Diffraction Phenomena produced by an Aperture on a Curved Surface.

By

H. Nagaoka.

In ordinary problems on diffraction of light produced by apertures of various shapes, the diffracting apertures are supposed to lie on a plane. The more general problem of diffraction produced by apertures on a known geometrical surface has not, so far, been touched. It has been my object to fill in this gap, although the expression for the intensity of diffracted light is integrable only in a few particular cases.

In the following, I give a general expression for the intensity of light diffracted by an aperture on a known surface, both for Fraunhofer's and Fresnel's diffraction phenomena. The expression is then applied to find the distribution of light after its passage through a small slit cut perpendicular to the generating line of a right circular cylinder.

Expression for the Intensity of the Diffracted Light.*

Let L be a source of light, and A B an aperture on a known geometrical surface. The ray of light propagated from L is diffracted by the aperture A B, and the diffraction phenomena thus produced may be seen either projected on a screen at D (Fig. 2), or observed by

* In the deduction of the expression for the intensity, I follow F. Neumann's method.

Fig. 1.

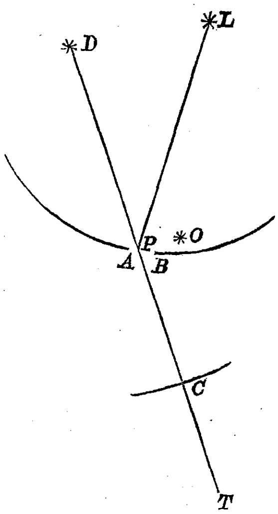
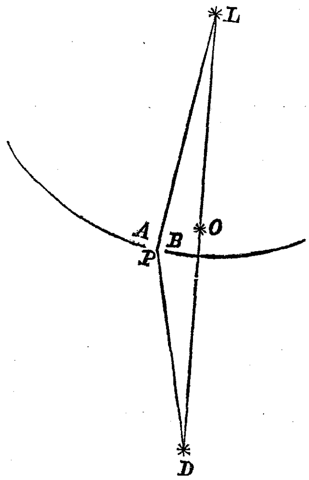


Fig. 2.



means of a telescope placed at T, and so focussed that the observer sees a distant point D (Fig. 1); in other words, D is the so-called diffraction point.

In order to find the general expression for the intensity of light after it is diffracted by an aperture on a curved surface, I shall assume that the diffracting aperture is very small compared with its distance from the source of light, and from the point at which the intensity of diffracted light is considered. Consequently the amplitude of vibration of the light coming from different points of the aperture will not vary at the point considered.

I shall first discuss Fraunhofer's (telescopic) diffraction phenomena. Referring to Fig. 1, let the vibration at L be represented by $\cos 2\pi \frac{t}{T}$; then, at any point P on the diffracting aperture, it will be proportional to

$$\cos\left(\frac{t}{T} - \frac{LP}{\lambda}\right) 2\pi.$$

Now considering the ray in the direction DP, the vibration at any point C in the line PT due to the small element $d\sigma$ at P is proportional to

$$d\sigma \cos\left(\frac{t}{T} - \frac{LP}{\lambda} - \frac{PC}{\lambda}\right) 2\pi.$$

Describe a sphere with D as centre, and passing through C; then the time taken by the ray to go from the spherical surface to the eye will be constant, provided D be sufficiently distant. Let this constant time be denoted by τ; the vibration at T is thus proportional to

$$d\sigma \cos\left(\frac{t-\tau}{T} - \frac{LP}{\lambda} - \frac{PC}{\lambda}\right) 2\pi.$$

A similar expression holds for the light propagated from every element of the aperture, so that the total effect at T will be given by the integral

$$(1) \qquad \int d\sigma \cos\left(\frac{t-\tau}{T} - \frac{LP}{\lambda} - \frac{PC}{\lambda}\right) 2\pi,$$

where the integration extends over the whole aperture.

Taking any point O near the aperture, we may write

$$\begin{aligned} PC &= DC - DP, \\ &= DC + (DO - DP) - DO, \\ LP &= LO - (LO - LP). \end{aligned}$$

Denoting the constant distances LO, DO by R and R' respectively, let $LO - LP = \varDelta R$, and $DO - DP = \varDelta R'$.

Introducing these symbols in the expressions for PC and LP, we find

$$\begin{aligned} PC &= DC - R' + \varDelta R', \\ LP &= R - \varDelta R. \end{aligned}$$

Substituting these in (1), we get for the vibration at T the integral

$$(2) \qquad \int d\sigma \cos\left(\frac{t-\tau}{T} - \frac{DC}{\lambda} - \frac{R-R'}{\lambda} + \frac{\varDelta R - \varDelta R'}{\lambda}\right) 2\pi.$$

Since τ, DC, $R - R'$ are all constant, we can put

$$\frac{t-\tau}{T} - \frac{DC}{\lambda} - \frac{R-R'}{\lambda} = \vartheta,$$

and the above expression for the vibration becomes

$$(3) \qquad \int d\sigma \cos\left(\vartheta + \frac{\varDelta R - \varDelta R'}{\lambda}\right)2\pi.$$

The intensity of light at T is, therefore, given by the expression

$$I = \left[\int d\sigma \cos\left(\frac{\varDelta R - \varDelta R'}{\lambda}\right) 2\pi\right]^2 + \left[\int d\sigma \cos\left(\frac{\varDelta R - \varDelta R'}{\lambda}\right) 2\pi\right]^2;$$

or more simply by

$$(\mathrm{I}) \qquad I = Mod.^2 \int d\sigma \, e^{\,i\frac{2\pi}{\lambda}(\varDelta R - \varDelta R')}$$

When the diffraction point is situated on the other side of the surface from the source of light, and the phenomenon is seen projected on a screen at D, we must slightly modify the expression for the intensity of light.

Proceeding in exactly the same way as before, the vibration at D due to a small element $d\sigma$ at P (Fig. 2), will be proportional to

$$d\sigma \cos\left(\frac{t}{T} - \frac{LO}{\lambda} - \frac{PD}{\lambda}\right)2\pi,$$

which can be written

$$d\sigma \cos\left(\frac{t}{T} - \frac{LP}{\lambda} + \frac{LO - LP}{\lambda} + \frac{OD - PD}{\lambda}\right)2\pi.$$

Putting as before

$$\frac{t}{T} - \frac{LD}{\lambda} = \vartheta,$$

$$LO - LP = \varDelta R,$$

$$OD - CD = \varDelta R',$$

we get

$$\cos\left(\frac{t}{T} - \frac{LP}{\lambda} - \frac{PD}{\lambda}\right)2\pi = \cos\left(\vartheta + \frac{\varDelta R + \varDelta R'}{\lambda}\right)2\pi.$$

Consequently, the total effect at D is given by

$$\int d\sigma \cos\left(\vartheta + \frac{\varDelta R + \varDelta R'}{\lambda}\right)2\pi.$$

Thus, the intensity of diffracted light at D is given by

$$I = \left[\int d\sigma \cos\left(\frac{\Delta R + \Delta R'}{\lambda}\right)2\pi\right]^2 + \left[\int d\sigma \sin\left(\frac{\Delta R + \Delta R'}{\lambda}\right)2\pi\right]^2;$$

or more briefly by

$$\text{(II)} \qquad I = Mod^2 \int d\sigma\, e^{\,i\frac{2\pi}{\lambda}(\Delta R + \Delta R')}.$$

The above expression gives the intensity of diffracted light for Fresnel's diffraction phenomena.

To evaluate the integrals given in (I) and (II), assume O as the origin of three rectangular co-ordinate axes x, y, z. Let the coordinates of the points L, D, P referred to these axes be denoted thus :—

$$\begin{aligned} L&: \ a, b, c, \\ D&: \ a', b', c', \\ P&: \ x, y, z, \end{aligned}$$

and let the equation of the surface referred to the same axes be

$$F(x, y, z) = const.$$

Thus, we have

$$\begin{aligned} LO &= \sqrt{a^2 + b^2 + c^2} = R, \\ OD &= \sqrt{a'^2 + b'^2 + c'^2} = R', \\ LP &= \sqrt{(a - x)^2 + (b - y)^2 + (c - z)^2} = R - \Delta R, \\ PD &= \sqrt{(a' - x)^2 + (b' - y)^2 + (c' - z)^2} = R' - \Delta R'. \end{aligned}$$

Expanding the expressions for LP and PD by means of the binomial theorem, we have

$$LP = R - \frac{ax + by + cz}{R} - \frac{1}{2R^3}(ax + by + cz)^2 + \frac{x^2 + y^2 + z^2}{2R} + \ldots\ldots,$$

$$PD = R' - \frac{a'x + b'y + c'z}{R'} - \frac{1}{2R'^3}(a'x + b'y + c'z)^3 + \frac{x^2 + y^2 + z^2}{2R'} + \ldots\ldots,$$

or

$$
(4)\quad \begin{cases} \Delta R = \dfrac{ax+by+cz}{R} + \dfrac{1}{2R^3}(ax+by+cz)^2 - \dfrac{x^2+y^2+z^2}{2R}, \\ \Delta R' = \dfrac{a'x+b'y+c'z}{R'} + \dfrac{1}{2R'^3}(a'x+b'y+c'z)^2 - \dfrac{x^2+y^2+z^2}{2R'}, \end{cases}
$$

Let the direction cosines of OL be $\varkappa$, μ, ν, and those of OD be $\varkappa'$, μ', ν'; then (4) becomes

$$
(4')\quad \begin{cases} \Delta R = (\varkappa x + \mu y + \nu z) + \dfrac{1}{2R}(\varkappa x + \mu y + \nu z)^2 - \dfrac{x^2+y^2+z^2}{2R}, \\ \Delta R' = (\varkappa' x + \mu' y + \nu' z) + \dfrac{1}{2R'}(\varkappa' x + \mu' y + \nu' z)^2 - \dfrac{x^2+y^2+z^2}{2R'}. \end{cases}
$$

In Fraunhofer's diffraction phenomena, R and R' are supposed to be very large compared with x, y, z, so that we can neglect the terms containing R or R' in the denominator. Thus

$$
\Delta R - \Delta R' = (\varkappa - \varkappa')x + (\mu - \mu')y + (\nu - \nu')z.
$$

Writing

$$
\frac{2\pi}{\lambda}(\varkappa - \varkappa') = l,
$$

$$
\frac{2\pi}{\lambda}(\mu - \mu') = m,
$$

$$
\frac{2\pi}{\lambda}(\nu - \nu') = n,
$$

the expression for the intensity of the diffracted light becomes

$$
(\mathrm{I}')\qquad I = Mod^2. \int d\sigma\, e^{i(lx+my+nz)}
$$

where the integration extends over the whole aperture.

In Fresnel's diffraction phenomena, we can no longer neglect the terms $\frac{1}{R}$ and $\frac{1}{R'}$. Thus the expression for $\Delta R + \Delta R'$ becomes very complicated. It is, however, somewhat simplified by taking O in the line LD as shown in Fig. 2. Thereby $\varkappa' = -\varkappa$, $\mu' = -\mu$, $\nu' = -\nu$, because OL and OD are in one line. Thus

$$\Delta R + \Delta R' = \left[(\varkappa x + \mu y + \nu z)^2 - (x^2 + y^2 + z^2)\right]\left(\frac{1}{2R} + \frac{1}{2R'}\right).$$

Introducing this value in (II), we get for the intensity of light at D

$$(\text{II}') \qquad I = Mod^2. \int d\sigma \, e^{i\frac{\pi}{\lambda}\left[(\kappa x + \mu y + \nu z)^2 - (x^2 + y^2 + z^2)\right]\left(\frac{1}{R} + \frac{1}{R'}\right)},$$

where the integration extends over the whole aperture.

Thus the problem of the diffraction of light produced by an aperture on a curved surface is reduced to the integration of expressions (I') and (II') for Fraunhofer's and Fresnel's diffraction phenomena respectively.

Fraunhofer's Diffraction Phenomena produced by a narrow Slit on a cylindrical Surface.

Let us now discuss Fraunhofer's diffraction phenomena produced by a narrow slit cut on a right circular cylinder and perpendicular to the generating line of the cylinder.

In order to calculate the intensity of light for different positions of the telescope, drop a perpendicular on the axis of the cylinder from the centre of the slit. Assume the centre as the origin of co-ordinate axes. Let the x axis be parallel to the axis of the cylinder, and the z axis perpendicular thereto, both drawn through the centre of the slit.

The axes being thus fixed, we have, by (I'), to find the integral

$$\int d\sigma e^{i(lx + my + nz)},$$

where the integration extends over the whole aperture, and l, m, n are determined by the directions of the incident light and of the observing telescope referred to the rectangular axes above specified, and by the wave length of light employed in the observation. In addition to this, there is the equation of condition

$$y^2 + z^2 - 2\,a\,z = 0$$

expressing the fact that the aperture lies on a cylinder of radius a.

In actual calculation, it is more convenient to use polar co-ordinates. In the right circular section of the cylinder, assume polar co-ordinates with the pole on the axis, and take

$$y = a \sin \vartheta\,, \qquad z = a\,(1 - \cos \vartheta),$$

Then
$$d\sigma = a\, dx\, d\vartheta.$$

Thus
$$\int e^{i(lx+my+nz)} d\sigma = a e^{ia} \int_{-b}^{+b} dx\, e^{ilx} \int d\vartheta\, e^{ia(m \sin \vartheta - n \cos \vartheta)}.$$

where $2b$ denotes the breadth of the slit.

The integral

$$\int_{-b}^{+b} dx\, e^{ilx} = \frac{e^{ilb} - e^{-ilb}}{il},$$

$$= \frac{2 \sin lb}{l}.$$

It thus remains to find the integral

$$\int d\vartheta\, e^{ia(m \sin \vartheta - n \cos \vartheta)}$$

taken between proper limits.

Introduce an auxiliary angle ϑ', such that

$$a\,m = \xi \sin \vartheta', \qquad a\,n = \xi \cos \vartheta'$$

where
$$\xi = a\sqrt{m^2 + n^2}.$$

Then $$a\,(m \sin \vartheta - n \cos \vartheta) = \xi \cos (\vartheta + \vartheta') = \xi \cos \varphi,$$

where φ stands for $\vartheta + \vartheta'$.

Thus $$\int d\vartheta\, e^{ia(m \sin \vartheta\, -\, n \cos \vartheta)} = \int d\varphi\, e^{i\,\xi \cos \varphi}.$$

The limits of integration with respect to φ are found from ϑ' and the known limits with respect to ϑ.

The difficulty of the problem lies simply in finding the integral

$$J = \int d\varphi\, e^{i\,\xi \cos \varphi}.$$

I shall henceforth put $J = K + i\,L$, where

$$K = \int \cos (\xi \cos \varphi)\, d\varphi,$$

$$L = \int \sin (\xi \cos \varphi)\, d\varphi.$$

Evaluation of the Integral $J = \int d\varphi\, e^{i\,\xi \cos \varphi}$.

There are various ways of evaluating the above integral. The simplest way would be to find a differential equation which is satisfied by J, and by this means to expand it in a series proceeding according to ascending powers of ξ.

Since every integral of the form $\int d\varphi\, e^{i\,\xi \cos \varphi}$ between known limits can be decomposed into a sum of two separate integrals of the form $\int_0^a d\varphi\, e^{i\,\xi \cos \varphi}$, I shall only consider

$$J = \int_0^a e^{i\,\xi \cos \varphi}\, d\varphi.$$

Putting $$\cos \varphi = u, \cos a = c,$$

$$J = -\int_1^c \frac{e^{i\,\xi\, u}}{\sqrt{1 - u^2}}\, du.$$

Differentiating with respect to ξ, we have

$$\frac{dJ}{d\xi} = -i\int_1^c \frac{u\,e^{i\xi u}}{\sqrt{1-u^2}}\,du,$$

$$\frac{d^2J}{d\xi^2} = \int_1^c \frac{e^{i\xi u}}{\sqrt{1-u^2}}\,du - \frac{1}{i\xi}\Big[(1-c^2)^{\frac{1}{2}}e^{i\xi u}\Big]_1^c - \frac{1}{i\xi}\int_1^c \frac{e^{i\xi u}u}{\sqrt{1-u^2}}\,du$$

$$= -J - \frac{1}{\xi}\frac{dJ}{d\xi} + \frac{i\,s\cos(c\xi)}{\xi} - \frac{s\sin(c\xi)}{\xi}$$

where s stands for $\sqrt{1-c^2} = \sin a$.

Thus the differential equations satisfied by K and L are respectively

$$\frac{d^2K}{d\xi^2} + \frac{1}{\xi}\frac{dK}{d\xi} + K = -\frac{s\sin(c\xi)}{\xi},$$

and

$$\frac{d^2L}{d\xi^2} + \frac{1}{\xi}\frac{dL}{d\xi} + L = \frac{s\cos(c\xi)}{\xi}.$$

To find the expression for K and L, assume a series proceeding according to ascending powers of ξ. Differentiating and properly choosing the constants, we easily find that

$$(a)\quad K = a - \frac{sp}{2^2}\xi^2 + \frac{s\xi^4}{4^2}\left(\frac{c^3}{3^2} + \frac{p}{2^2}\right) - \frac{s\xi^6}{6^2}\left(\frac{c^5}{5!} + \frac{c^3}{3!\,4^2} + \frac{p}{2^2.\,4^2}\right)$$

$$+ \cdots\cdots + (-1)^{n-1}\frac{s\xi^{2n}}{(2n)^2}\left(\frac{c^{2n-1}}{(2n-1)!} + \frac{c^{2n-3}}{(2n-3)!\,(2n-2)^2} + \cdots\cdots\right.$$

$$\left. + \frac{p}{2^2.\,4^2.\,6^2\ldots(2n-2)^2}\right) \pm \cdots\cdots$$

$$(b)\quad L = s\Big[\xi - \frac{\xi^3}{3^2}\left(\frac{c^2}{2^2} + 1\right) + \frac{\xi^5}{5^2}\left(\frac{c^4}{4!} + \frac{c^2}{2!\,3^2} + \frac{1}{3^2}\right)$$

$$- \frac{\xi^7}{7^2}\left(\frac{c^6}{6!} + \frac{c^4}{4!\,5^2} + \frac{c^2}{2!\,3^2.\,5^2} + \frac{1}{3^2.\,5^2}\right) + \cdots\cdots$$

$$+ (-1)^n\frac{\xi^{2n+1}}{(2n+1)^2}\left(\frac{c^{2n}}{2n!} + \frac{c^{2n-2}}{(2n-2)!\,(2n-1)^2} + \cdots\cdots\right.$$

$$\left. + \frac{1}{3^2.5^2\ldots(2n-1)^2}\right) \pm \cdots\cdots\Big].$$

where p stands for $\frac{1}{s}(cs + a)$.

It is to be remarked that when $a = \pi$, K becomes equal to $\pi J^{\circ}(\xi)$, where J° denotes Bessel's function of the first kind with index 0. Thus the above expression for K reduces to

$$K = \pi\left(1 - \frac{\xi^2}{2^2} + \frac{\xi^4}{2^2.\,4^2} - \frac{\xi^6}{2^2.\,4^2.\,6^2} + \frac{\xi^8}{2^2.\,4^2.\,6^2.\,8^2} - \cdots\cdots\cdots\right).$$

The expression within the bracket is the well-known form for $J^{\circ}(\xi)$.

It is easily seen that the above two series for K and L converge rapidly so long as ξ is small; but when ξ becomes large, it would be advantageous to employ other expressions for K and L.

The usual process of calculating $\int_0^a e^{i\,\xi\,\cos\varphi}\,d\varphi$ is to expand $e^{i\xi\,\cos\varphi}$ in a Fourier series, and integrate each term of the series separately. Thus

$$e^{i\,\xi\,\cos\varphi} = J^{\circ}(\xi) + 2\sum_1^{\infty} i^n J^n(\xi)\cos n\varphi.$$

$$\therefore \qquad \int_0^a e^{i\,\xi\,\cos\varphi}\,d\varphi = J^{\circ}(\xi)\,a + 2\sum_1^{\infty} i^n J^n(\xi)\,\frac{\sin(n\,a)}{n}.$$

Equating the real and imaginary parts to K and L respectively, we have

$$(c) \qquad K = a\,J^{\circ}(\xi) + 2\sum_1^{\infty}(-1)^n J^{2n}(\xi)\,\frac{\sin(2\,n\,a)}{2\,n},$$

$$(d) \qquad L = 2\sum_0^{\infty}(-1)^{n+1} J^{2n+1}(\xi)\,\frac{\sin(2\,n+1)\,a}{2\,n+1}.$$

The form given above is not rapidly convergent. The values of $J^n(\xi)$ can be easily calculated from the values of $J^{\circ}(\xi)$ and $J^1(\xi)$ given in the tables of Hansen and Meissel up to certain values of the argument ξ. But for higher values of ξ, we should have to calculate $J^{\circ}(\xi)$ and $J^1(\xi)$. Moreover, when n exceeds ξ, the value of $J^n(\xi)$ deduced successively from $J^{\circ}(\xi)$ and $J^1(\xi)$ becomes inaccurate, and we are thus compelled to undertake the calculation separately. These considerations make the formulae just given less convenient for calculation than the formulae given below.

As already mentioned, the form of the integral J shows that when the limits lie from o to π, it becomes equal to $\pi J^{\circ}(\xi)$. Thus J includes Bessel's function of the first kind with index 0 as a particular case. By a special transformation, J can be made to depend on $J^{\circ}(n\pi)$ as will now be shown.

Putting $u = \cos\varphi$, we have

$$J = -\int \frac{e^{i\xi u}}{\sqrt{1-u^2}} du.$$

Expanding $\frac{1}{\sqrt{1-u^2}}$ in a Fourier series,

$$\frac{1}{\sqrt{1-u^2}} = \frac{1}{2} + \sum_1^{\infty} J^{\circ}(n\pi) \cos n\pi u.$$

Multiplying this by $e^{i\xi u}$, and integrating

$$\int \frac{e^{i\xi u}}{\sqrt{1-u^2}} du = -\frac{i}{2}\left[\frac{e^{i\xi u}}{\xi} + \sum_1^{\infty} J^{\circ}(n\pi)\left(\frac{e^{i(\xi+n\pi)u}}{\xi+n\pi} + \frac{e^{i(\xi-n\pi)u}}{\xi-n\pi}\right)\right].$$

After a simple reduction, we have

$$\int \frac{e^{i\xi u}}{\sqrt{1-u^2}} du = -\frac{i}{2} e^{i\xi u}\left(\frac{1}{\xi} + 2\sum_1^{\infty} J^{\circ}(n\pi)\frac{\xi \cos n\pi - i n\pi \sin n\pi u}{\xi^2 - n^2\pi^2}\right).$$

Equating the real and imaginary parts of both sides of the equation

$$(e) \quad \int \frac{\cos(\xi u)}{\sqrt{1-u^2}} du = \frac{1}{2}\left[\frac{\sin\xi u}{\xi} + 2\xi \sin\xi u \sum_1^{\infty} \frac{J^{\circ}(n\pi)\cos(n\pi u)}{\xi^2 - n^2\pi^2}\right.$$
$$\left. - 2\pi \cos\xi u \sum_1^{\infty} \frac{n J^{\circ}(n\pi)\sin(n\pi u)}{\xi^2 - n^2\pi^2}\right],$$

$$(f) \qquad \int \frac{\sin(\xi u)}{\sqrt{1-u^2}}\,du = -\frac{1}{2}\Bigg[\frac{\cos \xi u}{\xi} + 2\,\xi \cos(\xi u) \sum_{1}^{\infty} \frac{J^{\circ}(n\pi)\cos n\pi u}{\xi^2 - n^2\pi^2} + 2\,\pi \sin(\xi u) \sum_{1}^{\infty} \frac{n\,J^{\circ}(n\pi)\sin(n\pi u)}{\xi^2 - n^2\pi^2}\Bigg].$$

These two expressions (e) and (f) are equal to $-K$ and $-L$ respectively.

Thus K and L are made to depend on $J^{\circ}(n\pi)$, which can be calculated once for all; the rest involving simple arithmetical and trigonometrical calculations.

The expressions (e) and (f) above deduced for K and L require special consideration when ξ is a multiple of π, since both then contain terms of the form $\frac{0}{0}$.

Let us suppose that $\xi = m\pi = \gamma$. Then the expressions for K and L assume following forms.

$$K = -\frac{1}{2}\Bigg[\frac{\sin \xi u}{\xi} + 2\,\xi \sin \xi u \sum_{1}^{m-1} \frac{J^{\circ}(n\pi)\cos(n\pi u)}{\xi^2 - n^2\pi^2} + 2\,\xi \sin \xi u \sum_{m+1}^{\infty} \frac{J^{\circ}(n\pi)\cos(n\pi u)}{\xi^2 - n^2\pi^2} - 2\,\pi \cos \xi u \sum_{1}^{m-1} \frac{nJ^{\circ}(n\pi)\sin(n\pi u)}{\xi^2 - n^2\pi^2} - 2\,\pi \cos(\xi u) \sum_{m+1}^{\infty} \frac{nJ^{\circ}(n\pi)\sin(n\pi u)}{\xi^2 - n^2\pi^2} + 2\,K' J^{\circ}(m\pi)\Bigg],$$

where

$$K' = Lt_{\xi=\gamma} \frac{\xi \sin(\xi u)\cos(\gamma u) - \gamma \cos(\xi u)\sin(\gamma u)}{\xi^2 - \gamma^2}.$$

$$L = \frac{1}{2}\Bigg[\frac{\cos \xi u}{\xi} + 2\,\xi \cos \xi u \sum_{1}^{m-1} \frac{J^{\circ}(n\pi)\cos(n\pi u)}{\xi^2 - n^2\pi^2} + 2\xi \cos \xi u \sum_{m+1}^{\infty} \frac{J^{\circ}(n\pi)\cos n\pi u}{\xi^2 - n^2\pi^2} + 2\,\pi \sin(\xi u) \sum_{1}^{m-1} \frac{J^{\circ}(n\pi)\sin(n\pi u)}{\xi^2 - n^2\pi^2} + 2\,\pi \sin(n\pi u) \sum_{m+1}^{\infty} \frac{J^{\circ}(n\pi)\sin n\pi u}{\xi^2 - n^2\pi^2} + 2\,L' J^{\circ}(m\pi)\Bigg],$$

where

$$L' = Lt_{\xi=\gamma} \frac{\xi cos(\xi u_2) cos(\gamma u_2) + \gamma sin(\xi u_2) sin(\gamma u_2) - \xi cos(\xi u_1) cos(\gamma u_1) - \gamma sin(\xi u_1) sin(\gamma u_1)}{\xi^2 - \gamma^2}$$

u_1, u_2 denoting the limits of integration with respect to u.

Evaluating these two indeterminate forms K' and L', we find

$$K' = \frac{2\gamma u + sin(2\gamma u)}{4\gamma},$$

$$L' = -\frac{sin\,\gamma(u_1 + u_2)\, sin\,\gamma(u_2 - u_1)}{2}.$$

Thus the expressions above deduced can be employed for calculating K and L for all values of ξ.

I may here remark, though it has nothing to do with the question of diffraction, that a more general integral of the form

$$\int d\varphi\, e^{i\xi\cos\varphi} sin^{2\nu}\varphi,$$

the limits lying between π and $-\pi$ can be made to depend on $J^{\nu}(n\pi)$, by exactly the same process as above given. In fact, Bessel's function of the first kind with index ν can be expressed by means of the following formula

$$J^{\nu}(\xi) = \xi^{\nu}\left(\frac{1.3.5. ..(2\nu-1)}{2.4.6.......2\nu}\frac{\sqrt{\pi}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}\frac{sin\,\xi}{\xi} + \frac{2\,\xi\,sin\,\xi}{\pi^{\nu}}\sum_1^{\infty}(-1)^{n+1}\frac{J^{\nu}(n\pi)}{n^{\nu}(n^2\pi^2-\xi^2)}\right)$$

For convenience of calculation, the following values of $J^{\circ}(n\pi)$, for successive values of n, have been calculated and tabulated.

n	$J^{\circ}(n\pi)$	$log\ J^{\circ}(n\pi)$	n	$J^{\circ}(n\pi)$	$log\ J^{\circ}(n\pi)$
1	−0.304242	(−) $\bar{1}$.483219	26	+0.062329	(+) $\bar{2}$.794694
2	+0.220277	(+) $\bar{1}$.342969	27	−0.061168	(−) $\bar{2}$.786523
3	−0.181212	(−) $\bar{1}$.258186	28	+0.060069	(+) $\bar{2}$.778650
4	+0.157507	(+) $\bar{1}$.197300	29	−0.059027	(−) $\bar{2}$.771051
5	−0.141182	(−) $\bar{1}$.149779	30	+0.058038	(+) $\bar{2}$.763710
6	+0.129064	(+) $\bar{1}$.110804	31	−0.057096	(−) $\bar{2}$.756608
7	−0.119609	(−) $\bar{1}$.077765	32	+0.056199	(+) 2.749732
8	+0.111968	(+) $\bar{1}$.049093	33	−0.055343	(−) $\bar{2}$.743066
9	−0.105625	(−) $\bar{1}$.023768	34	+0.054525	(+) $\bar{2}$.736599
10	+0.100251	(+) $\bar{1}$.001089	35	−0.053743	(−) $\bar{2}$.730320
11	−0.095621	(−) $\bar{2}$.980555	36	+0.052993	(+) $\bar{2}$.724216
12	+0.091579	(+) $\bar{2}$.961796	37	−0.052273	(−) $\bar{2}$.718280
13	−0.088010	(−) $\bar{2}$.944530	38	+0.051582	(+) $\bar{2}$.712501
14	+0.034827	(+) $\bar{2}$.928535	39	−0.050918	(−) $\bar{2}$.706872
15	−0.081967	(−) $\bar{2}$.913638	40	+0.050279	(+) $\bar{2}$.701386
16	+0.079378	(+) $\bar{2}$.899697	41	−0.049663	(−) $\bar{2}$.696035
17	−0.077019	(−) $\bar{2}$.886597	42	+0.049070	(+) $\bar{2}$.690812
18	+0.074859	(+) $\bar{2}$.874243	43	−0.048497	(−) $\bar{2}$.685712
19	−0.072871	(−) $\bar{2}$.862554	44	+0.047944	(+) $\bar{2}$ 680729
20	+0.071033	(+) 2.851462	45	−0.047409	(−) $\bar{2}$.675858
21	−0.069328	(−) $\bar{2}$.840910	46	+0.046892	(+) $\bar{2}$.671094
22	+0.067740	(+) $\bar{2}$.830846	47	−0.046391	(−) $\bar{2}$.666432
23	−0.066257	(−) $\bar{2}$.821228	48	+0.045906	(+) $\bar{2}$.661868
24	+0.064863	(+) $\bar{2}$.812017	49	−0.045436	(−) $\bar{2}$.657398
25	−0.063560	(−) $\bar{2}$.803183	50	+0.044980	(+) $\bar{2}$.653018

Returning to our problem on Fraunhofer's diffraction phenomena, we get for the expression of the intensity

$$I = 4\, a^2 \frac{\sin^2 lb}{l^2} (K^2 + L^2)$$

With a homogeneous source of light, the intensity always vanishes whenever $l\, b$ is a multiple of π. The fringes arising from the term $\sin^2 l\, b$ are exactly the same as those given by the plane slit. When the surface on which the slit is cut is cylindrical, the additional factor $K^2 + L^2$ enters into the expression for the intensity of the diffracted light. This factor has maxima and minima for different positions of the telescope, and moreover depends on the length of the slit. Thus, when the limits of integration lie from 0 to π, $K = \pi\, J^{\circ}(\xi)$ and $L = 0$, and there would be places of darkness for such positions of the telescope as are determined by the values of ξ corresponding to the roots of $J^{\circ}(\xi)$.

For a great number of equidistant slits, the expression for the intensity would be the same as that for ordinary grating, multiplied by the factor $K^2 + L^2$.

The case which calls for special attention is when the ray is normally incident, and the telescope turned so as always to lie in the plane xy. Then $\varkappa = 0$, $\mu = 0$, $\nu = 1$, and $\mu' = 0$. Thus $l = \frac{2\pi}{\lambda} \sin \omega$, where ω is the angle made by the axis of the telescope with z axis. The places of darkness are given by

$$\sin \omega = \frac{n}{2} \cdot \frac{\lambda}{b}$$

The maxima and minima arising from the term $K^2 + L^2$ must be separately determined for the particular slit in question.

Fresnel's Diffraction Phenomena produced by a Slit on a Cylindrical Surface.

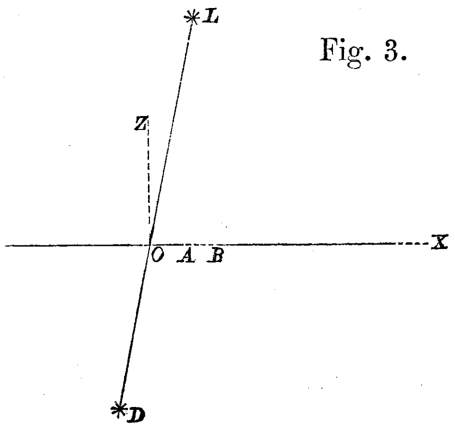


Fig. 3.

Let $A\ B$ be a section of the slit, cut by a plane passing through the source of light L, and the point D at which the illumination is required. I shall suppose that the point D is not very far from the line joining any point on the slit with the source of light. Also, the problem will be still further simplified if the plane $L\,A\,B$ is made to contain the axis of the cylinder.

For calculating the intensity of the diffracted light, assume the point O where $L\ D$ meets the cylindrical surface as the origin of coordinates. Let the x axis be parallel to the axis of the cylinder, and y perpendicular to the plane $L\ A\ B$.

In this case, $\varkappa = \cos LOA = \vartheta$,

where ϑ is very small,

$$\mu = 0, \qquad \text{and} \qquad \nu = 1 \quad \text{nearly}.$$

Thus

$$(\varkappa x + \mu y + \nu z)^2 = 2\,\vartheta\,x\,z + z^2$$

neglecting ϑ^2 upwards.

Recurring to formula (II),

$$\Delta R + \Delta R' = \Big[2\,\vartheta\,x\,z - (x^2 + y^2)\Big]\left(\frac{1}{2R} + \frac{1}{2R'}\right).$$

Therefore, by formula (II′),

$$(\mathrm{I}) \qquad I = Mod.^2\, a \int d\sigma\; e^{i\frac{\pi}{\lambda}\left[2\vartheta x z - (x^2 + y^2)\right]\left(\frac{1}{R} + \frac{1}{R'}\right)}.$$

Since ϑ, x, and z are all very small, we can write

$$e^{i\frac{\pi}{\lambda}\left(\frac{1}{R}+\frac{1}{R'}\right)2\vartheta x z} = 1 + i\,2\,\vartheta\,\xi\, x\, z$$

where ξ stands for $\frac{\pi}{\lambda}\left(\frac{1}{R} + \frac{1}{R'}\right)$.

Taking polar coordinates in the right circular section of the cylinder with the pole on the axis, we may write

$$d\sigma = a\, dx\, d\varphi,$$

$$y = a \sin\varphi, \qquad z = a\,(1 - \cos\varphi).$$

Introducing these expressions in (1), we get for the intensity of the diffracted ray

$$(2) \qquad I = Mod.^2\, a \iint dx\, d\varphi\; e^{i\,\xi\,(x^2 + a^2 \sin^2\varphi)}\left[1 + i\,2\,a\,\vartheta\,\xi\, x\,(1 - \cos\varphi)\right]$$

In integrating the above expression with respect to x, we must distinguish two cases according as D lies within or without the geometrical shadow.

Let $OA = \beta$, and $AB = b$; then the integration with respect to x must extend, when D lies within the geometrical shadow, from

$$\beta \quad \text{to} \quad \beta + b.$$

When D is outside the geometrical shadow, the limits of integration must be from

$$-\beta \quad \text{to} \quad b - \beta.$$

The integration with respect to φ must extend over the whole length of the slit.

I shall first perform the integration with respect to φ. We can write

$$\int e^{i\xi a^2 \sin^2 \varphi} d\varphi = \int e^{i\frac{\xi a^2}{2}(1 - \cos 2\varphi)} d\varphi = \frac{1}{2} e^{i\frac{\xi a^2}{2}} \int e^{-i\frac{\xi a^2}{2}\cos 2\varphi} d(2\varphi).$$

The integral thus obtained corresponds to J, which was already investigated in connection with Fraunhofer's diffraction phenomena. I shall, therefore, write for simplicity

$$(3) \qquad \int e^{i\xi a^2 \sin^2 \varphi} d\varphi = K + iL.$$

Again

$$\int e^{i\xi a^2 \sin^2 \varphi} \cos\varphi \, d\varphi = \frac{1}{a\sqrt{\xi}} \Big[\int \cos (a\sqrt{\xi} \sin\varphi)^2 d(a\sqrt{\xi}\sin\varphi)$$

$$+ i \int \sin (a\sqrt{\xi}\sin\varphi)^2 d(a\sqrt{\xi}\sin\varphi) \Big].$$

But $\int {}^{\cos}_{\sin} (a\sqrt{\xi}\sin\varphi)^2 d(a\sqrt{\xi}\sin\varphi)$ are derivable in terms of Fresnel's integrals, for which the series obtained by Knochenhauer, Gilbert, Cauchy, or Lommel can be used for calculation. I shall, therefore, put

$$(4) \qquad \int e^{i\xi a^2 \sin^2 \varphi} \cos\varphi \, d\varphi = \Gamma + i\Sigma$$

Next performing the integration with respect to x, we have to find

$$\int e^{i\xi x^2} dx \text{ and } \int e^{i\xi x^2} x\, dx.$$

The first is an ordinary Fresnel integral; and can, therfore, be written

$$(5) \qquad \int e^{i\xi x^2} dx = C + iS,$$

where

$$C=\frac{1}{\sqrt{\xi}}\int \cos(\sqrt{\xi}\,x)^2\, d(\sqrt{\xi}\,x),\quad S=\frac{1}{\sqrt{\xi}}\int \sin(\sqrt{\xi}\,x)^2\, d(\sqrt{\xi}\,x).$$

The second integral is integrable ; thus,

$$(6)\qquad \int e^{i\xi x^2}\, x\, dx = \frac{1}{2\xi}\, e^{i\xi x^2},$$

$$= \gamma + i\sigma.$$

Introducing the expressions (3) (4) (5) (6) in (2), we find for the intensity,

$$I = Mod^2\, a\Big[(C+iS)(K+iL)+i\,2\,a\,\vartheta\,\xi\,(\gamma+i\sigma)\Big\{(K+iL)-(\Gamma+i\Sigma)\Big\}\Big].$$

In finding $Mod.^2$, we can neglect the terms involving ϑ^2.

Thus, we get for the expression of the intensity

$$(7)\quad I = a^2\Big[(C^2+S^2)(K^2+L^2) + 4\,a\,\theta\,\xi\Big\{C(P\gamma-Q\sigma)+S(Q\gamma+P\sigma)\Big\}\Big].$$

where

$$P = K\Sigma - L\Gamma.$$

$$Q = K(K-\Gamma) + L(L-\Sigma).$$

The expression for the intensity of light diffracted by a slit on a circular cylinder differs from that for the plane slit by the introduction of the factor $K^2 + L^2$, and a small additional term multiplied by ϑ. Both K and L remain constant provided the distances of the slit from the source of light and the point at which the intensity is required do not change. If we observe the fringes in a plane parallel to the axis of the cylinder, K and L will remain sensibly constant. Neglecting the term multiplied by ϑ, the positions of maxima and minima will be the same as those produced by plane slit of the same breadth.

If the observer approaches or recedes from the slit, the intensity of light at a point directly opposite the slit will differ from that of the plane slit, for the intensity is affected by the factor K^2+L^2, which is no longer constant.

Observation shews that the small additional term is of very small effect. Calculating the minima of C^2+S^2 by means of Knochenhauer's series, I find that the agreement of calculation with observation is quite close, except when the point considered lies outside the geometrical shadow.

In order to test the result of calculation with observation, the following experiments were made with a slit of 90° aperture, cut on a right circular cylinder of 5.0 mm. radius. Sunlight was admitted into a darkened room. After passing through a small vertical slit, and a lens, it was analysed by a prism. The spectrum thus formed was projected on the slit of a spectrometer. The slit, however, was closed by thick paper, and only a small hole was pierced, through which light was passed to the slit under examination. The spectrum was so distinctly formed, that one could easily make the light corresponding to any one of the principal Fraunhofer's line illuminate the slit. The following observations were made for the positions of zero intensity of the fringes formed by the slit.

Width of the slit $2b = 0.5745$ mm.
Wave length of light $\lambda = 0.0004861$ mm.

	Observed Angle of Deviation.	Calculated Angle of Deviation.	Obs.—Calc.
1st Min.	94.3″	93.3″	+ 1.0″
2nd ,,	191.9	186.6	+ 5.3
3rd ,,	283.6	279.9	+ 3.4
4th ,,	377.8	373.2	+ 4.6
5th ,,	473.8	466.6	+ 7.2
6th ,,	566.9	559.8	+ 7.1
7th ,,	652.7	653.1	− 0.4

In observing Fresnel's diffraction phenomena, the optical bench was used. The intervals between the fringes were measured by means of a micrometer. The following table gives the observed numbers.

$R_0 = 324.0$, $R'_0 = 285.4$ mm.

$2b = 0.347$; $\lambda = 0.000486$ mm.

	Obs. Distance.	Calcul. Distance.	Obs.—Calc.
1st Min.	1.43 mm.	1.60 mm.	−0.17
2nd "	3.18	3.19	−0.01
3rd "	4.76	4.80	−0.04
4th "	6.39	6.40	−0.01
5th "	8.03	8.00	+0.03
6th "	9.58	9.59	−0.01
7th "	11.22	11.19	+0.03
8th "	12.81	12.79	+0.02
9th "	14.45	14.39	+0.06
10th "	16.06	15.99	+0.07
11th "	17.61	17.59	+0.02
12th "	19.24	19.19	+0.05