} = (-1)^{m+1}\begin{vmatrix} 1, z, z^2, \ldots & z^m, & s, & sz, & \ldots\ldots sz^{m-1} \\ & 1, 2z, \ldots mz^{m-1}, & Ds, & Dsz, & \ldots\ldots Dsz^{m-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ & m!\,, & D^m s, & D^m sz, & \ldots D^m sz^{m-1} \\ & & D^{m+1}s, & D^{m+1}sz, & \ldots D^{m+1}sz^{m-1} \\ & & D^{m+2}s, & D^{m+2}sz, & \ldots D^{m+2}sz^{m-1} \\ & & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ & & D^{2m}s, & D^{2m}sz & \ldots D^{2m}sz^{m-1} \end{vmatrix}$$

$$= (-1)^{m+1}\, m\,!!\; \Delta\,.$$

Substituting the values of P_0 and P_{m+1} just found in the first of equations (8.) and dividing by $z^{\frac{2m+1}{2}}$, we get

$$(14.)\quad \sqrt{\zeta} = (-1)^{m+1}\sqrt{z}\,\frac{\Delta - (m+1)\left\{D^m\left(\frac{s}{z}\right).\Delta_1 + D^m s.\Delta_2 + \ldots + D^m sz^{m-2}.\,\Delta_m\right\}}{\Delta}\,.$$

The expression for Δ given in (9.) may be greatly simplified. By an easy reduction, we find

$$(15.)\quad \Delta = \begin{vmatrix} D^{m+1}s, & (m+1)D^m s, & (m+1)D^{m-1}s, & \ldots\ldots\{(m+1)m\ldots3\}D^2 s \\ D^{m+2}s, & (m+2)D^{m+1}s, & (m+2)(m+1)D^m s, & \ldots\ldots\{(m+2)(m+1)\ldots4\}D^3 s \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ D^{2m}s, & 2mD^{2m-1}s, & 2m(2m-1)D^{2m-2}s, & \ldots\ldots\{2m(2m-1)\ldots(m+2)\}D^{m+1}s \end{vmatrix},$$

that is,

$$(16.)\quad \Delta=(-1)^{\frac{m(m-1)}{2}}\frac{(2m)!!}{m!!}\begin{vmatrix} \frac{1}{2!}D^2s, & \frac{1}{3!}D^3s, \ldots\ldots\ldots & \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3!}D^3s, & \frac{1}{4!}D^4s, \ldots\ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

The expression

$$(m+1)\left\{D^m\frac{s}{z}.\Delta_1+D^ms.\Delta_2+\ldots+D^msz^{m-2}.\Delta_m\right\}$$

which occurs on the righthand side of (12.), admits of being simplified in a similar manner; namely denoting, for a moment, this expression by Δ', we have

$$\Delta'=(m+1)\begin{vmatrix} D^m\frac{s}{z}, & D^ms, \ldots\ldots\ldots & D^msz^{m-2} \\ D^{m+2}s, & D^{m+2}sz, \ldots\ldots & D^{m+2}sz^{m-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ D^{2m}s, & D^{2m}sz, \ldots\ldots & D^{2m}sz^{m-1} \end{vmatrix}$$

$$=(-1)^{\frac{m(m-1)}{2}}\frac{(2m)!!}{m!!}\begin{vmatrix} D\frac{s}{z}, & \frac{1}{2!}D^2\frac{s}{z}, \ldots\ldots\ldots\ldots & \frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{3!}D^3s, & \frac{1}{4!}D^4s, \ldots\ldots\ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

Leaving the factor $(-1)^{m+1}\sqrt{z}\,.\,(-1)^{\frac{m(m-1)}{2}}\frac{(2m)!!}{m!!}$ out of consideration, the numerator of the righthand side of equation (8.) becomes

$$\begin{vmatrix} \frac{1}{2\,!}D^2s, & \frac{1}{3\,!}D^3s, \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}$$

$$-\begin{vmatrix} D\frac{s}{z}, & \frac{1}{2\,!}D^2\frac{s}{z}, \ldots\ldots & \frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

The two determinants may be compounded together; thus we obtain

$$(17.)\qquad z\begin{vmatrix} \frac{1}{2\,!}D^2\left(\frac{s}{z}\right), & \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right) \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

A slightly different form might be given to this expression, viz:

$$(18.)\qquad z^m\begin{vmatrix} \frac{1}{2\,!}D^2\left(\frac{s}{z}\right), & \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right) \\ \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), & \frac{1}{4\,!}D^4\left(\frac{s}{z}\right), \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}\left(\frac{s}{z}\right) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right), & \frac{1}{(m+2)!}D^{m+2}\left(\frac{s}{z}\right), \ldots & \frac{1}{(2m)!}D^{2m}\left(\frac{s}{z}\right) \end{vmatrix}.$$

Let this determinant (omitting the factor z^m) be called M_1 and the determinant on the righthand side of (16.) M. Observe that M and $\triangle$ differ from each other only by a mumerical factor. Equation (14.) now assumes the form :

$$\sqrt{\zeta} = (-1)^{m+1} z^{\frac{n}{2}} \frac{M_1}{M}. \tag{19.}$$

§. 4

Consider now

$$P_0 + P_1 + \dots + P_{m+1}$$

which may be written in the form of a determinant, viz.

$$\sum_{\mu=0}^{\mu=m+1} P_\mu = \left| \begin{array}{cccccccc} 1, & 1, & 1, \dots\dots\dots & 1, & 0, & 0, & \dots\dots & 0 \\ 1 & z & z^2 \dots\dots\dots & z^{m+1}, & s, & sz, & \dots\dots & sz^{m-1} \\ & 1 & 2z \dots\dots\dots & (m+1)z^m, & Ds, & Dsz, & \dots\dots & Dsz^{m-1} \\ & \dots & \dots\dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ & & & (m+1)!, & D^{m+1}s, & D^{m+1}sz, & \dots\dots & D^{m+1}sz^{m-1} \\ & & & & D^{m+2}s, & D^{m+2}sz, & \dots\dots & D^{m+2}sz^{m-1} \\ & & & & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ & & & & D^{2m}s, & D^{2m}sz, & \dots\dots & D^{2m}sz^{m-1} \end{array} \right|. \tag{20.}$$

This determinant may be reduced in exactly the same manner as P_0, and then we find

$$\sum_{\mu=0}^{\mu=m+1} P\mu = \Delta \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 1 \\ 1, z, & z^2, \ldots\ldots & z^m, z^{m+1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, (m+1)z^m \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, (m+1)!z \end{vmatrix}$$

$$-(m+1)!\,\Delta_1 \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, s \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Ds \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m s \end{vmatrix}$$

$$-(m+1)!\,\Delta_2 \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Dsz \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m sz \end{vmatrix}$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(m+1)!\,\Delta_m \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz^{m-1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Dsz^{m-1} \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m sz^{m-1} \end{vmatrix}.$$

Now

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 1 \\ 1, z, & z^2, \ldots\ldots & z^m, z^{m+1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, (m+1)z^m \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, (m+1)!z \end{vmatrix} = (-1)^{m+1} m!!(1-z)^{m+1},$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, & 0 \\ 1, z, & z^2, \ldots\ldots & z^m, & s \\ & 1, 2z, \ldots\ldots & mz^{m-1}, & Ds \\ & \ldots\ldots\ldots\ldots & & \\ & & m\,!\,, & D^m sz \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{s}{1-z},$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, & 0 \\ 1, z, & z^2, \ldots\ldots & z^m, & sz \\ & 1, 2z, \ldots\ldots & mz^{m-1}, & Dsz \\ & \ldots\ldots\ldots\ldots & & \\ & & m\,!\,, & D^m sz \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{sz}{1-z},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, & 0 \\ 1, z, & z^2, \ldots\ldots & z^m, & sz^{m-1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, & Dsz^{m-1} \\ & \ldots\ldots\ldots\ldots & & \\ & & m\,!\,, & D^m s^{m-1} \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{sz^{m-1}}{1-z}.$$

Hence

$$\sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{m+1} m!!(1-z)^{m+1} \Big\{ \Delta + (m+1) \Big[D^m \frac{s}{1-z} \cdot \Delta_1 + D^m \frac{sz}{1-z} \cdot \Delta_2 + \ldots + D^m \frac{sz^{m-1}}{1-z} \cdot \Delta_m \Big] \Big\}$$

$$= (-1)^{m+1} m!!(1-z)^{m+2} \begin{vmatrix} D^{m+1}\frac{s}{1-z}, & D^{m+1}\frac{sz}{1-z}, & \ldots\ldots & D^{m+1}\frac{sz^{m-1}}{1-z} \\ D^{m+2}s, & D^{m+2}sz, & \ldots\ldots & D^{m+2}sz^{m-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ D^{2m}s, & D^{2m}sz, & \ldots\ldots & D^{2m}sz^{m-1} \end{vmatrix}.$$

Further, we deduce without much difficulty

$$(21.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{\frac{m(m-1)}{2}}(-1)^{m+1}(2m)!!(1-z)^{m+2}$$

$$\times \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \cdots\cdots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3 s, & \frac{1}{4\,!}D^4 s, & \cdots\cdots & \frac{1}{(m+2)!}D^{m+2}s \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots\cdots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix},$$

or

$$(22.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{\frac{m(m-1)}{2}}(-1)^{m+1}(2m)!!(1-z)^{2m+1}$$

$$\times \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \cdots\cdots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3\frac{s}{1-z}, & \frac{1}{4\,!}D^4\frac{s}{1-z}, & \cdots\cdots & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z}, & \cdots & \frac{1}{(2m)!}D^{2m}\frac{s}{1-z} \end{vmatrix}.$$

Denoting the determinant on the righthand side of (22.) by M_2 and remembering

$$P_{m+1} = (-1)^{m+1}(2m)\,!!\,M,$$

we get by division

$$\frac{P_0 + P_1 + \ldots + P_{m+1}}{P_{m+1}} = (1-z)^{2m+1}\frac{M_2}{M}\,.$$

Substituting this in the second of equations (8.). we obtain

$$(23.)\qquad \sqrt{1-\zeta} = (1-z)^{\frac{n}{2}}\,\frac{M_2}{M}\,.$$

§. 5.

Let us consider the determinant

$$(24.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = \begin{vmatrix} k^{2m+2}, & k^{2m}, & k^{2m-2}, & \dots & 1, & 0, & 0, & \dots 0 \\ 1, & z, & z^2, & \dots & z^{m+1}, & s, & sz, & \dots sz^{m-1} \\ & 1, & 2z, & \dots & (m+1)z^m, & Ds, & Dsz, & \dots Dsz^{m-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & (m+1)!, & D^{m+1}s, & D^{m+1}sz, & \dots D^{m+1}sz^{m-1} \\ & & & & & D^{m+2}s, & D^{m+2}sz, & \dots D^{m+2}sz^{m-1} \\ & & & & & \dots & \dots & \dots \\ & & & & & D^{2m}s, & D^{2m}sz, & \dots D^{2m}sz^{m-1} \end{vmatrix}.$$

This determinant is of the same form as the one in (20.), and we find likewise

$$\sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = (-1)^{m+1} m\,!!\,(1-k^2z)^{m+1}\Big\{\Delta+(m+1)k^2\Big[D^m\frac{s}{1-k^2z}\cdot\Delta_1$$

$$+D^m\frac{sz}{1-k^2z}\cdot\Delta_2+\dots+D^m\frac{sz^{m-1}}{1-k^2z}\cdot\Delta_m\Big]\Big\}$$

$$=(-1)^{m+1}m!!(1-k^2z)^{m+2}\begin{vmatrix} D^{m+1}\dfrac{s}{1-k^2z}, & D^{m+1}\dfrac{sz}{1-k^2z}, & \dots D^{m+1}\dfrac{sz^{m-1}}{1-k^2z} \\ D^{m+2}s, & D^{m+2}sz, & \dots D^{m+2}sz^{m-1} \\ \dots & \dots & \dots \\ D^{2m}s, & D^{2m}sz, & \dots D^{2m}sz^{m-1} \end{vmatrix};$$

whence

$$(25.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = (-1)^{m+1}(2m)!!(1-k^2z)^{m+2}$$

$$\begin{vmatrix} \dfrac{1}{2\,!}D^2\dfrac{s}{1-k^2z}, & \dfrac{1}{3\,!}D^3\dfrac{s}{1-k^2z}, & \dots \dfrac{1}{(m+1)!}D^{m+1}\dfrac{s}{1-k^2z} \\ \dfrac{1}{3\,!}D^3s, & \dfrac{1}{4\,!}D^4s, & \dots \dfrac{1}{(m+2)!}D^{m+2}s \\ \dots & \dots & \dots \\ \dfrac{1}{(m+1)!}D^{m+1}s, & \dfrac{1}{(m+2)!}D^{m+2}s, & \dots \dfrac{1}{(2m)!}D^{2m}s \end{vmatrix},$$

or

$$(26.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu}=(-1)^{m+1}(-1)^{\frac{m(m-1)}{2}}(2m)\,!!\,(1-k^2z)^{2m+1}$$

$$\times\begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-k^2z}, & \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \cdots\cdots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z} \\ \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \frac{1}{4\,!}D^4\frac{s}{1-k^2z}, & \cdots\cdots & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z}, & \cdots & \frac{1}{(2m)!}D^{2m}\frac{s}{1-k^2z} \end{vmatrix}.$$

Call the determinant on the righthand side of (26.) M_3, we have

$$\frac{P_0k^{2m+2}+P_1k^{2m}+\ldots+P_{m+1}}{P_{m+1}}=(1-k^2z)^{2m+1}\frac{M_3}{M}\,.$$

Substituting this in the last of equations (8.) we obtain

$$(27.)\qquad \sqrt{1-k^2\zeta}=(1-k^2z)^{\frac{n}{2}}\frac{M_3}{M}\,.$$

§. 6.

In case n is even, say $n=2m$, put $m=q-1$, and, referring to §. 1., let one of the $2q$ points in which $P+Qs$ vanishes, coincide with the point $\{\zeta\mid\sigma\}$, another point with $\{0\mid 0\}$, and the remaining $(2q-2)$ points with the point $\{z\mid s\}$. Then

$$(28.)\qquad \frac{d\zeta}{\sigma}+n\frac{dz}{s}=0\quad,\quad n\text{ even,}$$

and

$$P+Qs=(a_1z+a_2z^2+\ldots+a_{m+1}z^{m+1})+(b_0+b_1z+\ldots+b_{m-1}z^{m-1})s,$$

a_0 being zero. If, as will be convenient, we write

$$\frac{\sigma}{\zeta}=\sqrt{\frac{1-\zeta\,.\,1-k^2\zeta}{\zeta}}=\phi\,,\ \frac{s}{z}=\sqrt{\frac{1-z\,.\,1-k^2z}{z}}=u,$$

then $\dfrac{P+Qs}{z}=(a_1+a_2z+\ldots+a_{m+1}z^m)+(b_0+b_1z+\ldots+b_{m-1}z)\,u\,,$

and

$$(29.)\left\{\begin{array}{l}a_1+a_2\zeta+a_3\zeta^2+\ldots+a_{m+1}\zeta^m+b_0\phi+b_1(\phi\zeta)+\ldots+b_{m-1}(\phi\zeta^{m-1})=0,\\ a_1+a_2z+a_3z^2+\ldots+a_{m+1}z^m+b_0u+b_1(uz)+\ldots+b_{m-1}(uz^{m-1})=0,\\ \quad a_2+a_3.2z+\ldots+a_{m+1}.mz^{m-1}+b_0Du+b_1D(uz)+\ldots+b_{m-1}D(uz^{m-1})=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \qquad a_{m+1}.\,m\,!\ +b_0D^m\,u+b_1D^m\,(uz)+\ldots\\ \qquad\qquad +b_{m-1}D^m\,(uz^{m-1})=0,\\ \qquad b_0D^{m+1}\,u+b_1D^{m+1}\,(uz)+\ldots\\ \qquad\qquad +b_{m-1}D^{m+1}\,(uz^{m-1})=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \qquad b_0D^{2m-1}u+b_1D^{2m-1}(uz)+\ldots\\ \qquad\qquad +b_{m-1}D^{2m-1}(uz^{m-1})=0\,;\end{array}\right.$$

whence

$$(30.)\quad\left|\begin{array}{llllll}1,\ \zeta,\ \zeta^2,\ldots & \zeta^m, & \phi, & \phi\zeta, & \ldots\,\phi\zeta^{m-1}\\ 1,\ z,\ z^2,\ldots & z^m, & u, & uz, & \ldots\,uz^{m-1}\\ \quad 1,\ 2z,\ldots & mz^{m-1}, & Du, & Duz, & \ldots\,Duz^{m-1}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ & m\,!, & D^mu, & D^muz, & \ldots\,D^muz^{m-1}\\ & & D^{m+1}u, & D^{m+1}uz, & \ldots\,D^{m+1}uz^{m-1}\\ & & D^{m+2}u, & D^{m+2}uz, & \ldots\,D^{m+2}uz^{m-1}\\ & & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ & & D^{2m-1}u, & D^{2m-1}uz, & \ldots\,D^{2m-1}uz^{m-1}\end{array}\right|=0\,.$$

The expansion of this determinant according to the elements of the first row may be written :

$$(31.)\quad \{P_1+P_2\zeta+\ldots+P_{m+1}\zeta^m\}+\{Q_0+Q_1\zeta+\ldots+Q_{m-1}\zeta^{m-1}\}\,\phi=0\,.$$

Hence

$$(32.)\quad Z\{P_1+P_2Z+\ldots+P_{m+1}Z^m\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2(1-Z)(1-k^2Z$$
$$\equiv P_{m+1}^2\,(Z-\zeta)(Z-z)^{2m}\,.$$

Herein putting $Z=0$, $Z=1$, $Z=\frac{1}{k^2}$ successively, we obtain, after sligh) reduction,

$$(33.)\quad \begin{cases} \sqrt{\zeta}\;z^m = \dfrac{Q_0}{P_{m+1}}\,, \\[2ex] \sqrt{1-\zeta}\,(1-z)^m = \dfrac{P_1+P_2+\ldots+P_{m+1}}{P_{m+1}}\,, \\[2ex] \sqrt{1-k^2\zeta}\,(1-k^2z)^m = \dfrac{P_1k^{2m}+P_2k^{2m-2}+\ldots+P_{m+1}}{P_{m+1}}\,. \end{cases}$$

§. 7.

The reduction of the expressions which occur on the righthand side of (33.) runs on the same line as the reduction of the corresponding expressions when n is odd, discussed somewhat in detail in preceding sections. We may therefore at once write down the results:

$$(34.)\quad P_{m+1}=(-1)^{\frac{m(m+1)}{2}}(2m-1)!!\begin{vmatrix} \frac{1}{1!}D\frac{s}{z}, & \frac{1}{2!}D^2\frac{s}{z}, & \ldots\frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{2!}D^2\frac{s}{z}, & \frac{1}{3!}D^3\frac{s}{z}, & \ldots\frac{1}{(m+1)!}D^{m+1}\frac{s}{z} \\ \cdots & \cdots & \cdots \\ \frac{1}{m!}D^m\frac{s}{z}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z}, & \ldots\frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z} \end{vmatrix},$$

$$(35.)\quad Q_0=(-1)^{\frac{(m-1)(m-2)}{2}}(2m-1)!!\begin{vmatrix} \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, & \cdots \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{4\,!}D^4s, & \frac{1}{5\,!}D^5s, & \cdots \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots \frac{1}{(2m-1)!}D^{2m-1}s \end{vmatrix},$$

$$(36.)\quad \sum_{\mu=1}^{\mu=m+1} P_\mu=(-1)^{\frac{m(m-1)}{2}}(2m-1)!!(1-z)^{2m}$$

$$\times\begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-z)} \end{vmatrix},$$

$$(37.)\quad \sum_{\mu=1}^{\mu=m+1} P_\mu k^{2m+2-2\mu}=(-1)^{\frac{m(m-1)}{2}}(2m-1)!!(1-k^2z)^{2m}$$

$$\times\begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-k^2z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-k^2z)} \\ \frac{1}{2\,!}D^3\frac{s}{z(1-k^2z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-k^2z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-k^2z)} \end{vmatrix}.$$

Calling the determinants on the right-hand sides of equations (34–37.) N, N_1, N_2, N_3 in order, and, substituting in (33.), we obtain

$$(38.)\quad \begin{cases} \sqrt{\zeta} = (-1)^{m-1}\dfrac{1}{z^m}\dfrac{N_1}{N}, \\ \sqrt{1-\zeta} = (1-z)^m\dfrac{N_2}{N}, \\ \sqrt{1-k^2\zeta} = (1-k^2z)^m\dfrac{N_3}{N}. \end{cases}$$

§. 8.

Writing, as is usual,

$$2u = \int_0^z \frac{dz}{s},$$

so that

$$\sqrt{z} = \operatorname{sn} u,$$

we have, in virtue of equations (3.) and (28.),

$$\begin{aligned} \sqrt{\zeta} &= -\operatorname{sn} nu, \\ \sqrt{1-\zeta} &= \operatorname{cn} nu, \\ \sqrt{1-k^2\zeta} &= \operatorname{dn} nu, \end{aligned}$$

where n is any even or odd integer.

When n is odd, say $n = 2m + 1$, we have from (16.), (19.), (23.) and (27.)

$$(39.) \qquad \left\{ \begin{aligned} \operatorname{sn} nu &= (-1)^m z^{\frac{n}{2}} \frac{M_1}{M}, \\ \operatorname{cn} nu &= (1-z)^{\frac{n}{2}} \frac{M_2}{M}, \\ \operatorname{dn} nu &= (1-k^2 z)^{\frac{n}{2}} \frac{M_3}{M}, \end{aligned} \right.$$

where

$$(40.) \qquad M_1 = \begin{vmatrix} \frac{1}{2!}D^2\frac{s}{z}, & \frac{1}{3!}D^3\frac{s}{z}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z} \\ \frac{1}{3!}D^3\frac{s}{z}, & \frac{1}{4!}D^4\frac{s}{z}, & \cdots \frac{1}{(m+2)!}D^{m+2}\frac{s}{z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{z}, & \cdots \frac{1}{(2m)!}D^{2m}\frac{s}{z} \end{vmatrix},$$

$$(41.) \quad M_2 = \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3\frac{s}{1-z}, & \frac{1}{4\,!}D^4\frac{s}{1-z}, & \cdots \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z}, & \cdots \frac{1}{(2m)!}D^{2m}\frac{s}{1-z} \end{vmatrix},$$

$$(42.) \quad M_3 = \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-k^2z}, & \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z} \\ \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \frac{1}{4\,!}D^4\frac{s}{1-k^2z}, & \cdots \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z}, & \cdots \frac{1}{(2m)!}D^{2m}\frac{s}{1-k^2z} \end{vmatrix},$$

$$(43.) \quad M = \begin{vmatrix} \frac{1}{2\,!}D^2s, & \frac{1}{3\,!}D^3s, & \cdots\cdots \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, & \cdots\cdots \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots\cdots \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

In case n is even, say $n=2m$, we have, in virtue of (38.),

$$(44.) \quad \begin{cases} \operatorname{sn} nu = (-1)^m \frac{1}{z^m}\,\frac{N_1}{N}, \\ \operatorname{cn} nu = (1-z)^m\,\frac{N_2}{N}, \\ \operatorname{dn} nu = (1-k^2z)^m\,\frac{N_3}{N}, \end{cases}$$

where

$$(45.)\qquad N_1 = \begin{vmatrix} \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, & \cdots \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{4\,!}D^4s, & \frac{1}{5\,!}D^5s, & \cdots \frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots \frac{1}{(2m-1)!}D^{2m-1}s \end{vmatrix},$$

$$(46.)\qquad N_2 = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-z)} \end{vmatrix},$$

$$(47.)\qquad N_3 = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-k^2z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-k^2z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1\text{-}k^2z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1\text{-}k^2z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1\text{-}k^2z)} \end{vmatrix},$$

$$(48.)\qquad N = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z}, & \frac{1}{2\,!}D^2\frac{s}{z}, & \cdots \frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{2\,!}D^2\frac{s}{z}, & \frac{1}{3\,!}D^3\frac{s}{z}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z} \end{vmatrix}.$$

It will be observed that the formulæ of $\operatorname{sn} nu$ for $n=2,\ 3,\ 4,\ 5,$ are those given by Jacobi in the Memoir referred to, which appeared on the opening page of the present paper. I was not aware of the existence of Jacobi's formulæ when I first found the formula for $\mathrm{sn}\, nu$ which was, in fact, then obtained in a slightly different form as furnished by (17.), and which is perhaps in some respects preferable to the one here given. I have, however, given it its present shape, in order that, for the particular values of n, it may exactly coincide with the formulæ given by the illustrious mathematician whose memory is sacred to every student of the theory of elliptic functions.

§. 9.

Before proceeding further with the reduction of the multiplication-formulæ, it is necessary to give a few formulæ relating to the differentiation of composite functions, to which frequent reference will subsequently be made.

Let u be a function of y and y a function of x. It is required to find the n^{th} differential coefficient of u with respect to x. By actual differentiation, we find

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx},$$

$$\frac{d^2u}{dx^2} = \frac{du}{dy}\frac{d^2y}{dx^2} + \frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2,$$

$$\frac{d^3u}{dx^3} = \frac{du}{dy}\frac{d^3y}{dx^3} + 3\,\frac{d^2u}{dy^2}\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^3,$$

$$\frac{d^4u}{dx^4} = \frac{du}{dy}\frac{d^4y}{dx^4} + \frac{d^2u}{dy^2}\left\{4\,\frac{dy}{dx}\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^2\right\} + 6\,\frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}$$
$$+ \frac{d^4u}{dy^4}\left(\frac{dy}{dx}\right)^4,$$

$$\frac{d^5u}{dx^5} = \frac{du}{dy}\frac{d^5y}{dx^5} + \frac{d^2u}{dy^2}\left\{5\frac{dy}{dx}\frac{d^4y}{dx^4} + 10\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\right\}$$

$$+ \frac{d^3u}{dy^3}\left\{10\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3} + 15\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2\right\}$$

$$+ 10\frac{d^4u}{dy^4}\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2} + \frac{d^5u}{dy^5}\left(\frac{dy}{dx}\right)^5,$$

....;

and, generally,

$$\frac{d^nu}{dx^n} = \frac{du}{dy}X_1 + \frac{d^2u}{dy^2}X_2 + \ldots + \frac{d^ru}{dy^r}X_r + \ldots + \frac{d^{n-1}u}{dy^{n-1}}X_{n-1} + \frac{d^nu}{dy^n}X_n, \tag{49}$$

where $X_1, X_2, \ldots X_n$ denote as yet unknown functions which contain only the differential coefficients of y with respect to x. A few of the initial and end coefficients may at once be written down:

$$(50)\quad \left\{\begin{aligned}
X_1 &= \frac{d^ny}{dx^n},\\
X_2 &= n\frac{dy}{dx}\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{2\,!}\frac{d^2y}{dx^2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots\\
&\qquad + \left\{\begin{aligned}
&\frac{n(n-1)\ldots(n-\frac{n-1}{2}+1)}{\left(\frac{n-1}{2}\right)!}\,\frac{d^{\frac{n-1}{2}}y}{dx^{\frac{n-1}{2}}}\cdot\frac{d^{\frac{n+1}{2}}y}{dx^{\frac{n+1}{2}}}, \ (n \text{ odd})\\
&\frac{n(n-1)\ldots(n-\frac{n}{2}+1)}{2\,.\left(\frac{n}{2}\right)!}\left(\frac{d^{\frac{n}{2}}y}{dx^{\frac{n}{2}}}\right)^2, \quad (n \text{ even})
\end{aligned}\right.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
X_{n-2} &= \frac{n(n-1)(n-2)}{3\,!}\left(\frac{dy}{dx}\right)^{n-3}\frac{d^3y}{dx^3}\\
&\qquad + \frac{n(n-1)(n-2)(n-3)}{2^3}\left(\frac{dy}{dx}\right)^{n-4}\left(\frac{d^2y}{dx^2}\right)^2,\\
X_{n-1} &= \frac{n(n-1)}{2\,!}\left(\frac{dy}{dx}\right)^{n-2}\frac{d^2y}{dx^2},\\
X_n &= \left(\frac{dy}{dx}\right)^n.
\end{aligned}\right.$$

To find X_r, observe that X_r, being a function of the differential coefficients of y with respect to x only, is independent of u as a function of y. Hence, putting $u=y, y^2, \ldots y^n$ successively, we obtain

$$\left\{\begin{aligned}
\frac{d^n y}{dx^n} &= X_1 ,\\
\frac{d^n(y^2)}{dx^n} &= 2yX_1+2X_2 ,\\
\frac{d^n(y^3)}{dx^n} &= 3y^2X_1+3.2yX_2+3.2X_3 ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{d^r(y^r)}{dx^n} &= ry^{r-1}X_1+r(r-1)y^{r-2}X_2+\ldots+r\,!\,X_r ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{d^n(y^n)}{dx^n} &= ny^{n-1}X_1+n(n-1)y^{n-2}X_2+\ \ldots\ldots\ldots\ +n\,!\,X_n .
\end{aligned}\right.$$

To obtain $X_1, X_2, \ldots$, we may solve these equations as was done by Bertrand* ; but it is shorter to proceed as follows:—Put $u=e^{\lambda y}$, then

$$e^{-\lambda y}\frac{d^n e^{\lambda y}}{dx^n} = \lambda X_1+\lambda^2X_2+\ldots+\lambda^rX_r+\ldots+\lambda^nX_n ,$$

and, on the other hand,

$$e^{-\lambda y} = 1-\lambda y+\frac{\lambda^2y^2}{2\,!}-\frac{\lambda^3y^3}{3\,!}+\ldots+(-1)^n\frac{\lambda^ny^n}{n\,!}+\ldots ,$$

$$\frac{d^ne^{\lambda y}}{dx^n} = \lambda\frac{d^ny}{dx^n}+\frac{\lambda^2}{2\,!}\,\frac{d^ny^2}{dx^n}+\frac{\lambda^3}{3\,!}\,\frac{d^ny^3}{dx^n}+\ldots+\frac{\lambda^n}{n\,!}\,\frac{d^ny^n}{dx^n}+\ldots .$$

Multiplying the last two equations together and equating the coefficients of the like powers of λ, we obtain

* Bertrand, Traité de Calcul Différential et de Calcul Integral, T. I, p. 139.

$$
(51)\quad \left\{
\begin{aligned}
X_1 &= \frac{d^ny}{dx^n},\\
X_2 &= \frac{1}{2\,!}\,\frac{d^n(y^2)}{dx^n} - y\,\frac{d^ny}{dx^n},\\
&\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\\
X_r &= \frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n} - \frac{y}{(r-1)!\,1!}\,\frac{d^n(y^{r-1})}{dx^n} + \frac{y^2}{(r-2)!\,2!}\,\frac{d^n(y^{r-2})}{dx^n} - \ldots\\
&\qquad + (-1)^{r-1}\frac{y^{r-1}}{1!\,(r-1)!}\,\frac{d^ny}{dx^n},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
X_n &= \frac{1}{n\,!}\,\frac{d^n(y^n)}{dx^n} - \frac{y}{(n-1)!\,1!}\,\frac{d^n(y^{n-1})}{dx^{n-1}} + \frac{y^2}{(n-2)!\,2!}\,\frac{d^ny^{n-2}}{dx^n} - \ldots\\
&\qquad + (-1)^{n-1}\frac{y^{n-1}}{1!\,(n-1)!}\,\frac{d^ny}{dx^n}.
\end{aligned}\right.
$$

It may be worth while to notice that the expression on the opposite side of X_r is identically zero for all values of r greater than n.

From the mode of derivation it is evident that X_r contains differential coefficients of y but not y itself. Hence, if $\frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n}$ be broken up into two parts such that one part contains all the terms independent of explicit y and the other part, terms having y as a factor, then

$$X_r = \text{that part of } \frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n} \text{ which is independent of explicit } y$$

and that part of $\frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n}$ which contains explicit y together with

$$-\frac{y}{(r-1)!\,1!}\,\frac{d^n(y^{r-1})}{dx^n} + \frac{y^2}{(r-2)!\,2!}\,\frac{d^n(y^{r-2})}{dx^n} - \ldots + (-1)^n\frac{y^{r-1}}{1!\,(r-1)!}\,\frac{d^ny}{dx^n}$$

is identically zero. Now

$$\frac{1}{n\,!}\,\frac{d^n(y^{r-1})}{dx^n} = \text{coefficient of } h^n \text{ in } \left\{y + h\frac{dy}{dx} + \frac{h^2}{2\,!}\,\frac{d^2y}{dx^2} + \ldots\right\}^r.$$

Hence that part of $\frac{1}{r!}\frac{d^n(y^r)}{dx^n}$ which is independent of explicit y is equal to the coefficient of h^n in

$$\frac{n!}{r!}\left\{h\frac{dy}{dx}+\frac{h^2}{2!}\frac{d^2y}{dx^2}+\dots\right\}^r,$$

that is, in

$$\frac{n!}{r!}\left\{f(x+h)-f(x)\right\}^r,$$

where $y=f(x)$.

Thus we obtain

$$X_r = \frac{1}{r!}\left[\left(\frac{d}{dh}\right)^n\left\{f(x+h)-f(x)\right\}\right]_{h=0}. \tag{52}$$

This form of X_r has been obtained in a different manner by U. Meyer.* Bertrand gives the following form of X_r which is substantially the same as (51):**

$$X_r = \frac{y^r}{r!}\frac{d^n\left(\frac{y}{\alpha}-1\right)^r}{dx^n},$$

where α is to be regarded as constant during differentiation and is afterwards to be replaced by y. Again making use of the form of X_r given by (51), we get

$$\frac{d^nu}{dx^n}=\sum_{r=1}^{r=n}\frac{1}{r!}\frac{d^ru}{dy^r}\frac{d^ny^r}{dx^n}-y\sum_{r=2}^{r=n}\frac{1}{(r-1)!}\frac{d^ru}{dy^r}\frac{d^ny^{r-1}}{dx^n}$$
$$+y^2\sum_{r=3}^{r=n}\frac{1}{(r-2)!}\frac{d^ru}{dy^r}\frac{d^ny^{r-2}}{dx^n}-\dots+(-1)^{n-1}y^{n-1}\frac{d^nu}{dy^n}\frac{d^ny}{dx^n},$$

* Grunert's Archiv der Mathematik, Bd. IX.

** Loc. cit. p. 140.

which agrees with the form of $\frac{d^n u}{dx^n}$ given by R. Hoppe.*

Put $u=y^p$, then $\frac{d^r u}{dy^r} = p_r\, y^{p-r}$, where p_r denotes the continued product of r quantities $p, p-1, p-2, \ldots (p-r+1)$. Multiplying equations (51) by py^{p-1}, $p(p-1)y^{p-2}, \ldots$ in order and adding, we obtain

$$(53)\quad \frac{d^n y^p}{dx^n} = \frac{p(n-p)_n}{n!\, y^{n-p}}\left\{-\frac{n_1}{p-1} y^{n-1}\frac{d^n y}{dx^n} + \frac{n_2}{p-2}\,\frac{y^{n-2}}{2!}\,\frac{d^n y^2}{dx^n} - \ldots \right.$$

$$\left. + (-1)^r \frac{n_r}{p-r}\,\frac{y^{n-r}}{r!}\,\frac{d^n y^r}{dx^n} + \ldots + (-1)^n \frac{1}{p-n}\,\frac{d^n y^n}{dx^n}\right\}.$$

For $p=\frac{1}{2}$, we have

$$(54)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{1}{y^{n-\frac{1}{2}}}\left\{\frac{n_1}{1}\, y^{n-1}\frac{d^n y}{dx^n} - \frac{n_2}{2!}\,\frac{y^{n-2}}{3}\,\frac{d^n y^2}{dx^n} + \ldots \right.$$

$$\left. + (-1)^{r-1}\frac{n_r}{r!}\,\frac{y^{n-r}}{2r-1}\,\frac{d^n y^r}{dx^n} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\,\frac{d^n y^n}{dx^n}\right\}.$$

If y be a rational integral function of x of the third degree, $\frac{d^n y^r}{dx^n}$ vanishes for all values of r for which $3r < n$. In this case, denoting by l either $\frac{n}{3}$ or the integer next above $\frac{n}{3}$ according as n is a multiple of 3 or not, we have

$$(55)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{1}{y^{n-\frac{1}{2}}}\left\{(-1)^{l-1}\frac{n_l}{l!}\,\frac{y^{n-l}}{2l-1}\,\frac{d^n y^l}{dx^n} + \ldots \right.$$

$$\left. + (-1)^{r-1}\frac{n_r}{r!}\,\frac{y^{n-r}}{2r-1}\,\frac{d^n y^r}{dx^n} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\,\frac{d^n y^n}{dx^n}\right\}.$$

Again

$$(56)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{1}{2^n y^{n-\frac{1}{2}}}\left\{(2y)^{n-1}X_1 - (2y)^{n-2}X_2 + \ldots \right.$$

$$\left. + (-1)^{r-1}.\,1.3.\ldots(2r-3)(2y)^{n-r}X_r + \ldots + (-1)^n.\,1.3.\ldots(2n-3)X_n\right\},$$

* Crelle's Journal, Bd. XXXIII.

and if y be a rational integral function of the third degree, $\frac{d^4y}{dx^4}$ and all the higher differential coefficients vanish, so that

$$X_r = \text{coef. of } h^n \text{ in } \frac{n\,!}{r\,!}\left\{h\frac{dy}{dx}+\frac{1}{2}h^2\frac{d^2y}{dx^2}+\frac{1}{6}h^3\frac{d^3y}{dx^3}\right\}^r,$$

or

$$X_{n-r} = \text{coef. of } h^r \text{ in } \frac{n\,!}{(n-r)!}\left\{\frac{dy}{dx}+\frac{1}{2}h\frac{d^2y}{dx^2}+\frac{1}{6}h^2\frac{d^3y}{dx^3}\right\}^{n-r};$$

whence follows that X_{n-r} vanishes for all values of r for which $r > 2n-2r$, that is, $3r > 2n$. Denoting the integral part of $\frac{2n}{3}$ by i, we have

$$(57)\quad \frac{d^n\sqrt{y}}{dx^n} = (-1)^{n-1}\frac{1}{2^n y^{n-\frac{1}{2}}}\Big\{(-1)^i.1.3.\ldots(2n-2i-3)(2y)^iX_{n-i}+\ldots$$

$$+(-1)^r.1.3.\ldots(2n-2r-3)(2y)^rX_{n-r}+\ldots+1.3.\ldots(2n-3)X_n\Big\},$$

where

$$X_{n-r} = n\,!\left\{\frac{1}{(n-2r)!\,r!}\left(\frac{dy}{dx}\right)^{n-2r}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^r\right.$$

$$+\frac{1}{(n-2r+1)!\,(r-2)!\,1!}\left(\frac{dy}{dx}\right)^{n-2r-1}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^{r-2}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)$$

$$\left.+\frac{1}{(n-2r+2)!\,(r-4)!\,2!}\left(\frac{dy}{dx}\right)^{n-2r-2}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^{r-4}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)^2+\ldots\right\},$$

the last term being

$$\frac{1}{\left(n-\frac{3}{2}r\right)!\left(\frac{r}{2}\right)!}\left(\frac{dy}{dx}\right)^{n-\frac{3r}{2}}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)^{\frac{r}{2}}$$

or

$$\frac{1}{\left(n-\frac{3r+1}{2}\right)!\,1\,!\left(\frac{r-1}{2}\right)!}\left(\frac{dy}{dx}\right)^{n-\frac{3r+1}{2}}\left(\frac{1}{2}\,\frac{d^2y}{dx^2}\right)\left(\frac{1}{6}\,\frac{d^3y}{dx^3}\right)^{\frac{r-1}{2}},$$

according as r is even or odd.

Similarly

$$(58)\quad \frac{d^ny^{-\frac{1}{2}}}{dx^n} = \frac{(-1)^n}{2^ny^{n+\frac{1}{2}}}\Big\{(-1)^i.1.3.\ldots(2n-2i-1)(2y)^iX_{n-i}+\ldots$$

$$+(-1)^r.1.3.\ldots(2n-2r-1)(2y)^rX_{n-r}+\ldots+1.3.\ldots(2n-1)X_n\Big\},$$

where X_{n-r} has the same signification as in (57).

$$\text{Let}\quad f = s^2 = z(1-z)(1-k^2z) = z-(1+k^2)z^2+k^2z^3\,,$$

$$f_1 = \frac{df}{dz} = 1-2(1+k^2)z+3k^2z^2\,,$$

$$f_2 = \frac{1}{2}\,\frac{d^2f}{dz^2} = -(1+k^2)+3k^2z\,,$$

$$f_3 = \frac{1}{6}\,\frac{d^3f}{dz^3} = k^2\,.$$

Now write

$$(59)\qquad \frac{1}{n\,!}D^n\,s = \frac{(-1)^{n-1}}{2^n\,s^{2n-1}}S_n\,,$$

$$(60)\qquad \frac{1}{n\,!}D^n\frac{1}{s} = \frac{(-1)^n}{2^n\,s^{2n+1}}R_n\,;$$

then by (57) and (58)

$$(61)\quad S_n = (-1)^i.1.3.\ldots(2n-2i-3)(2f)^iZ_{n-i}+\ldots$$

$$+(-1)^r.1.3.\ldots(2n-2r-3)(2f)^rZ_{n-r}+\ldots+1.3.\ldots(2n-3)Z_n\,,$$

$$(62)\quad R_n = (-1)^i.1.3.\ldots(2n-2i-1)(2f)^i Z_{n-i}+\cdots$$
$$+(-1)^r.1.3.\ldots(2n-2r-1)(2f)^r Z_{n-r}+\ldots+1.3.\ldots(2n-1)Z_n\,,$$

where

$$(63)\quad Z_{n-r}=\frac{f_1^{n-2r}f_2^r}{(n-2r)!\,r!}+\frac{f_1^{n-2r-1}f_2^{r-2}f_3}{(n-2r+1)!(r-2)!\,1!}+\frac{f_1^{n-2r-2}f_2^{r-4}f_3^2}{(n-2r+2)!\,(r-4)!\,2!}+\cdots,$$

the last term being

$$\frac{f_1^{n-\frac{3r}{2}}f_3^{\frac{r}{2}}}{\left(n-\frac{3}{2}r\right)!\left(\frac{r}{2}\right)!}\quad\text{or}\quad\frac{f_1^{n-\frac{3r+1}{2}}f_2 f_3^{\frac{r-1}{2}}}{\left(n-\frac{3r+1}{2}\right)!\left(\frac{r-1}{2}\right)!}$$

according as r is even or odd.

Observe that both S_n and R_n are rational integral functions of z of the degree $2n$.

§ 10.

The determinant-expressions of § 8, simple as they may appear, are not convenient for introducing S_n and R_n of the last section. It is desirable to derive the multiplication-formulæ in forms slightly different from those given in § 8.

Consider first the case n odd. Referring to § 2, we might have taken $\frac{P}{s}+Q$ instead of $P+Qs$ and then we shall have in place of (5) the following determinant:

$$
(64)\qquad
\begin{vmatrix}
1, \zeta, \zeta^2, \dots & \zeta^{m-1}, & \frac{1}{\sigma}, & \frac{\zeta}{\sigma}, & \frac{\zeta^2}{\sigma}, & \dots & \frac{\zeta^{m+1}}{\sigma} \\
1, z, z^2, \dots & z^{m-1}, & \frac{1}{s}, & \frac{z}{s}, & \frac{z^2}{s}, & \dots & \frac{z^{m+1}}{s} \\
1, 2z, \dots & (m-1)z^{m-2}, & D\frac{1}{s}, & D\frac{z}{s}, & D\frac{z^2}{s}, & \dots & D\frac{z^{m+1}}{s} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 & (m-1)!\,, & D^{m-1}\frac{1}{s}, & D^{m-1}\frac{z}{s}, & D^{m-1}\frac{z^2}{s}, & \dots & D^{m-1}\frac{z^{m+1}}{s} \\
 & & D^{m}\frac{1}{s}, & D^{m}\frac{z}{s}, & D^{m}\frac{z^2}{s}, & \dots & D^{m}\frac{z^{m+1}}{s} \\
 & & D^{m+1}\frac{1}{s}, & D^{m+1}\frac{z}{s}, & D^{m+1}\frac{z^2}{s}, & \dots & D^{m+1}\frac{z^{m+1}}{s} \\
 & & \dots & \dots & \dots & \dots & \dots \\
 & & D^{2m}\frac{1}{s}, & D^{2m}\frac{z}{s}, & D^{2m}\frac{z^2}{s}, & \dots & D^{2m}\frac{z^{m+1}}{s}
\end{vmatrix} = 0\,.
$$

Now put

$$
(65)\qquad \Omega = \begin{vmatrix}
D^{m-1}\frac{1}{s}, & D^{m-1}\frac{z}{s}, & \dots & D^{m-1}\frac{z^{m+1}}{s} \\
D^{m}\frac{1}{s}, & D_{m}\frac{z}{s}, & \dots & D^{m}\frac{z^{m+1}}{s} \\
\dots & \dots & \dots & \dots \\
D^{2m}\frac{1}{s}, & D^{2m}\frac{z}{s}, & \dots & D^{2m}\frac{z^{m+1}}{s}
\end{vmatrix},
$$

and let the expansion of this determinant according to the elements of the first row be written :

$$
\Omega = D^{m-1}\frac{1}{s}.\ \Omega_1 + D^{m-1}\frac{z}{s}.\ \Omega_2 + \dots + D^{m-1}\frac{z^{m+1}}{s}.\ \Omega_{m+2}\,.
$$

We find

$$
\sqrt{\zeta}\; z^{\frac{2m+1}{2}} = \frac{\Omega_1}{\Omega_{m+2}}\,,
$$

$$
\sqrt{1-\zeta}\;(1-z)^{\frac{2m+1}{2}} = \frac{\Omega_1 + \Omega_2 + \dots + \Omega_{m+2}}{\Omega_{m+2}}\,,
$$

$$
\sqrt{1-k^2\zeta}\;(1-k^2 z)^{\frac{2m+1}{2}} = \frac{\Omega_1 k^{2m+2} + \Omega_2 k^{2m} + \dots + \Omega_{m+2}}{\Omega_{m+2}}\,,
$$

and thence, n being odd,

$$
(66) \qquad \begin{cases} \operatorname{sn} nu = (-1)^m \; z^{-\frac{n}{2}} \; \dfrac{M_1'}{M'}, \\ \operatorname{cn} nu = \qquad (1-z)^{-\frac{n}{2}} \; \dfrac{M_2'}{M'}, \\ \operatorname{dn} nu = \qquad (1-k^2 z)^{-\frac{n}{2}} \; \dfrac{M_3'}{M'}, \end{cases}
$$

where

$$
(67) \qquad M' = \begin{vmatrix} \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \cdots \dfrac{1}{m!} D^m \dfrac{1}{s} \\ \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{1}{s}, & \cdots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{1}{m!} D^m \dfrac{1}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s}, & \cdots \dfrac{1}{(2m)!} D^{2m} \dfrac{1}{s} \end{vmatrix},
$$

$$
(68) \qquad M_1' = \begin{vmatrix} \dfrac{z}{s}, & \dfrac{1}{1\,!} D \dfrac{z}{s}, & \cdots \dfrac{1}{m!} D^m \dfrac{z}{s} \\ \dfrac{1}{1\,!} D \dfrac{z}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{z}{s}, & \cdots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{z}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{1}{m!} D^m \dfrac{z}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{z}{s}, & \cdots \dfrac{1}{(2m)!} D^{2m} \dfrac{z}{s} \end{vmatrix},
$$

$$
(69) \qquad M_2' = \begin{vmatrix} (1-z)^{m+1}, & (1-z)^m, & (1-z)^{m-1}, & \cdots 1 \\ 0, & \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \cdots \dfrac{1}{m!} D^m \dfrac{1}{s} \\ \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{1}{s}, & \cdots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{1}{(m-1)!} D^{m-1} \dfrac{1}{s}, & \dfrac{1}{m!} D^m \dfrac{1}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s}, & \cdots \dfrac{1}{(2m)!} D^{2m} \dfrac{1}{s} \end{vmatrix},
$$

$$(70)\quad M_3' = \begin{vmatrix} (1-k^2z)^{m+1}, & k^2(1-k^2z)^m, & k^4(1-k^2z)^{m-1}, & \ldots & k^{2m+2} \\ 0, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \ldots & \frac{1}{m!}D^m\frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \ldots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \ldots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{vmatrix}.$$

Observe that the expression of sn nu as given by (67) and (68) is that which has been indicated by Jacobi. It will, however, be more convenient to write M_1' as follows:

$$(71)\quad M_1' = \begin{vmatrix} z^{m+1}, & -z^m, & z^{m-1}, & \ldots & (-1)^{m+1} \\ 0, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \ldots & \frac{1}{m!}D^m\frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \ldots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \ldots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{vmatrix}.$$

We may here point out that the comparison of the preceding formulæ with the corresponding expressions of § 8, gives some elegant theorems in determinants. Take for example M and M'. We find easily

$$M = C\, s^{2m+1}\, M',$$

where C denotes a constant which may be a function of m. To determine C we may take some particular values of m. For $m = 1$, we see at once that

$$\frac{1}{2!}D^2 s = \quad -s^3 \begin{vmatrix} \frac{1}{s}, & D\frac{1}{s}, \\ D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \end{vmatrix}.$$

For $m=2$, we have

$$\begin{vmatrix} \frac{1}{s}, & D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{4!}D^4\frac{1}{s} \end{vmatrix} = s^2 \begin{vmatrix} \frac{1}{s}, & D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{s}D\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{4!}D^4\frac{1}{s} \end{vmatrix}$$

$$= s\begin{vmatrix} \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}-\left(D\frac{1}{s}\right)^2, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}-D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{s}\cdot\frac{1}{2!}D^3\frac{1}{s}-D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{4!}D^4\frac{1}{s}-\left(\frac{1}{2}D^2\frac{1}{s}\right)^2 \end{vmatrix}.$$

Multiply the first row by $D\frac{1}{s}$ and subtract the product from the second row, and then, multiply the first column by $D\frac{1}{s}$ and subtract the product from the second column ; and, observing

$$\frac{1}{2!}D^2s = s^3\left(D\frac{1}{s}\right)^2-s^2.\frac{1}{2!}D^2\frac{1}{s},$$

$$\frac{1}{3!}D^3s = -s^4\left(D\frac{1}{s}\right)^3+2s^3D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}-s^2\frac{1}{3!}D^3\frac{1}{s},$$

$$\frac{1}{4!}D^4s = s^5\left(D\frac{1}{s}\right)^4-3s^4\left(D\frac{1}{s}\right)^2.\frac{1}{2!}D^2\frac{1}{s},$$

$$+s^3\left\{2D\frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}\left(\frac{1}{2!}D^2\frac{1}{s}\right)^2\right\}-s^2\frac{1}{4!}D^4\frac{1}{s},$$

we get

$$\begin{vmatrix} \frac{1}{2!}D^2s, & \frac{1}{3!}D^3s \\ \frac{1}{3!}D^3s, & \frac{1}{4!}D^4s \end{vmatrix} = s^5\begin{vmatrix} \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{4!}D^4\frac{1}{s} \end{vmatrix}.$$

Thus we find $C=(-1)^m$ and thence

$$(72)\quad \begin{vmatrix} \frac{1}{2\,!}D^2 s, & \cdots \frac{1}{(m+1)!}D^{m+1}s \\ \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \cdots \frac{1}{(2m)!}D^{2m}s \end{vmatrix}$$

$$=(-1)^m s^{2m+1}\begin{vmatrix} \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \cdots & \frac{1}{m\,!}D^m\frac{1}{s} \\ \frac{1}{1\,!}D\frac{1}{s}, & \frac{1}{2\,!}D^2\frac{1}{s}, & \cdots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \cdots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{vmatrix},$$

which identity may also be proved directly in the manner exemplified by the particular case $m=2$.

§ 11.

Consider next the case n even. Referring to § 6, if we take $\frac{P}{s}+Q$ instead of $P+Qs$, we shall have in place of (30) the following determinant:

$$
(73)\qquad
\begin{vmatrix}
1, \zeta, \zeta^2, \dots & \zeta^{m-1}, & \dfrac{\zeta}{\sigma}, & \dfrac{\zeta^2}{\sigma}, & \dots\dots & \dfrac{\zeta^{m+1}}{\sigma} \\
1, z, z^2, \dots & z^{m-1}, & \dfrac{z}{s}, & \dfrac{z^2}{s}, & \dots\dots & \dfrac{z^{m+1}}{s} \\
1, 2z, \dots & (m-1)\, z^{m-2}, & D\dfrac{z}{s}, & D\dfrac{z^2}{s}, & \dots\dots & D\dfrac{z^{m+1}}{s} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
 & (m-1)\,!\,, & D^{m-1}\dfrac{z}{s}, & D^{m-1}\dfrac{z^2}{s}, & \dots\dots & D^{m-1}\dfrac{z^{m+1}}{s} \\
 & & D^{m}\dfrac{z}{s}, & D^{m}\dfrac{z^2}{s}, & \dots\dots & D^{m}\dfrac{z^{m+1}}{s} \\
 & & D^{m+1}\dfrac{z}{s}, & D^{m+1}\dfrac{z^2}{s}, & \dots\dots & D^{m+1}\dfrac{z^{m+1}}{s} \\
 & & \dots & \dots & \dots & \dots \\
 & & D^{2m-1}\dfrac{z}{s}, & D^{2m-1}\dfrac{z^2}{s}, & \dots & D^{2m-1}\dfrac{z^{m+1}}{s}
\end{vmatrix} = 0,
$$

say $$Q_0 + Q_1\zeta + Q_2\zeta^2 + \dots + Q_{m-1}\zeta^{m-1} + P_1\frac{\zeta}{\sigma} + P_2\frac{\zeta^2}{\sigma} + \dots + P_{m+1}\frac{\zeta^{m+1}}{\sigma} = 0.$$

Now

$$\sqrt{\zeta}\, z^m = \frac{Q_0}{P_{m+1}},$$

$$\sqrt{1-\zeta}\,(1-z)^m = \frac{P_1 + P_2 + \dots + P_{m+1}}{P_{m+1}},$$

$$\sqrt{1-k^2\zeta}\,(1-k^2z)^m = \frac{P_1 k^{2m} + P_2 k^{2m-2} + \dots + P_{m+1}}{P_{m+1}},$$

and thence, n being even,

$$
(74)\qquad
\begin{cases}
\operatorname{sn} nu = (-1)^m\, z^{\frac{n}{2}}\, \dfrac{N'_1}{N'}, \\[2ex]
\operatorname{cn} nu = (1-z)^{-\frac{n}{2}}\, \dfrac{N'_2}{N'}, \\[2ex]
\operatorname{dn} nu = (1-k^2z)^{-\frac{n}{2}}\, \dfrac{N'_3}{N'},
\end{cases}
$$

where

$$(75)\quad N' = \begin{vmatrix} \frac{1}{1\,!} D \frac{z}{s}, & \frac{1}{2\,!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{m!} D^m \frac{z}{s} \\ \frac{1}{2\,!} D^2 \frac{z}{s}, & \frac{1}{3\,!} D^3 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)\,!} D^{m+1} \frac{z}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m\,!} D^m \frac{z}{s}, & \frac{1}{(m+1)\,!} D^{m+1} \frac{z}{s}, & \cdots\cdots & \frac{1}{(2m-1)!} D^{2n-1} \frac{z}{s} \end{vmatrix},$$

$$(76)\quad N_1' = \begin{vmatrix} 0, & \frac{1}{s}, & \cdots\cdots & \frac{1}{(m-1)\,!} D^{m-1} \frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1\,!} D \frac{1}{s}, & \cdots\cdots & \frac{1}{m\,!} D^m \frac{1}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m-1)\,!} D^{m-1} \frac{1}{s}, & \frac{1}{m\,!} D^m \frac{1}{s}, & \cdots\cdots & \frac{1}{(2m-1)\,!} D^{2m-1} \frac{1}{s} \end{vmatrix},$$

$$(77)\quad N_2' = \begin{vmatrix} (1-z)^m, & (1-z)^{m-1}, & \cdots\cdots & 1 \\ \frac{z}{s}, & \frac{1}{1\,!} D \frac{z}{s}, & \cdots\cdots & \frac{1}{m\,!} D^m \frac{z}{s} \\ \frac{1}{1\,!} D \frac{z}{s}, & \frac{1}{2\,!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)\,!} D^{m+1} \frac{z}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m-1)\,!} D^{m-1} \frac{z}{s}. & \frac{1}{m\,!} D^m \frac{z}{\varepsilon}, & \cdots\cdots & \frac{1}{(2m-1)\,!} D^{2m-1} \frac{z}{s} \end{vmatrix},$$

$$(78)\quad N_3' = \begin{vmatrix} (1-k^2z)^m, & k^2(1-k^2z)^{m-1}, & \cdots\cdots & k^{2m} \\ \frac{z}{s}, & \frac{1}{1\,!} D \frac{z}{s}, & \cdots\cdots & \frac{1}{m\,!} D^m \frac{z}{s} \\ \frac{1}{1\,!} D \frac{z}{s}, & \frac{1}{2\,!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)\,!} D^{m+1} \frac{z}{s} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{)m-1)\,!} D^{m-1} \frac{z}{s}, & \frac{1}{m\,!} D^m \frac{z}{s}, & \cdots\cdots & \frac{1}{(2m-1)\,!} D^{2m-1} \frac{z}{s} \end{vmatrix}.$$

N', N'_2, N'_3 may also be written as follows :

$$(79)\quad N'=\begin{vmatrix} z^m, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \cdots\cdots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ -z^{m-1}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \cdots\cdots & \frac{1}{m!}D^m\frac{1}{s} \\ z^{m-2}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \cdots\cdots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^m, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \cdots\cdots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix},$$

$$(80)\quad N'_2=\begin{vmatrix} 0, & (1-z)^m, & (1-z)^{m-1}, & \cdots\cdots & 1 \\ -z^m, & 0, & \frac{1}{s}, & \cdots\cdots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ z^{m-1}, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \cdots\cdots & \frac{1}{m!}D^m\frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{m+1}, & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \cdots\cdots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix},$$

$$(81)\quad N'_3=\begin{vmatrix} 0, & (1-k^2z)^m, & k^2(1-k^2z)^{m-1}, & \cdots\cdots & k^{2m} \\ -z^m, & 0, & \frac{1}{s}, & \cdots\cdots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ z^{m-1}, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \cdots\cdots & \frac{1}{m!}D^m\frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{m+1}, & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \cdots\cdots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix}.$$

§ 12.

We are now in a position to introduce R_n and S_n into the multiplication-formulæ. In some cases, we might have introduced S_n rather than R_n ; but, for the sake of uniformity, we have chosen those forms which are expressible by R_n. On introducing R_n, it will be seen that there comes out a certain power of s as a factor in the numerator as well as in the denominator, which may be cancelled against each other, leaving rational integral functions of $\sqrt{z} = \operatorname{sn} u$ both in the numerator and the denominator. The final results are:

$$n \text{ odd}, \quad m = \frac{n}{2},$$

$$(82)\quad \operatorname{sn} nu = (-1)^m \sqrt{z} \begin{vmatrix} 1, & 2.1-z.1-k^2z, & (2.1-z.1-k^2z)^2, & \ldots & (2.1-z.1-k^2z)^{m+1} \\ 0, & 1, & R_1, & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(83)\quad \operatorname{cn} nu = \sqrt{1-z} \begin{vmatrix} 1, & -2.z.1-k^2z, & (2.z.1-k^2z)^2, & \ldots & (-2.z.1-k^2z)^{m+1} \\ 0, & 1, & R_1 & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(84)\quad \operatorname{dn} nu = \sqrt{1-k^2z} \begin{vmatrix} 1, & -2.k^2z.1-z, & (-2.k^2z.1-z)^2, & \ldots & (-2.k^2z.1-z)^{m+1} \\ 0, & 1, & R_1, & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(85)\qquad \text{denom.} = \begin{vmatrix} 1, & R_1, & \ldots & R_m \\ R_1, & R_2, & \ldots & R_{m+1} \\ \ldots & \ldots & \ldots & \ldots \\ R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix},$$

$$n \text{ even}, \quad m = \frac{n-1}{2},$$

$$\text{(86)} \qquad \operatorname{sn} nu = (-1)^m 2s \begin{vmatrix} 0, & 1, \dots R_{m-1} \\ 1, & R_1, \dots R_m \\ \dots\dots & \dots\dots \\ R_{m-1}, & R_m, \dots R_{2m-1} \end{vmatrix}, \div$$

$$\text{(87)} \qquad \operatorname{cn} nu = \begin{vmatrix} 0, & 1, & -2.z.1-k^2z, & \dots & (-2.z.1-k^2z)^m \\ 1, & 0, & 1, & \dots & R_{m-1} \\ 2.1-z.1-k^2z, & 1, & R_1, & \dots & R_m \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (2.1-z.1-k^2z)^m, & R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\text{(88)} \qquad \operatorname{dn} nu = \begin{vmatrix} 0, & 1, & -2.k^2z.1-z, & \dots & (-2.k^2z.1-z)^m \\ 1, & 0, & 1, & \dots & R_{m-1} \\ 2.1-z.1-k^2z, & 1, & R_1, & \dots & R_m \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (2.1-z.1-k^2z)^m, & R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\text{(89)} \qquad \text{denom.} = -\begin{vmatrix} 1, & 1, & R_1, & \dots R_{m-1} \\ 2.1-z.1-k^2z, & R_1, & R_2, & \dots R_m \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (2.1-z.1-k^2z)^m, & R_m, & R_{m+1}, & \dots R_{2m-1} \end{vmatrix}.$$

Part Second.

§ 13.

To avoid confusion, we shall adopt once for all the following notation :

$$x = \operatorname{sn} u, \quad \xi = \sqrt{k}\operatorname{sn} u, \quad z = \operatorname{sn}^2 u.$$

Following Jacobi in his lectures, we write*

$$\theta(u) = 1 + 2\sum_{n=1}^{n=\infty}(-1)^n q^{n^2}\cos\frac{n\pi u}{K}, \qquad \theta_2(u) = 2\sum_{n=1}^{n=\infty} q^{\left(\frac{2n-1}{2}\right)^2}\cos\frac{(2n-1)\pi u}{2K},$$

$$\theta_1(u) = 2\sum_{n=1}^{n=\infty}(-1)^{n-1} q^{\left(\frac{2n-1}{2}\right)^2}\sin\frac{(2n-1)\pi u}{2K}, \qquad \theta_3(u) = 1 + 2\sum_{n=1}^{n=\infty} q^{n^2}\cos\frac{n\pi u}{K};$$

and then

$$\sqrt{k}\operatorname{sn} u = \frac{\theta_1(u)}{\theta(u)}, \quad \sqrt{\frac{k}{k'}}\operatorname{cn} u = \frac{\theta_2(u)}{\theta(u)}, \quad \frac{1}{\sqrt{k'}}\operatorname{dn} u = \frac{\theta_3(u)}{\theta(u)}.$$

Put $\omega = -\log q = \dfrac{\pi K'}{K}$ **, then

$$\frac{d\theta}{d\omega} = \frac{\pi^2}{K^2}\frac{d^2\theta}{du^2} - \frac{u}{K}\frac{dK}{d\omega}\frac{d\theta}{du};$$

but

$$\frac{d\omega}{dK} = -\frac{2}{k(1-k^2)\left(\dfrac{2K}{\pi}\right)^2} = -\frac{\pi^2}{2kk'^2K^2} \;\dagger,$$

hence

$$(90) \qquad \frac{d^2\theta}{du^2} + 2kk'^2\frac{d\log K}{dk}\,u\,\frac{d\theta}{du} + 2kk'^2\frac{d\theta}{dk} = 0.$$

Observe that the same differential equation is satisfied by θ_1, θ_2 and θ_3.

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* Jacobi's gesammelte Werke, Bd. I, pp. 501, 511, and 512.

** Ditto p. 259.

† Ditto p. 260.

Equation (90) may also be written in the form

$$(91)\qquad \frac{d^2\log\theta}{du^2}+\left(\frac{d\log\theta}{du}\right)^2+2kk'^2\frac{d\log K}{dk}u\frac{d\log\theta}{du}+2kk'^2\frac{d\log\theta}{dk}=0\,;$$

similarly $$\frac{d^2\log\theta_1}{du^2}+\left(\frac{d\log\theta_1}{du}\right)^2+2kk'^2\frac{d\log K}{dk}u\frac{d\log\theta}{du}+2kk'^2\frac{d\log\theta}{dk}=0\,.$$

Subtracting the former equation from the latter and replacing $\frac{\theta_1(u)}{\theta(u)}$ by ξ, we get

$$(92)\qquad \frac{d^2\log\xi}{du^2}+\left(\frac{d\log\xi}{du}\right)^2+2\left\{\frac{d\log\xi}{du}+kk'^2\frac{d\log K}{dk}u\right\}\frac{d\log\xi}{du}+2kk'^2\frac{d\log\xi}{dk}=0.$$

Now

$$\sqrt{k}\,\mathrm{sn}\,nu=\frac{\theta_1(nu)}{\theta(nu)},\ \sqrt{\frac{k}{k'}}\,\mathrm{cn}\,nu=\frac{\theta_2(nu)}{\theta(nu)},\ \sqrt{k'}\,\mathrm{dn}\,nu=\frac{\theta_3(nu)}{\theta(nu)}\,;$$

multiplying the numerators and denominators on the right-hand side by $\frac{\theta^{n^2-1}(0)}{\theta^{n^2}(u)}$,

$$\sqrt{k}\,\mathrm{sn}\,nu=\frac{V_1}{V},\ \sqrt{\frac{k}{k'}}\,\mathrm{cn}\,nu=\frac{V_2}{V},\ \sqrt{k'}\,\mathrm{dn}\,nu=\frac{V_3}{V},$$

where

$$(93)\qquad \left\{\begin{aligned} V&=\frac{\theta(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}, & V_2&=\frac{\theta_2(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)},\\ V_1&=\frac{\theta_1(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}, & V_3&=\frac{\theta_3(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}.\end{aligned}\right.$$

Put nu instead of u in equation (91), thus :—

$$\frac{d^2\log\theta(nu)}{du^2}+\left(\frac{d\log\theta(nu)}{du}\right)^2+2n^2kk'^2\frac{d\log K}{dk}u\frac{d\log\theta(nu)}{du}+2n^2kk'^2\frac{d\log\theta(nu)}{dk}=0\,.$$

Now

$$\log \theta(nu) = \log V + n^2 \log \theta(u) - (n^2-1)\log \theta(0).$$

Substituting in (91), we get

$$(94)\quad \frac{d^2\log V}{du^2} + \left(\frac{d\log V}{du}\right)^2 + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{d\log V}{du}$$
$$+ 2n^2kk'^2\frac{d\log V}{dk} - n^2(n^2-1)\left\{\frac{d^2\log\theta(u)}{du^2} + 2kk'^2\frac{d\log\theta(0)}{dk}\right\} = 0.$$

Again, $\dfrac{d\log\theta(u)}{du} = Z(u)$, $\quad \theta(0) = \sqrt{\dfrac{2k'K}{\pi}}$ *;

differentiating, and observing $\dfrac{dK}{dk} = \dfrac{E-k'^2K}{kk'^2}$, we obtain

$$(95)\quad \frac{d^2\log\theta(u)}{du^2} + 2kk'^2\frac{d\log\theta(0)}{dk} = -k^2\mathrm{sn}^2u.$$

Substituting this in (94),

$$(96)\quad \frac{d^2\log V}{du^2} + \left(\frac{d\log V}{du}\right)^2 + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{d\log V}{du}$$
$$+ 2n^2kk'^2\frac{d\log V}{dk} + n^2(n^2-1)k^2\mathrm{sn}^2u = 0,$$

or

$$(97)\quad \frac{d^2V}{du^2} + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{dV}{du}$$
$$+ 2n^2kk'^2\frac{dV}{dk} + n^2(n^2-1)k^2\mathrm{sn}^2u\,V = 0.$$

Introducing $\xi = \sqrt{k}\,\mathrm{sn}\,u$ as the new independent variable, and observing (92),

* Jacobi's gesammelte Werke, Bd. I, pp. 198 and 235.

$$\left(\frac{d\xi}{du}\right)^2\frac{d^2V}{d\xi^2}-(n^2-1)\frac{d^2\xi}{du^2}\frac{dV}{d\xi}+2n^2kk'^2\frac{dV}{dk}+n^2(n^2-1)k\xi^2V=0\,.$$

Since

$$\left(\frac{d\xi}{du}\right)^2=k\left\{1-\left(k+\frac{1}{k}\right)\xi^2+\xi^4\right\},\ \frac{d^2\xi}{du^2}=k\left\{-\left(k+\frac{1}{k}\right)\xi+2\xi^3\right\},$$

equation (97) takes the form

$$(98)\quad \left\{1-\left(k+\frac{1}{k}\right)\xi^2+\xi^4\right\}\frac{d^2V}{d\xi^2}-(n^2-1)\left\{-\left(k+\frac{1}{k}\right)\xi+2\xi^3\right\}\frac{dV}{d\xi}$$
$$+2n^2k'^2\frac{dV}{dk}+n^2(n^2-1)\xi^2V=0$$

This equation is also satisfied by V_1, V_2, and V_3.

With Jacobi, we may put $k+\dfrac{1}{k}=\alpha$ and then equation (98) becomes

$$(99)\quad (1-\alpha\xi^2+\xi^4)\frac{d^2V}{d\xi^2}-(n^2-1)(-\alpha\xi+2\xi^3)\frac{dV}{d\xi}+2(4-\alpha^2)n^2\frac{dV}{d\alpha}$$
$$+n^2(n^2-1)\xi^2V=0\,,$$

in which form, the partial differential equation was given for the first time by Jacobi.

The above demonstration of Jacobi's partial differential equation is substantially the same as the one given by Briot and Bouquet* in their well-known treatise on elliptic functions, the only difference being that, in their demonstration, the intermediary steps are conducted by means of Weierstrass' function Al.

Briot and Bouquet give also the following forms of the partial differential equation :—

* Loc. cit. p. 529.

$$(100)\quad \left(1-\frac{1-2k^2}{kk'}\eta^2-\eta^4\right)\frac{d^2V}{d\eta^2}+(n^2-1)\left(\frac{1-2k^2}{kk'}\eta+2\eta^3\right)\frac{dV}{d\eta}$$
$$+2n^2k'\frac{dV}{dk}+n^2(n^2-1)\left(\frac{k}{k'}-\eta^2\right)V=0\,,$$

$$(101)\quad \left(1-\frac{2-k^2}{k'}\zeta^2+\zeta^4\right)\frac{d^2V}{d\zeta^2}+(n^2-1)\left(\frac{2-k^2}{k'}\zeta-2\zeta^3\right)\frac{dV}{d\zeta}$$
$$-2n^2kk'\frac{dV}{dk}-n^2(n^2-1)\left(\frac{1}{k'}-\zeta^2\right)V=0\,,$$

where $\eta=\sqrt{\frac{k}{k'}}\,\mathrm{cn}\,u$ and $\zeta=\frac{1}{\sqrt{k'}}\,\mathrm{dn}\,u.$

§ 14.

We shall have, by suitably determining the constants,

$$(102)\quad \begin{cases} V_1=\sqrt{k}\,x\,A(x^2), & A(0)=n, \\ V_2=\sqrt{\frac{k}{k'}}\sqrt{1-x^2}\,B(x^2), & B(0)=1, \\ V_3=\frac{1}{\sqrt{k'}}\sqrt{1-k^2x^2}\,C(x^2), & C(0)=1, \\ V\;=D(x^2), & D(0)=1, \end{cases}\qquad n \text{ odd},$$

$$(103)\quad \begin{cases} V_1=\sqrt{k}\,x\sqrt{1-x^2}\sqrt{1-k^2x^2}\,A(x^2), & A(0)=n, \\ V_2=\sqrt{\frac{k}{k'}}\,B(x^2), & B(0)=1, \\ V_3=\frac{1}{\sqrt{k'}}\,C(x^2), & C(0)=1, \\ V\;=D(x^2), & D(0)=1, \end{cases}\qquad n \text{ even},$$

where A, B, C, D are rational integral functions of x^2.

By division, we obtain

$$
\begin{array}{llll}
 & n \text{ odd}, & & n \text{ even}, \\
 & \operatorname{sn} nu = \dfrac{xA(x^2)}{D(x^2)}, & & \operatorname{sn} nu = \dfrac{x\sqrt{1-x^2}\sqrt{1-k^2x^2}A(x^2)}{D(x^2)}, \\
(104) & \operatorname{cn} nu = \dfrac{\sqrt{1-x^2}B(x^2)}{D(x^2)}, & & \operatorname{cn} nu = \dfrac{B(x^2)}{D(x^2)}, \\
 & \operatorname{dn} nu = \dfrac{\sqrt{1-k^2x^2}C(x^2)}{D(x^2)}, & & \operatorname{dn} nu = \dfrac{C(x^2)}{D(x^2)}.
\end{array}
$$

When n is odd, A, B, C, D are all of the degree n^2-1 in x, and when n is even, A is of the degree n^2-4 while A, B, C are all of the degree n^2 in x. Moreover, A, B, C, D are rational integral functions of k^2 whose coefficients are integral numbers. The coefficients of the highest power of x in A, B, C, D are respectively

$$(-1)^{\frac{n-1}{2}} k^{\frac{n^2-1}{2}}, \quad k^{\frac{n^2-1}{2}}, \quad k^{\frac{n^2-1}{2}}, \quad (-1)^{\frac{n-1}{2}} n\, k^{\frac{n^2-1}{2}},$$

or

$$(-1)^{\frac{n-2}{2}} n\, k^{\frac{n^2-4}{2}}, \quad k^{\frac{n^2}{2}}, \quad k^{\frac{n^2}{2}}, \quad (-1)^{\frac{n}{2}} k^{\frac{n^2}{2}},$$

according as n is odd or even.

Write

$$
(105) \quad
\begin{aligned}
A &= \Sigma A_{2m}x^{2m} = \Sigma A'_{2m}(1-x^2)^m = \Sigma A''_{2m}(1-k^2x^2)^m, \\
B &= \Sigma B_{2m}x^{2m} = \Sigma B'_{2m}(1-x^2)^m = \Sigma B''_{2m}(1-k^2x^2)^m, \\
C &= \Sigma C_{2m}x^{2m} = \Sigma C'_{2m}(1-x^2)^m = \Sigma C''_{2m}(1-k^2x^2)^m, \\
D &= \Sigma D_{2m}x^{2m} = \Sigma D'_{2m}(1-x^2)^m = \Sigma D''_{2m}(1-k^2x^2)^m;
\end{aligned}
$$

then, we have the well-known relations*

* Compare the work of Briot et Bouquet already referred to, or Baehr, *Sur les formules pour la multiplication des fonctions elliptiques de la première espèce*, Grunert's Archiv der Mathematik, Bd. XXXVI, pp. 125-176.

$$A_{2m}(k)=k^{2m}A_{2m}\left(\frac{1}{k}\right),\quad A''_{2m}(k)=A'_{2m}\left(\frac{1}{k}\right),$$

(106) $$C_{2m}(k)=k^{2m}B_{2m}\left(\frac{1}{k}\right),\quad C''_{2m}(k)=B'_{2m}\left(\frac{1}{k}\right),\quad B''_{2m}(k)=C'_{2m}\left(\frac{1}{k}\right),$$

$$D_{2m}(k)=k^{2m}D_{2m}\left(\frac{1}{k}\right),\quad D''_{2m}(k)=D'_{2m}\left(\frac{1}{k}\right).$$

Observe that A_{2m}, B_{2m}, C_{2m}, D_{2m}, A'_{2m}, B'_{2m}, C'_{2m}, D'_{2m} are integral functions of k^2 of the degree at most equal to m.

§ 15.

Consider first the case where n is an odd number, and put $n^2-1=4p=8q$.

Introducing the new variable $\xi=\sqrt{k}\,x$ in

$$\operatorname{sn} nu=\frac{x\,\Sigma'\,A_{2m}\,x^{2m}}{\Sigma'\,D_{2m}\,x^{2m}},$$

we get

(107) $$\sqrt{k}\operatorname{sn} nu=\frac{\xi\,\Sigma'\,a_{2m}\,\xi^{2m}}{\Sigma'\,d_{2m}\,\xi^{2m}},$$

where a_{2m}, d_{2m} are rational integral functions of $\alpha=k+\dfrac{1}{k}$.

We may suppose A and B to be arranged according to the powers of α, thus :—

(108) $$\begin{aligned} A&\equiv\Sigma A_{2m}x^{2m}\equiv\Sigma a_{2m}\xi^{2m}\equiv\Sigma E_m\alpha^m,\\ D&\equiv\Sigma D_{2m}x^{2m}\equiv\Sigma d_{2m}\xi^{2m}\equiv\Sigma H_m\alpha^m.\end{aligned}$$

From the well known relations

$$(109)\quad A(x,k)=(-1)^{\frac{n-1}{2}}D\left(\frac{1}{kx},k\right)k^{2p}x^{4p},\quad D(x,k)=(-1)^{\frac{n-1}{2}}A\left(\frac{1}{kx},k\right)k^{2p}x^{4p},$$

which may also be written in the form

$$(110)\quad A(\alpha,\xi)=(-1)^{\frac{n-1}{2}}D\left(\alpha,\frac{1}{\xi}\right)\xi^{4p},\qquad D(\alpha,\xi)=(-1)^{\frac{n-1}{2}}A\left(\alpha,\frac{1}{\xi}\right)\xi^{4p},$$

we see that A and D are of the same degree in α and that, moreover,

$$(111)\qquad E_m(\xi)=(-1)^{\frac{n-1}{2}}H_m\left(\frac{1}{\xi}\right)\xi^{4p},\quad H_m(\xi)=(-1)^{\frac{n-1}{2}}E_m\left(\frac{1}{\xi}\right)\xi^{4p}.$$

When the multiplicator n is required to be put in evidence, we shall write $A[n]$ and $D[n]$. Now the terms containing the highest power of α in $A[3]$, $A[5]$, $A[7]$ are $-2^2\xi^2\alpha$, $2^6\xi^{14}\alpha^3$, $-2^{12}\xi^{20}\alpha^6$, and those in $D[3]$, $D[5]$, $D[7]$ are $2^2\xi^6\alpha$, $2^6\xi^{10}\alpha^3$, $2^{12}\xi^{28}\alpha^6$. By virtue of the relation

$$(112)\quad A[n+2]A[n-2]=D^2[2]A^2[n]-(1-\alpha\xi^2+\xi^4)A^2[2]D^2[n],$$

which is easily deducible from the addition-equation, we conclude by applying mathematical induction that, generally, the terms involving the highest power of α in $A[n]$, $D[n]$ are

$$(113)\qquad E_q\alpha^q=(-1)^{\frac{n-1}{2}}2^p\xi^{\lambda_n}\alpha^q,\quad H_q\alpha^q=2^p\xi^{\mu_n}\alpha^q,$$

where $\lambda_n+\mu_n=n^2-1$.

To find λ_n, μ_n, we deduce from (112)

$$\lambda_{n+2}+\lambda_{n-2}=2\mu_n+2=2n^2-2\lambda_n,$$

which may be written

$$\left\{\lambda_{n+2}-\frac{(n+2)^2}{2}+1\right\}+2\left\{\lambda_n-\frac{n^2}{2}+1\right\}+\left\{\lambda_{n-2}-\frac{(n-2)^2}{2}+1\right\}=0,$$

or, putting for a moment $\left(\lambda_n-\frac{n^2}{2}+1\right)\equiv\psi_n$,

$$\psi_{n+2}+2\psi_n+\psi_{n-2}=0\,.$$

Observing $\psi_1=\frac{1}{2}$, $\psi_3=-\frac{3}{2}$, we find

$$\psi_n=\text{ coef. of } x^n \text{ in } \frac{1}{2}\frac{x-x^3}{(1+x^2)^2}=(-1)^{\frac{n-1}{2}}\frac{n}{2}\,.$$

Hence,

$$\lambda_n=\frac{n^2}{2}+(-1)^{\frac{n-1}{2}}\frac{n}{2}-1\,,\quad \mu_n=\frac{n^2}{2}-(-1)^{\frac{n-1}{2}}\frac{n}{2}\,, \tag{114}$$

that is,

$$\lambda_n=\frac{1}{2}n(n-1)-1\,,\quad \mu_n=\frac{1}{2}n(n+1)\,,$$

or

$$\lambda_n=\frac{1}{2}n(n+1)-1\,,\quad \mu_n=\frac{1}{2}n(n-1)\,,$$

according as $\frac{n-1}{2}$ is odd or even.

This premised, we now substitute $D=\sum\limits_{m=0}^{m=q}H_m\alpha^m$ in (99), which may be written in the form

$$\alpha\xi^2\frac{d^2V}{d\xi^2}-(n^2-1)\alpha\xi\frac{dV}{d\xi}+2n^2\alpha^2\frac{dV}{d\alpha}=(1+\xi^4)\frac{d^2V}{d\xi^2}-2(n^2-1)\xi^3\frac{dV}{d\xi}$$
$$+n^2(n^2-1)\xi^2V+8n^2\frac{dV}{d\alpha}\,,$$

and obtain

$$\xi^2\frac{d^2H_q}{d\xi^2}-(n^2-1)\xi\frac{dH_q}{d\xi}+2n^2qH_q=0\,, \tag{115}$$

$$\xi^2\frac{d^2H_{q-1}}{d\xi^2}-(n^2-1)\xi\frac{dH_{q-1}}{d\xi}+2n^2(q-1)H_{q-1}=(1+\xi^4)\frac{d^2H_q}{d\xi^2}-2(n^2-1)\xi^3\frac{dH_q}{d\xi} \tag{116}$$
$$+n^2(n^2-1)\xi^2H_q\,,$$

and, generally,

$$\xi^2\frac{d^2H_m}{d\xi^2}-(n^2-1)\xi\frac{dH_m}{d\xi}+2n^2mH_m=U_m\,,\quad m=q-2,\ q-3,\ \dots 0\,, \tag{117}$$

where $U_m=(1+\xi^4)\dfrac{d^2H_{m+1}}{d\xi^2}-2(n^2-1)\xi^3\dfrac{dH_{m+1}}{d\xi}+n^2(n^2-1)\xi^2H_{m+1}$

$$+8n^2(m+2)H_{m+2}\,.$$

It will be seen that equation (115) is satisfied by ξ^{μ_n}; indeed we might have determined μ_n by means of this equation, for if the operator $\xi\dfrac{d}{d\xi}$ be denoted by ϑ, then the differential equation may be written

$$(\vartheta^2-n^2\vartheta+2n^2q)\,H_q=0\,,$$

the general solution of which is

$$C'\xi^{\frac{1}{2}n(n-1)}+C''\xi^{\frac{1}{2}n(n+1)},$$

where C', C'' are arbitrary constants; and, as H_q can contain only even powers of ξ, $C'=0$, or $C''=0$, according as $\dfrac{n-1}{2}$ is odd or even.

Let us now investigate equation (116). For this purpose, it will be convenient to distinguish two cases according as $\dfrac{n-1}{2}$ is even or odd. When $\dfrac{n-1}{2}$ is even, $\mu_n=\dfrac{1}{2}n(n-1)$ and $H_q=2^p\xi^{\frac{1}{2}n(n-1)}$, further

$$(1+\xi^4)\frac{d^2H_q}{d\xi^2}-2(n^2-1)\xi^3\frac{dH_q}{d\xi}+n^2(n^2-1)\xi^2H_q$$

$$=2^{p-2}\{(n-2)(n-1)n(n+1)\xi^{\frac{1}{2}n(n-1)-2}+(n-1)n(n+1)(n+2)\xi^{\frac{1}{2}n(n-1)+2}\},$$

so that equation (116) now assumes the form

$$(\vartheta^2-n^2\vartheta+2n^2(q-1))H_{q-1}$$

$$=2^{p-2}\{(n-2)(n-1)n(n+1)\xi^{\frac{1}{2}n(n-1)-2}+(n-1)n(n+1)(n+2)\xi^{\frac{1}{2}n(n-1)+2}\}.$$

Now, q being equal to $\frac{n^2-1}{8}$,

$$\vartheta^2-n^2\vartheta+2n^2(q-1)=\left(\vartheta-\frac{n^2}{2}\right)^2-\left(\frac{3n}{2}\right)^2.$$

The complementary function is thus

$$C'\xi^{\frac{1}{2}n(n+3)}+C''\xi^{\frac{1}{2}n(n-3)};$$

but, as $\frac{1}{2}n(n-3)$ is odd and H_{q-1} contains only even powers of ξ, $C''=0$. Again, a particular integral is $-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-2}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+2}$, which, together with the complementary function just found, gives

$$H_{q-1}=-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-2}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+2}+\Gamma_1\,\xi^{\frac{1}{2}n(n+3)},$$

where Γ_1 is an as-yet undetermined function of n.

When $\frac{n-1}{2}$ is odd, we find likewise

$$H_{q-1}=\Gamma_1\,\xi^{\frac{1}{2}n(n-3)}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)}-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+1)}.$$

By virtue of (111),

$$E_{q-1}=\Gamma_1\,\xi^{\frac{1}{2}n(n-3)-1}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)-3}-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+1)+1},\quad \frac{n-1}{2}\text{ even,}$$

$$E_{q-1}=2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-3}+2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+1}-\Gamma_1\,\xi^{\frac{1}{2}n(n+3)-1},\quad \frac{n-1}{2}\text{ odd.}$$

We may put the multiplicator n in evidence by writing

$$A[n]\equiv\Sigma\overset{n}{E}_m\alpha^m,\quad D[n]\equiv\Sigma\overset{n}{H}_m\alpha^m,\quad q\equiv q_n,\quad p\equiv p_n,$$

and then equation (112) may be written

$$(118)\quad \Sigma\overset{n+2}{E}_m\alpha^m.\,\Sigma\overset{n-2}{E}_m\alpha^m=(1-\xi^4)^2\left[\Sigma\overset{n}{E}_m\alpha^m\right]^2-4(1-\alpha\xi^2+\xi^4)\left[\Sigma\overset{n}{H}_m\alpha^m\right]^2,$$

whence, equating the coefficients of the second highest power of α,

$$(119)\quad \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1}+\overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1}=(1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2-4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2$$
$$+8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1}.$$

If $\frac{n-1}{2}$ be odd, so that $\frac{n\pm 2-1}{2}$ is even, then the term containing the lowest power of ξ

$$
\begin{array}{llll}
\text{in} & \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1} & \text{is} & 2^{p_{n+2}}\overset{n-2}{\Gamma_1}\,\xi^{n^2-n+6}, \\
,, & \overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1} & ,, & 2^{p_{n-2}}\overset{n+2}{\Gamma_1}\,\xi^{n^2-n-2}, \\
,, & (1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2 & ,, & 2^{2p_n}\xi^{n^2-n-2}, \\
,, & -4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2 & ,, & -2^{2p_n+2}\xi^{n^2+n}, \\
,, & 8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1} & ,, & 2^{p_n+3}\overset{n}{\Gamma_1}\,\xi^{n^2-n+2}.
\end{array}
$$

Equating the coefficients of the lowest power of ξ (that is ξ^{n^2-n-2}) on the two sides of equation (119), we get

$$\overset{n+2}{\Gamma_1}=2^{2p_n-p_{n-2}}=2^{\frac{(n-1)(n+5)}{4}}, \qquad \frac{n-1}{2} \text{ odd},$$

or, writing n instead of $n+2$,

$$\overset{n}{\Gamma_1}=2^{\frac{(n-3)(n+3)}{4}}=2^{p_n-2}=2^{2(q_n-1)}, \qquad \frac{n-1}{2} \text{ even}.$$

Again, when $\frac{n-1}{2}$ is even, so that $\frac{n\pm 2-1}{2}$ is odd, the term containing the highest power of ξ

$$
\begin{array}{llll}
\text{in} & \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1} & \text{is} & 2^{p_{n+2}}\overset{n-2}{\Gamma_1}\,\xi^{n^2+n-2}, \\
,, & \overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1} & ,, & 2^{p_{n-2}}\overset{n+2}{\Gamma_1}\,\xi^{n^2+n+6}, \\
,, & (1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2 & ,, & 2^{2p_n}\xi^{n^2+n+6}, \\
,, & -4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2 & ,, & -2^{2p_n+2}\xi^{n^2-n+4}, \\
,, & 8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1} & ,, & 2^{p_n+3}\xi^{n^2+n+2}.
\end{array}
$$

Equating the coefficients of the highest power of ξ (that is ξ^{n^2+n+6}) on the two sides of equation (119), we obtain

$$\overset{n+2}{\Gamma}_1 = 2^{2p_n - p_{n-2}} = 2^{\frac{(n-1)(n+5)}{4}}, \quad \frac{n-1}{2} \text{ odd,}$$

or, writing n instead of $n+2$,

$$\overset{n}{\Gamma}_1 = 2^{\frac{(n-3)(n+3)}{4}} = 2^{p_n - 2} = 2^{2(q_n - 1)}, \quad \frac{n-1}{2} \text{ even.}$$

Thus we find, whether $\frac{n-1}{2}$ is odd or even,

$$\overset{n}{\Gamma}_1 = 2^{p_n - 2} = 2^{2(q_n - 1)} \tag{120}$$

and thence, when $\frac{n-1}{2}$ is even,

$$H_{q-1} = -2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-2} - 2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+2} + 2^{p-2}\xi^{\frac{1}{2}n(n+3)}, \tag{121}$$

$$E_{-1} = 2^{p-2}\xi^{\frac{1}{2}n(n-3)-1} - 2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)-3} - 2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+2)+1}, \tag{122}$$

and, when $\frac{n-1}{2}$ is odd,

$$H_{q-1} = 2^{p-2}\xi^{\frac{1}{2}n(n-3)} - 2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)-2} - 2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+1)+2}, \tag{123}$$

$$E_{q-1} = 2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-3} + 2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+1} - 2^{p-2}\xi^{\frac{1}{2}n(n+3)-1}. \tag{124}$$

Consider next H_{q-2}, whereby we suppose $\frac{n-1}{2}$ to be even. H_{q-2} satisfies differential equation (117)

$$\{\vartheta^2 - n^2\vartheta + 2n^2(q-2)\}H_{q-2} = U_{q-2},$$

where

$$U_{q-2} = -2^{p-5}(n-1)n(n^2-n-4)(n^2-n-6)\xi^{\frac{1}{2}n(n-1)-4} - 2^{p-4}n^2(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n-1)}$$
$$- 2^{p-5}n(n+1)(n^2+n-4)(n^2+n-6)\xi^{\frac{1}{2}n(n-1)+4} + 2^{p-4}n(n+3)(n^2+3n-2)\xi^{\frac{1}{2}n(n+3)-2}$$
$$+ 2^{p-4}n(n-3)(n^2-3n-2)\xi^{\frac{1}{2}n(n+3)+2}.$$

The complementary function is

$$C'\xi^{\frac{n^2-\sqrt{17}\,n}{2}}+C''\xi^{\frac{n^2+\sqrt{17}\,n}{2}},$$

and, as H_{q-2} can not contain irrational powers of ξ, we must have $C'=0$, $C''=0$. Hence, $\frac{n-1}{2}$ being even,

$$\begin{aligned}(125)\quad H_{q-2}=&\ 2^{p-7}(n-1)n(n^2-n-6)\xi^{\frac{1}{2}n(n-1)-4}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n-1)}\\ &+2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n-1)+4}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n+3)-2}\\ &-2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n+3)+2}\end{aligned}$$

and thence,

$$\begin{aligned}(126)\quad E_{q-2}=&-2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n-3)-3}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n-3)+1}\\ &+2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n+1)-5}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n+1)-1}\\ &+2^{p-7}(n-1)n(n^2-n-6)\xi^{\frac{1}{2}n(n+1)+3}.\end{aligned}$$

When $\frac{n-1}{2}$ is odd, we find likewise (or, more simply, by writing $-n$ in place of n in (125) and (126),)

$$\begin{aligned}(127)\quad H_{q-}=&-2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n-3)-2}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n-3)+2}\\ &+2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n+1)-4}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n+1)}\\ &+2^{p-7}n(n-1)(n^2-n-6)\xi^{\frac{1}{2}n(n+1)+4},\end{aligned}$$

$$\begin{aligned}(128)\quad E_{q-2}=&-2^{p-7}n(n-1)(n^2-n-6)\xi^{\frac{1}{2}n(n-1)-5}-2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n-1)-1}\\ &-2^{p-7}n(n-1)(n^2+n-6)\xi^{\frac{1}{2}n(n-1)+3}+2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n+3)-3}\\ &+2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n+3)+1}.\end{aligned}$$

Next, consider H_{q-3} which satisfies the differential equation

$$(\vartheta-\tfrac{1}{2}n(n+5))(\vartheta-\tfrac{1}{2}n(n-5))\ H_{q-3}=U_{q-3},$$

where

$$U_{q-3}=\frac{d^2H_{q-2}}{d\xi^2}+\xi^2(\vartheta-n^2)(\vartheta-n^2+1)H_{q-2}+\frac{n^2(n-3)(n+3)}{4}H_{q-1}.$$

When $\frac{n-1}{2}$ is even,

$$\begin{aligned}U_{q-3}= \; & 2^{p-9}(n-1)n(n-3)(n+2)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-6}\\ &+2^{p-9}(n-1)n(n-3)(3n+2)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-2}\\ &+2^{p-9}n(n+1)(n+3)(3n-2)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+2}\\ &+2^{p-9}n(n+1)(n+3)(n-2)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+6}\\ &-2^{p-7}n(n+3)(n+4)(n-1)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-4}\\ &-2^{p-6}n^2(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)}\\ &-2^{p-7}n(n-3)(n-4)(n+1)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+4}.\end{aligned}$$

A particular integral may easily be found. The complementary function is $\Gamma_3\,\xi^{\frac{1}{2}n(n-5)}$, where Γ_3 is a function of n. Hence we get, $\frac{n-1}{2}$ being even,

$$\begin{aligned}(129)\quad H_{q-3}= \; & \Gamma_3\xi^{\frac{1}{2}n(n-5)}\\ &-2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-6}\\ &-2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-2}\\ &-2^{p-10}n(n+3)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+2}\\ &-2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+6}\\ &+2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-4}\\ &+2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)}\\ &+2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+4},\end{aligned}$$

$$\begin{aligned}(130)\quad E_{q-3}= \; & 2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n-3)-5}\\ &+2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n-3)-1}\\ &+2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n-3)+3}\\ &-2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n+1)-7}\\ &-2^{p-10}n(n+3)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n+1)-3}\\ &-2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n+1)+1}\\ &-2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n+1)+5}\\ &+\Gamma_3\xi^{\frac{1}{2}n(n+5)-1}.\end{aligned}$$

When $\frac{n-1}{2}$ is odd, we get, by writing $-n$ instead of n in (129),

$$(131)\quad H_{q-3}= 2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n-3)-4}$$
$$+2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n-3)}$$
$$+2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n-3)+4}$$
$$-2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n+1)-6}$$
$$-2^{p-10}n(n+3)(n^4-2n^3+21n^2+42n+60)\xi^{\frac{1}{2}n(n+1)-2}$$
$$-2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n+1)+2}$$
$$-2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n+1)+6}$$
$$+\Gamma_3\xi^{\frac{1}{2}n(n+5)},$$

and thence, by virtue of (111),

$$(132)\quad E_{q-3}=-\Gamma_3\xi^{\frac{1}{2}n(n-5)-1}$$
$$+2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-7}$$
$$+2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-3}$$
$$+2^{p-10}n(n+3)(n^4-2n^3+21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+1}$$
$$+2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+5}$$
$$-2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-5}$$
$$-2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)-1}$$
$$-2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+3}.$$

Equating the coefficients of $\alpha^{\frac{n^2-9}{4}}$ on the two sides of (118), we obtain

$$\overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-3}+\overset{n+2}{E}_{q_{n+2}-1}\overset{n-2}{E}_{q_{n-2}-2}+\overset{n+2}{E}_{q_{n+2}-2}\overset{n-2}{E}_{q_{n-2}-1}+\overset{n+2}{E}_{q_{n+2}-3}\overset{n-2}{E}_{q_{n-2}}$$
$$=(1-\xi^4)^2\left(2\overset{n}{E}_{q_n}\overset{n}{E}_{q_n-2}+\overset{n}{E}_{q_n-1}\overset{n}{E}_{q_n-1}\right)$$
$$-4(1+\xi^4)\left(2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-2}+\overset{n}{H}_{q_n-1}\overset{n}{H}_{q_n-1}\right)+8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1}.$$

When $\frac{n-1}{2}$ is even, the lowest power of ξ in the above equation is ξ^{n^2-3n-2}. Equating the coefficients of this power of ξ on the two sides, we get $\overset{n+2}{\Gamma_3}\,2^{p_n-2}=2^{2(p_n-2)}$, whence

$$\overset{n+2}{\Gamma'_3} = 2^{\frac{(n+2)^2-1}{4}-6}, \quad \frac{n-1}{2} \text{ even}$$

or, writing n in place of $n+2$,

$$\overset{n}{\Gamma_3} = 2^{\frac{n^2-1}{4}-6} = 2^{2(q-3)}, \quad \frac{n-1}{2} \text{ odd.}$$

In like manner, we may shew that $\overset{n}{\Gamma_3}$ has the same value in the case where $\frac{n-1}{2}$ is even. Thus, whether n be odd or even,

(133) $$\Gamma_3 = 2^{2(q-3)}.$$

In determining $H_{q-\mu}$, every time $8\mu+1$ is of the form r^2, where r denotes an integer, there comes in the term $\overset{n}{\Gamma_\mu}\xi^{\frac{1}{4}n(n\pm r)}$. Indeed all the terms of H and E may readily be expressed in terms of Γ. On the other hand, Γ_μ may be determined by means of (118), as has been actually done in the cases $\mu=1$, 3; but when μ is a large number, this becomes very laborious. Now in the series

$$\Gamma_0, \ \Gamma_1, \ \Gamma_3, \ \Gamma_6, \ \Gamma_{10}, \ \Gamma_{15}, \ \Gamma_{21}, \ldots\ldots,$$

$$\Gamma_0 = 2^{2q}, \quad \Gamma_1 = 2^{2(q-1)}, \quad \Gamma_3 = 2^{2(q-3)},$$

so that most probably $\Gamma_\mu = 2^{2(q-\mu)}$; but I have not thus far succeeded in proving this generally.

For a given particular value of n, the constants Γ may be determined in a different manner. For example, take the case $n=9$.

$$\text{Sin } 9x = x(9-120x^2+432x^4-576x^6+256x^8).$$

Again $\overset{n}{E}_{q-6}$ contains the term $\overset{n}{\Gamma_6}\xi^{\frac{1}{4}n(n-7)-1}$ and $\overset{n}{H}_{q-10}$ the term $\overset{n}{\Gamma_{10}}\xi^{\frac{1}{4}n(n-9)}$. If we put $k=0$, α becomes infinite but in the same manner as $\frac{1}{k}$. Thus $\sin 9x$ contains the term $\overset{9}{\Gamma_6}x^9$, whence follows $\overset{9}{\Gamma_6} = 256$. Obviously $\overset{9}{\Gamma_{10}}=1$.

The elliptic functions of nu for $n = 2, 3, 4, 5, 6, 7, 8$ have been calculated by Baehr and others* by the primitive method of successively applying the addition-equation. Having found the values of $\overset{9}{\Gamma}$, $\operatorname{sn} 9u$ may be calculated by the above method without much difficulty.

§ 16.

As regards $\operatorname{sn} nu$, n being odd, the analysis contained in the preceding section, whereby the variables are taken to be $\sqrt{k}\operatorname{sn} u$ and α, leaves nothing to be desired; yet for the other functions, this is not the case, and it is better to have as the variables $\operatorname{sn} u$ and k. For the sake of uniformity, therefore, we shall once more investigate $\operatorname{sn} nu$, but this time consider it as a function of $\operatorname{sn} u$ and k.

Changing the variable from ξ to x, equation (98) takes the form

$$(134)\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2V}{dx^2}+\{((2n^2-1)k^2-1)x-2(n^2-1)k^2x^3\}\frac{dV}{dx}$$
$$+2n^2k(1-k^2)\frac{dV}{dk}+n^2(n^2-1)k^2x^2\,V=0\,,$$

and the equation is satisfied by the numerators and denominator of $\sqrt{k}\operatorname{sn} nu$, $\sqrt{\frac{k}{k'}}\operatorname{cn} nu$, $\frac{1}{\sqrt{k'}}\operatorname{dn} nu$.

The numerator of $\operatorname{sn} nu$, that is, $xA(x^2)$, satisfies the differential equation

* See the paper of Baehr already referred to, and Proceedings of the Royal Society of London, Vol. XXXIII (1882) pp. 480-489.

$$(135)\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2V}{dx^2}+\{((2n^2-1)k^2-1)x-2(n^2-1)k^2x^3\}\frac{dV}{dx}$$
$$+2n^2k(1-k^2)\frac{dV}{dk}+\{n^2(n^2-1)k^2x^2+n^2(1-k^2)\}\ V=0\,.$$

We may suppose the numerator of $\operatorname{sn} nu$ to be arranged according to the powers of k^2 and write

$$(136)\qquad xA(x^2)=\sum_{m=0}^{m=p}P_{2m}k^{2m}.$$

Then, denoting the operator $x\dfrac{d}{dx}$ by ϑ, we have

$$(137)\quad \frac{d^2P_{2m}}{dx}+(n^2(4m+1)-\vartheta^2)P_{2m}=((4m-3)n^2-2n^2\vartheta+\vartheta^2)P_{2m-2}$$
$$-x^2(n^2-\vartheta)(n^2-1-\vartheta)P_{2m-2}\,.$$

For $m=0$, we have

$$\frac{d^2P_0}{dx^2}+(n^2-\vartheta^2)\,P_0=0,$$

and, if we put

$$P_0=\sum_{r=0}\beta_r\,x^{2r+1},$$

then

$$\beta_r\,(2r+1)2r=-\beta_{r-1}(n^2-(2r-1)^2)\,.$$

And, since $\beta_0=n$, $\beta_r=(-1)^r\dfrac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-1)^2)}{(2r+1)\,!}$; thus we obtain the well-known series

$$(138)\qquad P_0=\sum_{r=0}(-1)^r\frac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-1)^2)}{(2r+1)\,!}\,x^{2r+1}\,.$$

For $m=1$, we have

$$\frac{d^2P_2}{dx^2}+(5n^2-\vartheta^2)P_2=(n^2-2n\vartheta+\vartheta^2)P_0+x^2(n^2-\vartheta)(n^2-1-\vartheta)P_0\,.$$

Since the lowest power of x in P_2 is 3, we write

$$(139) \qquad P_2=\sum_{r=1}\gamma_r\,x^{2r+1}\,,$$

then

$$2r(2r+1)\gamma_r+(5n^2-(2r^2-1))\,\gamma_{r-1}$$
$$=(n^2-2n^2(2r-1)+(2r-1)^2)\beta_{r-1}-(n^2-2r+3)(n^2-2r+2)\beta_{r-2}\,.$$

We find

$$\gamma_1=-\frac{n(n^2-1)}{3\,!},\quad \gamma_2=\frac{n(n^2-1)}{5\,!}2(2n^2-3),\quad \gamma_3=-\frac{n(n^2-1)(n^2-9)}{7\,!}3(3n^2-5),$$

....

Generally,

$$2r(2r+1)\gamma_r+(5n^2-(2r-1)^2)\gamma_{r-1}$$

$$=(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}\left\{\begin{array}{l}(4r^2-10r+5)n^4\\ -(12r^2-30r+16)n^2\\ +(2r-3)(2r-1)\end{array}\right\}$$

$$=(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}\left\{\begin{array}{l}-r(rn^2-(2r-1))\,(n^2-(2r-3)^2)\\ +(5n^2-(2r-1)^2)\,(r-1)\\ \qquad\qquad((r-1)n^2-(2r-3))\end{array}\right\}$$

$$=(-1)^r\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)\,(n^2-(2r-3)^2)}{(2r-1)\,!}r(rn^2-(2r-1))$$
$$+(5n^2-(2r-1)^2)(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}(r-1)$$
$$((r-1)n^2-(2r-3))\,,$$

or,

$$2r(2r+1)\left[\gamma_r-(-1)^r\frac{n(n^2-1)\ldots(n^2-(2r-3)^2)}{(2r+1)\,!}r(rn^2-(2r-1))\right]$$
$$=-(5n^2-(2r-1)^2)\left[\gamma_{r-1}-(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}(r-1)\right.$$
$$\left.((r-1)n^2-(2r-3))\right]\,;$$

whence follows

$$(140) \qquad \gamma_r = (-1)^r \frac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-3)^2)}{(2r+1)!} r(rn^2-(2r-1)).$$

For $m=2$, we have

$$\frac{d^2P_4}{dx^2}+(9n^2-\vartheta^2)P_4=(5n^2-2n^2\vartheta+\vartheta^2)P_2-x^2(n^2-\vartheta)(n^2-\vartheta-1)P_2.$$

Since the lowest power of x in P_4 is 5, we write

$$(141) \qquad P_4 = \sum_{r=2} \delta_r x^{2r+1};$$

then, $\quad 2r(2r+1)\delta_r+(9n^2-(2r-1)^2)\delta_{r-1}$

$$=((2r-1)^2-n^2(4r-7))\gamma_{r-1}-(n^2-2r+3)(n^2-2r+2)\gamma_{r-2},$$

whence we find

$$(142) \qquad \begin{aligned} \delta_2 &= \frac{n(n^2-1)(n^2-9)}{5!}, \\ \delta_3 &= -\frac{n(n^2-1)(n^2-9)}{7!}3(3n^2-5), \\ \delta_4 &= -\frac{n(n^2-1)}{9!}6(n^6+85n^4-671n^2+945), \\ \delta_5 &= \frac{n(n^2-1)(n^2-9)}{11!}2(247n^6+325n^4-13757n^2+23625). \end{aligned}$$

Thus the general law is not obvious as in the case of γ, and it seems to be impracticable to proceed further in this way.

Reverting to the formulæ

$$A = \sum_{m=0}^{m=2p} A_{2m}x^{2m}, \quad D = \sum_{m=0}^{m=2p} D_{2m}x^{2m},$$

since, by virtue of (109),

$$A_{2m} = (-1)^{\frac{n-1}{2}} D_{4p-2m}k^{-(2p-2m)}, \quad D_{2m} = (-1)^{\frac{n-1}{2}} A_{4p-2m}k^{-(2p-2m)},$$

we need only determine either A or D. Let us take D and apply (134). We obtain $D_0=1$, $D_2=0$, $D_{4p}=(-1)^{\frac{n-1}{2}}nk^{2p}$, and, generally,

$$(143)\quad (2m+1)(2m+2)D_{2m+2}+4m\{(n^2-m)k^2-m\}D_{2m}+2n^2k(1-k^2)\frac{dD_{2m}}{dk} \\ +\{(n^2-2m+1)(n^2-2m+2)\}k^2D_{2m-2}=0\,.$$

When $m\leqq p$,

$$(144)\quad D_{2m}=D_{2m,\,0}(1+k^{2m})+D_{2m,\,2}(k^2+k^{2m-2})+\ldots+D_{2m,\,2r}(k^{2r}+k^{2m-2r})+\ldots,$$

the last term being $D_{2m,\,m}k^m$ or $D_{2m,\,m-1}(k^{m-1}+k^{m+1})$ according as m is even or odd. Substituting this in (143), we obtain

$$\begin{aligned}
&(2m+1)(2m+2)\{D_{2m+2,\,0}(1+k^{2m+2})+D_{2m+2,\,2}(k^2+k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m+2,\,2r}(k^{2r}+k^{2m-2r+})+\ldots\} \\
&+\ 4m\,(n^2-m)\ \{D_{2m,\,0}(k^2+k^{2m+2})+D_{2m,\,2}(k^4+k^{2m})+\ldots \\
&\qquad\qquad +D_{2m,\,2r}(k^{2r+2}+k^{2m-2r+2})+\ldots\} \\
&-\qquad 4m^2\qquad \{D_{2m,\,0}(1+k^{2m})+D_{2m,\,2}(k^2+k^{2m-2})+\ldots+D_{2m,\,2r}(k^{2r}+k^{2m-2r})+\ldots\} \\
&+\qquad 2n^2\qquad \{D_{2m,\,0}(*+2mk^{2m})+D_{2m,\,2}(2k^2+(2m-2)k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m,\,2r}(2rk^{2r}+(2m-2r)k^{2m-2r})+\ldots\} \\
&-\qquad 2n^2\qquad \{D_{2m,\,0}(*+2mk^{2m+2})+D_{2m,\,2}(2k^4+(2m-2)k^{2m})+\ldots \\
&\qquad\qquad +D_{2m,\,2r}(2rk^{2r+2}+(2m-2r)k^{2m-2r+2})+\ldots\} \\
&+(n^2-2m+1)(n^2-2m+2)\{D_{2m-2,\,0}(k^2+k^{2m})+D_{2m-2,\,2}(k^4+k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m-2,\,2r}(k^{2r+2}+k^{2m-2r})+\ldots\} \\
&\qquad\qquad =0\,.
\end{aligned}$$

Hence

$$\begin{aligned}
(2m+1)(2m+2)D_{2m+2,\,0} &= 4m^2D_{2m,\,0}\,,\quad D_{2m,\,0}=0, \\
(2m+1)(2m+2)D_{2m+2,\,2} &= -4(n^2-m^2)D_{2m,\,2} \\
(2m+1)(2m+2)D_{2m+2,\,4} &= -4\{(m-1)n^2-m^2\}D_{2m,\,2}-4(2n^2-m^2)D_{2m,\,4}\,, \\
&\qquad -(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,2}\,, \\
(2m+1)(2m+2)D_{2m+2,\,6} &= -4\{(m-2)n^2-m^2\}D_{2m,\,4}-4(3n^2-m^2)D_{2m,\,6}\,, \\
&\qquad -(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,4}\,,
\end{aligned}$$

and, generally,

$$(145)\quad (2m+1)(2m+2)D_{2m+2,\,2r} = -4\{(m-r+1)n^2-m^2\}D_{2m,\,2r-2}$$
$$-4(rn^2-m^2)D_{2m,\,2r}-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,2r-2}\,,$$

the last coefficient being given by

$$(2m+1)(2m+2)D_{2m+2,\,m+1} = -8\left\{\frac{m+1}{2}n^2-m^2\right\}D_{2m,\,m-1}$$
$$-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,m-1}\,,$$

or

$$(2m+1)(2m+2)D_{2m+2,\,m} = -2\{(m+2)n^2-2m^2\}D_{2m,\,m-2}$$
$$-4\left(\frac{m}{2}n^2-m^2\right)D_{2m,\,m}-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,m-2}\,,$$

according as m is odd or even.

By means of (145), we find

(146)

$$D_0 = 1,\quad D_2 = 0,\quad D_4 = -\frac{n^2(n^2-1)}{4\,!}2k^2,$$
$$D_6 = \frac{n^2(n^2-1)(n^2-4)}{6\,!}8(k^2+k^4),$$
$$D_8 = -\frac{n^2(n^2-1)(n^2-4)}{8\,!}4\{8(n^2-9)(k^2+k^6)+(17n^2-69)k^4\},$$
$$D_{10} = \frac{n^2(n^2-1)(n^2-4)(n^2-9)}{10\,!}32\{4(n^2-16)(k^2+k^8)+15(n^2-4)(k^4+k^6)\},$$
$$D_{12} = -\frac{n^2(n^2-1)(n^2-4)(n^2-9)}{12\,!}8\{64(n^2-16)(n^2-25)(k^2+k^{10})$$
$$+8(n^2-16)(47n^2-185)(k^4+k^8)+15(45n^4-569n^2+1544)k^6\},$$
$$D_{14} = \frac{n^2(n^2-1)(n^2-4)(n^2-9)}{14\,!}32\{64(n^2-16)(n^2-25)(n^2-36)(k^2+k^{12})$$
$$+16(n^2-16)(34n^4-982n^2+3300)(k^4+k^{10})$$
$$+(1549n^6-43925n^4+357196n^2-815040)(k^6+k_8)\}.$$

Again equation (143) may be written in the form

$$(147)\quad (2m+2)(2m+3)k^2D_{4p-2m-2}+2n^2k(1-k^2)\frac{dD_{4p-2m}}{dk}$$
$$+(n^2-2m-1)\{(n^2+2m+1)k^2-(n^2-2m-1)\}D_{4p-2m}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2}=0\,.$$

Now, m being less than p,

$$(148)\quad D_{4p-2m}=D_{4p-2m,\,2p-2m}(k^{2p-2m}+k^{2p})+D_{4p-2m,\,2p-2m+2}(k^{2p-2m+2}+k^{2p-2})+\ldots$$
$$+D_{4p-2m,\,2p-2m+2r}(k^{2p-2m+2r}+k^{2p-2r})+\ldots\,,$$

the last term being

$$D_{4p-2m,\,2p-m}\,k^{2p-m}\quad\text{or}\quad D_{4p-2m,\,2p-m-1}(k^{2p-m-1}+k^{2p-m+1})\,,$$

according as m is even or odd.

From (147) and (148), we obtain

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m-2}+\{n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m}=0\,,$$

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m}+\{5n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2}$$
$$+\{(4m+1)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+2}=0\,,$$

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m+2}+\{9n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+4}$$
$$+\{(4m-3)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+4}=0\,,$$

generally,

$$(149)\quad (2m+2)(2m+3)D_{4p-2m-2,\,2p-2m+2r-2}$$
$$+\{(4r+1)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2r}$$
$$+\{(4m-4r+5)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2r-2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+2r}=0\,,$$

and, lastly,

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-m-2}+(2m+1)(n^2-2m-1)D_{4p-2m,\,2p-m}$$
$$+\{(2m+5)n^2-(2m+1)^2\}D_{4p-2m,\,2p-m-2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-m}=0\,,$$

or

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-m-1}+2\{(2m+3)n^2-(2m+1)^2\}D_{4p-2m,\,2p-m-1}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-m+1}=0\,,$$

according as m is even or odd.

From the above equations, we find

(150)

$$D_{4p-12}=(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)(n^2-9)}{13\,!}k^{2p-12}\{(n^2-25)(n^2-49)(n^2-81)(n^2-121)(1+k^{12})$$
$$+6(n^2-25)(n^2-49)(n^2-81)(6n^2-11)(k^2+k^{10})$$
$$-3(1927n^8-33068n^6-962n^4$$
$$+1033308n^2-1819125)(k^4+k^8)$$
$$-4(3046n^8-38037n^6+32799n^4$$
$$+708647n^2-1299375)k^6\}\,,$$

$$D_{4p-10}=-(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)(n^2-9)}{11\,!}k^{2p-10}\{(n^2-25)(n^2-49)(n^2-81)(1+k^{10})$$
$$+5(n^2-25)(n^2-49)(5n^2-9)(k^2+k^8)$$
$$-2(247n^6+325n^4-13757n^2+23625)(k^4+k^6)\}\,,$$

$$D_{4p-8}=(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)}{9\,!}k^{2p-8}\{(n^2-9)(n^2-25)(n^2-49)(1+k^8)$$
$$+4(n^2-9)(n^2-25)(4n^2-7)(k^2+k^6)$$
$$-6(n^6+85n^4-671n^2+945)k^4\}\,,$$

$$D_{4p-6} = -(-1)^{\frac{n-1}{2}} \frac{n(n^2-1)(n^2-9)}{7\,!} k^{2p-6}\{(n^2-25)(1+k^6)+3(3n^2-5)(k^2+k^4)\},$$

$$D_{4p-4} = (-1)^{\frac{n-1}{2}} \frac{n(n^2-1)}{5\,!} k^{2p-4}\{(n^2-9)(1+k^4)+2(2n^2-3)k^2\},$$

$$D_{4p-2} = -(-1)^{\frac{n-1}{2}} \frac{n(n^2-1)}{3\,!} k^{2p-2}(1+k^2),$$

$$D_{4p} = (-1)^{\frac{n-1}{2}} n\, k^{2p}.$$

The first five coefficients were given by Jacobi himself. The first and the last six coefficients of all the four functions have been found by Baehr by a different method.

§ 17.

Let us now consider the rational integral functions of x^2, B, C which enter into the numerators of $\operatorname{cn} nu$ and $\operatorname{dn} nu$, n being odd.

From the well-known relations

$$(151) \qquad B(x, k) = C\left(\frac{1}{kx}, k\right) k^{2p} x^{4p}, \quad C(x, k) = B\left(\frac{1}{kx}, k\right) k^{2p} x^{4p},$$

$$(152) \qquad B\left(kx, \frac{1}{k}\right) = C(x, k); \qquad C\left(kx, \frac{1}{k}\right) = B(x, k),$$

we deduce

$$(153) \qquad B\left(kx, \frac{1}{k}\right) = B\left(\frac{1}{kx}, k\right) k^{2p} x^{4p},$$

whence

$$(154) \qquad B_{4p-2m}(k) = B_{2m}\left(\frac{1}{k}\right) k^{2p}, \quad B_{2m}(k) = B_{4p-2m}\left(\frac{1}{k}\right) k^{2p}.$$

Hence, if we put

$$B = \sum_{m=0}^{m=2p} B_{2m}(k)\, x^{2m},$$

then

$$C = \sum_{m=0}^{m=2p} B_{2m}\left(\frac{1}{k}\right) x^{2m},$$

so that we need only determine B_{2m} and this only for the initial values of m in view of (154).

Now (134) may easily be modified in such a manner that the resulting equation is satisfied by B. We find

$$(155)\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2B}{dx^2}+\{[(2n^2-1)k^2-3]\,x-2(n^2-2)k^2x^3\}\frac{dB}{dx}$$
$$+2n^2k(1-k^2)\frac{dB}{dk}+(n^2-1)\{1+(n^2-2)k^2x^2\}\,B=0\,,$$

whence follows

$$(156)\quad (2m+1)(2m+2)B_{2m+2}+\{[n^2-(2m+1)^2]+4m(n^2-m)k^2\}B_{2m}$$
$$+2n^2k(1-k^2)\frac{dB_{2m}}{dk}+(n^2-2m)(n^2-2m+1)k^2B_{2m-2}=0\,.$$

Further we find, for $m \leqq p$,

$$(2m+1)(2m+2)B_{2m+2,\,0}+\{n^2-(2m+1)^2\}B_{2m,\,0}=0\,,$$

$$B_{2m,\,0}=0\,, \qquad n-1<2m\leqq 2p,$$

$$(2m+1)(2m+2)B_{2m+2,\,2}+\{5n^2-(2m+1)^2\}B_{2m,\,2}$$
$$+4m(n^2-m)B_{2m,\,0}+(n^2-2m)(n^2-2m+1)B_{2m-2,\,0}=0\,,$$

$$(2m+1)(2m+2)B_{2m+2,\,4}+\{9n^2-(2m+1)^2\}B_{2m,\,4}$$
$$+4\{(m-1)n^2-m^2\}B_{2m,\,2}+(n^2-2m)(n^2-2m+1)B_{2m-2,\,2}=0\,,$$

generally,

$$(175)\quad (2m+1)(2m+2)B_{2m+2,\,2r}+\{(4r+1)n^2-(2m+1)^2\}B_{2m,\,2r}$$
$$+4\{(m-r+1)n^2-m^2\}B_{2m,\,2r-2}$$
$$+(n^2-2m)(n^2-2m+1)B_{2m-2,\,2r-2}=0\,,$$

and, lastly,

$$\{(4m+1)n^2-(2m+1)^2\}B_{2m,\,2m}+(n^2-2m)(n^2-2m+1)B_{2m-2,\,2m-2}=0\,,$$

$$B_{2m,\,2m}=0\,,\qquad m\leqq p\,.$$

By means of the above equations, we find

(158)

$$B_0=1,\quad B_2=-\frac{n^2-1}{2\,!},$$

$$B_4=\frac{n^2-1}{4\,!}\{(n^2-9)+2n^2k^2\},$$

$$B_6=-\frac{n^2-1}{6\,!}\{(n^2-9)(n^2-25)+6n^2(n^2-9)k^2+8n^2(n^2-4)k^4\},$$

$$B_8=\frac{n^2-1}{8\,!}\{(n^2-9)(n^2-25)(n^2-49)+12n^2(n^2-9)(n^2-25)k^2$$
$$+4n^2(n^2-4)(15n^2-107)k^4+32n^2(n^2-4)(n^2-9)k^6\},$$

$$B_{10}=-\frac{(n^2-1)(n^2-9)}{10\,!}\{n^2-25)(n^2-49)(n^2-81)+20n^2(n^2-25)(n^2-49)k^2$$
$$+12n^2(n^2-4)(29n^2-329)k^4+32n^2(n^2-4)(14n^2-89)k^6$$
$$+128n^2(n^2-4)(n^2-16)k^8\},$$

$$B_{12}=\frac{(n^2-1)(n^2-9)}{12\,!}\{(n^2-25)(n^2-49)(n^2-81)(n^2-121)$$
$$+30n^2(n^2-25)(n^2-49)(n^2-81)k^2$$
$$+4n^2(593n^6-17082n^4+179517n^2-482708)k^4$$
$$+8n^2(n^2-4)(575n^4-10111n^2+44276)k^6$$
$$+192n^2(n^2-4)(n^2-16)(15n^2-89)k^8$$
$$+512n^2(n^2-4)(n^2-16)(n^2-25)k^{10}\}.$$

§ 18.

Consider next the case where n is even. To begin with, take $\operatorname{sn} nu$. Here D is exactly of the same form as in the case where n is odd, so that we may restrict ourselves to the consideration of A alone.

Now A is of the degree $n^2-4=4p$ say, and

(159) $$A\,(x, k) = (-1)^{\frac{n-2}{2}} A\left(\frac{1}{kx}, k\right) k^{2p}\, x^{4p},$$

and, therefore,

(160) $$A_{4p-2m} = (-1)^{\frac{n-2}{2}} A_{2m}\, k^{4p-2m}.$$

In consequence of (104) and (134), $A = \sum\limits_{m=0}^{m=2p} A_{2m} x^{2m}$ satisfies the differential equation

(161) $$\{x-(1+k^2)x^3+k^2x^5\}\frac{d^2A}{dx^2}+\{2+[(2n^2-5)k^2-5]\,x^2-2(n^2-4)k^2x^4\}\frac{dA}{dx}$$
$$+2n^2k(1-k^2)x\frac{dA}{dk}+(n^2-4)\{(1+k^2)x+(n^2-3)k^2x^3\}A=0\,,$$

whence follows,

(162) $$(2m+2)(2m+3)A_{2m+2}+\{n^2-4(m+1)^2+[(4m+1)n^2-4(m+1)^2]\,k^2\}\,A_{2m}$$
$$+2n^2k(1-k^2)\frac{dA_{2m}}{dk}+(n^2-2m-2)(n^2-2m-1)k^2A_{2m-2} = 0\,.$$

By virtue of (160), we need only determine the first half of the coefficients A_{2m}. Again, A_{2m} is of the form

$$A_{2m} = A_{2m,\,0}(1+k^{2m})+A_{2m,\,2}\,(k^2+k^{2m-2})+\ldots+A_{2m,\,2r}\,(k^{2r}+k^{2m-2r})+\ldots\,,$$

the last term being $A_{2m,\,m}\, k^m$ or $A_{2m,\,m-1}\,(k^{m-1}+k^{m+1})$, according as m is even or odd. Substituting this in (162), we get

$$(2m+2)(2m+3)A_{2m+2,0}+\{n^2-4(m+1)^2\}A_{2m,0}=0\,,$$

$$A_{2m,0}=(-1)^m\frac{n(n^2-4)(n^2-16)\ldots(n^2-4m^2)}{(2m+1)\,!}\,,$$

$$\begin{aligned}(2m+2)(2m+3)A_{2m+2,2}&+\{5n^2-4(m+1)^2\}A_{2m,2}\\&+\{(4m+1)n^2-4(m+1)^2\}A_{2m,0}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,0}=0\,,\end{aligned}$$

$$\begin{aligned}(2m+2)(2m+3)A_{2m+2,4}&+\{9n^2-4(m+1)^2\}A_{2m,4}\\&+\{(4m-3)n^2-4(m+1)^2\}A_{2m,2}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,2}=0\,,\end{aligned}$$

$$\begin{aligned}(2m+2)(2m+3)A_{2m+2,6}&+\{13n^2-4(m+1)^2\}A_{2m,6}\\&+\{(4m-7)n^2-4(m+1)^2\}A_{2m,4}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,4}=0\,,\end{aligned}$$

generally,

$$\begin{aligned}(163)\quad(2m+2)(2m+3)A_{2m+2,2r}&+\{(4r+1)n^2-4(m+1)^2\}A_{2m,2r}\\&+\{(4m-4r+5)n^2-4(m+1)^2\}A_{2m,2r-2}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,2r-2}=0\,,\end{aligned}$$

and, lastly,

$$\begin{aligned}(2m+2)(2m+3)A_{2m+2,m+1}&+2\{(2m+3)n^2-4(m+1)^2\}A_{2m,m-1}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,m-1}=0\,,\end{aligned}$$

or

$$\begin{aligned}(2m+2)(2m+3)A_{2m+2,m}&+\{(2m+1)n^2-4(m+1)^2\}A_{2m,m}\\&+\{(2m+5)n^2-4(m+1)^2\}A_{2m,m-2}\\&+(n^2-2m-2)(n^2-2m-1)A_{2m-2,m-2}=0\,,\end{aligned}$$

according as m is odd or even.

By means of the above equations, we find

(164)

$$A_0 = n, \quad A_2 = -\frac{n(n^2-4)}{3!}(1+k^2),$$

$$A_4 = \frac{n(n^2-4)}{5!}\{(n^2-16)(1+k^4)+2(2n^2-7)k^2\},$$

$$A_6 = -\frac{n(n^2-4)(n^2-16)}{7!}\{(n^2-36)(1+k^6)+3(3n^2-10)(k^2+k^4\},$$

$$A_8 = \frac{n(n^2-4)}{9!}\{(n^2-16)(n^2-36)[(n^2-64)(1+k^8)+4(4n^2-13)(k^2+k^6)] \\ -6(n^6+196n^4-2114n^2+4752)k^4\},$$

$$A_{10} = -\frac{n(n^2-4)}{11!}\{(n^2-16)(n^2-36)(n^2-64)[(n^2-100)(1+k^{10})+5(5n^2-16)(k^2+k^8)] \\ -2(247n^8-882n^6-72102n^4+661112n^2-1368000)(k^4+k^6)\},$$

$$A_{12} = \frac{n(n^2-4)}{13!}\{(n^2-16)(n^2-36)(n^2-64)(n^2-100)(n^2-144)(1+k^{12}) \\ +6(n^2-16)(n^2-36)(n^2-64)(n^2-100)(6n^2-19)(k^2+k^{10}) \\ +(-5781n^{10}+170472n^8-490140n^6-22744752n^4 \\ +193986816n^2-383754240)(k^4+k^8) \\ +4(-3046n^{10}+75579n^8-260532n^6-5901554n^4 \\ +48907368n^2-93493440)k^6\}.$$

§ 19.

Lastly we consider the numerators of $\operatorname{cn} nu$ and $\operatorname{dn} nu$, n being even. In this case, put $n^2 = 4p$.

As in the case where n is odd, $C_{2m}(k) = B_{2m}\left(\frac{1}{k}\right)k^{2m}$ by (106), so that we may here also restrict ourselves to the consideration of B. Moreover, since in this case

$$(165)\qquad B(x, k) = B\left(\frac{1}{kx}, k\right) k^{2p} x^{4p},$$

it follows,

$$(166)\qquad B_{4p-2m} = B_{2m} k^{2p-2m},$$

and we need only determine the first half of the coefficients B_{2m}.

Now B satisfies the differential equation

$$(167)\qquad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2B}{dx^2}+\{[(2n^2-1)k^2-1]x-2(n^2-1)k^2x^3\}\frac{dB}{dx}$$
$$+2n^2k(1-k^2)\frac{dB}{dk}+\{n^2(n^2-1)k^2x^2+n^2\} B = 0,$$

whence we deduce

$$(168)\qquad (2m+1)(2m+2)B_{2m+2}+\{(n^2-4m^2)+4m(n^2-m)k^2\}B_{2m}$$
$$+2n^2k(1-k^2)\frac{dB_{2m}}{dk}+(n^2-2m+1)(n^2-2m+2)k^2B_{2m-2} = 0.$$

Substituting

$$B_{2m} = B_{2m,0} + B_{2m,2}\, k^2+\ldots+B_{2m,2r}\, k^{2r}+\ldots+B_{2m,2m}\, k^{2m},$$
$$m \leqq p,$$

in (168), we get

$$(2m+1)(2m+2)B_{2m+2,0} +(n^2-4m^2)B_{2m,0} = 0,$$
$$B_{2m,0} = 0, \qquad n < 2m \leqq 2p,$$

$$(2m+1)(2m+2)B_{2m+2,2} +(5n^2-4m^2)B_{2m,2}$$
$$+4m(n^2-m)B_{2m,0} +(n^2-2m+1)(n^2-2m+2)B_{2m-2,0} = 0,$$

$$(2m+1)(2m+2)B_{2m+2,\,4}+(9n^2-4m^2)B_{2m,\,4}$$
$$+4\{(m-1)n^2-m^2\}B_{2m,\,2}+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2}=0\,,$$

generally,

$$(169)\quad (2m+1)(2m+2)B_{2m+2,\,2r}+\{(4r+1)n^2-4m^2\}B_{2m,\,2r}$$
$$+4\{(m-r+1)n^2-m^2\}B_{2m,\,2r-2}$$
$$+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2r-2}=0\,,$$

and, lastly,

$$(2m+1)(2m+2)B_{2m+2,\,2m+2}+\{(4m+1)n^2-4m^2\}B_{2m,\,2m}$$
$$+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2m-2}=0\,,$$

By means of the above equations, we find

(170)

$$B_0=1,\quad B_2=-\frac{n^2}{2\,!},$$

$$B_4=\frac{n^2}{4!}\{(n^2-4)+2(n^2-1)k^2\},$$

$$B_6=-\frac{n^2(n^2-4)}{6\,!}\{(n^2-16)+6(n^2-1)k^2+8(n^2-1)k^4\},$$

$$B_8=\frac{n^2(n^2-4)}{8\,!}\{(n^2-16)(n^2-36)+12(n^2-1)(n^2-16)k^2+$$
$$+4(n^2-1)(15n^2-51)k^4+32(n^2-1)(n^2-9)k^6\},$$

$$B_{10}=-\frac{n^2(n^2-4)}{10\,!}\{(n^2-16)(n^2-36)(n^2-64)+20(n^2-1)(n^2-16)(n^2-36)k^2$$
$$+12(n^2-1)(n^2-9)(29n^2-104)k^4$$
$$+64(n^2-1)(n^2-9)(7n^2-22)k^6$$
$$+128(n^2-1)(n^2-9)(n^2-16)k^8\},$$

$$
\begin{aligned}
B_{12} = \frac{n^2(n^2-4)}{12\,!}\{&(n^2-16)(n^2-36)(n^2-64)(n^2-100)\\
&+30(n^2-1)(n^2-16)(n^2-36)(n^2-64)k^2\\
&+4(n^2-1)(593n^6-14305n^4+113972n^2-257760)k^4\\
&+40(n^2-1)(n^2-9)(115n^4-1283n^2+2968)k^6\\
&+2880(n^2-1)(n^2-9)(n^2-16)(n^2-3)k^8\\
&+512(n^2-1)(n^2-9)(n^2-16)(n^2-25)k^{10}\}.
\end{aligned}
$$

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