

## Researches on the Distribution of the Mean Motions of the Asteroids.

By

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In accordance with the views of some astronomers the fall of meteors, the zodiacal light and the *gegenschein* suggest the possibility of some kind of resistance to the planetary and satellite motions. It may be exceedingly small for the major planets. But for small bodies like asteroids or satellites<sup>1)</sup>, it may not be entirely negligible if the interval of time be sufficiently long, say hundreds or thousands of years.

Last year while at Yale University I considered the theoretical effect of a resisting medium, supposedly motionless, on the libration of asteroids, and tried to explain the gaps of the asteroid distribution on that hypothesis. But I did not succeed.

Recently I have worked on the supposition of another kind of resistance suggested by Prof. E. W. Brown.<sup>2)</sup> According to this, resisting materials having the size of ordinary meteors are supposed to move around the central body in circular orbits. The result of my study seems satisfactory to explain the gaps in the first approximation. I shall present the course of my investigation in the following chapters.

The numerical computations throughout this investigation were duplicated by Mr. S. Terada, to whom the writer desires to express his sincere thanks.

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1) For comets see § 8.

2) Sir G. Darwin seems to have had similar idea. See A. N. 184, p. 263.

## Chapter I.

Effect on the Elliptic Motion, of Resisting Materials supposed to move around the Central Body in Circular Orbits.

1. I shall assume that *the resisting particles and the asteroid move in one and the same plane*. The components of the velocities of two bodies at a common point, referred to the center of the sun and a system of fixed axes, are

|                   | Asteroid                          | Particle |
|-------------------|-----------------------------------|----------|
| $\frac{dr}{dt}$   | $\frac{nae}{\sqrt{1-e^2}} \sin w$ | 0        |
| $r \frac{dw}{dt}$ | $\frac{na^2 \sqrt{1-e^2}}{r}$     | $n_0 r$  |

where  $r$ ,  $w$ ,  $n$ ,  $a$  and  $e$  are respectively the common radius vector, the true anomaly, the mean motion, the semi-major axis and the eccentricity of the asteroid and  $n_0$ , the circular mean motion of the particle. We have

$$n^2 a^3 = n_0^2 r^3$$

or

$$n_0 r = na \sqrt{\frac{a}{r}}$$

Hence the components of the relative velocity of the asteroid are

$$(1) \quad \begin{cases} V_s = \frac{nae}{\sqrt{1-e^2}} \sin w \\ V_t = na \left( \frac{a\sqrt{1-e^2}}{r} - \sqrt{\frac{a}{r}} \right) = na \frac{1+e \cos w - \sqrt{1+e \cos w}}{\sqrt{1-e^2}} \end{cases}$$

Assuming that *the resistance is proportional to the  $h$ th power of the relative velocity*, and denoting by  $S$  and  $T$ , the components of the resistance in the direction of the radius vector and of the perpendicular to it, we may write

$$(2) \quad S = -c\rho V^{h-1} V_s \quad T = -c\rho V^{h-1} V_t$$

where  $c$  is a constant depending only on the size of the asteroid,  $\rho$ , the density of the particles and  $V$ , the resultant relative velocity of the asteroid. The constants  $c$  and  $\rho$  are essentially positive.

2. Let  $\rho$  be a *continuous function* of  $r$ , so that it may be developed in the convergent series,

$$\rho = \rho_0 + \frac{d\rho}{da} (r-a) + \frac{d^2\rho}{da^2} \frac{(r-a)^2}{2} + \dots$$

Put

$$k_1 = \frac{a}{\rho_0} \frac{d\rho}{da}, \quad k_2 = \frac{1}{2} \frac{a^2}{\rho_0} \frac{d^2\rho}{da^2}, \quad \dots$$

then

$$\rho = \rho_0 \left\{ 1 + k_1 \frac{r-a}{a} + k_2 \left( \frac{r-a}{a} \right)^2 + \dots \right\}$$

Or developing  $\frac{r-a}{a}$  in powers of  $e$ ,

$$(3) \quad \rho = \rho_0 (1 - k_1 e \cos w - k_1 e^2 \sin^2 w + k_2 e^2 \cos^2 w + \dots)$$

3. The equations for the variations of the elements are<sup>1)</sup>

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}} \left[ Se \sin w + T(1+e \cos w) \right] \\ \frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na} \left[ S \sin w + T \left( \cos w + \frac{\cos w + e}{1+e \cos w} \right) \right] \\ e \frac{d\varpi}{dt} &= \frac{\sqrt{1-e^2}}{na} \left[ -S \cos w + T \left( 1 + \frac{1}{1+e \cos w} \right) \sin w \right] \\ \frac{d\varepsilon}{dt} &= -\frac{2(1-e^2)}{na(1+e \cos w)} S + \frac{e^2}{1+\sqrt{1-e^2}} \frac{d\varpi}{dt} \end{aligned}$$

where  $\varpi$  and  $\varepsilon$  are the mean longitude of perihelion and the mean longitude at the epoch. Since  $V$  contains the 1st power of  $e$  as factor,  $S$  and  $T$  will contain  $e^h$  as factor. Hence, if we confine ourselves to the order of  $e^{h+1}$  in the development of the differential coefficients, we may neglect  $e^2$  terms in the coefficients of  $S$  and  $T$ . Thus,

1) Tisserand, *Mécanique Céleste*, I p. 433 and IV p. 218.

$$(4) \quad \begin{cases} \frac{da}{dt} = \frac{2}{n} [Se \sin w + T(1 + e \cos w)] \\ \frac{de}{dt} = \frac{1}{na} [S \sin w + T(2 \cos w + e \sin^2 w)] \\ e \frac{d\varpi}{dt} = \frac{1}{na} [-S \cos w + T(2 - e \cos w) \sin w] \\ \frac{d\varepsilon}{dt} = -\frac{2}{na} S(1 - e \cos w) + \frac{e^2}{2} \frac{d\varpi}{dt} \end{cases}$$

The equations (1) become, keeping two orders of  $e$ ,

$$V_s = nae \sin w \quad V_t = \frac{1}{2} nae \cos w \left(1 + \frac{e}{4} \cos w\right)$$

whence

$$V = nae \sqrt{1 - \frac{3}{4} \cos^2 w + \frac{e}{8} \cos^3 w}$$

or neglecting  $e^3$

$$(5) \quad V = nae \left(1 + \frac{e}{16} \frac{\cos^3 w}{1 - \frac{3}{4} \cos^2 w}\right) \sqrt{1 - \frac{3}{4} \cos^2 w}$$

The equation (3) becomes simply

$$(6) \quad \rho = \rho_0 (1 - k_1 e \cos w)$$

4. We have to change the independent variable from  $t$  to  $w$  in the equations (4). The relation between the differentials is

$$dt = \frac{(1 - e^2)^{\frac{3}{2}}}{n(1 + e \cos w)^2} dw$$

or neglecting  $e^2$

$$(7) \quad dt = \frac{1}{n} (1 - 2e \cos w) dw$$

5. Combining the equations (2), (4), (5), (6) and (7), and putting

$W =$

$$-c\rho_0 n^{h-2} a^{h-1} e^h (1 - 2e \cos w)(1 - k_1 e \cos w) \left(1 - \frac{3}{4} \cos^2 w\right)^{\frac{h-1}{2}} \left(1 + \frac{1}{16} \frac{e \cos^3 w}{1 - \frac{3}{4} \cos^2 w}\right)^{h-1}$$

I obtain

$$\begin{aligned}\frac{da}{dw} &= Wa \left[ 2e \sin^2 w + \cos w \left( 1 + \frac{e}{4} \cos w \right) \left( 1 + e \cos w \right) \right] \\ \frac{de}{dw} &= W \left[ \sin^2 w + \cos w \left( 1 + \frac{e}{4} \cos w \right) \left( \cos w + \frac{e}{2} \sin^2 w \right) \right] \\ e \frac{d\varpi}{dw} &= W \left[ -\sin w \cos w + \sin w \cos w \left( 1 + \frac{e}{4} \cos w \right) \left( 1 - \frac{e}{2} \cos w \right) \right] \\ \frac{d\varepsilon}{dw} &= -2W \sin w (1 - e \cos w) + \frac{e^2}{2} \frac{d\varpi}{dw}\end{aligned}$$

Or neglecting  $e^{h+2}$  and simplifying

$$\begin{aligned}W &= -c\rho_0 n^{h-2} a^{h-1} e^h \left( 1 - \frac{3}{4} \cos^2 w \right)^{\frac{h-1}{2}} \left\{ 1 - \left( 2 + k_1 - \frac{h-1}{16} \frac{\cos^2 w}{1 - \frac{3}{4} \cos^2 w} \right) e \cos w \right\} \\ (8) \quad \begin{cases} \frac{da}{dw} = Wa (\cos w + 2e - \frac{3}{4} e \cos^2 w) \\ \frac{de}{dw} = W (1 + \frac{e}{2} \cos w - \frac{e}{4} \cos^3 w) \\ e \frac{d\varpi}{dw} = -W \frac{e}{4} \sin w \cos^2 w \\ \frac{d\varepsilon}{dw} = -2W (\sin w - e \sin w \cos w) + \frac{e^2}{2} \frac{d\varpi}{dw} \end{cases}\end{aligned}$$

6. To find the secular variations of the elements in a unit interval of time we have to evaluate

$$\left[ \frac{da}{dt} \right] = \frac{n}{2\pi} \int_0^{2\pi} \frac{da}{dw} dw \quad \text{etc.} \quad \text{etc.}$$

Since the effect of the resistance is supposed to be very small, we may neglect  $c^2 \rho_0^2$  within a single revolution, so that the quantities  $n$ ,  $a$  and  $e$  may be integrated as constants.

Now if we put

$$X = (1 - q \cos^2 w)^{\frac{s}{2}} \sin^i w \cos^j w$$

where  $q < 1$  and  $s, i, j$  are any positive integers, it may be easily seen that

$$\int_0^{2\pi} X dw = \{1 + (-1)^i\} \{1 + (-1)^j\} \int_0^{\frac{\pi}{2}} X dw$$

Hence the integral vanishes when either  $i$  or  $j$  is odd, and becomes

$$4 \int_0^{\frac{\pi}{2}} X dw$$

when both are even.

7. Applying this result to the equations (8), I obtain after some simplifications

$$(9) \quad \begin{cases} \left[ \frac{d\alpha}{dt} \right] = -c\rho_0 n^{h-1} \alpha^h e^{h+1} \left\{ 2I_1 - \left( \frac{11}{4} + k_1 \right) I_2 + \frac{h-1}{16} I_3 \right\} \\ \left[ \frac{de}{dt} \right] = -c\rho_0 n^{h-1} \alpha^{h-1} e^h I_1 \\ \left[ \frac{d\varpi}{dt} \right] = 0 \quad \quad \quad \left[ \frac{d\varepsilon}{dt} \right] = 0 \end{cases}$$

where

$$I_1 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left( 1 - \frac{3}{4} \cos^2 w \right)^{\frac{h-1}{2}} dw$$

$$I_2 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left( 1 - \frac{3}{4} \cos^2 w \right)^{\frac{h-1}{2}} \cos^2 w dw$$

$$I_3 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left( 1 - \frac{3}{4} \cos^2 w \right)^{\frac{h-3}{2}} \cos^4 w dw$$

Taking the first integral and expanding

$$I_1 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left\{ 1 - \frac{h-1}{2} \left( \frac{3}{4} \right) \cos^2 w + \frac{(h-1)(h-3)}{2 \cdot 4} \left( \frac{3}{4} \right)^2 \cos^4 w - \dots \right\} dw$$

$$\text{Now} \quad \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^i w dw = \frac{1 \cdot 3 \dots (i-1)}{2 \cdot 4 \dots i}$$

when  $i$  is even. Therefore

$$\begin{aligned} I_1 = 1 - \frac{h-1}{2} \frac{1}{2} \left( \frac{3}{4} \right) + \frac{(h-1)(h-3)}{2 \cdot 4} \frac{1 \cdot 3}{2 \cdot 4} \left( \frac{3}{4} \right)^2 \\ - \frac{(h-1)(h-3)(h-5)}{2 \cdot 4 \cdot 6} \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left( \frac{3}{4} \right)^3 + \dots \end{aligned}$$

Similarly

$$I_2 = \frac{1}{2} - \frac{h-1}{2} \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{3}{4}\right) + \frac{(h-1)(h-3)}{2 \cdot 4} \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{3}{4}\right)^2 - \dots$$

$$I_3 = \frac{1 \cdot 3}{2 \cdot 4} - \frac{h-3}{2} \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{3}{4}\right) + \frac{(h-3)(h-5)}{2 \cdot 4} \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \left(\frac{3}{4}\right)^2 - \dots$$

These series stop at a finite number of terms when  $h$  is odd, and continue infinitely when  $h$  is even. If we put

$$(10) \quad \alpha_1 = 2I_1 - \frac{11}{4}I_2 + \frac{h-1}{16}I_3 \quad \alpha_2 = I_2 \quad \beta_1 = I_1$$

we get finally

$$(11) \quad \begin{cases} \left[ \frac{da}{dt} \right] = -(a_1 - a_2 k_1) c \rho_0 n^{h-1} a^h e^{h+1} & \left[ \frac{d\varpi}{dt} \right] = 0 \\ \left[ \frac{de}{dt} \right] = -\beta_1 c \rho_0 n^{h-1} a^{h-1} e^h & \left[ \frac{d\varepsilon}{dt} \right] = 0 \end{cases}$$

The numerical values of  $\alpha_1$ ,  $\alpha_2$ , and  $\beta_1$ , for  $h=1, 2, 3$  are computed as follows:—

| $h$ | $\alpha_1$ | $\alpha_2$ | $\beta_1$ |
|-----|------------|------------|-----------|
| 1   | 0.62       | 0.50       | 1.00      |
| 2   | 0.69       | 0.32       | 0.77      |
| 3   | 0.70       | 0.22       | 0.62      |

8. If we put  $h=2$  in (11), as it seems most natural,  $\left[ \frac{de}{dt} \right]$  is proportional to  $e^2$ , and  $\left[ \frac{da}{dt} \right]$  to  $e^3$ . Hence the effect must be very small for the bodies whose orbits have small eccentricities. Or in other words if the effect be appreciable in the motion of asteroids which have small eccentricities in general, it must be remarkably great for the motion of the comets. This appears to be almost fatal to our assumption, supposing the resistance still to exist, because the effect of the resistance on the cometary motion, whatever may be its law, is known to be very small if it exists at all.

But there is an answer to this objection. The comet, as far as we know, is not a single body rigidly bound like a planet. It

seems to be a loose aggregation of small bodies<sup>1)</sup>, perhaps composed of the same kind of material as meteors, with rare gaseous envelope. Most of the particles which are supposed to effect the resistance will pass freely through this meteoric swarm. Rarely it may occur that some resisting particle strikes a cometary particle. Then the latter will be projected outside of the swarm and take on an individual motion. Gradual degeneration will follow this action if repeated frequently but there is no effect on the motion of the main body. Yet one more thing is conceivable, viz. an indirect effect coming in through the gaseous envelope. This, however, would be very small owing to the tenuity of the latter.

9. The assumption that the particles move in circular orbits in a definite plane is nothing but an imaginary convention to make the problem simpler. Practically this may be said to be that at a point in or near the plane, the resultant composite velocity of the particles passing through that point is equal to the velocity of the circular motion.

As for the density of the particles it is natural to assume a certain amount of decrease as the distance from the sun increases, that is, to assume a negative value for  $k_1$ . This is not all, for there is some reason to believe that the particles are not numerous near the path of the major planets. They cannot move in orbits of small eccentricity in the neighborhood of the planets. If they did, they would be disturbed a great deal by the action of the latter, or they might even be swept up, except those moving about the triangular equilibrium points.

## Chapter II.

Motion of the Asteroids whose Mean Motions are nearly Commensurable to the Mean Motion of Jupiter.

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1) Young compares it with "pin-heads several hundred feet apart."



10. In this chapter I shall develop a simple theory of planetary librations according to the method of Prof. Brown.<sup>1)</sup>

Assuming that *the asteroid moves in the plane of the orbit of Jupiter and that Jupiter moves in a circular orbit*, let  $n$ ,  $a$ ,  $e$ ,  $\varepsilon$  and  $\varpi$  be the elements of the asteroid as before,  $n'$ ,  $a'$ , and  $\varepsilon'$ , the circular elements of Jupiter and  $R$  the perturbative function, as usual. Take the mass of the sun as unity and the unit of length, so as to make

$$(1) \quad n^2 a^3 = 1,$$

neglecting the mass of the asteroid, and also

$$n'^2 a'^3 = 1 + m'$$

$m'$  being the mass of Jupiter which is about 1/1047.

11. Let  $n_0$  be the mean motion commensurable to  $n'$  and let

$$(2) \quad n = n_0(1+x) \quad \text{or} \quad a = a_0(1+x)^{-\frac{2}{3}}$$

The quantity  $x$  is supposed to be small, of the same order as  $e^2$  at most. Let also

$$(3) \quad \frac{n'}{n_0} = \frac{s'}{s}$$

where  $s$  and  $s'$  are positive integers prime to each other.

12. As a consequence of the assumptions mentioned above, any argument of long period-terms, or critical argument as usually called, takes the form

$$is'l - is'l' + jw$$

where  $i$  and  $j$  are any integers positive or negative, and  $l$  and  $l'$  are mean longitudes. By the properties of  $R$  we have

$$is' - is + j = 0 \quad \text{or} \quad j = i(s - s')$$

Hence the critical argument is

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1) Month. Notices of the R. A. S., lxxii, p. 609.

$$(4) \quad i\{s'l - sl' + (s - s')\varpi\} \equiv i\theta$$

The corresponding term will be factored by  $e^{i(s-s')}$ , so that for the principal term we have to put  $i=1$ . Now

$$l = \int n dt + \varepsilon = n_0 t + n_0 \int x dt + \varepsilon$$

by (2) and  $l' = n't + \varepsilon'$

whence

$$(5) \quad \theta = s'n_0 \int x dt + s'\varepsilon - s\varepsilon' + (s - s')\varpi$$

13. According to the theory of perturbations we have

$$(6) \quad \begin{cases} \frac{da}{dt} = (a, \varepsilon) \frac{\partial R}{\partial \varepsilon} & \frac{d\varepsilon}{dt} = (\varepsilon, a) \frac{\partial R}{\partial a} + (\varepsilon, e) \frac{\partial R}{\partial e} \\ \frac{de}{dt} = (e, \varepsilon) \frac{\partial R}{\partial \varepsilon} + (e, \varpi) \frac{\partial R}{\partial \varpi} & \frac{d\varpi}{dt} = (\varpi, e) \frac{\partial R}{\partial e} \end{cases}$$

in which

$$(7) \quad \begin{cases} (a, \varepsilon) = -(\varepsilon, a) = 2na^2 \\ (e, \varepsilon) = -(\varepsilon, e) = -\frac{na\sqrt{1-e^2}}{e}(1 - \sqrt{1-e^2}) \\ (e, \varpi) = -(\varpi, e) = -\frac{na\sqrt{1-e^2}}{e} \end{cases}$$

Neglecting all short period-terms in  $R$ , let

$$R = R_0 + R_e$$

and

$$R_0 = \frac{m'}{a'} \varphi\left(\frac{a}{a'}, e^2\right)$$

$$R_e = \frac{m'}{a'} \phi_1\left(\frac{a}{a'}, e^2\right) e^{s-s'} \cos \theta + \frac{m'}{a'} \phi_2\left(\frac{a}{a'}, e^2\right) e^{2(s-s')} \cos 2\theta + \dots$$

The functions  $\varphi, \phi_1, \phi_2, \dots$  are developable in terms of powers and products of  $\frac{a}{a'}$  and  $e^2$ . The second and succeeding terms of  $R_e$  are not important, being of higher orders with respect to  $e$ . Since  $\varepsilon$  and  $\varpi$  are contained in  $R$  through  $\theta$ , we have

$$\frac{\partial R}{\partial \varepsilon} = s' \frac{\partial R}{\partial \theta} \quad \frac{\partial R}{\partial \varpi} = (s - s') \frac{\partial R}{\partial \theta}$$

The elements  $\varepsilon$  and  $\varpi$  may be eliminated by these relations; viz.

$$(8) \quad \begin{cases} \frac{da}{dt} = s'(a, \varepsilon) \frac{\partial R}{\partial \theta} & \frac{de}{dt} = [s'(e, \varepsilon) + (s - s')(e, \varpi)] \frac{\partial R}{\partial \theta} \\ \frac{d\theta}{dt} = s'n_0x + s' \frac{d\varepsilon}{dt} + (s - s') \frac{d\varpi}{dt} \\ \quad = s'n_0x + s'(\varpi, a) \frac{\partial R}{\partial a} + [s'(\varepsilon, e) + (s - s')(\varpi, e)] \frac{\partial R}{\partial e} \end{cases}$$

14. The first two equations give

$$\frac{de}{dt} = \frac{s'(e, \varepsilon) + (s - s')(e, \varpi)}{s'(a, \varepsilon)} \frac{da}{dt}$$

or

$$2e \frac{de}{dt} = - \frac{\sqrt{1-e^2}}{s'a} (s - s'\sqrt{1-e^2}) \frac{da}{dt} = \frac{2}{3} \frac{n_0}{n} \sqrt{1-e^2} \frac{s - s'\sqrt{1-e^2}}{s'} \frac{dx}{dt}$$

whence

$$\frac{2s'}{\sqrt{1-e^2}(s - s'\sqrt{1-e^2})} e \frac{de}{dt} = \frac{2}{3} \frac{n_0}{n} \frac{dx}{dt}$$

Expanding the coefficient of  $e \frac{de}{dt}$  in terms of  $e^2$ , and  $n$ , in terms of  $x$ , and integrating we get

$$(9) \quad \frac{s'}{s - s'} e^2 = \frac{2}{3} x + \text{Const.}$$

in which  $x^2$  and  $e^4$ , and higher powers are neglected.

15. From the three equations of (8), we get

$$\frac{dR}{dt} = \frac{\partial R}{\partial a} \frac{da}{dt} + \frac{\partial R}{\partial e} \frac{de}{dt} + \frac{\partial R}{\partial \theta} \frac{d\theta}{dt} = s'n_0x \frac{\partial R}{\partial \theta} = \frac{n_0}{(a, \varepsilon)} x \frac{da}{dt}$$

But

$$\frac{da}{dt} = - \frac{2}{3} a \frac{n_0}{n} \frac{dx}{dt}$$

whence

$$\frac{dR}{dt} = -\frac{1}{3a} \frac{n_0^2}{n^2} x \frac{dx}{dt}$$

Developing  $n$  and  $a$  in terms of powers of  $x$  and integrating

$$R = -\frac{x^2}{6a_0} + \text{Const.}$$

in which  $x^3$  and higher powers are neglected. Or

$$x^2 + 6a_0R_0 + 6a_0R_c = \text{Const.}$$

$R_0$  is a function of  $a$  and  $e^2$ , and  $e^2$  is a function of  $x$  containing an arbitrary constant. Hence developing

$$R_0 = (R_0)_0 + \left( \frac{\partial R_0}{\partial a} \frac{da}{dx} + \frac{\partial R_0}{\partial e^2} \frac{de^2}{dx} \right)_0 x + \dots$$

where the suffix 0 outside of the parentheses means the value for  $x=0$ . The terms with  $x^2$  and higher powers may be neglected, since they are multiplied by  $m'$ . Now

$$\left( \frac{da}{dx} \right)_0 = -\frac{2}{3} a_0 \quad \left( \frac{de^2}{dx} \right)_0 = \frac{2}{3} \frac{s-s'}{s'}$$

whence

$$R_0 = (R_0)_0 - \frac{2}{3} a_0 \left( \frac{\partial R_0}{\partial a} \right)_0 x + \frac{2}{3} \frac{s-s'}{s'} \left( \frac{\partial R_0}{\partial e^2} \right)_0 x$$

Substituting this expression in the integral we get

$$x^2 - 4a_0^2 \left( \frac{\partial R_0}{\partial a} \right)_0 x + 4 \frac{s-s'}{s'} a_0 \left( \frac{\partial R_0}{\partial e^2} \right)_0 x + 6a_0R_c = \text{Const.} - 6a_0(R_0)_0$$

or putting

$$(10) \quad x_0 = 2a_0^2 \left( \frac{\partial R_0}{\partial a} \right)_0 - 2 \frac{s-s'}{s'} a_0 \left( \frac{\partial R_0}{\partial e^2} \right)_0$$

the integral becomes

$$(11) \quad (x-x_0)^2 + 6a_0R_c = C$$

where  $C$  is an arbitrary constant.

16. By the last equation of (8)

$$\frac{d\theta}{dt} = s'n_0x - 2s'na^2 \frac{\partial R}{\partial a} + \frac{na\sqrt{1-e^2}}{e} (s-s'\sqrt{1-e^2}) \frac{\partial R}{\partial e}$$

Neglecting  $m'x$ ,  $m'e$  and higher powers we may write

$$\frac{d\theta}{dt} = s'n_0x - 2s'n_0a_0^2 \left( \frac{\partial R_0}{\partial a} \right)_0 + 2(s-s')n_0a_0 \left( \frac{\partial R_0}{\partial e^2} \right)_0 + (s-s') \frac{n_0a_0}{e} \frac{\partial R_e}{\partial e}$$

or introducing  $x_0$  by (10)

$$(12) \quad \frac{d\theta}{dt} = s'n_0(x-x_0) + (s-s') \frac{n_0a_0}{e} \frac{\partial R_e}{\partial e}$$

The equation (9) may be written also

$$(13) \quad e^2 = E + \frac{2}{3} \frac{s-s'}{s'} (x-x_0)$$

where  $E$  is an arbitrary constant.

17. The constant  $x_0$  is a quantity depending on  $\frac{a_0}{a'}$  and  $E$ , multiplied by  $m'$ . Now  $E$  is supposed to be small, of the same order as  $e^2$ . Hence, neglecting  $m'E$  in  $x_0$ , the latter becomes a constant depending only on  $\frac{a_0}{a'}$ .

Since  $x$  is always diminished by  $x_0$ , if we change the origin by that amount, we may write  $x$  for  $x-x_0$  and  $x_0$  disappears in the equations (11), (12) and (13). Thus

$$(14) \quad \begin{cases} x^2 = C - 6a_0R_e \\ e^2 = E + \frac{2}{3} \frac{s-s'}{s'} x \\ \frac{1}{n_0} \frac{d\theta}{dt} = s'x + (s-s') \frac{a_0}{e} \frac{\partial R_e}{\partial e} \end{cases}$$

18. The expression of  $x_0$  in terms of the powers of  $\frac{a_0}{a'}$  may be obtained easily by the usual process. The result is

$$\begin{aligned}\frac{x_0}{m'} &= \alpha_0^2 \frac{db^{(0)}}{da_0} - \frac{\nu \alpha_0^2}{2} \left( \frac{db^{(0)}}{da_0} + \frac{\alpha_0}{2} \frac{d^2b^{(0)}}{da_0^2} \right) \\ &= (4-3\nu) \left( \frac{1}{2} \right)^2 \alpha_0^3 + 2(4-5\nu) \left( \frac{1.3}{2.4} \right)^2 \alpha_0^5 + 3(4-7\nu) \left( \frac{1.3.5}{2.4.6} \right) \alpha_0^7 + \dots\end{aligned}$$

where  $\alpha_0 = \frac{a_0}{a'}$  and  $\nu = \frac{s-s'}{s'}$

The numerical values are

| $s/s'$ | $x_0 \times 10^2$ |
|--------|-------------------|
| 2/1    | -0.005            |
| 3/2    | +0.027            |
| 3/1    | -0.012            |

which are too small to be considered in our problem.

The equations (14) represent the motion in the librating region with remarkable simplicity. This will be shown geometrically.

19. *General Case of the First Order,  $s-s'=1$ .* The principal term of  $R_e$ , becomes in this case

$$\alpha_0 R_e = -p_1^{(s)} e \cos \theta$$

in which 
$$p_1^{(s)} = \frac{m' a_0}{2} \left( 2s b^{(s)} + \alpha_0 \frac{db^{(s)}}{da_0} \right)$$

The three equations of (14) become

$$(15) \quad x^2 = C + 6p_1^{(s)} e \cos \theta$$

$$(16) \quad e^2 = E + \frac{2}{3s'} x$$

$$(17) \quad \frac{1}{n_0} \frac{d\theta}{dt} = s'x - \frac{p_1^{(s)} \cos \theta}{e}$$

Suppose the quantities  $x$  and  $e$  are two rectangular coordinates. The equation (16), then, represents a *parabola* whose axis is the

axis of  $x$ . Let this parabola be designated by  $A$ . The equation (15), also, if we put a definite value for  $\theta$ , represents a parabola whose axis is the axis of  $e$ . For the limiting values of  $\cos\theta$ ,  $\pm 1$ , we have two *limiting parabolas* whose equations are

$$(18) \quad x^2 = C \pm 6p_1 e^{11}$$

Let these limiting parabolas be designated by  $B_0$  and  $B_1$ , respectively.  $B_0$  and  $B_1$  meet at the same points  $(\pm\sqrt{C}, 0)$  with the axis of  $x$ . Two straight lines passing through these points and parallel to the axis of  $e$  represent the equation (15) for  $\theta = \frac{\pi}{2}$  and  $\frac{3\pi}{2}$ .

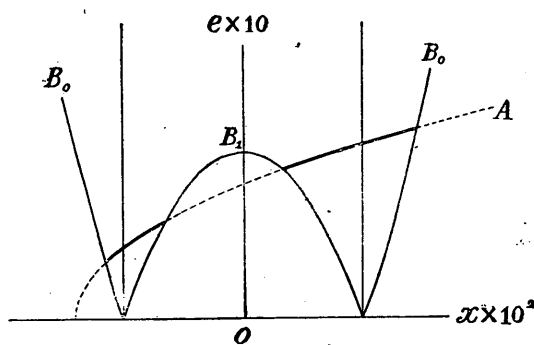


Fig. 1

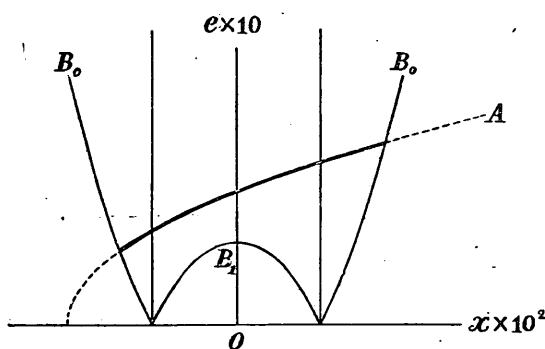


Fig. 2

1) Writing  $p_1$  for  $p_1^{(s)}$  and  $p_{s-s'}$  for  $p_{s-s'}^{(s)}$ , generally, for the sake of brevity.

The parts, or the part as the case may be, of  $A$  inside of  $B_0$  and outside of  $B_1$  are the real paths of the point  $(x, e)$ . In Fig. 1 two parts, one on the negative side of  $x$  and the other on the positive side, are separate. The argument  $\theta$  may take any value in each part.

But when  $E$  becomes larger or  $C$  becomes smaller, these two parts approach each other until  $A$  touches  $B_1$ , in which case they are connected. After this  $A$  does not intersect  $B_1$ , as in Fig. 2, and the argument  $\theta$  never takes the value  $\pi$  or  $-\pi$ . It increases from 0 to the maximum value  $\theta_0$  and decreases, passing through 0 again, until it reaches the minimum value  $-\theta_0$ . Then it increases again and so forth. This is a kind of *libration*.<sup>1)</sup>

20. We can distinguish *six types of the motion*, namely:—

- 1 *Revolution* on the negative side (of  $x$ ),
- 2 „ extending on both sides,
- 3 „ on the positive side,
- 1 *Libration* on the negative side,
- 2 „ extending on both sides,
- 3 „ on the positive side.

21. In order to determine the limits of  $C$  and  $E$  for each type, we need to consider some singular cases.

1. Condition of the contact of  $A$  and  $B_0$  or  $B_1$ . Differentiating the equations (16) and (18) with respect to  $x$  and equating  $\frac{de}{dx}$  we get the condition

$$(19) \quad ex = \pm \frac{p_1}{s'}$$

The same equation results from (17) putting  $\cos\theta = \pm 1$  and  $\frac{d\theta}{dt} = 0$  in it. To find the relation between  $C$  and  $E$ , eliminating  $x$  and  $e$  from the equations (16), (18) and (19), we get

---

1) Libration is defined as a case of motion in which the value of the argument is limited. This definition is slightly different from that of Prof. Brown (Month. Not. lxxii p. 618). The term *revolution* is used after M. Callandreau (Annal. de l'Obs. de Paris, Memoires Tome XXII).



$$(20) \quad 12s'^2 p_1^2 E^3 - \frac{s'^2}{3} E^2 C^2 - 6p_1^2 EC + \frac{9}{s'^2} p_1^4 + \frac{4C^3}{27} = 0$$

2. Condition of the contact of the second order of  $A$  and  $B_0$  or  $B_1$ . The equation (20) gives three real roots for  $E$  when  $C$  is great. Let them be denoted by  $E_1$ ,  $E_2$  and  $E_3$  in which  $E_1$  is the smallest and  $E_3$ , the greatest, algebraically.  $E_1$  is always real.  $E_2$  and  $E_3$  are positive when real and may be equal. The condition of the equal roots is

$$(21) \quad C = 3 \left( \frac{3}{s'} p_1^2 \right)^{\frac{2}{3}} \equiv C_1 \quad E = \frac{1}{s'} \left( \frac{3}{s'} p_1^2 \right)^{\frac{1}{3}}$$

which is the condition of the contact of the second order, geometrically. When  $C$  is smaller than  $C_1$  two roots  $E_2$  and  $E_3$  become imaginary.

3. Case in which  $A$  and  $B_0$  or  $B_1$  intersect on the axis of  $x$ . Putting  $e=0$  in the equations (16) and (18), and equating  $x$ , we get

$$(22) \quad E = \pm \frac{2}{3s'} \sqrt{C} \equiv \pm F_1$$

4. Case in which  $A$  and  $B_0$  or  $B_1$  intersect on the axis of  $e$ .

$$(23) \quad E = \left( \frac{C}{6p_1} \right)^{\frac{2}{3}} \equiv F_0$$

5. Singular case in which  $A$  and  $B_0$  or  $B_1$  intersect on the axis of  $x$  and touch each other, simultaneously. Putting the condition (22) in the equation (20) we get easily

$$(24) \quad C = \frac{9}{4} \left( \frac{6}{s'} p_1^2 \right)^{\frac{2}{3}} \equiv C_2 \quad E = \frac{1}{s'} \left( \frac{6}{s'} p_1^2 \right)^{\frac{1}{3}}$$

22. The following table is constructed to show how different types of the motion are related to the magnitudes of the arbitrary constants  $C$  and  $E$ .

| Type     | Limiting<br>Curves | Limits<br>of $x$ | Limits of $C$  |                 |                 |                |
|----------|--------------------|------------------|----------------|-----------------|-----------------|----------------|
|          |                    |                  | $-\infty$ 0    | 0 $C_1$         | $C_1$ $C_2$     | $C_2 + \infty$ |
|          |                    |                  | Limits of $E$  |                 |                 |                |
| Revol. 1 | $B_0 B_1$          | — —              | impossible     | impossible      | impossible      | $+F_1$ $E_3$   |
| „ 2      | $B_1 B_0$          | — +              | „              | $F_0 + F_1$     | $F_0 + F_1$     | $F_0$ $E_3$    |
| „ 3      | „                  | + +              | „              | $-F_1$ $F_0$    | $-F_1$ $F_0$    | $-F_1$ $F_0$   |
| Libr. 1  | $B_1 B_1$          | — —              | „              | impossible      | $E_2$ $E_3$     | $E_2 + F_1$    |
| „ 2      | $B_0 B_0$          | — +              | $F_0 + \infty$ | $+F_1 + \infty$ | $+F_1 + \infty$ | $E_1 + \infty$ |
| „ 3      | „                  | + +              | $E_1$ $F_0$    | $E_1 - F_1$     | $E_1 - F_1$     | $E_1 - F_1$    |

This will be very easily understood by the geometrical representation of the motion.

23. The sign of  $\frac{d\theta}{dt}$  for different types will be discussed now. The point at which  $\frac{d\theta}{dt}$  changes its sign is determined by the equation

$$s'ex - p_1 \cos \theta = 0$$

combined with the equations (15) and (16). Eliminating  $e$  and  $\cos \theta$ , we get easily

$$x^2 + 2s'Ex + \frac{C}{3} = 0$$

or

$$(25) \quad x = -s'E \pm \sqrt{s'^2 E^2 - \frac{C}{3}}$$

The corresponding values of  $e$  are given by

$$(26) \quad e^2 = \frac{E}{3} \pm \frac{2}{3s'} \sqrt{s'^2 E^2 - \frac{C}{3}}$$

Since  $e$  must be positive, two points, real or imaginary, will be determined by these equations. Now evidently only an odd number of zero-points in the librations and an even number in the revolutions, are possible. Accordingly, since there cannot be more than two, *one and only one zero-point of  $\frac{d\theta}{dt}$  will exist in the case of the librations.*

The conditions necessary in order that two zero-points may be real, are

$$E > 0, \quad \frac{C}{3s'^2} < E^2 < \frac{4}{9s'^2} C$$

by (25) and (26). Hence  $C$  must be positive and therefore both values of  $x$  must be negative. Accordingly there exists no zero-point in the revolution of Type 3. For the revolution of Type 1, we have the condition

$$E > +F_1 \left( = \frac{2}{3s'} \sqrt{C} \right)$$

which contradicts with one of the above conditions. Hence no zero-point exists in the revolution of Type 1, and therefore the *only type of the motion, in which two zero-points of  $\frac{d\theta}{dt}$  may exist, is the revolution of Type 2.*

Evidently the sign of  $\frac{d\theta}{dt}$  is *negative in the revolution of Type 1* and *positive in the revolution of Type 3.*

24. *General Case of the Second Order,  $s-s'=2$ .* We have in this case

$$a_0 R_e = p_2 e^2 \cos \theta$$

where 
$$p_2 = p_1^{(s)} = \frac{m' a_0}{8} \left[ (4s^2 - 5s) b^{(s)} + (4s - 2) a_0 \frac{db^{(s)}}{da_0} + a_0^2 \frac{d^2 b^{(s)}}{da_0^2} \right]$$

Putting 
$$e^2 = y$$

we have by (14)

$$(27) \quad \begin{cases} x^2 = C - 6p_2 y \cos \theta \\ y = E + \frac{4}{3s'} x \\ \frac{1}{n_0} \frac{d\theta}{dt} = s'x + 4p_2 \cos \theta \end{cases}$$

The curve  $A$  becomes a *straight line* in this case. The condition of the contact of  $A$  and  $B_0$  or  $B_1$  is

$$x = \mp \frac{4p_2}{s'}$$

and the relation between  $C$  and  $E$  becomes simply

$$C \mp 6p_2 E + \frac{16}{s'^2} p_2^2 = 0$$

There is *no contact of the second order*. The condition that  $A$  and  $B_0$  or  $B_1$  may touch each other on the axis of  $x$  is

$$C = \frac{16}{s'^2} p_2^2 \equiv C_2 \quad E = \pm \frac{16}{3s'^2} p_2$$

25. The limits of  $C$  and  $E$  for different types of the motion are as follows:—

| Type   |   | Limiting<br>Curves | Limits<br>of $x$ | Limits of $C$   |                  |            |           |
|--------|---|--------------------|------------------|-----------------|------------------|------------|-----------|
|        |   |                    |                  | $-\infty$       | 0                | 0          | $+\infty$ |
|        |   |                    |                  | Limits of $E$   |                  |            |           |
| Revol. | 1 | $B_1 \ B_0$        | — —              | impossible      | impossible       | $+F_1$     | $E_3$     |
| „      | 2 | $B_0 \ B_1$        | — +              | „               | $F_0 \ +F_1$     | $F_0$      | $E_3$     |
| „      | 3 | „                  | + +              | „               | $-F_1 \ F_0$     | $-F_1$     | $F_0$     |
| Libr.  | 1 | —                  | —                | „               | impossible       | impossible |           |
| „      | 2 | $B_1 \ B_1$        | — +              | $F_0 \ +\infty$ | $+F_1 \ +\infty$ | $F_3$      | $+\infty$ |
| „      | 3 | „                  | + +              | $E_1 \ F_0$     | $E_1 \ -F_1$     | impossible |           |

The limits are simpler than in the case of the first order. *The libration of Type 1 is impossible throughout* and the librations are restricted to that about  $\pi$ .

26. The point at which  $\frac{d\theta}{dt}$  changes its sign is given by the equations

$$x = -\frac{3s'}{4}E \pm \sqrt{\left(\frac{3s'}{4}E\right)^2 - C}$$

$$e^2 = \pm \sqrt{E^2 - \left(\frac{4}{3s'}\right)^2 C}$$

Only one point is possible on the positive side of  $e$ . Hence, there is *no zero-point in the revolutions* and one and only one point, in the librations.

27. *General Case of the Third or Higher Orders,  $s-s' > 2$ .* We have in this case

$$a_0 R_c = (-1)^{s-s'} p_{s-s'} e^{s-s'} \cos \theta$$

where  $p_{s-s'}$  is a function of  $a_0$  with a positive value. Putting

$$y = e^{s-s'}$$

we have by (14)

$$(28) \quad \begin{cases} x^2 = C - (-1)^{s-s'} 6 p_{s-s'} y \cos \theta \\ y = \left( E + \frac{2}{3} \frac{s-s'}{s'} x \right)^{\frac{s-s'}{2}} \\ \frac{1}{n_0} \frac{d\theta}{dt} = s'x + (-1)^{s-s'} (s-s')^2 p_{s-s'} y^{\frac{s-s'-2}{s-s'}} \cos \theta \end{cases}$$

The curve  $A$  becomes a parabola of various degrees with the axis parallel to the axis of  $y$ , and the vertex on the axis of  $x$ . *The constant  $C_2$  becomes 0.* The table of the limits is much simplified by this change. The libration of Type 1 is always impossible and the librations are restricted to those about 0 when  $s-s'$  is odd and those about  $\pi$  when  $s-s'$  is even. The libration of Type 3

becomes impossible when  $C > 0$ , in the cases of the fourth and higher orders.

### Chapter III.

#### Theoretical Effect of the Resistance on the Motion of the Asteroids in the Librating Region.

28. Putting  $h=2$ , the equations (11) of Chapter I become

$$(1) \quad \begin{cases} \frac{1}{a} \left[ \frac{da}{dt} \right] = -ae^3 & \left[ \frac{de}{dt} \right] = -\beta e^2 \\ \left[ \frac{d\varepsilon}{dt} \right] = 0 & \left[ \frac{d\varpi}{dt} \right] = 0 \end{cases}$$

where

$$\alpha = (\alpha_1 - \alpha_2 k_1) c \rho_0 n a \quad \beta = \beta_1 c \rho_0 n a$$

Now, by the equations (1) and (2) of Chapter II, we have

$$\frac{1}{a} \left[ \frac{da}{dt} \right] = -\frac{2}{3} \frac{1}{n} \left[ \frac{dn}{dt} \right]$$

and

$$\left[ \frac{dx}{dt} \right] = \frac{1}{n_0} \left[ \frac{dn}{dt} \right]$$

Since  $n$  and  $n_0$  are supposed to be very nearly equal

$$\frac{1}{a} \left[ \frac{da}{dt} \right] = -\frac{2}{3} \left[ \frac{dx}{dt} \right]$$

Hence

$$(2) \quad \left[ \frac{dx}{dt} \right] = \frac{3\alpha}{2} e^3$$

We have, also, by (5) of Chapter II

$$\frac{d\theta}{dt} = s' n_0 x + s' \frac{d\varepsilon}{dt} - s \frac{d\varepsilon'}{dt} + (s - s') \frac{d\varpi}{dt}$$

and, since  $x$  is not affected directly by the resistance, we have by (1)

$$(3) \quad \left[ \frac{d\theta}{dt} \right] = 0$$

The differential coefficients of  $x$ ,  $e$  and  $\theta$  are composed of two parts, viz; the differential coefficients due to the perturbation and those due to the resistance which are denoted by  $\left[ \frac{dx}{dt} \right]$ , etc. Hence, if we denote the former by  $\left( \frac{dx}{dt} \right)$ , etc.

$$\frac{dx}{dt} = \left( \frac{dx}{dt} \right) + \left[ \frac{dx}{dt} \right] \quad \text{etc.}$$

Or, substituting the expressions of  $\left[ \frac{dx}{dt} \right]$ , etc. we get

$$(4) \quad \begin{cases} \frac{dx}{dt} = \left( \frac{dx}{dt} \right) + \frac{3\alpha}{2} e^3 \\ \frac{de}{dt} = \left( \frac{de}{dt} \right) - \beta e^2 \\ \frac{d\theta}{dt} = \left( \frac{d\theta}{dt} \right) \end{cases}$$

Since  $x$  is supposed to be very small, we can write

$$\alpha = (a_1 - a_2 k_1) c \rho_0 n_0 a_0 \quad \beta = \beta_1 c \rho_0 n_0 a_0$$

As it will be seen in Chapter I,  $\beta$  is essentially *positive* and  $\alpha$  may be positive or negative according as  $k_1$  is less or greater than  $\frac{a_1}{a_2}$  which is positive. We shall consider the case in which  $k_1$  is negative and consequently  $\alpha$  is *positive*.

29. *General Case of the First Order.* Differentiating the equations (15) and (16) of Chapter II completely with respect to  $t$ , we get

$$2x \frac{dx}{dt} = \frac{dC}{dt} + 6p_1 \cos \theta \frac{de}{dt} - 6p_1 e \sin \theta \frac{d\theta}{dt}$$

$$2e \frac{de}{dt} = \frac{dE}{dt} + \frac{2}{3s'} \frac{dx}{dt}$$

Differentiating the same equations with respect to  $t$ , supposing the effect of the resistance is null,

$$\begin{aligned} 2x\left(\frac{dx}{dt}\right) &= 6p_1 \cos \theta \left(\frac{de}{dt}\right) - 6p_1 e \sin \theta \left(\frac{d\theta}{dt}\right) \\ 2e\left(\frac{de}{dt}\right) &= \frac{2}{3s'} \left(\frac{dx}{dt}\right) \end{aligned}$$

Substituting the expressions of  $\frac{dx}{dt}$ , etc. by (4) and taking the differences between the two sets of equations, we obtain

$$(5) \quad \begin{cases} \frac{dC}{dt} = 3ae^3x + 6\beta p_1 e^3 \cos \theta \\ \frac{dE}{dt} = -\left(\frac{\alpha}{s'} + 2\beta\right)e^3 \end{cases}$$

which are the equations for the variations of the arbitrary constants. These equations show that  $E$  *always decreases* while  $C$  increases or decreases depending on  $x$  and  $\cos \theta$ . Now the signs of  $x$  and  $\cos \theta$  are both negative in the libration of Type 1 and both positive in the libration of Type 3. Hence  $C$  *always decreases in the libration of Type 1 and increases in that of Type 3*.

30. There are three more cases in which the sign of  $\frac{dC}{dt}$  may be determined easily, namely; the libration of Type 2 when  $C$  is negative, the revolution of Type 1, and the revolution of Type 3 when  $E$  is negative. In the first of these cases the motion being a kind of libration about 0,  $\theta$  may increase from 0 to an angle  $\theta_0$  and decrease to  $-\theta_0$ . The limit  $\theta_0$  is not greater than  $\frac{\pi}{2}$  so long as  $C$  is negative and consequently the second term of  $\frac{dC}{dt}$  is always positive. Now, we have by the equation (17) of Chapt. II

$$n_0 \int e^3 x dt = \int \frac{e^4 x d\theta}{s' e x - p_1 \cos \theta}$$

and



$$n_0 \int_0^T e^3 x dt = \left( \int_0^{\theta_0} + \int_{\theta_0}^0 + \int_0^{-\theta_0} + \int_{-\theta_0}^0 \right) \frac{e^4 x d\theta}{s' e x - p_1 \cos \theta}$$

where  $T$  is the period of the libration. Writing  $x'$  and  $e'$  for  $x$  and  $e$  in the part in which  $\frac{d\theta}{dt}$  is negative, we have

$$n_0 \int_0^T e^3 x dt = 2 \int_0^{\theta_0} \left( \frac{e^4 x}{s' e x - p_1 \cos \theta} - \frac{e'^4 x'}{s' e' x' - p_1 \cos \theta} \right) d\theta \equiv 2 \int_0^{\theta_0} X d\theta$$

$X$  is evidently positive when both  $x$  and  $x'$  are positive. Also,

$$X = \frac{s'(e^3 - e'^3) e e' x x' - p_1 \cos \theta (e^4 x - e'^4 x')}{(s' e x - p_1 \cos \theta)(s' e' x' - p_1 \cos \theta)}$$

The denominator is negative,  $e^3 - e'^3$  is positive,  $\cos \theta$  and  $e^4 x - e'^4 x'$  are also positive and therefore  $X$  is positive when  $x'$  is negative. Hence the integral is always positive and accordingly  $\int_0^T \frac{dC}{dt} dt$  is positive. Thus  $C$  increases algebraically when negative.

31. The sign of the first term of  $\frac{dC}{dt}$  is exclusively negative in the revolution of Type 1 and positive in that of Type 3. For the second term we have

$$n_0 \int_0^T e^3 \cos \theta dt = \pm \int_0^{2\pi} \frac{e^3 \cos \theta}{s' e x - p_1 \cos \theta} d\theta$$

in which  $T$  is the period of the revolution and the double sign corresponds to the positive and negative values of  $\frac{d\theta}{dt}$ , i. e., to Type 3 and Type 1 respectively. We may write

$$\begin{aligned} \int_0^{2\pi} \frac{e^3 \cos \theta d\theta}{s' e x - p_1 \cos \theta} &= 2 \int_0^{\frac{\pi}{2}} \left( \frac{e^3}{s' e x - p_1 \cos \theta} - \frac{e'^3}{s' e' x' + p_1 \cos \theta} \right) \cos \theta d\theta \\ &\equiv 2 \int_0^{\frac{\pi}{2}} Y \cos \theta d\theta \end{aligned}$$

in which  $x$  and  $e$  are written  $x'$  and  $e'$  where  $\cos \theta$  is negative, and

$$Y = \frac{s'ee'(e^2x' - e'^2x) + p_1 \cos \theta (e^3 + e'^3)}{(s'ex - p_1 \cos \theta)(s'e'x' + p_1 \cos \theta)}$$

$$= \frac{s'ee'(x' - x)E + p_1 \cos \theta (e^3 + e'^3)}{(s'ex - p_1 \cos \theta)(s'e'x' + p_1 \cos \theta)}$$

The denominator is positive in both cases. The quantity  $x' - x$  is positive in Type 1 and negative in Type 3.  $E$  is positive in Type 1 and may be positive or negative in Type 3. Hence, for the *revolution of Type 1* taking the negative sign for the double sign, the integral becomes negative and consequently  $C$  decreases. For the *revolution of Type 3*, if  $E$  is negative or less than a certain positive value,  $Y$  is positive and, taking the positive sign for the double sign, the integral becomes positive and therefore  $C$  increases.

32. Since  $e$  cannot be zero permanently in the general case of the first order,  $E$  decreases without limit so that it will become negative after a certain epoch. Now, when  $E$  is negative, the type of the motion is limited to two kinds, namely; the revolution and the libration, of Type 3. Hence we see that *the revolutions and the librations, of Type 1 and Type 2, are not permanent forms of the motion; but will change ultimately either to the revolution or the libration, of Type 3*. Also, since  $C$  increases algebraically when negative, *it will become positive if it was initially negative, and it cannot become negative if it was initially positive*; but will decrease to some limiting value which is positive and then will increase without limit.

33. It becomes necessary before proceeding further to find the limiting values of  $x$  and  $e$  in the libration or the revolution of Type 2, in which  $\frac{d\theta}{dt} = 0$  for  $\theta = \pi$ , supposing the effect of the resistance is null. Let  $x_1$  and  $e_1$  be the values of  $x$  and  $e$  at the zero-point of  $\frac{d\theta}{dt}$ , then

$$x_1^2 = C - 6p_1e_1 \quad e_1^2 = E + \frac{2}{3s'}x_1$$

and

$$s'e_1x_1 + p_1 = 0$$

Evidently, the limiting values of  $x$  and  $e$  will be attained when  $\theta=0$  or  $\pi$ . Denoting these values by  $\bar{x}$  and  $\bar{e}$ , we have

$$\bar{x}^2 = C \pm 6p_1\bar{e} \quad \bar{e}^2 = E + \frac{2}{3s'}\bar{x}$$

in which the upper sign corresponds to  $\theta=0$  and the lower to  $\theta=\pi$ . Eliminating  $C$ ,  $E$  and  $\bar{e}$ , we have

$$(\bar{x}^2 - x_1^2)^2 - 12p_1e_1(\bar{x}^2 - x_1^2) - \frac{24}{s'}p_1^2(\bar{x} - x_1) = 0$$

which becomes by the use of the relation  $s'e_1x_1 + p_1 = 0$

$$(\bar{x} - x_1)^2 \{ (\bar{x} + x_1)^2 - 12p_1e_1 \} = 0$$

Hence,

$$\bar{x} = x_1 \quad \text{or} \quad -x_1(\pm)\sqrt{12p_1e_1}$$

that is,

$$\bar{x} = -\frac{p_1}{s'e_1} \quad \text{or} \quad \frac{p_1}{s'e_1}(\pm)\sqrt{12p_1e_1}$$

the double sign enclosed in the parentheses meaning that it is independent of the value of  $\theta$ . Similarly, we obtain

$$\bar{e} = \mp e_1 \quad \text{or} \quad \pm e_1(\pm)\sqrt{\frac{4p_1}{3s'^2e_1}}$$

The limiting values of  $x$  and  $e$  will thus be tabulated as follows:—

(6)

| $\theta$ | $x$                                    | $e$  |
|----------|--|--|
| $0, \pi$ | $+\frac{p_1}{s'e_1} - \sqrt{12p_1e_1}$ | $\pm\left(e_1 - \sqrt{\frac{4p_1}{3s'^2e_1}}\right)$ |
| $\pi$    | $-\frac{p_1}{s'e_1}$                   | $e_1$  |
| $0$      | $+\frac{p_1}{s'e_1} + \sqrt{12p_1e_1}$ | $+e_1 + \sqrt{\frac{4p_1}{3s'^2e_1}}$                |

The following conditions become necessary:—

$$\begin{aligned} 0 < e_1 - \sqrt{\frac{4p_1}{3s'^2e_1}} & \quad \text{for libration} \\ 0 < \sqrt{\frac{4p_1}{3s'^2e_1}} - e_1 < e_1 & \quad \text{for revolution} \end{aligned}$$

that is,

$$(7) \quad \begin{cases} 4p_1 < 3s'^2e_1^3 & \text{for libration} \\ p_1 < 3s'^2e_1^3 < 4p_1 & \text{for revolution} \end{cases}$$

It may be remarked that when  $3s'^2e_1^3 = p_1$ , then  $C = C_1$ , and when  $3s'^2e_1^3 = 4p_1$ , then  $C = C_2$ , according to the equations (21) and (24) of Chapter II.

34. Let  $\bar{x}$  and  $\bar{e}$  be the values of  $x$  and  $e$  corresponding to a definite value  $\bar{\theta}$  of  $\theta$  and put

$$H = \bar{e} \bar{x}$$

Supposing  $\bar{x}$  and  $\bar{e}$  are functions of  $C$  and  $E$ , and differentiating  $H$ ,

$$\frac{dH}{dt} = \left( \bar{e} \frac{\partial \bar{x}}{\partial C} + \bar{x} \frac{\partial \bar{e}}{\partial C} \right) \frac{dC}{dt} + \left( \bar{e} \frac{\partial \bar{x}}{\partial E} + \bar{x} \frac{\partial \bar{e}}{\partial E} \right) \frac{dE}{dt}$$

$$\text{Now,} \quad \bar{x}^2 = C + 6p_1 \bar{e} \cos \bar{\theta} \quad \bar{e}^2 = E + \frac{2}{3s'} \bar{x}$$

and therefore,

$$\begin{aligned} 2\bar{x} \frac{\partial \bar{x}}{\partial C} &= 1 + 6p_1 \cos \bar{\theta} \frac{\partial \bar{e}}{\partial C} & 2\bar{e} \frac{\partial \bar{e}}{\partial C} &= \frac{2}{3s'} \frac{\partial \bar{x}}{\partial C} \\ 2\bar{x} \frac{\partial \bar{x}}{\partial E} &= 6p_1 \cos \bar{\theta} \frac{\partial \bar{e}}{\partial E} & 2\bar{e} \frac{\partial \bar{e}}{\partial E} &= 1 + \frac{2}{3s'} \frac{\partial \bar{x}}{\partial E} \end{aligned}$$

whence

$$2(s'\bar{e}\bar{x} - p_1 \cos \bar{\theta}) \left( \bar{e} \frac{\partial \bar{x}}{\partial C} + \bar{x} \frac{\partial \bar{e}}{\partial C} \right) = s'\bar{e}^2 + \frac{\bar{x}}{3}$$

$$2(s'\bar{e}\bar{x} - p_1 \cos \bar{\theta}) \left( \bar{e} \frac{\partial \bar{x}}{\partial E} + \bar{x} \frac{\partial \bar{e}}{\partial E} \right) = s'(\bar{x}^2 + 3p_1 \bar{e} \cos \bar{\theta})$$

Substituting these expressions and the expressions of  $\frac{dC}{dt}$  and  $\frac{dE}{dt}$  of (5) in the equation of  $\frac{dH}{dt}$ ,

$$2(s'\bar{e}\bar{x} - p_1 \cos \bar{\theta}) \frac{dH}{dt} = (3s'\bar{e}^2 + \bar{x})(ae^3x + \varepsilon\beta p_1 e^2 \cos \theta) - (\bar{x}^2 + 3p_1 \bar{e} \cos \bar{\theta})(a + 2s'\beta)e^3$$

Or, reducing by the known relations we obtain finally,

$$(8) \quad 2 \frac{dH}{dt} = M\bar{e}e + 3(a + 2s'\beta)\bar{e}e(e^2 - \bar{e}^2) + \frac{N}{2}[Me(e^2 - \bar{e}^2) + \frac{3}{2}(2a + s'\beta)e(e^2 - \bar{e}^2)^2]$$

where

$$M = 3a\bar{e}^2 - 2\beta\bar{x} \quad N = \frac{s'(3s'\bar{e}^2 + \bar{x})}{s'\bar{e}\bar{x} - p_1 \cos \bar{\theta}}$$

35. Putting  $\bar{\theta} = \pi$  and changing  $x$ ,  $e$  and  $H$  to  $x_1$ ,  $e_1$  and  $H_1$  respectively, apply the above result to the case of the revolutions of Type 1 and Type 2 very near to the libration of Type 2. In this case, (a)  $s'e_1x_1 + p_1$  is very small, negative in Type 1 and positive in Type 2, (b)  $e^2 - e_1^2$ , is always negative in Type 1 and positive in Type 2, (c)  $M$  is positive by (a), (d) the quantity

$$3s'e_1^2 + x_1 = \frac{3s'^2e_1^3 - p_1}{s'e_1}$$

is positive by (7) and therefore (e)  $N$  is negative in Type 1 and positive in Type 2. Since  $N$  is very great, we may write

$$\frac{dH_1}{dt} = \frac{N}{4} \left[ Me(e^2 - e_1^2) + \frac{3}{2}(2a + s'\beta)e(e^2 - e_1^2)^2 \right]$$

In the revolution of Type 2,  $M$ ,  $N$  and  $e^2 - e_1^2$ , are all positive and therefore  $\frac{dH_1}{dt}$  is always positive. In the revolution of Type 1,  $N$  and  $e^2 - e_1^2$ , are negative. Writing

$$(9) \quad \frac{dH_1}{dt} = \frac{3N}{8}e(e^2 - e_1^2) \left[ (2a + s'\beta)e^2 - s'\beta e_1^2 - \frac{4}{3}\beta x_1 \right]$$

and denoting the smallest value of  $e$  by  $e_0$ ,  $\frac{dH_1}{dt}$  is always positive if

$$(2\alpha + s'\beta)e_0^2 > s'\beta e_1^2 + \frac{4}{3}\beta x_1$$

But

$$x_1 = -\frac{p_1}{s'e_1}$$

and

$$e_0 = e_1 - \sqrt{\frac{4p_1}{3s'^2e_1}}$$

by (6) approximately. Hence in the revolution of Type 1,  $\frac{dH_1}{dt}$  is always positive if the condition

$$e_1^3 > \frac{4p_1}{3s'^2} \left(1 + \frac{s'\beta}{\alpha}\right)^2 \quad \text{or} \quad \frac{e_1 - e_0}{e_0} < \frac{\alpha}{s'\beta}$$

be fulfilled.

36. Since  $s'e_1x_1 + p_1$  is zero when the path  $A$  touches the curve  $B_1$ , and negative or positive according as the revolution belongs to Type 1 or Type 2, the quantity  $H_1$  is equal to  $-\frac{p_1}{s'}$  when the path is in contact with  $B_1$  and is less or greater algebraically than  $-\frac{p_1}{s'}$  according as the revolution belongs to Type 1 or Type 2. Now it has been proved that  $H_1$  increases in the revolution of Type 2, and in that of Type 1 if  $e_1$  is greater than a certain positive value. Hence the path  $A$  recedes from the position of the contact in the revolution of Type 2, and approaches to that position in the revolution of Type 1 if  $e_1$  is greater than a certain positive value.

37. To see how the librations are changed by the resistance, let us find the variation of the minimum or the maximum value  $u$  of  $\cos\theta$ . Let  $\bar{x}$  and  $\bar{e}$  be the values of  $x$  and  $e$  corresponding to  $\cos\theta = u$ . We have then

$$s'\bar{e}\bar{x} - p_1u = 0$$

$$\text{and} \quad x^2 - \bar{x}^2 = 6p_1(\bar{e} \cos \theta - \bar{e}u) \quad e^2 - \bar{e}^2 = \frac{2}{3s'}(x - \bar{x})$$

Differentiating these equations and eliminating  $\frac{dx}{dt}$  and  $\frac{d\bar{e}}{dt}$ , we get

$$-p_1\bar{e}\frac{du}{dt} = \frac{x - \bar{x}}{3} \left[ \frac{dx}{dt} \right] + (s'\bar{e}x - p_1 \cos \theta) \left[ \frac{d\bar{e}}{dt} \right]$$

Or substituting the expressions of  $\left[ \frac{dx}{dt} \right]$  and  $\left[ \frac{d\bar{e}}{dt} \right]$  by (4) and eliminating  $x$  and  $\cos \theta$  by the known relations, we get

$$(10) \quad -4p_1\bar{e}\frac{du}{dt} = s'Me(e^2 - \bar{e}^2) + \frac{3}{2}s'(2\alpha + s'\beta)e(e^2 - \bar{e}^2)^2$$

where  $M = 3a\bar{e}^3 - 2\beta\bar{x}$   
as before.

38. The second term of the right hand member of (10) is positive for all values of  $e$ . The sign of the first term depends on the sign of  $M$  and  $e^2 - \bar{e}^2$ . Now, if we confine ourselves to the libration of Type 2 in which  $u$  is very nearly equal to  $-1$ ,  $M$  is positive since  $x$  is negative, and the integral

$$\int_0^T e(e^2 - \bar{e}^2) dt$$

may be proved to be positive supposing the quantities  $C$  and  $E$  are constants within a single revolution. To prove this I shall proceed as follows:—

We have, using the symbol of § 28,

$$\frac{1}{n_0} \left( \frac{de}{dt} \right) = -p_1 \sin \theta$$

and therefore

$$n_0 \int (e^2 - \bar{e}^2) dt = - \int \frac{e^2 - \bar{e}^2}{p_1 \sin \theta} de$$

Now, since  $\bar{\theta}$  is very nearly equal to  $\pi$ ,

$$x^2 - \bar{x}^2 = 6p_1 e \cos \theta + 6p_1 \bar{e} \quad e^2 - \bar{e}^2 = \frac{2}{3s'}(x - \bar{x})$$

and

$$s' \bar{e} \bar{x} = -p_1$$

Accordingly

$$-\frac{e^2 - \bar{e}^2}{p_1 \sin \theta} = \mp \frac{4e}{s' \sqrt{12p_1 \bar{e} - (x + \bar{x})^2}}$$

in which the negative sign will be taken if  $e^2 - \bar{e}^2$  and  $\sin \theta$  have the same sign, and the positive sign if otherwise. Hence

$$n_0 \int_0^T (e^2 - \bar{e}^2) dt = \frac{4}{s'} \left( \int_{\bar{e}}^{e_0} - \int_{e_0}^{\bar{e}} + \int_{\bar{e}}^{e'_0} - \int_{e'_0}^{\bar{e}} \right) \frac{e de}{\sqrt{12p_1 \bar{e} - (x + \bar{x})^2}}$$

where  $e_0$  and  $e'_0$  are the limiting values of  $e$  determined by the equations

$$e_0 = \bar{e} + \sqrt{\frac{4p_1}{3s'^2 \bar{e}}} \quad e'_0 = \bar{e} - \sqrt{\frac{4p_1}{3s'^2 \bar{e}}}$$

Putting

$$f(e) = 12p_1 \bar{e} - (x + \bar{x})^2$$

and

$$e = \bar{e} \pm \epsilon \quad \epsilon > 0$$

we have

$$\begin{aligned} n_0 \int_0^T (e^2 - \bar{e}^2) dt &= \frac{8}{s'} \int_0^{\sqrt{\frac{4p_1}{3s'^2 \bar{e}}}} \left( \frac{\bar{e} + \epsilon}{\sqrt{f(\bar{e} + \epsilon)}} - \frac{\bar{e} - \epsilon}{\sqrt{f(\bar{e} - \epsilon)}} \right) d\epsilon \\ &= \frac{8}{s'} \int_0^{\sqrt{\frac{4p_1}{3s'^2 \bar{e}}}} \frac{(\bar{e} + \epsilon)^2 f(\bar{e} - \epsilon) - (\bar{e} - \epsilon)^2 f(\bar{e} + \epsilon)}{\sqrt{f(\bar{e} + \epsilon) f(\bar{e} - \epsilon)} \{ (\bar{e} + \epsilon) \sqrt{f(\bar{e} - \epsilon)} + (\bar{e} - \epsilon) \sqrt{f(\bar{e} + \epsilon)} \}} d\epsilon \end{aligned}$$

Now

$$f(e) = \frac{6p_1}{\bar{e}} (e^2 + \bar{e}^2) - \frac{9s'^2}{4} (e^2 - \bar{e}^2) - \frac{4p_1^2}{s'^2 \bar{e}^2}$$



and

$$f(\bar{e} \pm \epsilon) = \frac{6p_1}{\bar{e}}(2\bar{e}^2 \pm 2\bar{e}\epsilon + \epsilon^2) - \frac{9s'^2}{4}\epsilon^2(4\bar{e}^2 \pm 4\bar{e}\epsilon + \epsilon^2) - \frac{4p_1^2}{s'^2\bar{e}^2}$$

whence

$$\begin{aligned} & (\bar{e} + \epsilon)^2 f(\bar{e} - \epsilon) - (\bar{e} - \epsilon)^2 f(\bar{e} + \epsilon) \\ &= \frac{s'^2\epsilon}{\bar{e}} \left( 6\bar{e}^3 - 3\bar{e}\epsilon^2 - \frac{4p_1}{s'^2} \right) \left( \frac{4p_1}{s'^2} - 3\bar{e}\epsilon^2 \right) \\ &= \frac{s'^2\epsilon}{\bar{e}} \left( 6\bar{e}^3 - \frac{8p_1}{s'^2} + \frac{4p_1}{s'^2} - 3\bar{e}\epsilon^2 \right) \left( \frac{4p_1}{s'^2} - 3\bar{e}\epsilon^2 \right) \end{aligned}$$

The quantity  $\frac{4p_1}{s'^2} - 3\bar{e}\epsilon^2$  is positive within the limits and  $3\bar{e}^3 - \frac{4p_1}{s'^2}$  is also positive by (7). Hence the integral

$$\int_0^T (e^2 - \bar{e}^2) dt$$

is positive. Now

$$\int_0^T e(e^2 - \bar{e}^2) dt = \bar{e} \int_0^T (e^2 - \bar{e}^2) dt + \int_0^T (e - \bar{e})(e^2 - \bar{e}^2) dt$$

Hence

$$\int_0^T e(e^2 - \bar{e}^2) dt > 0$$

and therefore

$$\int_0^T \frac{du}{dt} dt < 0$$

This result shows that the quantity  $u$ , when it is very nearly equal to  $-1$ , approaches to  $-1$ . Hence we may conclude that *the libration of Type 2 in which the maximum value of  $\theta$  is very nearly equal to  $\pi$  changes to the revolution of Type 1 or Type 2.*

Now it has been proved that the revolution of Type 1, when  $e_1$  is greater than a certain positive value, approaches the position of contact, and that the revolution of Type 2 recedes from that position. Hence it may be seen that *the revolution of Type 1 changes to the libration of Type 2, when  $e_1$  is greater than a certain positive value, and immediately later to the revolution of Type 2.*

39. The equation (8) becomes, when  $\bar{e}$  is very small and negligible,

$$\frac{dH}{dt} = -\frac{s'\bar{x}}{4p_1 \cos \bar{\theta}} \left[ -2\beta\bar{x} + \frac{3}{2}(2\alpha + s'\beta)e^2 \right] e^3$$

Accordingly, if  $\bar{x}$  be negative, the sign of  $\frac{dH}{dt}$  is determined by the sign of  $\cos \bar{\theta}$ . Applying this result to the limiting cases of the revolution and the libration, of Type 1, we see at once that the quantity  $H$  at the point very near to the axis of  $x$ , increases in the revolution and decreases in the libration. Now, the quantity  $H$  is very small and negative in both cases. Accordingly it approaches 0 in the revolution and recedes from 0 in the libration. Hence we conclude that the *revolution of Type 1 in its limiting case changes to the libration of Type 1*.

The same result may be obtained in the limiting cases of the revolution and the libration, of Type 2, when  $C$  is positive and less than  $C_2$ . Thus we see that the *libration of Type 2 changes to the revolution of Type 2, in this case*.

In the limiting cases of the revolution and the libration, of Type 3, the sign of the quantity in the brackets is not definite. It becomes positive when  $\beta$  is very small compared with  $\alpha$ . In this case  $H$  increases in the revolution and decreases in the libration, and, since  $H$  is very small and positive, it approaches 0 in the libration and recedes in the revolution. Hence, *if  $\beta$  be very small compared with  $\alpha$ , the libration of Type 3 changes to the revolution of Type 3*.

40. Let us next consider the extreme case of the libration of Type 1 in which  $\frac{d\theta}{dt}$  is very small for  $\theta=\pi$ . We have by (9)

$$\frac{dH_1}{dt} = \frac{3}{8}Ne(e^2 - e_1^2) \left[ (2\alpha + s'\beta)e^2 - s'\beta e_1^2 - \frac{4}{3}\beta x_1 \right]$$

Now  $s'e_1x_1 + p_1$  is negative,  $3s'e_1^2 + x_1$  is positive by (7), and therefore  $N$  is negative. The quantity  $e_2 - e_1^2$  is negative for all values of  $e$ . The lower limit of  $e$  is  $\sqrt{\frac{4p_1}{3s'^2e_1}} - e_1$  by (6), and

$$\begin{aligned}
 (2\alpha + s'\beta) \left( \sqrt{\frac{4p_1}{3s'^2e_1}} - e_1 \right)^2 - s'\beta e_1^2 - \frac{4}{3}\beta x_1 \\
 = 2\alpha \left( \sqrt{\frac{4p_1}{3s'^2e_1}} - e_1 \right)^2 + 2s'\beta \sqrt{\frac{4p_1}{3s'^2e_1}} \left( \sqrt{\frac{4p_1}{3s'^2e_1}} - e_1 \right) > 0
 \end{aligned}$$

Hence  $\frac{dH_1}{dt} > 0$ , and therefore the *libration of Type 1 changes to the revolution of Type 2*.

41. When the motion changes from the revolution or the libration, of Type 1 to the revolution of Type 2, the limits of  $x$  and  $e$  increase discontinuously. Referring back to the table (6), it may be seen that the lower limits change from the second row to the third row, and the upper limits from the third row to the fourth row. Hence the lower limits of  $x$  and  $e$  increase by the amounts

$$\sqrt{12p_1e_1} - \frac{2p_1}{s'e_1} \quad \text{and} \quad e_1 \mp \left( e_1 - \sqrt{\frac{4p_1}{3s'^2e_1}} \right)$$

respectively, and the upper limits by

$$\sqrt{12p_1e_1} + \frac{2p_1}{s'e_1} \quad \text{and} \quad \sqrt{\frac{4p_1}{3s'^2e_1}}$$

respectively.

The upper limits become minimum when  $3s'^2e_1^3 = p_1$ , and the minimum values of  $x$  and  $e$  are

$$3 \left( \frac{3p_1^2}{s'} \right)^{\frac{1}{3}} \quad \text{and} \quad 3 \left( \frac{p_1}{3s'^2} \right)^{\frac{1}{3}} \quad \text{respectively.}$$

The corresponding lower limits are

$$- \left( \frac{3p_1^2}{s'} \right)^{\frac{1}{3}} \quad \text{and} \quad \left( \frac{p_1}{3s'^2} \right)^{\frac{1}{3}}$$

Numerical values of these limits for different cases of the first order are as follows:—

| $s/s'$ | $p_1 \times 10^3$ | $-\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \times 10^2$ | $3\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \times 10^2$ | $\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}} \times 10$ | $3\left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}} \times 10$ |
|--------|-------------------|---|---|--|---|
| 2/1    | 0.716             | -1.15   | 3.46  | 0.62   | 1.86  |
| 3/2    | 1.474             | -1.48   | 4.45  | 0.50   | 1.49  |
| 4/3    | 2.237             | -1.71   | 5.13  | 0.45   | 1.34  |

The variations of the constants,  $C$  and  $E$ , become rapid by this increase of the eccentricity.

42. *The revolution of Type 2 will change to the revolution of Type 3 sooner or later, since  $E$  decreases and  $C$  cannot become negative. It has been proved that  $C$  increases when  $E$  is less than a certain positive value. Hence  $C$  begins to increase without limit just as  $E$  decreases.*

43. The equation (10) may be integrated by expanding in series when  $\bar{\theta}$  is small. Let us first take the integral

$$\int_0^T e(e^2 - \bar{e}^2) dt$$

where  $T$  is the period of the libration as before. Putting

$$\xi = x - \bar{x} \quad \eta = e - \bar{e}$$

we have 
$$e^2 - \bar{e}^2 = 2\bar{e}\eta + \eta^2 = \frac{2}{3s'}\xi$$

and 
$$x^2 - \bar{x}^2 = 2\bar{x}\xi + \xi^2 = 6p_1(e \cos \theta - \bar{e} \cos \bar{\theta})$$

Eliminating  $\xi$  and making use of the relation

$$s'\bar{e}\bar{x} = p_1 \cos \bar{\theta}$$

we get

$$s'\bar{x}\eta^2 + \frac{3s'^2}{4}(2\bar{e}\eta + \eta^2)^2 = 4pe\left(\sin^2 \frac{\bar{\theta}}{2} - \sin^2 \frac{\theta}{2}\right)$$

whence, neglecting  $\eta^3$ ,  $\eta\theta^2$ ,  $\eta\bar{\theta}^2$  and higher powers, we obtain

$$\eta^2 = \frac{p_1 \bar{e}(\bar{\theta}^2 - \theta^2)}{3s'^2 \bar{e}^2 + s' \bar{x}}$$

Now

$$n_0 \int e(\bar{e}^2 - \bar{e}^2) dt = \int \frac{e^2(e^2 - \bar{e}^2)}{s'ex - p_1 \cos \theta} d\theta$$

and

$$s'ex - p_1 \cos \theta = s'(3s'\bar{e}^2 + \bar{x})\eta + \frac{s'}{2\bar{e}}(6s'\bar{e}^2 - \bar{x})\eta^2 + \dots$$

Also

$$e^2(e^2 - \bar{e}^2) = 2\bar{e}^3\eta + 5\bar{e}^2\eta^2 + \dots$$

Hence

$$\begin{aligned} \frac{e^2(e^2 - \bar{e}^2)}{s'ex - p_1 \cos \theta} &= \frac{2\bar{e}^3 + 5\bar{e}^2\eta + \dots}{s'(3s'\bar{e}^2 + \bar{x}) + \frac{s'}{2\bar{e}}(6s'\bar{e}^2 - \bar{x})\eta + \dots} \\ &= \frac{2\bar{e}^3}{s'(3s'\bar{e}^2 + \bar{x})} \left( 1 + \frac{3}{2} \frac{3s'\bar{e}^2 + 2\bar{x}}{3s'\bar{e}^2 + \bar{x}} \frac{\eta}{\bar{e}} \right) \end{aligned}$$

neglecting  $\eta^2$ . Accordingly we can write

$$n_0 \int e(e^2 - \bar{e}^2) dt = P \int d\theta \pm Q \int \sqrt{\bar{\theta}^2 - \theta^2} d\theta$$

Since  $\bar{x}$  is positive in this case the coefficients  $P$  and  $Q$  are finite and positive. The double sign will be taken the same as that of  $\eta$ . Now

$$\int_0^T \frac{d\theta}{dt} dt = 0$$

and the sign of  $\eta$  is the same as that of  $\frac{d\theta}{dt}$  throughout. Hence

$$n_0 \int_0^T e(e^2 - \bar{e}^2) dt = 2Q \int_{-\bar{\theta}}^{\bar{\theta}} \sqrt{\bar{\theta}^2 - \theta^2} d\theta = Q\pi\bar{\theta}^2$$

Again

$$n_0 \int e(e^2 - \bar{e}^2)^2 dt = \int \frac{e^2(e^2 - \bar{e}^2) d\theta}{s' ex - p_1 \cos \theta} = \frac{4\bar{e}^4}{s'(3s'\bar{e}^2 + \bar{x})} \int \eta d\theta$$

neglecting  $\eta^2$ . Hence

$$n_0 \int e(e^2 - \bar{e}^2)^2 dt = \pm Q' \int \sqrt{\bar{\theta}^2 - \theta^2} d\theta$$

in which  $Q'$  is a quantity finite and positive, and the double sign will be taken the same as that of  $\eta$ . Consequently

$$n_0 \int_0^T e(e^2 - \bar{e}^2)^2 dt = 2Q' \int_{-\bar{\theta}}^{\bar{\theta}} \sqrt{\bar{\theta}^2 - \theta^2} d\theta = Q' \pi \bar{\theta}^2$$

and

$$n_0 \int_0^T \frac{du}{dt} dt = -S\pi \bar{\theta}^2$$

where  $S$  is a finite quantity, positive when  $M$  is positive. Also

$$n_0 \int dt = \int \frac{e d\theta}{s' ex - p_1 \cos \theta} = \frac{\bar{e}}{s'(3s'\bar{e}^2 + \bar{x})} \int \frac{d\theta}{\eta} = \pm P' \int \frac{d\theta}{\sqrt{\bar{\theta}^2 - \theta^2}}$$

where  $P'$  is a quantity finite and positive, and

$$n_0 T = n_0 \int_0^T dt = 2P' \int_{-\bar{\theta}}^{\bar{\theta}} \frac{d\theta}{\sqrt{\bar{\theta}^2 - \theta^2}} = 2\pi P'$$

Hence

$$\frac{1}{T} \int_0^T \frac{du}{dt} dt = -\frac{S}{2P'} \bar{\theta}^2 = -S' \bar{\theta}^2$$

and

$$\frac{1}{T} \int_0^T \frac{d\bar{\theta}}{dt} dt = S' \bar{\theta}$$

Consequently the variation of  $\bar{\theta}$  is slow when  $\bar{\theta}$  is small. The same proposition may be proved for the libration of Type 1 when  $\bar{\theta}$  is very nearly equal to  $\pi$ . Hence in general *the amplitude of the libration varies very slowly when the amplitude is very small.*

44. Now when the amplitude of the libration is very small  $x$  and  $e$  are connected approximately by the relation  $s'ex = p_1$  in the libration of Type 3. Hence the point  $(x, e)$  moves in a rectangular hyperbola, and therefore when  $e$  is great the point moves down nearly parallel to the axis of  $e$  until  $e$  becomes small and consequently the variation becomes very slow.

In the case of the libration of Type 1 the point  $(x, e)$  will move in another rectangular hyperbola represented by the equation  $s'ex = p_1$ . It will move nearly parallel to the axis of  $x$  with a small eccentricity until reaches at the point

$$x = -\left(\frac{3p_1^2}{s'}\right)^{\frac{1}{3}} \quad e = \left(\frac{p_1}{3s'^2}\right)^{\frac{1}{3}}$$

and the motion abruptly changes to the revolution of Type 2.

45. The conclusions for the general case of the first order may be stated as follows:—

1. The revolution of Type 1 will change to the revolution of Type 2 either directly or indirectly and finally to that of Type 3. The limits of the mean motion and eccentricity increase discontinuously when the motion changes to the revolution of Type 2, whence the variations of the constants become rapid, and the asteroids of this class will not stay long near the critical point.

2. The libration of Type 2, when  $C$  is not negative and relatively great, changes to the revolution of Type 2 either directly or indirectly and finally to that of Type 3.

3. The libration of Type 2, when  $C$  is negative and relatively great, changes to the libration of Type 3 which may change to the revolution of Type 3 if the constant  $\beta$  be

sufficiently small compared with  $a$ . The amplitude of the libration varies very slowly when the amplitude is small, and the asteroids of this class will stay long near the critical point with small eccentricities.

46. *General Case of the Second Order.* Putting  $\theta = \pi + \theta'$  in the equations (27) of Chapter II, we have

$$(11) \quad \begin{cases} x^2 = C + 6p_2 e^2 \cos \theta' & e^2 = E + \frac{4}{3s'} x \\ \frac{1}{n_0} \frac{d\theta'}{dt} = s' x - 4p_2 \cos \theta' \end{cases}$$

The equations for the variations of the constants become

$$\frac{dC}{dt} = 3ae^3 x + 12\beta p_2 e^3 \cos \theta' \quad \frac{dE}{dt} = -2\left(\frac{a}{s'} + \beta\right)e^3$$

The quantity  $E$  always decreases as in the case of the first order.  $C$  increases in the libration of Type 3, since  $x$  and  $\cos \theta'$  are always positive. We can prove also that  $C$  increases if negative, that it decreases in the revolution of Type 1 and that it increases in the revolution of Type 3 if  $E$  be less than a certain positive value.

47. The limits of the mean motion and eccentricity, when the path  $A$  is in contact with the limiting curve  $B_0$ , may be obtained as follows:—

| $\theta'$ | $x$                                  | $e^2$  |
|-----------|--------------------------------------|--|
| $\pi$     | $-\frac{4p_2}{s'}$                   | $E - \frac{16}{3s'^2} p_2$                                   |
| 0         | $\frac{4p_2}{s'} \pm \sqrt{12p_2 E}$ | $E + \frac{16}{3s'^2} p_2 \pm \sqrt{\frac{64}{3s'^2} p_2 E}$ |

For the limiting case in which  $e_0 = e_1 = 0$  we have



| $\theta'$ | $x$                | $e^2$                 |
|-----------|--------------------|-----------------------|
| $\pi, 0$  | $-\frac{4p_2}{s'}$ | 0                     |
| 0         | $\frac{12p_2}{s'}$ | $\frac{64p_2}{3s'^2}$ |

Numerical values of these limits for different cases of the second order become as follows:—

| $s/s'$ | $p_2 \times 10^2$ | $-\frac{4p_2}{s'} \times 10^2$ | $\frac{12}{s'} p_2 \times 10^2$ | $\left(\frac{64}{3s'^2} p_2\right)^{\frac{1}{2}} \times 10$ |
|--------|-------------------|--------------------------------|---------------------------------|---|
| 3/1    | 0.0275            | -0.110                         | 0.330                           | 0.776   |
| 5/3    | 0.222             | -0.296                         | 0.888                           | 0.725   |

48. The equation corresponding to (8) becomes

$$\frac{d\bar{x}}{dt} = \frac{3}{2} \left\{ a\bar{e}^2 + (a + s'\beta)(e^2 - \bar{e}^2) \right\} e$$

$$+ \frac{9s'^2}{8(s'\bar{x} - 4p_2 \cos \bar{\theta}')} \left\{ a\bar{e}^2 + \left( a + \frac{s'\beta}{2} \right) (e^2 - \bar{e}^2) \right\} e(e^2 - \bar{e}^2)$$

Putting  $\bar{\theta}' = \pi$  and applying this equation to the case of the revolutions very near to the contact, it may be seen that  $\bar{x}$  increases in the revolution of Type 2 and that it also increases in the revolution of Type 1 if

$$\bar{e}^2 > \frac{s'\beta}{a} \left( 1 + \frac{1}{2} \frac{s'\beta}{a} \right) \frac{32}{3s'^2} p_2$$

Since the libration of Type 1 is impossible in the case of the second order, the revolution of Type 1, if it does not change to the revolution of Type 2, will remain unchanged.

49. The equation corresponding to (10) becomes in this case

$$\frac{8}{3} p_2 \bar{e}^3 \frac{du}{dt} = -s' a \bar{e}^2 e(e^2 - \bar{e}^2) - s' \left( a + \frac{s'\beta}{2} \right) e(e^2 - \bar{e}^2)^2$$

where  $u = \cos \bar{\theta}'$  is the minimum value of  $\cos \theta'$  and  $\bar{e}$  is the corresponding value of  $e$ . Now

$$\begin{aligned} n_0 \int_0^T (e^2 - \bar{e}^2) dt &= \frac{4n_0}{3s'} \int_0^T (x - \bar{x}) dt \\ &= \frac{8}{3n'} \int_0^{\bar{\theta}'} \left( \frac{x - \bar{x}}{s'x - 4p_2 \cos \theta'} - \frac{x' - \bar{x}}{s'x' - 4p_2 \cos \theta'} \right) d\theta \\ &\equiv \frac{8}{3s'} \int_0^{\bar{\theta}'} Z d\theta \end{aligned}$$

$x'$  corresponding to the portion in which  $\frac{d\theta}{dt}$  is negative, and

$$Z = \frac{(s'\bar{x} - 4p_2 \cos \theta')(x - x')}{(s'x - 4p_2 \cos \theta')(s'x' - 4p_2 \cos \theta')} = \frac{-4p_2(\cos \theta' - \cos \bar{\theta}')(x - x')}{(s'x - 4p_2 \cos \theta')(s'x' - 4p_2 \cos \theta')}$$

The denominator is negative. The differences  $x - x'$  and  $\cos \theta' - \cos \bar{\theta}'$  are positive and therefore  $Z$  is always positive. Hence

$$\int_0^T (e^2 - \bar{e}^2) dt > 0$$

Now

$$\int_0^T e(e^2 - \bar{e}^2) dt = \int_0^T (e - \bar{e})(e^2 - \bar{e}^2) dt + \bar{e} \int_0^T (e^2 - \bar{e}^2) dt$$

and therefore

$$\int_0^T e(e^2 - \bar{e}^2) dt > 0$$

Hence

$$\int_0^T \frac{du}{dt} dt < 0$$

always. Accordingly *the librations ultimately change to revolutions*. The libration of Type 2 changes to the revolution of Type 2 or to that of Type 1, and the libration of Type 3 to the revolution of Type 3.

50. We can prove also, as in the case of the first order, that

$$\frac{1}{T} \int_0^T \frac{d\bar{\theta}'}{dt} dt = S' \bar{\theta}'$$

where  $S'$  is a quantity finite and *positive*. *The amplitude of the libration therefore increases very slowly when the amplitude is very small.*

51. The conclusions for the general case of the second order are as follows:—

1. The revolution of Type 1 may change to the revolution of Type 2 and finally to that of Type 3. The limits of the mean motion and eccentricity increase discontinuously when the motion changes to the revolution of Type 2, whence the variations of the constants become rapid and the asteroids of this class will not stay long near the critical point.

2. The libration of Type 2, when  $C$  is not negative and relatively great, changes either to the revolution of Type 1 or to the revolution of Type 2 which changes finally to that of Type 3.

3. The libration of Type 2, when  $C$  is negative and relatively great, changes to the libration of Type 3 and finally to the revolution of Type 3. The amplitude of the libration increases slowly when the amplitude is small, and the asteroids of this class will stay long near the critical point with small eccentricities.

52. *General Case of the Third or Higher Orders.* Restoring  $e$  and putting

$$s - s' = i \quad \text{and} \quad \theta = (i-1)\pi + \theta'$$

in the equations (28) of Chapter II, we get

$$(12) \quad \begin{cases} x^2 = C + 6p_i e \cos \theta' & e^2 = E + \frac{2i}{3s'} x \\ \frac{1}{n_0} \frac{d\theta'}{dt} = s'x - i^2 p_i e^{-2} \cos \theta' \end{cases}$$

The equations for the variations of the constants become

$$\frac{dC}{dt} = 3ae^3x + 6\beta ip_i e^{i+1} \cos \theta' \quad \frac{dE}{dt} = -\left(\frac{ia}{s'} + 2\beta\right)e^3$$

*E* always decreases, as in the previous cases. We can prove also that *C* increases if negative, that it decreases in the revolution of Type 1 and that it increases in the revolution of Type 3 if *E* be less than a certain positive value.

53. We have to consider the orders of the small quantities in this case. The eccentricity *e* may be supposed to be a quantity of the order of  $10^{-}$ , and the constant  $p_i$ , being a quantity multiplied by  $m' = 1/1047$ , may therefore be supposed to be the third order of *e*. Hence, assuming that the orders of the three terms in the first equation of (12) are the same, *x* becomes a quantity of the order  $\frac{i+3}{2}$ . Consequently if  $i > 3$ , the order of *x* will be higher than that of  $e^2$  by a unit order at least, so that *e* may be supposed to be a constant in the first approximation. In the second member of the third equation of (12), the order of  $p_i e^{i-2}$ , being  $i+1$ , is higher than that of *x* by a unit order at least, if  $i > 3$ . Hence we may write

$$\frac{1}{n_0} \frac{d\theta'}{dt} = s'x - g \cos \theta'$$

where *g* is a constant.

54. The equation corresponding to (8) becomes in this case

$$\begin{aligned} \frac{d}{dt}(\bar{x}\bar{e}^{-i+2}) &= \left[ \frac{3a}{2}\bar{e}^2 + \frac{3a}{2}(e^2 - \bar{e}^2) + \beta(i-2)\bar{x} + \frac{3s'\beta}{i}(e^2 - \bar{e}^2) \right] \bar{e}^{-i+2}e \\ &+ \frac{N}{2} \left[ \frac{3a}{2}\bar{e}^2 + \frac{3a}{2}(e^2 - \bar{e}^2) + \beta(i-2)\bar{x} + \frac{3s'\beta}{4}(e^2 - \bar{e}^2) \right] e(e^2 - \bar{e}^2) \end{aligned}$$

in which

$$N = \frac{s'}{i} \frac{3s'\bar{e}^2 - i(i-2)\bar{x}}{s'\bar{x} - i^2 p_i \bar{e}^{i-2} \cos \theta'} \bar{e}^{-i}$$

Or neglecting the quantities of higher orders we get

$$\frac{d}{dt}(\bar{x}\bar{e}^{-i+2}) = \frac{3a}{2}\bar{e}^{-i+2}e + \frac{3a}{4}N\bar{e}^2e(e^2 - \bar{e}^2)$$

where

$$N = \frac{s'}{i} \frac{3s'\bar{e}^{-i+2}}{s'\bar{x} - i^2p_i\bar{e}^{i-2} \cos \theta'}$$

If we put  $\bar{\theta}' = \pi$  in this equation,  $N(e^2 - \bar{e}^2)$  is always positive, and hence

$$\frac{d}{dt}(\bar{x}\bar{e}^{-i+2}) > 0$$

Accordingly the revolution of Type 1 in its limiting case approaches the position of contact, and the revolution of Type 2 recedes from that position.

55. The equation corresponding to (10) takes the form

$$\begin{aligned} 2ip_i\bar{e}^i \frac{du}{dt} &= -s' \left[ \frac{3a}{2}\bar{e}^2 + \frac{3a}{2}(e^2 - \bar{e}^2) + \beta(i-2)\bar{x} + \frac{3s'\beta}{4}(e^2 - \bar{e}^2) \right] e(e^2 - \bar{e}^2) \\ &= -\frac{3s'}{2}a\bar{e}^2e(e^2 - \bar{e}^2) \end{aligned}$$

neglecting the quantities of higher orders. Now

$$n_0 \int e(e^2 - \bar{e}^2) dt = \int \frac{e(e^2 - \bar{e}^2)}{s'x - g \cos \theta'} d\theta'$$

Accordingly we can prove that

$$\int_0^T e(e^2 - \bar{e}^2) dt > 0$$

as in the case of the second order. Hence

$$\int_0^T \frac{du}{dt} dt < 0$$

and therefore, *the librations ultimately change to revolutions.*

We can prove also that *the amplitude of the libration increases very slowly when it is very small*, as in the previous case.

Thus we obtain the same conclusions as for the general case of the second order.

### Chapter IV.

#### Peculiarities in the Distribution of the Mean Motions of the Asteroids and their Possible Explanations.

56. We shall first examine the nature of the *gaps*. The ratios of the mean motions up to the seventh order and lying within the denser portion of the asteroids are as follows:—

| Order | $n_0/n'$ | Order | $n_0/n'$ | Order | $n_0/n'$ |
|-------|----------|-------|----------|-------|----------|
| 1     | 2/1      | 5     | 9/4      | 7     | 13/6     |
| 2     | 3/1      |       | 8/3      |       | 12/5     |
| 3     | 5/2      |       | 7/2      |       | 11/4     |
| 4     | 7/3      | 6     | 11/5     |       | 10/3     |

The simplest way of determining the width of the gaps is to find the difference of the two mean motions nearest to  $n_0$  in both directions. But this will be very rough, especially when one or both of the mean motions is not reliable, as in the case of (132) of the class 3/1. The following method will answer for this defect. Denoting by

$$n_{-15}, n_{-14}, \dots, n_{-2}, n_{-1}, n_1, n_2, \dots, n_{14}, n_{15}$$

the mean motions arranged in ascending order and interposing  $n_0$  between  $n_{-1}$  and  $n_1$ , the quantities

$$Q_1 = \frac{1}{6}(n_1 + n_2 + \dots + n_9 - n_{13} - n_{14} - n_{15}) - n_0$$

and

$$Q_{-1} = n_0 - \frac{1}{6}(n_{-1} + n_{-2} + \dots + n_{-9} - n_{-13} - n_{-14} - n_{-15})$$

may represent the width in the positive and negative directions of  $x$  respectively. The sum of these two quantities becomes zero when the distribution is uniform, and becomes negative when there is some condensation near  $n_0$ . The number of the mean motions in each direction may be taken more or less. But if this number be too small the result will be inaccurate, while if too many it may be affected by other irregularities of the distribution.

57. The width and position of the gaps thus determined, according to the *Berliner Jahrbuch* for 1917, are as follows:—

| $n_3/n'$ | Order | $n_0$   | $Q_{-1}$ | $Q_1$ | Width | Displacement<br>of Center |
|----------|-------|---------|----------|-------|-------|---------------------------|
| 2/1      | 1     | 598.26  | 18.76    | 9.58  | 23.34 | -4.59                     |
| 13/6     | 7     | 648.12  | 0.67     | 0.06  | 0.73  | -0.30                     |
| 11/5     | 6     | 653.09  | 3.43     | 0.81  | 4.24  | -1.31                     |
| 9/4      | 5     | 673.04  | 1.56     | 0.96  | 2.52  | -0.30                     |
| 7/3      | 4     | 697.97  | 2.66     | 5.73  | 8.39  | +1.54                     |
| 12/5     | 7     | 717.91  | 0.52     | 1.93  | 2.45  | +0.70                     |
| 5/2      | 3     | 747.82  | 6.17     | 4.33  | 10.50 | -0.92                     |
| 8/3      | 5     | 797.68  | 2.76     | 1.26  | 4.02  | -0.75                     |
| 11/4     | 7     | 822.61  | 0.17     | -0.99 | -0.82 | -0.58                     |
| 3/1      | 2     | 897.39  | 12.12    | 10.92 | 23.04 | -0.60                     |
| 10/3     | 7     | 997.10  | 0.80     | -3.22 | -2.42 | -2.01                     |
| 7/2      | 5     | 1046.96 | 1.69     | 5.24  | 6.93  | +1.73                     |

*The existence of the gaps up to the fifth order seems established beyond doubt.* It may also be observed that the width becomes narrower as the order advances and also as the denominator or the numerator of the ratio increases.

58. In the portion with smaller mean motions than 500'', though the number of the asteroids is very small we see at once that the character is *reversed* (PLATE). No asteroid can be found

except near the commensurable points in which the gaps are found in the other portion. The positions of the condensations and the number of the asteroids are as follows:—

|     | Order | $n_0$  | No. of Asteroids |
|-----|-------|--------|------------------|
| 1/1 | 0     | 299.13 | 4                |
| 4/3 | 1     | 398.84 | 1                |
| 3/2 | 1     | 448.70 | 6                |

If the distribution of the mean motions can be compared with the spectrum, the portion with greater mean motions than  $600''$  will correspond to an absorption spectrum, and that with smaller mean motions than  $500''$  to an emission spectrum. The portion intermediate between these portions may possibly belong to the former class, although the character is not distinct owing to the scarcity of the asteroids.

59. That the eccentricity of the asteroids near the gaps is smaller than its mean value was remarked by Prof. Brown.<sup>1)</sup> To verify this I have taken ten asteroids on each side of the gap and computed the mean angle of eccentricity as follows:—

| $n_0/n'$ | Order | $\varphi$ |       |      |       |
|----------|-------|-----------|-------|------|-------|
|          |       | Outer     | Inner | Mean | Diff. |
| 2/1      | 1     | 7.49      | 6.09  | 6.79 | +1.40 |
| 11/5     | 6     | 9.08      | 8.57  | 8.82 | +0.51 |
| 9/4      | 5     | 7.16      | 6.24  | 6.70 | +0.92 |
| 7/3      | 4     | 8.94      | 6.10  | 7.52 | +2.84 |
| 5/2      | 3     | 8.41      | 6.54  | 7.48 | +1.87 |
| 8/3      | 5     | 8.15      | 7.57  | 7.86 | +0.58 |
| 3/1      | 2     | 7.89      | 7.87  | 7.88 | +0.02 |
| 7/2      | 5     | 6.65      | 9.47  | 8.06 | -2.82 |
| Mean     |       | 7.97      | 7.31  | 7.64 | +0.66 |

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1) Science, Jan. 20, 1911.



The mean value of the angle of eccentricity of all the asteroids, 791 in number, is  $8^{\circ}50'$ . Accordingly the mean value in the outer portion is less than the total mean by  $0^{\circ}53'$  and that in the inner portion, by  $1^{\circ}19'$ . Thus it seems quite certain that *the eccentricity in the portion near the gaps, especially on the inner side, is smaller than that in the other portions.*

60. In order to determine the types of the motion near the commensurable points I have computed the quantities  $C$ ,  $E$ ,  $F_0$ ,  $F_1$  and  $E_3$  for each asteroid by the formulæ—

$$x = \frac{n}{n_0} - 1 \quad \theta = s'(l - l') + (s - s')(\varpi - l')$$

$$C = x^2 + (-1)^{s-s'} 6p_{s-s'}^{(s)} e^{s-s'} \cos \theta \quad E = e^2 - \frac{2}{3} \frac{s-s'}{s'} x$$

$$F_0 = \left( \frac{C}{6p_{s-s'}^{(s)}} \right)^{\frac{2}{s-s'}} \quad F_1 = \frac{2}{3} \frac{s-s'}{s'} \sqrt{C}$$

$$E_3 = F_0 + \frac{(s-s')^3}{3s'^2} p_{s-s'}^{(s)} \left( \frac{C}{6p_{s-s'}^{(s)}} \right)^{\frac{s-s'-2}{s-s'}} \quad (\text{approximately})$$

which may be easily deduced from the equations in Chapter II. The constants  $p_{s-s'}^{(s)}$  were computed by Leverrier's formulæ, and the elements of the asteroids,  $n$ ,  $e$ ,  $\varpi$  and  $l$ , according to the data in the *Berliner Jahrbuch* for 1917.

61. For the class  $2/1$  taking 41 asteroids with  $n - n_0$  within the limits  $-35''$  and  $+25''$ , the results are as follows:—

$$2/1 \quad n_0 = 598^{\prime\prime} 26 \quad p_1^{(2)} \times 10^3 = 0.716$$

| No.      | Epoch    | $\varpi$ | $l$   | $l'$  | $n$    | $x \times 10^2$ | $e \times 10$ | $\theta$ | $C \times 10^4$ | $E \times 10^2$ | $F_0 \times 10^2$ | $F_1 \times 10^2$ | Type           |
|----------|----------|----------|-------|-------|--------|-----------------|---------------|----------|-----------------|-----------------|-------------------|-------------------|----------------|
| 790 1914 | VII 10.5 | 235.6    | 274.0 | 319.0 | 564.31 | -5.68           | 1.48          | 281.6    | +30.99          | +5.97           | 52.03             | 3.71              | R <sub>1</sub> |
| .76 1911 | VII 6.0  | 87.5     | 309.6 | 227.5 | 564.54 | -5.64           | 1.73          | 302.1    | +27.86          | +6.76           | 42.06             | 3.52              | R <sub>1</sub> |

| No. | Epoch      | $\omega$          | $l$                | $l'$               | $n$                | $x \times 10^2$      | $e \times 10$ | $\theta$ | $C \times 10^4$    | $E \times 10^2$ | $F_0 \times 10^2$ | $F_1 \times 10^2$ | Type              |                |
|-----|------------|-------------------|--------------------|--------------------|--------------------|----------------------|---------------|----------|--------------------|-----------------|-------------------|-------------------|-------------------|----------------|
| 713 | 1911. IV   | 23 <sup>h</sup> 5 | 349 <sup>°</sup> 4 | 209 <sup>°</sup> 6 | 221 <sup>°</sup> 9 | 565 <sup>''</sup> 33 | -5.50         | 1.59     | 115 <sup>°</sup> 2 | +33.17          | + 6.20            | 59.61             | 3 <sup>h</sup> 84 | R <sub>1</sub> |
| 733 | 1912. IX   | 19.5              | 152.6              | 8.5                | 264.3              | 566.13               | -5.37         | 0.59     | 352.5              | +26.33          | + 3.93            | 37.56             | 3.42              | R <sub>1</sub> |
| 225 | 1903. XI   | 5.0               | 298.5              | 27.2               | 354.8              | 567.59               | -5.13         | 2.64     | 336.1              | +15.95          | +10.40            | 13.79             | 2.67              | R <sub>1</sub> |
| 528 | 1913. IX   | 13.0              | 52.2               | 9.6                | 294.0              | 567.84               | -5.09         | 0.21     | 193.8              | +26.78          | + 3.44            | 38.86             | 3.45              | L <sub>1</sub> |
| 566 | 1905. VI   | 1.5               | 17.0               | 260.3              | 42.6               | 570.18               | -4.69         | 1.36     | 192.1              | +27.70          | + 4.99            | 41.58             | 3.51              | R <sub>1</sub> |
| 692 | 1910. V    | 30.5              | 111.8              | 194.2              | 194.1              | 570.82               | -4.59         | 1.65     | 277.8              | +20.10          | + 5.78            | 21.89             | 2.99              | R <sub>1</sub> |
| 168 | 1899. V    | 29.0              | 23.8               | 242.2              | 220.2              | 571.69               | -4.44         | 0.76     | 185.6              | +22.97          | + 3.54            | 28.59             | 3.20              | R <sub>1</sub> |
| 466 | 1915. V    | 26.0              | 198.1              | 246.5              | 345.6              | 575.95               | -3.73         | 0.74     | 113.4              | +15.17          | + 3.04            | 12.47             | 2.60              | R <sub>1</sub> |
| 643 | 1907. IX   | 12.5              | 90.2               | 9.5                | 111.8              | 577.58               | -3.46         | 0.77     | 236.1              | +13.82          | + 2.90            | 10.35             | 2.48              | R <sub>1</sub> |
| 525 | 1904. III  | 18.5              | 47.4               | 116.7              | 6.0                | 581.34               | -2.83         | 3.71     | 152.1              | +22.09          | +15.66            | 26.44             | 3.13              | R <sub>1</sub> |
| 401 | 1913. III  | 17.0              | 239.3              | 164.5              | 279.0              | 584.39               | -2.32         | 0.49     | 205.8              | + 7.28          | + 1.79            | 2.87              | 1.80              | L <sub>1</sub> |
| 745 | 1913. III  | 7.5               | 129.2              | 152.6              | 278.2              | 606.78               | +1.42         | 0.90     | 85.4               | + 1.71          | - 0.13            |                   | 0.87              | R <sub>3</sub> |
| 781 | 1914. I    | 25.5              | 267.6              | 136.1              | 305.2              | 608.78               | +1.76         | 0.86     | 153.3              | + 6.40          | - 0.43            |                   | 1.69              | R <sub>3</sub> |
| 175 | 1914. I    | 11.0              | 330.5              | 90.4               | 303.9              | 609.57               | +1.89         | 1.87     | 173.1              | +11.55          | + 2.24            | 7.23              |                   | R <sub>3</sub> |
| 530 | 1911. IX   | 3.5               | 323.0              | 323.7              | 232.5              | 610.21               | +2.00         | 1.77     | 181.7              | +11.60          | + 1.80            | 7.29              |                   | R <sub>3</sub> |
| 777 | 1914. I    | 28.5              | 167.1              | 121.8              | 305.5              | 611.31               | +2.18         | 1.46     | 37.9               | - 0.20          | + 0.68            | 0.00              |                   | L <sub>2</sub> |
| 756 | 1908. IV   | 26.5              | 194.8              | 217.6              | 130.7              | 612.32               | +2.35         | 1.20     | 151.0              | +10.03          | - 0.12            |                   | 2.11              | R <sub>3</sub> |
| 758 | 1912. VI   | 9.5               | 56.2               | 256.2              | 255.8              | 612.61               | +2.40         | 1.12     | 160.8              | +10.31          | - 0.35            |                   | 2.14              | R <sub>3</sub> |
| 122 | 1911. V    | 7.0               | 189.9              | 214.4              | 222.5              | 614.37               | +2.69         | 0.56     | 319.3              | + 5.42          | - 1.48            |                   | 1.55              | R <sub>3</sub> |
| 581 | 1905. XII  | 24.5              | 63.5               | 92.1               | 59.7               | 615.96               | +2.96         | 0.44     | 36.2               | + 7.24          | - 1.78            |                   | 1.79              | R <sub>3</sub> |
| 300 | 1895. VII  | 10.0              | 325.4              | 302.1              | 102.3              | 617.27               | +3.18         | 0.43     | 62.9               | + 9.27          | - 1.93            |                   | 2.03              | R <sub>3</sub> |
| 318 | 1912. IV   | 11.0              | 78.4               | 186.5              | 250.7              | 617.67               | +3.24         | 0.59     | 123.5              | +11.91          | - 1.81            |                   | 2.30              | R <sub>3</sub> |
| 108 | 1911. IX   | 24.0              | 164.9              | 324.5              | 234.1              | 617.91               | +3.28         | 1.05     | 21.2               | + 6.55          | - 1.08            |                   | 1.70              | R <sub>3</sub> |
| 667 | 1908. VIII | 24.5              | 98.4               | 334.7              | 140.6              | 618.03               | +3.30         | 1.71     | 151.9              | +17.37          | + 0.73            | 16.35             |                   | R <sub>3</sub> |
| 325 | 1913. XII  | 2.0               | 60.4               | 69.9               | 300.6              | 618.24               | +3.34         | 1.65     | 249.1              | +13.68          | + 0.50            | 10.14             |                   | R <sub>3</sub> |
| 580 | 1906. II   | 12.5              | 54.9               | 86.8               | 63.8               | 618.61               | +3.40         | 1.33     | 14.1               | + 6.01          | - 0.50            |                   | 1.63              | R <sub>3</sub> |
| 755 | 1915. VIII | 14.0              | 217.5              | 316.2              | 352.2              | 619.88               | +3.61         | 1.27     | 189.3              | +18.42          | - 0.79            |                   | 2.86              | R <sub>3</sub> |
| 595 | 1906. V    | 18.5              | 289.5              | 221.1              | 71.7               | 620.18               | +3.66         | 0.75     | 7.2                | +10.20          | - 1.88            |                   | 2.13              | R <sub>3</sub> |
| 645 | 1907. IX   | 29.5              | 89.9               | 14.6               | 113.2              | 620.25               | +3.68         | 1.55     | 238.1              | +17.06          | - 0.05            |                   | 2.76              | R <sub>3</sub> |
| 491 | 1903. I    | 0.0               | 41.1               | 21.8               | 329.1              | 620.55               | +3.73         | 0.65     | 124.7              | +15.51          | - 2.06            |                   | 2.62              | R <sub>3</sub> |
| 381 | 1906. III  | 14.0              | 268.4              | 174.9              | 66.3               | 620.62               | +3.74         | 1.26     | 310.7              | +10.46          | - 0.90            |                   | 2.16              | R <sub>3</sub> |
| 236 | 1905. VI   | 7.0               | 32.8               | 244.8              | 43.0               | 620.63               | +3.75         | 0.13     | 191.6              | +14.61          | - 2.48            |                   | 2.55              | R <sub>3</sub> |
| 702 | 1910. VIII | 4.5               | 345.3              | 316.0              | 199.6              | 621.86               | +3.95         | 0.15     | 262.1              | +15.70          | - 2.61            |                   | 2.64              | R <sub>3</sub> |
| 696 | 1910. II   | 1.5               | 37.9               | 92.6               | 184.4              | 621.91               | +3.95         | 2.41     | 121.7              | +21.04          | + 3.18            | 23.99             |                   | R <sub>3</sub> |
| 618 | 1906. X    | 25.5              | 346.6              | 19.7               | 85.0               | 622.09               | +3.98         | 0.60     | 196.3              | +18.32          | - 2.29            |                   | 2.85              | R <sub>3</sub> |
| 436 | 1906. II   | 2.0               | 15.4               | 106.1              | 62.9               | 622.10               | +3.99         | 0.83     | 355.7              | +12.37          | - 1.97            |                   | 2.34              | R <sub>3</sub> |
| 184 | 1910. XII  | 18.0              | 191.0              | 75.6               | 210.8              | 622.48               | +4.05         | 0.60     | 205.0              | +18.74          | - 2.34            |                   | 2.89              | R <sub>3</sub> |
| 92  | 1904. II   | 13.0              | 323.4              | 105.9              | 3.1                | 622.68               | +4.08         | 0.94     | 63.1               | +14.84          | - 1.85            |                   | 2.57              | R <sub>3</sub> |
| 316 | 1912. V    | 1.0               | 75.4               | 229.0              | 252.4              | 623.00               | +4.14         | 1.29     | 159.6              | +22.33          | - 1.09            |                   | 3.15              | R <sub>3</sub> |

Two asteroids (401) and (528) seem to make the libration of Type 1, but as the difference of  $E$  and  $F_1$  is very small the motion may be changeable to the revolution of Type 1 depending on the amount of the smaller inequalities. The asteroid (777) has a negative value of  $C$  and the motion is the libration of Type 2. All the remaining 38 asteroids make the revolution of Type 1 or Type 3.

62. For the class  $3/2$  six asteroids only may be taken. The results of computation are as follows: —

$$3/2 \quad n_0 = 448.70 \quad p_1^{(3)} \times 10^3 = 1.47$$

| No. | Epoch | $\omega$ | $l$  | $l'$  | $n$   | $x \times 10^2$ | $e \times 10$ | $\theta$ | $C \times 10^4$ | $E \times 10^2$ | $F_0 \times 10^2$ | $F_1 \times 10^2$ | Type |       |
|-----|-------|----------|------|-------|-------|-----------------|---------------|----------|-----------------|-----------------|-------------------|-------------------|------|-------|
| 153 | 1911  | III      | 28.0 | 282.6 | 207.9 | 219.2           | 449.46        | +0.17    | 1.62            | 40.8            | -10.79            | +2.57             | 1.49 | $L_2$ |
| 748 | 1913  | III      | 8.5  | 103.0 | 160.9 | 278.3           | 451.35        | +0.59    | 1.36            | 309.9           | -7.34             | +1.65             | 0.69 | $L_2$ |
| 361 | 1914  | XI       | 27.0 | 93.4  | 161.6 | 330.6           | 453.60        | +1.09    | 2.08            | 304.8           | -9.26             | +3.96             | 1.10 | $L_2$ |
| 190 | 1910  | XI       | 8.0  | 103.7 | 71.0  | 207.5           | 453.69        | +1.11    | 1.67            | 343.2           | -12.87            | +2.42             | 2.13 | $L_2$ |
| 499 | 1911  | I        | 30.5 | 92.5  | 112.4 | 214.5           | 457.15        | +1.88    | 2.14            | 33.8            | -12.15            | +3.95             | 1.90 | $L_2$ |
| 334 | 1913  | IV       | 26.0 | 8.5   | 225.4 | 282.3           | 459.51        | +2.41    | 0.15            | 332.4           | +4.63             | -0.78             | 0.72 | $L_3$ |

Five asteroids out of six have negative values of  $C$  and make the libration of Type 2. The asteroid (334) makes the libration of Type 3 with a positive value of  $C$ .

63. Only one asteroid may be taken near the point  $4/3$  with the following result: —

$$4/3 \quad n_0 = 398.84 \quad p_1^{(4)} \times 10^3 = 2.24$$

| No. | Epoch | $\omega$ | $l$  | $l'$  | $n$   | $x \times 10^2$ | $e \times 10$ | $\theta$ | $C \times 10^4$ | $E \times 10^2$ | $F_0 \times 10^2$ | $F_1 \times 10^2$ | Type |   |
|-----|-------|----------|------|-------|-------|-----------------|---------------|----------|-----------------|-----------------|-------------------|-------------------|------|---|
| 279 | 1913  | VI       | 17.5 | 296.1 | 294.7 | 286.7           | 397.60        | -0.31    | 0.64            | 33.4            | -7.08             | +0.48             | 0.28 | L |

The type of the motion is the same as those of the five asteroids of the class  $3/2$ , i.e. the libration of Type 2 with a negative value of  $C$ .

64. For the class 3/1, taking 25 asteroids with  $n-n_0$  within the limits  $-25''$  and  $+25''$ , the following results are obtained:—

|     |       | 3/1      |      | $n_0=897.39$ |       | $p_2^{(3)} \times 10^2=0.0275$ |               |          |                 |                 |                   |                   |      |                |
|-----|-------|----------|------|--------------|-------|--------------------------------|---------------|----------|-----------------|-----------------|-------------------|-------------------|------|----------------|
| No. | Epoch | $\omega$ | $l$  | $l'$         | $n$   | $x \times 10^2$                | $e \times 10$ | $\theta$ | $C \times 10^4$ | $E \times 10^2$ | $F_0 \times 10^3$ | $F_1 \times 10^2$ | Type |                |
| 765 | 1913  | X        | 3.5  | 36.7         | 21.3  | 295.7                          | 874.04        | -2.60    | 2.81            | 287.6           | +7.16             | +11.36            | 43.4 | R <sub>1</sub> |
| 714 | 1911  | V        | 25.5 | 102.7        | 214.2 | 224.1                          | 874.17        | -2.59    | 0.45            | 107.3           | +6.70             | +3.65             | 40.6 | R <sub>1</sub> |
| 472 | 1908  | III      | 23.0 | 62.2         | 177.7 | 127.8                          | 875.74        | -2.41    | 0.98            | 278.7           | +5.83             | +4.17             | 35.3 | R <sub>1</sub> |
| 787 | 1914  | IV       | 22.5 | 309.5        | 230.8 | 312.4                          | 876.73        | -2.30    | 1.24            | 272.6           | +5.30             | +4.60             | 32.1 | R <sub>1</sub> |
| 355 | 1905  | I        | 2.5  | 86.9         | 99.3  | 30.1                           | 877.28        | -2.24    | 1.08            | 182.8           | +4.82             | +4.15             | 29.2 | R <sub>1</sub> |
| 695 | 1909  | XI       | 7.5  | 353.4        | 40.6  | 177.2                          | 877.30        | -2.24    | 1.56            | 215.8           | +4.69             | +5.42             | 28.4 | R <sub>1</sub> |
| 660 | 1908  | I        | 12.5 | 264.0        | 126.0 | 121.9                          | 877.99        | -2.16    | 1.03            | 288.3           | +4.72             | +3.94             | 28.6 | R <sub>1</sub> |
| 421 | 1912  | VIII     | 29.0 | 34.6         | 349.8 | 262.4                          | 878.56        | -2.10    | 2.92            | 351.8           | +5.79             | +11.32            | 35.1 | R <sub>1</sub> |
| 292 | 1902  | IV       | 4.0  | 331.4        | 206.7 | 306.6                          | 881.55        | -1.77    | 0.29            | 309.7           | +3.14             | +2.44             | 19.0 | R <sub>1</sub> |
| 46  | 1910  | XI       | 28.0 | 354.5        | 62.6  | 209.2                          | 884.45        | -1.44    | 1.67            | 144.0           | +1.71             | +4.71             | 10.4 | R <sub>1</sub> |
| 518 | 1903  | X        | 20.5 | 322.5        | 10.2  | 353.5                          | 885.77        | -1.29    | 2.20            | 314.7           | +2.24             | +6.56             | 13.6 | R <sub>1</sub> |
| 619 | 1906  | X        | 22.5 | 2.4          | 37.7  | 84.8                           | 886.62        | -1.20    | 0.75            | 148.1           | +1.35             | +2.16             | 8.2  | R <sub>1</sub> |
| 132 | 1895  | XI       | 30.5 | 152.4        | 123.2 | 114.2                          | 903.69        | +0.70    | 3.31            | 85.4            | +0.64             | +10.02            | 3.9* | L <sub>2</sub> |
| 495 | 1902  | XI       | 21.5 | 26.5         | 47.4  | 325.9                          | 910.12        | +1.42    | 1.47            | 202.7           | +1.68             | +0.27             | 10.2 | R <sub>3</sub> |
| 329 | 1901  | VIII     | 27.0 | 217.0        | 337.1 | 288.3                          | 912.13        | +1.64    | 0.28            | 266.2           | +2.69             | -2.11             | 2.2  | R <sub>3</sub> |
| 335 | 1906  | II       | 2.0  | 288.8        | 134.3 | 62.9                           | 912.66        | +1.70    | 1.80            | 163.2           | +2.38             | +0.97             | 14.4 | R <sub>3</sub> |
| 17  | 1911  | VII      | 26.0 | 263.0        | 290.0 | 229.1                          | 913.55        | +1.80    | 1.33            | 128.7           | +3.06             | -0.63             | 2.3  | R <sub>3</sub> |
| 248 | 1905  | VIII     | 6.0  | 247.8        | 319.5 | 48.0                           | 913.94        | +1.84    | 0.64            | 311.1           | +3.43             | -2.05             | 2.5  | R <sub>3</sub> |
| 556 | 1905  | I        | 16.5 | 101.0        | 116.6 | 31.2                           | 915.85        | +2.06    | 1.01            | 225.0           | +4.13             | -1.73             | 2.7  | R <sub>3</sub> |
| 752 | 1913  | V        | 10.5 | 105.8        | 212.5 | 283.6                          | 917.80        | +2.27    | 0.74            | 293.3           | +5.18             | -2.48             | 3.0  | R <sub>3</sub> |
| 623 | 1907  | II       | 5.5  | 71.7         | 123.0 | 93.6                           | 918.32        | +2.33    | 1.15            | 345.6           | +5.63             | -1.79             | 3.2  | R <sub>3</sub> |
| 650 | 1907  | X        | 4.5  | 31.7         | 34.8  | 113.6                          | 918.48        | +2.35    | 1.87            | 117.4           | +5.25             | +0.36             | 31.8 | R <sub>3</sub> |
| 732 | 1912  | IV       | 24.5 | 236.9        | 212.8 | 252.0                          | 919.07        | +2.42    | 0.46            | 290.6           | +5.87             | -3.02             | 3.2  | R <sub>3</sub> |
| 178 | 1910  | III      | 13.0 | 261.3        | 178.1 | 187.6                          | 919.41        | +2.45    | 0.45            | 137.9           | +5.98             | -3.07             | 3.3  | R <sub>3</sub> |
| 198 | 1910  | VII      | 31.0 | 356.4        | 310.6 | 199.2                          | 920.05        | +2.53    | 2.28            | 65.8            | +6.75             | +1.82             | 40.9 | R <sub>3</sub> |

\*  $E_3 \times 10^2 = 4.0$

All the asteroids except (132) make the revolution of Type 1 or Type 3. The asteroid (132) seems to make the libration of Type 2. But this asteroid has not been observed since its discovery in 1873 and is known as one of the lost planets.<sup>1)</sup> So the existence

1) Mr. D. Alter corrected the mean motion of this asteroid to  $883''.47$  on the assumption of its identity with the object observed at the Lowell Observatory in 1913. See the Lick Obs. Bull. Nos. 275 and 285.

of the librating asteroids near the point 3/1 is doubtful.

65. Only two asteroids will be taken near the point 5/3 with the following results:—

$$5/3 \quad n_0 = 498''.55 \quad p_2^{(5)} \times 10^2 = 0.2222$$

| No. | Epoch | $\omega$ | $l$  | $l'$ | $n$   | $x \times 10^2$ | $e \times 10$ | $\theta$ | $C \times 10^4$ | $E \times 10^2$ | $F_0 \times 10^2$ | $F_1 \times 10^2$ | Type               |
|-----|-------|----------|------|------|-------|-----------------|---------------|----------|-----------------|-----------------|-------------------|-------------------|--------------------|
| 522 | 1913  | IV       | 6.0  | 1.3  | 228.3 | 280.6           | 513.62        | +3.02    | 0.78            | 4.5             | +9.93             | -0.74             | 1.4 R <sub>3</sub> |
| 721 | 1911  | X        | 18.5 | 29.0 | 19.2  | 236.2           | 526.85        | +5.68    | 1.18            | 14.6            | +34.05            | -1.13             | 2.6 R <sub>3</sub> |

Both make the revolution of Type 3.

66. In the cases of the third or higher orders, the effect of the neglected terms being relatively great, it becomes very hard to determine the types of the motion accurately. As a rough approximation however the same method was applied for the classes 5/2 and 7/4, arriving at the following results:—

All the asteroids near the point 5/2 ( $n_0 = 747''.82$ ) make the revolution of Type 1 or Type 3, except (464) which seems to make the libration of Type 2. But this asteroid has not been observed since its discovery in 1901 and therefore full weight cannot be assigned to its result. For the class 7/4 ( $n_0 = 523''.48$ ) two asteroids (522) and (721) only will be taken. The former seems to make the revolution of Type 1 and the latter, the libration of Type 2 with a positive value of  $C$ .

67. It is a well known fact that all of the four asteroids of the class 1/1 make the libration about the triangular equilibrium points. Hence, including this case, the results will be summarized as follows:—

1. *All the asteroids with smaller mean motions than 500'' make the libration and form a series of the groups near the commensurable points 1/1, 3/2 and 4/3.*

2. *The asteroids with greater mean motions than 580'' do not make the libration (except a few cases which are mostly doubtful) and form a series of the gaps at the commensurable points 2/1, 3/1, 5/2 etc.*

68. The first of these remarkable peculiarities may be accounted for by a consideration of gravitation only. Taking the cases of the first order, we have

$$\theta = s(l-l') - (l - \varpi)$$

If we put

$$l' = l$$

in this equation, then

$$l = \varpi - \theta$$

which shows that the conjunction of the mean longitudes occurs at the point  $l = \varpi - \theta$ . Now in the libration of Type 2 with a negative value of  $C$  and in that of Type 3, the argument  $\theta$  oscillates about 0 and the amplitude of the oscillation is less than  $\frac{\pi}{2}$ . Hence in these cases the conjunction takes place near the point  $l = \varpi$ , that is, near the perihelion of the asteroid. Contrarily if the type of the motion be the libration of Type 1 or the revolution of any type, the conjunction may take place near the aphelion of the asteroid.

69. The linear distance of the asteroid from Jupiter when the conjunction occurs at the aphelion of the asteroid is

$$a' - a(1 + e)$$

This expression becomes zero when

$$\frac{a'}{a} = 1 + e = \left(\frac{n}{n'}\right)^{\frac{2}{3}}$$

The values of the eccentricity satisfying this condition are

|     |       |       |       |       |
|-----|-------|-------|-------|-------|
| $n$ | 350'' | 400'' | 450'' | 500'' |
| $e$ | 0.11  | 0.21  | 0.31  | 0.41  |

This shows how the asteroid with a moderate eccentricity approaches Jupiter when the conjunction occurs near the aphelion of the asteroid. The eccentricity of the asteroid is generally

smaller than  $0.20$ , but this will not remain always small if the orbit of the asteroid be sufficiently close to that of Jupiter. So the motion of the asteroids with smaller mean motions, something like  $450''$  or less, if the conjunctions occur near the aphelia, will be disturbed a great deal by the attraction of Jupiter.

In the cases of the asteroids which make the libration about zero with the amplitudes less than  $\frac{\pi}{2}$ , since the conjunctions always take place near the perihelia, they will not suffer large disturbances and the motions will be stable. Contrarily the asteroids librating about  $\pi$ , or making the revolution will suffer large disturbances, since the conjunctions may take place near the aphelia. So their motions will be unstable.

Theoretically speaking the libration of Type 3 is possible for any positive value of  $x$ . But if the value of  $x$  be great the range of  $E$  for the libration becomes very small. Consequently the libration may be changeable to the revolution by a slight variation of  $E$  due to the smaller inequalities. The libration of Type 3 is thus practically impossible when  $x$  is not small. The fact that the asteroids do not exist at the intermediate positions is thus explicable.

70. The asteroids of the class  $1/1$ , in spite of the proximity of their orbits to that of Jupiter, never approach very near to the latter and consequently their motions are stable. This fact is in perfect agreement with the above explanation.

An instance analogous to the librating asteroids of the  $3/2$  and  $4/3$  classes may be found in the Saturnian system. Hyperion, the seventh satellite of Saturn, has a period of revolution very nearly commensurable to that of Titan, the sixth and the largest satellite of Saturn. It has been shown in theory and by observations that the argument  $4l - 3l' - \varpi^1$  of Hyperion makes a libration about  $\pi$ , so that the conjunctions always take place near the aposaturnium of Hyperion. Consequently these satellites, in spite of the

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1) Here  $l$ ,  $\varpi$  and  $l'$  denote the mean longitude, the mean longitude of perisaturnium, of Hyperion, and the mean longitude of Titan, respectively.

proximity of their orbits, never approach very near to each other.

71. To explain the second character of the distribution of the mean motions, I shall introduce the hypothesis of resisting materials. Assuming the existence of the resisting materials moving around the sun in circular orbits, it was proved in Chapter III that, in the general cases of the second and higher orders, the librations ultimately change to revolutions, while the revolutions do not change to librations. In the general case of the first order it was proved that the revolutions do not change to librations except that the revolution of Type 1 may temporarily change to the libration of Type 1, that the librations ultimately change to the revolution or the libration, of Type 3, and also that, if the constant  $\alpha$  be sufficiently great compared with  $\beta$ , the libration of Type 3 changes to the revolution of the same type. Now in the class 2/1, the unique case of the first order with greater mean motions, we may naturally suppose, as it was noticed in § 9, that the density of the resisting materials rapidly decreases as the distance from the sun increases, so that  $\alpha$  becomes great in comparison with  $\beta$ . Hence in general, in the cases of greater mean motions, the librations ultimately change to revolutions, while the revolutions do not change to librations, except the revolution of Type 1 in the general case of the first order, which may temporarily change to the libration of Type 1. In the cases of smaller mean motions, we have sufficient reason to believe that the density of the resisting materials is very rare, as was noticed in § 9, so that the effect on the librating motions is insignificant.

The fact that the eccentricities of the asteroids near the gaps, especially on the inner side, are generally smaller than those in the other portions (§ 59), may be explained in the following manner:—On the negative (outer) side of  $x$ , the eccentricity cannot be great so long as the type of the motion is confined to the revolutions. Even when the eccentricity is moderate, the motion will sometimes change to the libration on account of the smaller inequalities and the asteroid will remove to the positive side of  $x$ . Hence in order that the revolution of Type 1 may be



stable, the eccentricity must always be very small. On the positive (inner) side of  $x$ , the asteroids which made the libration previously with negative value of  $C$  remain near the critical point with small eccentricities, as was shown in Chapter III. The asteroids which were removed from the negative side do not stay long near the critical point, and therefore will not affect the mean value of the eccentricity, although they may have unusually great values.

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In concluding the present paper I wish to express my gratitude to Prof. Brown of Yale University, and to Prof. Eichelberger of the Naval Observatory at Washington, both of whom read the first part of this paper and gave valuable and encouraging suggestions in connection with my investigation.

Astronomical Observatory, Tokyo, Nov. 6, 1917.

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## List of Notations Used.

Numbers refer to the articles in which the notations are introduced. The parentheses enclosing numbers indicate that the notations are used temporarily.

|              |  |   |   |  |  |
|--------------|--|---|---|--|--|
| $A$          | $\left. \begin{array}{l} Q_1 \\ B_0 \end{array} \right\} 19$     | $\left. \begin{array}{l} Q_1 \\ Q_{-1} \end{array} \right\} 56$ | $a$ 1   | $\left. \begin{array}{l} n_{-15} \\ \vdots \\ n_{-1} \\ n_1 \\ \vdots \\ n_{15} \end{array} \right\} 56$ | $a$ 28   |
| $B_0$        |  |   | $a'$ 10   |  | $a_0$ 18   |
| $B_1$        |  | $R$ 10  | $a_0$ 11  |  | $\left. \begin{array}{l} a_1 \\ a_2 \end{array} \right\} 7$          |
| $C$ 15       | $\left. \begin{array}{l} R_0 \\ C_1 \end{array} \right\} 21$     | $\left. \begin{array}{l} R_0 \\ R_c \end{array} \right\} 13$    | $b^{(s)}$ 18, 19  |  | $\beta$ 28   |
| $C_1$ 21     |  |   | $c$ 1   |  | $\beta_1$ 7  |
| $C_2$ 21, 24 | $S$ 1, (43)  | $S'$ (43), (50)   | $e$ 1   | $\left. \begin{array}{l} p_{s-s'} \\ p_{s-s'}^{(s)} \end{array} \right\} 19, 24, 27$                     | $\varepsilon$ 3, (38)  |
| $E$ 16       | $T$ 1, 30, (31)  |   | $e'$ (30), (31)   |  | $\varepsilon'$ 10  |
| $E_1$        | $\left. \begin{array}{l} E_1 \\ E_2 \end{array} \right\} 21$     | $V$   | $e_0$ 35  | $q$ (6)  | $\eta$ (43)  |
| $E_2$        |  | $\left. \begin{array}{l} V_s \\ V_t \end{array} \right\} 1$     | $e_1$ 33  | $r$ 1  | $\theta$ 12  |
| $E_3$        |  |   | $\bar{e}$ (33), 34, 37                                      | $s$ (6), 11  | $\theta'$ 46, 52   |
| $F_0$        | $\left. \begin{array}{l} F_0 \\ F_1 \end{array} \right\} 21, 60$ | $W$ 5   | $f$ (38)  | $s'$ 11  | $\theta_0$ 19  |
| $F_1$        |  | $X$ (6), (30)   | $g$ 53  | $t$ 1  | $\bar{\theta}$ 34  |
| $H$ 34       | $Y$ (31)   | $Z$ (49)  | $h$ 1   | $u$ 37   | $\bar{\theta}'$ 48   |
| $H_1$ 35     |  |   | $i$ (6), (12), 52   | $w$ 1  | $\nu$ (18)   |
| $I_1$        |  |   | $j$ (6), (12)   | $x$ 11   | $\xi$ (43)   |
| $I_2$        | $\left. \begin{array}{l} I_2 \\ I_3 \end{array} \right\} (7)$    |   | $\left. \begin{array}{l} k_1 \\ k_2 \end{array} \right\} 2$ | $x'$ (30), (31)  | $\varpi$ 3   |
| $I_3$        |  |   | $\left. \begin{array}{l} l \\ l' \end{array} \right\} 12$   | $x_0$ 15   | $\rho$ 1   |
| $M$ 34       |  |   | $m'$  | $x_1$ 33   | $\rho_0$ 2   |
| $N$ 34, 54   |  |   | $n'$  | $\bar{x}$ (33), 34, 37   | $\varphi$  |
| $P$          |  |   | $\left. \begin{array}{l} m' \\ n' \end{array} \right\} 10$  | $y$ (24), (27)   | $\left. \begin{array}{l} \phi_1 \\ \phi_2 \end{array} \right\} (13)$ |
| $P'$         | $\left. \begin{array}{l} P \\ P' \end{array} \right\} (43)$      |   | $n_0$ 1, 11   |  |  |
| $Q$          |  |   |   |  |  |
| $Q'$         |  |   |   |  |  |

