A Pocket Galvanometer.

By

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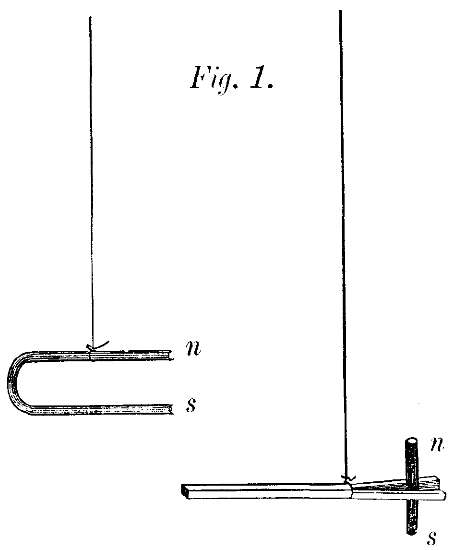
§ I. Coilless Pocket Galvanometer.

If a permanent magnet be fixed by an axis through its magnetic axis, it will be perfectly restrained from responding to external magnetic influences. But if the magnet be fixed by an axis which is only parallel to, and not coincident with, its magnetic axis, it will still be in neutral equilibrium in a uniform magnetic field. In other words, a magnet so supported cannot be distinguished from non-magnetic bodies whatever be the strength of the field so long as this remains uniform; induction being neglected for the time.

Fig. 1.



If one or any number of small bar magnets be vertically attached to a suspended piece of wood, or if a horse-shoe magnet be hung by a string as in *Fig. 1.*, such a system may be realized. The astatism is quite independent of the number of magnets, as each several magnet is in neutral equilibrium.

In such a state of equilibrium, if a pole of another magnet, or a wire conducting a current be brought near, the first magnet will be pulled or pushed as the case may be, so as to rotate round the fixed axis; the only necessary condition being that the forces acting on the two poles of the magnet shall be unequal, as will generally be the case in the neighbourhood of a straight conductor. The circumstances of rotation will depend upon the strengths of the magnet and of the current respectively.

Any small portion of a closed electric circuit may be looked upon as the edge of a large magnetic shell. Now, if two opposite poles of a small magnet be placed close to the edge of such a shell at equal distances from the edge, the electro-magnetic force acting on the magnet will be constant, provided the line joining the two poles always passes through the edge of the shell, no matter how the other portions of the shell may lie with regard to the magnet. If four poles in rigid connection be placed about such an edge, it will be possible to find such an arrangement of the poles that the force acting on the system of the poles will be sensibly constant when the edge is within a certain portion of space between the four poles.

Thus it becomes possible to construct an instrument which will measure the strength of such a shell without breaking its continuity, and that independently of the uniform field in which the shell may be situated. That is, an instrument which will measure the current without breaking the circuit and which can be used in any position and in any uniform field. This is the idea upon which the apparatus to be described is constructed. It is indeed simply a modified form of an astatic galvanometer.

The following is one particular form of the apparatus:—Four small bar magnets are fixed symmetrically at the four corners of a thin rectangular plate of wood, which can rotate as a whole about a fixed central axis perpendicular to its plane and parallel to the magnets, as shown in *Fig. 2.* What we want to measure is the couple about this axis. We may accomplish this by means of two springs, one attached to the frame, the other to an index, and both springs initially stretched very much as in an ordinary aneroid barometer. The motion of the index

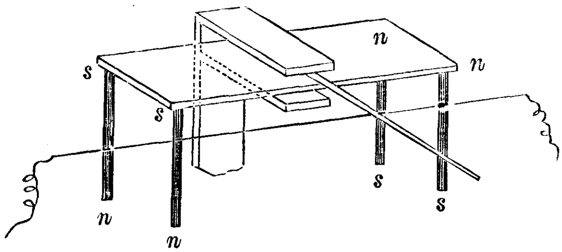
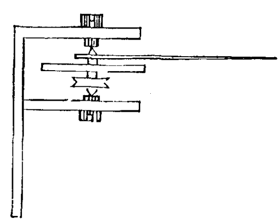


Fig. 2.

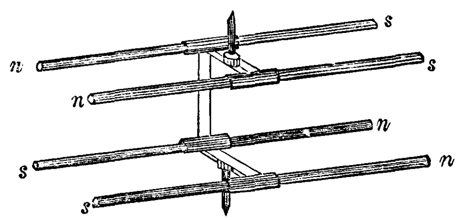


is magnified by means of a multiplying lever introduced between the index and its spring, as shown in *Fig. 4* below. As we cannot use steel springs, an alloy of gold and brass, which is used in making springs for *pince-nez* eyeglasses, answers here very well. Ordinary brass springs however are easier to make, although they are not quite so lasting.

The whole arrangement is then put in a case with a jaw-like vacant space hollowed out for receiving a conducting wire. The axis for the index projects above a graduated circular plate, and the index is fixed at right angles to the jaw, in such a way that when part of a circuit is slipped in within the jaw, the index will be moved toward that side to which the current is passing; in other words we may imagine the tip of the index to be carried by the current. The value of the indications depends upon the strengths of both the magnets and the springs, and has to be obtained by comparison with some standard galvanometer; and this calibration must be repeated occasionally as with other instruments of a similar description.

There is one inconvenience however in this system, namely, that the four magnets placed round the current have all their axes perpendicular to the direction of the current and consequently they are subject to the effect of induction. The induction will be positive for the two approaching magnets (that is their moments will be increased), and negative for the receding two. The total effect on the system will be nearly nil for small currents; but with large currents, the magnets may be permanently affected and may even be reversed in magnetization. We can get rid of this inconvenience by fixing the four magnets to an axis which is perpendicular to each axis of the magnets as in *Fig. 3.* It will be seen that the system is exactly the same as the previous one, if the four magnets be of equal moments or if they satisfy the condition $\sum M = 0$ with respect to the fixed axis. The former condition we can not hope to attain in practice, but the latter can be approximated to thus :—

Fig. 3.



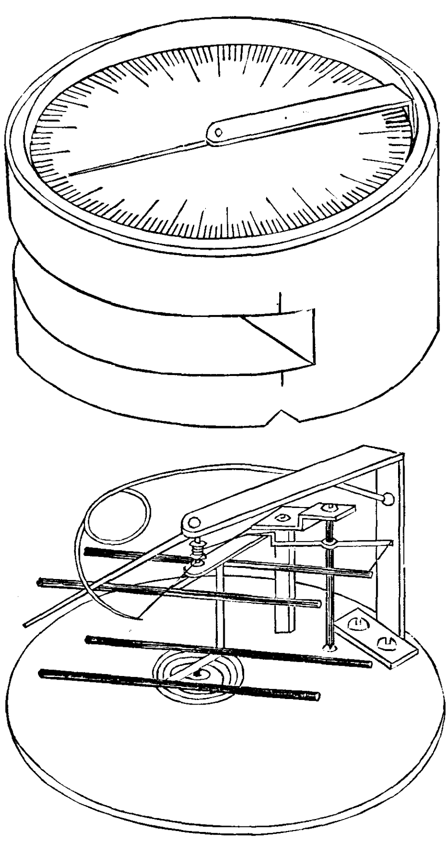
Make some 10 or 20 magnets of the same steel wire and leave them for some months or even for years, until the time-rate of the fall of moments becomes insensible. Then measure the moment of each magnet by any of the ordinary magnetometric methods, such as the deflection of a mirror magnetometer at a constant distance from each magnet, a knowledge of the relative strengths only being essential. The magnets I used in one arrangement had the following relative moments.

(*a*)	(*b*)	(*c*)	(*d*)
313.3	311.7	312.0	313.0

This gives $(a) + (b) = (c) + (d)$ within the errors of experiment, an arrangement which was accordingly adopted. We may notice here, that if the determination of any one of them were in error of 1 in the unit figure, its uncompensated effect would be $1/\sum M = 1/1250$ of the whole.

In other details of construction this form of apparatus is exactly the same as the previous one, except that the jaw is side-wise instead of vertical. This gives a greater precision to the instrument as will be seen immediately, and still more facilitates the introduction of the coils of fine wire for measuring large differences of potential and small currents in the way to be described further on. The following sketch shows one of the working instruments.

Fig. 4.



In calibrating the instrument, it is best to use a small current repeated many time instead of a single strong current which is so difficult to keep steady. For this, 200 turns of a fine insulated wire were wound round in the form of a circular ribbon having a circumference of about 1 metre. The wires were tied together with a string so that the external appearance was like one thick insulated wire of circular section. One portion of this was put into the jaw, the remaining portions being as far removed as possible from the instrument. A current from 20 Daniell cells was passed through the ribbon; a standard tangent galvanometer and a resistance box being in the circuit. Simultaneous readings of the standard galvanometer and the pocket instrument were taken for different strengths of current, obtained by varying the resistance. From these the value of the current corresponding to 1 division of the pocket instrument was easily calculated. The following table gives the results of the comparison thus made with the working apparatus.

Standard Galvanometer.		Pocket Galvanometer.	
Reading.	Reduced to Ampères.	Reading.	Value of 1 Division in Ampères.
12.0	.0350	12.4	.565
19.7	.0575	20.6	.558
28.9	.0844	30.5	.554
35.5	.1037	37.4	.555
		Mean	.558

As the instrument was graduated to 100 divisions we could measure with this up to 56 ampères, and, as will be seen later, by placing the wire below the instrument up to 168 (= 3 × 56) ampères. In making the comparison, we must place the standard galvanometer at a considerable distance from the ribbon, which becomes a strong magnetic shell when the current is passing. In fact it is best to put the two instruments on different tables, and make two series of observations, one with the current direct and the other with it reversed. For measuring a moderate current such as 1 or 2 ampères we can advantageously repeat the circuit three or four times by simply coiling the conductor so as to make a temporary ribbon like that just described.

By altering the points of support of the springs (as in the time rating of a watch) we may, if it is required, adjust the instrument so as to make one division correspond to some simple fraction of an ampère, say .5 or .2

Field of Force and Arrangement of Magnets.

In studying the field of force due to four magnets arranged as in *Fig. 4*, and a single straight conductor parallel to their axes, we have only to consider the action of a single straight conductor upon one set of four poles which lie in a plane perpendicular to the conductor, since the other set of poles is an exact counterpart of the one considered. As proved in Maxwell, the electromagnetic force at an external point of a straight cylindrical conductor of infinite length depends only upon its distance from the center of the section, for any concentric distribution of current; and since the action between a magnet and a current is mutual, if a compound cylindrical magnet consists of concentric layers of uniform intensities, its action upon an externally placed current running parallel to the axis of the magnet, must be reducible to the action of a single equivalent pole at its centre, whatever be the law of distribution of magnetism from layer to layer. As we make our magnets of cylindrical wire, and as most conductors are cylindrical, we may safely reduce the action to the centers of the sections of the magnets and the conductor, neglecting the pole-shifting effect due to the induction of magnets on each other.

Let $2a$ be the distance between like poles, and $2b$ that between unlike ones: and put r for $\sqrt{a^2 + b^2}$.

Let x, y be the co-ordinates of any point referred to the center of the rectangle $[2a, 2b]$, measured parallel to a and b respectively.

Let ξ η be the co-ordinates of any one of the poles, referred to *any position of the current as origin*, so that, if x, y be the co-ordinates of the current,

$\xi + x = a$ and $\eta + y = b$

Then $\rho = \sqrt{\xi^2 + \eta^2}$ is the distance of any pole from the current.

The potential energy of unit current and four unit poles is

$$V = 2\sum tan^{-1}\frac{\eta}{\xi} + const$$

$$= 2\sum\theta + const \qquad (1)$$

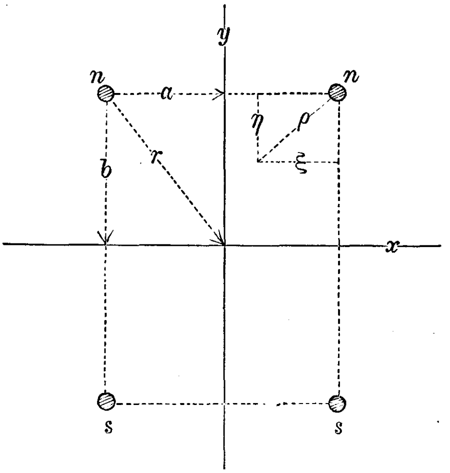


Fig. 5.

$\sum$ meaning summation for the four distinct poles, and θ being the angle between any of the polar distances and the axis of ξ. V then reduces to the algebraic sum of the two angles subtended by the two parallel lines of length $2\,b$ placed $2\,a$ apart. This is the same as the potential energy of unit pole and two parallel magnetic strips of infinite length placed $2\,a$ apart, the breadth of each being $2\,b$. The solid angles ω_1 and ω_2 of the usual notation become two spherical wedges. The two magnetic strips may be replaced by two equal pairs of parallel currents, placed along the edges of the strips, as will be evident *à priori*, since the action of a unit current upon four unit poles must be the same as that of four currents upon a unit pole so far as the dynamical aspect is concerned. This latter was the combination I used in making an experimental verification of the result.

From (1) the equation

$$\sum\theta = const.$$

gives the equipotential surfaces, and

$$\sum log\ \rho = const.$$

gives the lines of force. *Fig. 6* shows a special case of such a field.

To determine the best arrangement of the magnets so as to minimize the error due to the eccentric position of the current (which is the same thing as that due to the deflected position of the magnets) we may proceed to find the electro-magnetic forces acting upon the system of magnets, when the current is placed in any position x, y. Putting F' for this force and expanding it in terms of eccentric displacement ∂x, ∂y, we have,

$$F' = F'_0 + \frac{\partial F'}{\partial x}\partial x + \frac{\partial F'}{\partial y}\partial y$$
$$+ \frac{1}{\lfloor 2}\left\{\frac{\partial^2 F'}{\partial x^2}(\partial x)^2 + 2\frac{\partial^2 F'}{\partial x \partial y}(\partial x)(\partial y) + \frac{\partial^2 F'}{\partial y^2}(\partial y)^2\right\}$$
$$+ \frac{1}{\lfloor 3}\left\{\frac{\partial^3 F'}{\partial x^3}(\partial x)^3 + 3\frac{\partial^3 F'}{\partial x^2 \partial y}(\partial x)^2(\partial y) + 3\frac{\partial^3 F'}{\partial x \partial y^2}(\partial x)(\partial y)^2 + \frac{\partial^3 F'}{\partial y^3}(\partial y)^3\right\}$$
$$+ \cdots\cdots$$

where F_0 is the force at the origin. On account of the symmetry of configuration, the terms containing odd powers of either ∂x or ∂y vanish when taken for all the four magnets. The equation then reduces to

$$F' = F'_0 + \frac{1}{2}\left\{\frac{\partial^2 F'}{\partial x^2}(\partial x)^2 + \frac{\partial^2 F'}{\partial y^2}(\partial y)^2 + \frac{\partial^4 F'}{\partial x^4}(\partial x)^4 + \cdots\cdots\right\} \cdots\cdots (2)$$

where $$-F' = \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\, 2\sum tan^{-1}\frac{\eta}{\xi}$$

but since

$$\left.\begin{aligned} x &= a - \xi \\ y &= b - \eta \end{aligned}\right\}$$
$$\frac{\partial^n}{\partial x^n} = (-1)^n\frac{\partial^n}{\partial \xi^n}$$
$$\frac{\partial^n}{\partial y^n} = (-1)^n\frac{\partial^n}{\partial \eta^n}$$
$$F' = \frac{\partial}{\partial \xi}\sum tan^{-1}\frac{\eta}{\xi} = \sum\frac{-\eta}{\xi^2+\eta^2}$$

also

$$\frac{\partial^n}{\partial \xi^n}\left(\frac{\eta}{\xi^2+\eta^2}\right)=\frac{(-1)^n \lfloor n \, sin \, (n+1) \, \theta}{(\xi^2+\eta^2)^{\frac{n+1}{2}}}$$

$$\frac{\partial^n}{\partial \eta^n}\left(\frac{\eta}{\xi^2+\eta^2}\right)=-\frac{(-1)^n \lfloor n \, sin \, (n+1) \, \theta}{(\xi^2+\eta^2)^{\frac{n+1}{2}}}$$

$$tan \, \theta = \frac{\xi}{\eta} \qquad \text{as before}$$

Thus, taking only the increment of force, we get from equation (2)

$$\Delta^2 F' = -\sum\left(\frac{sin \, 3 \, \theta}{\rho^3} \, \partial x^2 - \frac{sin \, 3 \, \theta}{\rho^3} \partial y^2 + \cdots\cdots\cdots\right)$$

but when ∂x and ∂y are each small ρ becomes very nearly r, and θ may be regarded as measured from the center of the rectangle [$2a$, $2b$]. We may then dispense with the sign $\sum$ in discussing the configuration. Thus if

$$3\theta = \pi \quad \text{or} \quad n\pi$$

or

$$\theta = \frac{\pi}{3} \quad \text{or} \quad \frac{n\pi}{3}$$

the coefficients of $(\partial x)^2$ and $(\partial y)^2$ vanish simultaneously. In other words if the magnets are arranged occupying any four opposite corners of a regular hexagon, that is if a/b be nearly 4/7,* the error due to a small displacement will be eliminated up to the third order of the displacement inclusive.

The simultaneous disappearance of the coefficients of $(\partial x)^2$ and $(\partial y)^2$ might indeed have been anticipated from the general equation $\nabla^2 V = 0$ since V is here function of only x and y we have

$$\frac{\partial}{\partial x}\nabla^2 V = \frac{\partial^2}{\partial x^2}\frac{\partial V}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial V}{\partial x} = 0$$

now $\partial V/\partial x$ is F'. Hence when either of $\partial^2 F'/\partial x^2$ or $\partial^2 F'/\partial y^2$ vanishes the other must do so too.

* Practically it will be found better to make a/b somewhat larger than 4/7, so to obtain a greater range of uniformity in the field.

The uniformity of the field arrived at by this arrangement is shown by the following diagram of equipotential lines and lines of force. Curves representing the intensities of forces along the coordinate axes and the diagonal are also given.

FIELD OF FORCE.

$a^2 : b^2 : r^2 = 1 : 3 : 4$

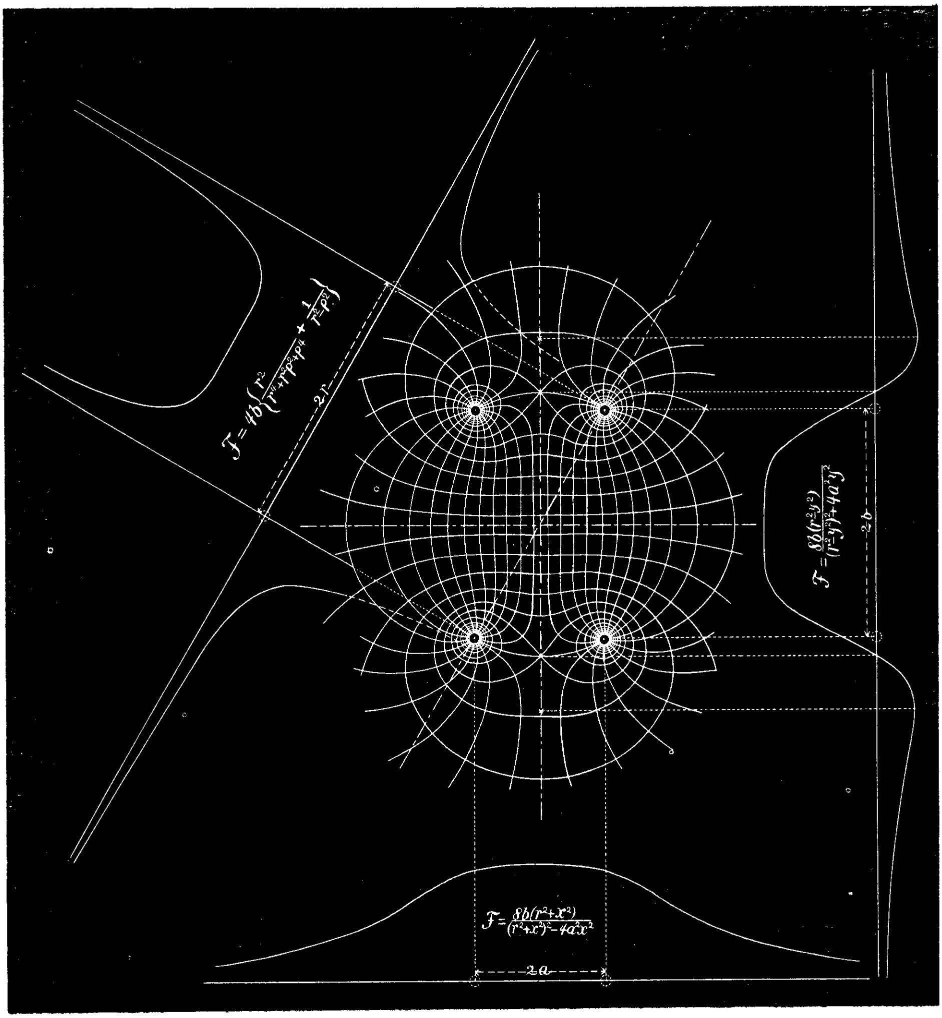


Fig. 6.

The general expression for the force is

$$-F=\frac{\partial}{\partial x}\ 2\sum tan^{-1}\frac{\xi}{\eta}$$

$$=2\left\{\frac{b-y}{(b-y)^2+(a-x)^2}+\frac{b-y}{(b-y)^2+(a+x)^2}+\frac{b+y}{(b+y)^2+(a-x)^2}+\frac{b+y}{(b+y)^2+(a+x)^2}\right\}$$

If, now, we make

$$\left.\begin{aligned} x&=0\\ y&=0\\ \rho&=x\frac{a}{r}=y\frac{b}{r}\end{aligned}\right\}\ \text{successively}$$

we obtain the expressions

$$\left.\begin{aligned} F&=\frac{8\,b\,(r^2+x^2)}{(r^2+x^2)^2-4\,a^2\,x^2}\cdots\cdots(\alpha)\quad \text{along } x\text{-axis}\\ F&=\frac{8\,b\,(r^2-y^2)}{(r^2-y^2)^2+4\,a^2\,y^2}\cdots\cdots(\beta)\quad ,,\quad y\text{-axis}\\ F&=\frac{4\,b\,r^2}{r^4+r^2\,\rho^2+\rho^4}+\frac{4\,b}{r^2-\rho^2}\ (\gamma)\quad ,,\ \text{diagonal}\end{aligned}\right\}\ \text{respectively}$$

The maxima and minima of these are given by

$$x=0$$
$$x=\pm\sqrt{r\,(-r\pm 2\,a)}\ \cdots\cdots\cdots\ (\alpha')$$
$$y=0$$
$$y=\pm\sqrt{r\,(\,r\pm 2\,a)}\ \cdots\cdots\cdots\ (\beta')$$
$$\rho=0$$
$$\rho=r\ \text{(others being imaginary)}\cdots\cdots(\gamma')$$

from which we see that the pairs of maxima are superposed at the origin if $r=2a$, which is equivalent to $\theta=\frac{\pi}{3}$. The minima along the y-axis at $y=\pm\sqrt{r(r+2a)}$ are always real for any ratio of a/r. In the present case, the values become $\pm 2\sqrt{\frac{2}{3}}b$. Substituting this value of y in (β) we get the value of F there $=-\frac{1}{3}F_0$. It will be seen that at each of these points, the lines of force begin to change their curvature, and the field is sensibly constant. By constructing the instrument, so that the base lies just above one of these points, we adapt it for the measurement of very strong currents, such as 100 ampères or greater. To facilitate such a measurement a V-groove is cut out along the base in the proper position, and into this the circuit bearing the current is received. To reduce the value thus obtained to what it would have been, had it been placed in the jaw, we have only to multiply the reading by -3.

The following table giving the proportional decrement of force for given displacements of the magnets, will show to what amount of displacement we may, without sensible error, assume the uniformity of the galvanometer constant as obtained by calibration for small displacements. Computing $(F-F_0)/F_0$ we have

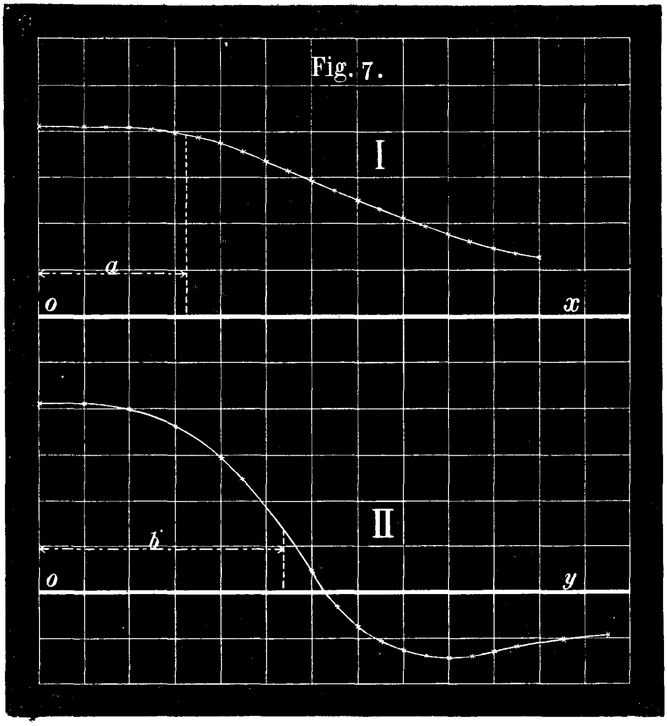
Along x-axis.		Along y-axis.		Along Diagonal.	
$\partial x/a$	$\partial F/F_0$	$\partial y/b$	$\partial F/F_0$	$\partial \rho/r$	$\partial F/F_0$
1/4	1/4161	1/4	1/435	1/4	1/445
1/3	1/1333	1/3	1/133	1/3	1/132
1/2	1/273	1/2	1/24	1/2	1/21
1	1/21				

As long as the deflection is within $\frac{1}{2}a$ the error will be less than 1/2 per cent. and when it is to the extreme limit of a, the error will be about 5 per cent. It is interesting to notice that ∂F is very nearly proportional to the fourth powers of ∂x, ∂y, $\partial \rho$, even when these are $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}r$. The above table enables us to adjust the range of the index and the jaw of the instrument so as to keep the magnitude of errors within any assigned limit. In the actual instrument constructed the width of the jaw was $\frac{1}{2}b$ and the deflection was limited to $\frac{1}{2}a$, so that the error fell within $\frac{1}{273}$ for any reading even if we suppose the eccentricity in the direction of y to be as much as $\frac{1}{4}b$.

To verify the results thus far obtained the following experiment was made in the Physics Laboratory of the Science College of the Imperial University :—Six blocks of wood, each 21 cm. high and 13 cm. wide were arranged in a row upon a long laboratory table extending through a space of 3.7 metres along the magnetic meridian. These blocks served simply as guides for the stretching of two rectangular coils of insulated wire, whose distance apart bore to the height of either the ratio required (tangent 30°). Each coil consisted of six turns. Thus were obtained two parallel magnetic strips of practically infinite length.

One of Thomson's graded galvanometers with its field magnet taken away, was placed in the space between the two central blocks, its V-groove lying along the magnetic east and west line. The height of the galvanometer was so adjusted that the centre of the four small magnets, belonging to the fan-shaped compass, was always in the plane half-way between the upper and the lower lines.

A current was run round the wire and was left for about 20 minutes till its flow became steady, and then the compass was slid along the V-groove and its position and deflection simultaneously observed at several positions. From these observations the curve I in *Fig. 7* was obtained. By a slight modification of the arrangement, providing a vertical V-groove, the compass was made to move along a vertical line; and from a similar series of observations the curve II was obtained. These should be compared with the curves of *Fig. 6*.

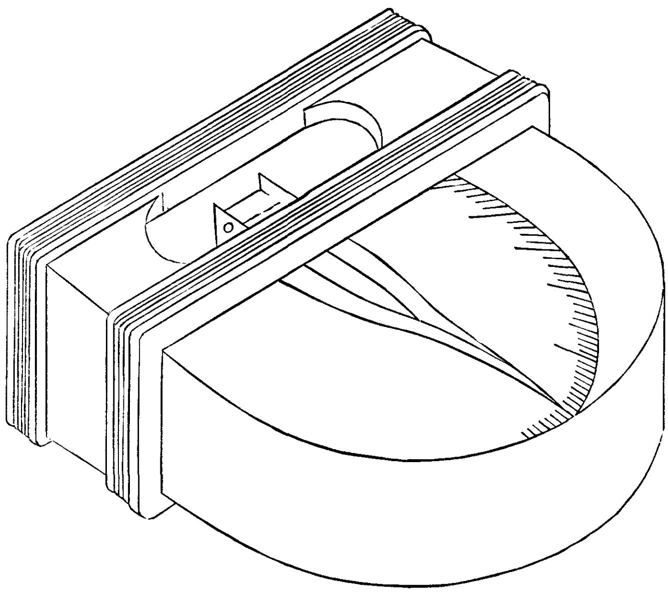


It will be seen that the system of magnets thus arranged is equivalent to Helmholtz's galvanometer of two circular coils, if we imagine the two coils to be deformed into two long rectangles of infinite length, that is, reduced to four parallel currents. For if we adjust these currents to proper positions, and replace each of them by a magnet and the central magnet by a straight current, we have the system just discussed.

§ II. Flat Coil Pocket Galvanometer.

I now pass to the description of the flat coil pocket galvanometer, which may in virtue of its compactness and simplicity of construction be found useful for some purposes, although it can not of course take the place of circular coils when absolute determinations of electro-magnetic constants are required. One such instrument is represented in *Fig. 8.* It is essentially the same as an old form of current

Fig. 8.



detectors, constructed however with due regard to the proportional dimensions which theory shows to be best.

The following calculation of the proper distance between two rectangular coils is made not so much for the sake of the flat coil galvanometer as for the sake of the internal coil mentioned above (page 279) and described in detail below (page 296).

Let $2a$ be the distance between the two coils,

,, $2b$,, ,, height of the coils

,, $2c$,, ,, length ,, ,, ,, .

Let x, y, z be the co-ordinates of any point referred to the centre of the coils as origin, axes being parallel to a, b, c.

Let ξ, η, ζ be the co-ordinates of any corner of the coils referred to any point x, y, z as origin, so that

$$\xi + x = a, \qquad \eta + y = b, \qquad \zeta + z = c.$$

Then the potential at any point due to unit current is the sum of the solid angles subtended by the coils, or

$$V = \sum sin^{-1} \frac{\eta\,\zeta}{\sqrt{\xi^2 + \eta^2}\sqrt{\xi^2 + \zeta^2}} = \sum \Omega \quad \text{say}$$

$\sum$ meaning the summation for 8 corners, there being four to each coil. Now,

$$\frac{\partial^n}{\partial x^n} = (-1)^n \frac{\partial^n}{\partial \xi^n}; \qquad \frac{\partial^n}{\partial y^n} = (-1)^n \frac{\partial^n}{\partial \eta^n}; \qquad \frac{\partial^n}{\partial z^n} = (-1)^n \frac{\partial^n}{\partial \zeta^n}.$$

Hence
$$F = -\frac{\partial V}{\partial x} = \sum \frac{-\eta\zeta}{\rho_0}\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right)$$

where $\rho_0 = \sqrt{\xi^2 + \eta^2 + \zeta^2}$; $\rho_1 = \sqrt{\xi^2 + \eta^2}$; $\rho_2 = \sqrt{\xi^2 + \zeta^2}$

The force at the center of the coils will be given by making $\xi, \eta, \zeta = a, b, c$ respectively and multiplying the result by n, if n be the number of turns of wire in the coil.

The increment of force at any displaced point $\partial x, \partial y, \partial z$, is given by the equation (odd terms disappearing as before),

$$\Delta^2 F = \frac{1}{\lfloor 2}\left\{\frac{\partial^2 F}{\partial x^2}(\partial x)^2 + \frac{\partial^2 F}{\partial y^2}(\partial y)^2 + \frac{\partial^2 F}{\partial z^2}(\partial z)^2\right\} + \frac{1}{\lfloor 4}\left\{\dots\dots\dots\right.$$

but, since

$$\frac{\partial}{\partial x}\nabla^2 V = \frac{\partial^2}{\partial x^2}\frac{\partial V}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial V}{\partial x} + \frac{\partial^2}{\partial z^2}\frac{\partial V}{\partial x} = 0$$

it is necessary and sufficient that any two of $\partial^2 F/\partial x^2$, $\partial^2 F/\partial y^2$, $\partial^2 F/\partial z^2$ should vanish simultaneously, in order that they all may vanish at the same time.

Thus we have

$$\left.\begin{aligned} \frac{\partial^2 F}{\partial y^2} &= \sum \frac{\eta\,\zeta}{\rho_0}\left\{\frac{3}{\rho_0^2}\left(1 - \frac{\eta^2}{\rho_0^2}\right)\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right) + \frac{6}{\rho_1^4} - \frac{4\eta^2}{\rho_0^2\rho_1^4} - \frac{8\eta^2}{\rho_1^6}\right\} = 0 \\ \frac{\partial^2 F}{\partial z^2} &= \sum \frac{\eta\,\zeta}{\rho_0}\left\{\frac{3}{\rho_0^2}\left(1 - \frac{\zeta^2}{\rho_0^2}\right)\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right) + \frac{6}{\rho_2^4} - \frac{4\zeta^2}{\rho_0^2\rho_2^4} - \frac{8\zeta^2}{\rho_2^6}\right\} = 0 \end{aligned}\right\} \text{(A)}$$

which at the center of the coils (ξ, η, ζ being a, b, c) reduces to

$$\frac{\partial^2 F}{\partial y^2} = \frac{b\,c}{(a^2+b^2+c^2)^{\frac{5}{2}}(a^2+b^2)^3}\Big\{12a^6+(21b^2+15c^2)a^4$$
$$+(6[b^2+c^2]^2-2b^2c^2)a^2-b^2(b^2+c^2)(3b^2+2c^2)\Big\}=0$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{b\,c}{(a^2+b^2+c^2)^{\frac{5}{2}}(a^2+c^2)^3}\Big\{12a^6+(21c^2+15b^2)a^4$$
$$+(6[b^2+c^2]^2-2b^2c^2)a^2-c^2(b^2+c^2)(3c^2+2b^2)\Big\}=0$$

Unless these two equations are simultaneously satisfied, the three partial differential coefficients will not vanish. Eliminating a between the two equations we find that the only admissible cases are when $b=c$, and either b or $c=\infty$. Thus it seems that $\Delta^2 F$ can not be made to vanish entirely except for the case of a square, and four infinite parallel straight currents. But when either b or c is great compared with the other, the partial differential coefficient with respect to the greater will be comparatively small as may be judged from the equations (A). Hence for such cases, if $\partial^2 F/\partial^2 x = 0$, both the others must be small. But since $\frac{\partial^2 F}{\partial x^2} = -\left(\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}\right)$* we find from the above two equations

$$\frac{\partial^2 F}{\partial x^2} \propto 24a^{12} + 72\,(b^2+c^2)\,a^{10} + [93\,(b^4+c^4) + 146\,b^2c^2]\,a^8$$
$$+ 8\,(b^2+c^2)\,[9\,(b^4+c^4) + 4b^2c^2]\,a^6$$
$$+ 3\,(b^2+c^2)^2[11\,(b^4+c^4) - 10b^2c^2]\,a^4$$
$$+ 2\,(b^2+c^2)^3[3\,(b^4+c^4) - 7b^2c^2]\,a^2$$
$$- b^2c^2\,(b^2+c^2)^2[2\,(b^4+c^4) + b^2c^2)\,]=0$$

from which a is to be found for any given value of b and c, the sides of the coils. Since the equation is homogeneous, we may take either b or c as unit of length and measure the other lengths in terms of it. Thus if we take b (half the height of the coils) as the unit, and express a and c in terms of it, we have all the possible cases brought out by varying c from 1 to ∞. Examining the above equation, we find that

* The equation $\partial^2 F/\partial y^2+\partial^2 F/\partial z^2=0$ shows that at the origin F, considered as a function of y and z, is a minimax with respect to those variables; when $b=c$, this becomes what may be called a *flat point*, and for $b=\infty$, a *flat line*.

there is only one variation of sign among the coefficients of a whatever b or c may be. Hence this equation has only one pair of real roots. Putting $a = \frac{1}{\sqrt{2}} b$, and $a = \frac{1}{2} b$ successively the expression changes sign once, so that the positive root of this equation lies between .5 and .71 whatever be the values of c/b. Dividing the equation by its last term we see that the higher powers of a rapidly converge when c increases. When $c = \infty$ the equation reduces to

$$3a^2 - b^2 = 0$$

which agrees with the previous result. When $b = c = 1$ it reduces to

$$4\,(a^2 + 1)^3\,(6a^6 + 18a^4 + 11a^2 - 5) = 0$$

as might be found by an independent calculation.

Since a knowledge of the solution of this equation will serve as a guide in the construction of such galvanometers, I give the following table of its roots for several values of c/b together with the values of the field and the proportional decrement of force at the point $\partial x = \frac{1}{3}a$. From these numbers we can at once judge of the uniformity of the field.

Value of c/b.	a (root).	$2\,tan^{-1}\frac{a}{b}$.	$F_0 = \frac{8bc}{r_0}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)$	$\Delta^2 F/F_0$ at $x = \frac{1}{3}a$.
1	.54451	57° 8′	8.144	.00085
$\sqrt{2} = 1.4$	.60040	61° 58′	7.151	.00081
2	.59736	61° 42′	6.681	.00077
3	.58393	60° 34′	6.365	.00074
4	.57980	60° 13′	6.222	.00074
5	.57843	60° 6′	6.148	.00074
10	.57742	60° 0′	6.039	.00074
∞	.57735	60° 0′	6.000	.00075 *

* For Helmholtz's arrangement $\Delta^2 F/F_0 =$ .00077.

The values of F_0 for any actual case are to be obtained by dividing the above number by the number expressing half the height of the coils in centimetres, and multiplying by the number of turns of wire in the coil. From the table it is seen that when the length c exceeds 5 times the height b the action of the coils is not far from that of the four infinite parallel currents already treated. This justifies the experiment described at the close of the first section of the paper. Further we see that *the assumption for a large magnetic shell made in the beginning of the paper is practically correct*, if the straight part of the circuit extends more than 5 times the distance between the upper and lower magnets ($c/b = 10$) on each side of the instrument, and if the length of each magnet is not less than $5c$.

In practice however the coils are of finite section whereas the above results refer to coils of infinitesimally small section. So long as the depth and width of the sections of the coils are small fractions (say 1/10) of the height of the coils or of their distance apart, we may, without sensible error take the centers of the sections for the positions of simple equivalent coils of infinitesimal section.

If the depth of the section is small and the width finite the force at the center of the coils is got by a process of ordinary integration. Thus if the number of turns of wire per centimeter be n, we have with the same notation as before

$$F = \sum \int_{a_1}^{a_2} \frac{\partial \Omega}{\partial x} n\, dx$$

$$= \sum n \Big[\Omega \Big]_{a_1}^{a_2}$$

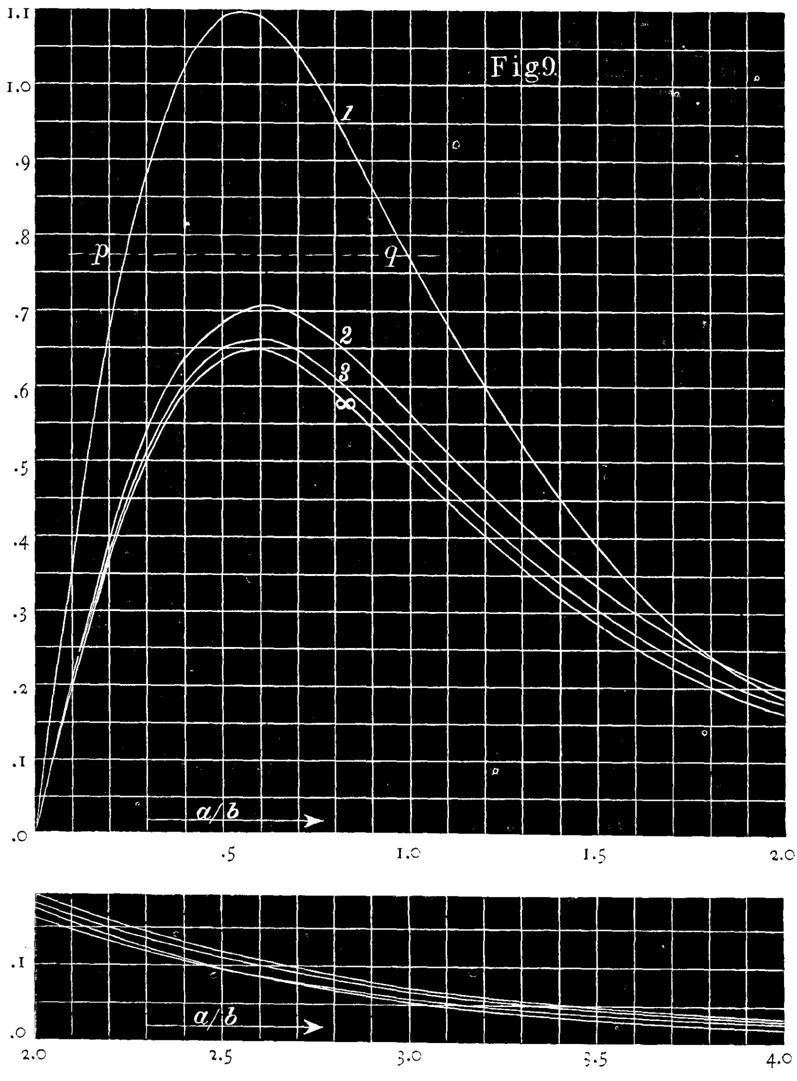
where a_1 and a_2 are the distances of the internal and external faces of either coil measured from the point half way between the two coils. Hence in order that the effect of eccentric displacement may be small we have

$$\frac{\partial^2 F}{\partial x^2} = \sum \left\{ \left(\frac{\partial^2 \Omega}{\partial x^2}\right)_{a_1} - \left(\frac{\partial^2 \Omega}{\partial x^2}\right)_{a_2} \right\} = 0$$

$$\frac{\partial^2 \Omega}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \sin^{-1} \frac{\eta \zeta}{\sqrt{\xi^2 + \eta^2}\sqrt{\xi^2 + \zeta^2}}$$

$$= \frac{\xi \eta \zeta}{\rho_0} \left\{ \frac{1}{\rho_0^2}\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right) + 2\left(\frac{1}{\rho_1^4} + \frac{1}{\rho_2^4}\right) \right\} \qquad (B)$$

but we have seen that $\frac{\partial}{\partial x}\frac{\partial^2 \Omega}{\partial x^2}$ vanishes for some value of a/b between



.5 and .71 so that $\frac{\partial^2\Omega}{\partial x^2}$ has a maximum between a/b = .5 and .71 for any ratio of c/b. Hence $\partial^2 F/\partial x^2$ can always be made to vanish by taking a_1 and a_2 on both sides of the maximum. The values of $\frac{\partial^2\Omega}{\partial x^2}$ are graphically represented in *Fig. 9* for $c/b = 1, 2, 3, \infty$.

At first these curves are in the order 1, 2, 3 &c, counting from above, but afterwards when they become distinctly asymptotic their order becomes reversed. This is indeed apparent from the equation.

An indefinite number of proper values for a_1 and a_2 may be got by the following simple construction. Draw any horizontal line cutting any particular curve in the points p, q; the x-coordinates of these points at once give a special pair of suitable values a_1 and a_2. In the case of simple coils p and q coincide at the top of the curve, which is the case already discussed.

§ III. Internal Coils for Measuring Large Differences of Potential and Small Currents.

We are now in a position to consider the dimensions which ought to be given to the internal coils spoken of in page 279 as necessary for the measurement of large differences of potential and small currents. For examining the above curves we see that $\partial^2\Omega/\partial x^2$ becomes very small when a/b is more than 3. Also putting (B) in the form

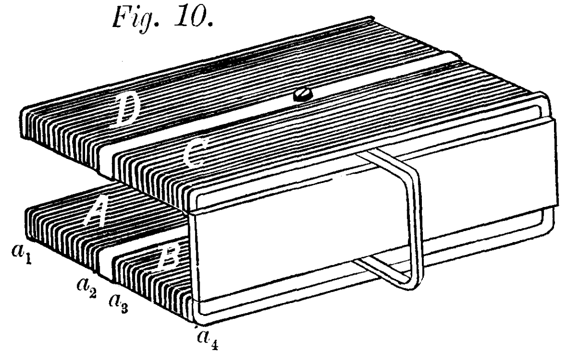
$$\frac{\partial^2\Omega}{\partial x^2}=\frac{\xi\eta\zeta}{\rho_0\rho_1^2}\left(\frac{1}{\rho_0^2}+\frac{2}{\rho_1^2}\right)+\frac{\xi\eta\zeta}{\rho_0\rho_2^2}\left(\frac{1}{\rho_0^2}+\frac{1}{\rho_2^2}\right)$$

and considering η and ζ as variables we may regard the action as due to two equivalent "Electro-magnetic strips."* The coils are shown in the

* See my paper on the 'Electro-magnetic Declinometer' published in the Proceedings of the Royal Society of Edinburgh (Vol. XII. P. 544 1882—4) or Rigakukyōkwai Zassi (Vol. II. P. 84). in which curves are shown very similar to those just given.

diagram below. There are four, one around each magnet. They are necessarily made of small height, in fact just enough to allow the wire magnets to move freely inside and yet to leave a good space for the jaw. For convenience of reference call these coils A, B, C, D, as indicated in the figure. From the symmetry of configuration

Fig. 10.



we may take any one of the eight poles and consider the action of four coils upon it. Take one of the poles inside A and call it P_A for the sake of definiteness. The variations in the actions of C and D upon P_A due to possible motions of the same may be regarded as the differentials of the actions of electromagnetic strips placed along the edges of C and D. These we may safely neglect in comparison with the variations in the actions of A and B. Thus we have only to consider the effect of the four faces of A and B. But since each coil extends a good way over the poles of the magnet, we may regard these coils as drawn out indefinitely in the direction of the magnets' axes without committing any sensible error as was shown in the last section. Hence, calling the distances of the faces of A and B from P_A, a_1, a_2, a_3, a_4, as in the diagram, we have

$$\Delta^2 F = \left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_1} + \left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_2} - \left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_3} + \left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_4} \qquad (C)$$

where $a_4 = a_1 + a_2 + a_3$ from symmetry of arrangement. But from the curves given above we see that when a/b is more than 4, the values of $\frac{\partial^2\Omega}{\partial x^2}$ become insensible. Therefore by simply extending the ends of the coils so as to have the least value of a/b more than

4, we make $\Delta^2 F$ practically vanish.

Theoretically there is an indefinite number of solutions of the equation (C) obtained by selecting sets of values of a_1, a_2, a_3, a_4 such that

$$\left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_1}+\left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_2}+\left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_4}=\left(\frac{\partial^2\Omega}{\partial x^2}\right)_{a_3}$$

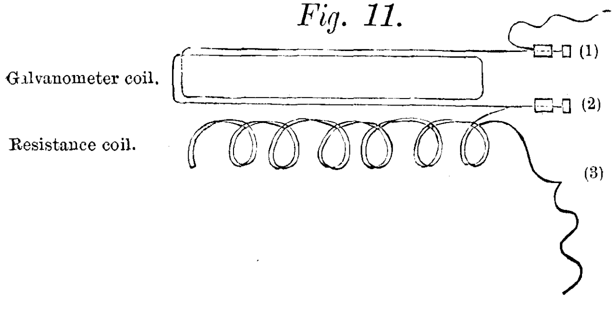
which is always possible since the curve has a maximum. But to apply such a mode of solution practically is dangerous on account of the steep rise of the curve toward the maximum. Any small errors in the values of a_1, a_2 &c. will give rise to an ultimate error greater than that caused by the neglect of the converging terms when a/b is made great. Hence in practice it is advisable to make $a/b > 4$, and limit the range of deflection so that the least value of ξ/b should not fall within 2 or 3. If the instrument were constructed on a large scale with all the dimensions determined accurately, the graphical solution just indicated might be applicable.

Fig. 10. above shows the actual coil belonging to the instrument represented in *Fig. 4.* which was purposely drawn without the coil so as to lay open its internal construction. Both these figures are drawn very nearly to full size. The central ridges which divide the upper and the lower coils into two parts serve for holding the pivots of the frame of the magnets.

The fittings of the levers and springs remain, of course, exactly the same as in *Fig. 4.*

The coil is wound with a thin copper wire in four layers and has a resistance of 40 ohms. An extra resistance of 5000 ohms is added by introducing a small bobbin of fine german silver wire not shown in *Fig. 1.*

Fig. 11.



into the back corner of the case of the instrument. The connection of the terminals is diagramatically shown in *Fig. 11.*

For the measurement of potential differences, the terminals (1) and (3) are used and for the measurement of small currents the terminals (1) and (2). To avoid the possibility of confusion as to which pair of terminals is to be used, a pair of binding screws are provided at (1) and (2) for the case of small currents, and a pair of wire ropes at (1) and (3) for the case of potential differences. An alternative construction would be to omit the pair of binding screws and provide a plug hole between (2) and (3) to shunt off the resistance when the instrument is to be used for small currents. For the measurement of very large potential differences or of moderate currents a system of shunts may be employed in the usual way. Various other like devices may be multiplied almost endlessly.

A separate calibration is required for this coil, but this is nothing more than the ordinary galvanometer gauging. The following table gives a comparison of the readings of an actual instrument with those of a standard galvanometer.

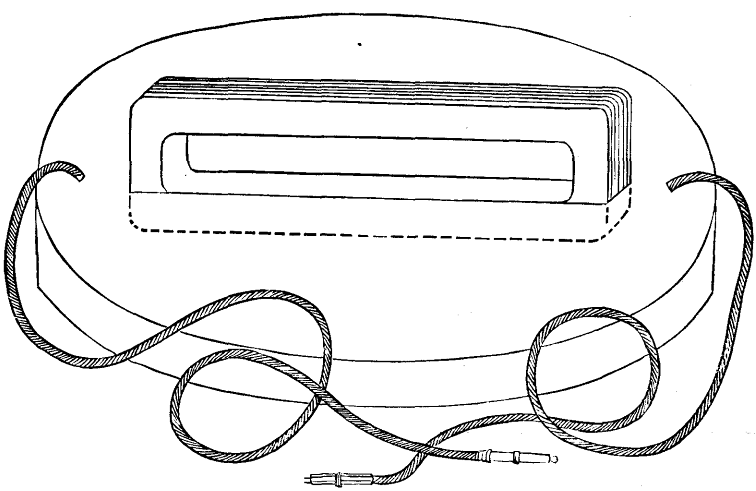
Standard Galvanometer.		Pocket Galvanometer.	
Reading.	Reduced to Ampères.	Reading.	Value of 1 Division in Ampères.
7.8	.02278	25.8	.000883
13.6	.03971	45.1	.000880
22.1	.06453	73.2	.000882
		Mean	.000882

From the above result we easily find, for the given extra resistance of 5000 ohms inserted, the value of the potential difference corresponding to 1 division of the scale. It is 4.4 volts (= .000882 × 5040). If a thinner wire be wound with a greater number of layers, it will be easy to make the above value 1 volt or less. Indeed this would have been done in the instrument now being described had it not been at the time impossible to obtain in this country a fine enough insulated copper wire. No doubt, the value required could be arrived at by simply reducing the resistance to little more than 1000 ohms. This however would be too small for the purpose; if possible 10,000* ohms or more would be desirable.

§ IV. **External Coil.**

A single rectangular coil consisting of several hundred turns of a fine copper wire (see *Fig. 12.*) may take the place of this internal coil just described, if one part of the coil be temporarily slipped in between the jaw in the same way as the so called ribbon was slipped in when the instrument was being gauged. In simplicity of construction, this plan is perhaps superior to that of introducing the internal

Fig. 12.



coil especially when the instrument is to be made on a small scale. But it is inferior inasmuch as it needs an extra piece of apparatus.

* Thomson's graded potential galvanometer has about 7000 ohms.

Of course, we cannot regard this coil as equivalent to a single magnetic shell of an infinite extent. But referring to the diagram of the field of force given in *Fig. 6*, and the table of page 293 we see that the variation of the action of the coil due to deflections of the magnet will be insensible, if the height of the coil be such that the part outside the jaw is just below the base of the instrument, and if the length of the coil be such that the bends are distant from the nearest poles of the magnets by more than five times the distance between the magnets.

Further we see that the part of the coil outside the jaw will add to the effect of the part inside by about 33 per cent the current necessarily being in opposite directions in these two parts.

The coil shown in *Fig. 12.* has 600 turns of copper wire and a resistance of 100 ohms.

Comparison with a standard galvanometer gave the following results.

Standard Galvanometer.		Pocket Galvanometer.	
Reading.	Reduced to Ampères.	Reading.	Value of 1 division in Ampères.
11.8	.03446	41.2	.000836
17.4	.05081	61.0	.000833
21.2	.06190	74.5	.000831
		Mean	.000833

Hence when an extra resistance of 5000 ohms is added 1 division of the scale will correspond to 4.25 volts ($=.000833 \times 5100$).

Thus we see that the present instrument measures from .001 ampères to 168 Ampères and from 4 volts to 400 volts with a probable error of 1 per cent. The galvanometer has of course three constants, one for the jaw, one for the internal coil, and one for the external coil if it has one. When these are once determind it is not necessary however to test all the three from time to time, since the ratios of the constants amongst themselves depend only upon the configuration of the instrument. These ratios being determined once for all, any change in the value of the constants due to a change in the moments of the magnets or the strengths of springs can be readily discovered by testing for any one of them, say, that for the internal coil.