JOURNAL OF THE COLLEGE OF SCIENCE, TOKYO IMPERIAL UNIVERSITY.
VOL. XLI., ART. 4.

Asymptotic Formulae for oscillating Dirichlet's Integrals and Coefficients of Power Series*

By

Motoji KUNIYEDA, *Rigakushi.*

Introduction.

1. G. H. Hardy, in his interesting papers† with the title "Oscillating Dirichlet's Integrals", has discussed almost completely the oscillating nature of an integral of the form

$$\int_0^{\xi} f(x)\frac{\sin \lambda x}{x}dx \qquad (\xi>0),$$

when λ tends to infinity. Hereby $f(x)$ is of the form

$$\rho(x)\, e^{i\sigma(x)},$$

where $\rho(x)$ and $\sigma(x)$ are logarithmico-exponential functions (or L-functions) and $\sigma(x)$ tends to infinity as $x \to 0$.

As he remarks, the problem is equivalent to that of investigating the convergence or divergence of the Fourier's series defined by a function which has a single oscillating discontinuity of the type specified by

$$\rho(x) \begin{matrix}\cos\\ \sin\end{matrix} \sigma(x).$$

It is also closely related to that of determining asymptotic formulae for the coefficients a_n of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

* This paper was worked out at the suggestion of Mr. Hardy, to whom I wish to express my sincere thanks for valuable advices.

† *Quarterly Journal*, Vol. XLIV, *pp. 1—40 and 242—263.* We shall refer to these papers as "*O. D. I. 1.*" and "*O. D. I. 2.*" respectively.

convergent for $|z|<1$, and representing a function $f(z)$ which has, on the circle of convergence, one singular point only, at $z=1$. In the investigation of this problem, we are often led to consider integrals of the types

$$\int_0^\xi f(x)\frac{\sin\lambda x}{x}dx, \qquad \int_0^\xi f(x)\frac{\cos\lambda x}{x}dx.$$

2. In Part I of this paper, I consider the integrals

$$\begin{aligned} &(1) \qquad S(\lambda)=\int_0^\xi \rho(x)e^{i\sigma(x)}\frac{\sin\lambda x}{x}dx \\ &(2) \qquad C(\lambda)=\int_0^\xi \rho(x)e^{i\sigma(x)}\frac{\cos\lambda x}{x}dx \end{aligned} \qquad (\xi>0),$$

where, ρ and σ denote L-functions and $\sigma\succ 1$* as $x\to 0$, ξ being a positive number chosen sufficiently small so as to ensure that ρ and σ are monotonic and continuous in the interval $0<x\leqq\xi$. These integrals (1) and (2) will be called "the *sine-integral*" and "the *cosine-integral*" respectively.

Hardy, following Du Bois-Reymond, distinguishes the following three cases:

$$\begin{aligned} &(A) \qquad \sigma(x)\prec l(1/x), \\ &(B) \qquad \sigma(x)\asymp l(1/x), \\ &(C) \qquad \sigma(x)\succ l(1/x). \end{aligned}$$

The results arrived at concerning the sine-integral are designated as theorems A, B, C' in his papers.

As will be seen from these results, Hardy has principally considered the cases in which the sine-integral oscillates as $\lambda\to\infty$; but he did not went into a minute discussion of the cases in which the integral tends to zero. It will be interesting to find asymptotic formulae for $S(\lambda)$ in the latter cases; and it appears quite natural that the formulae obtained by him are also available to a certain extent in such cases. I have succeeded in extending the range of validity of his formulae considerably—roughly speaking, to all cases in which the order of $S(\lambda)$ is greater than $\frac{1}{\lambda}$.

* Throughout this paper, I will entirely adopt the symbols and notations defined in Section II of "*O. D. I. 1.*"

As for the cosine-integral $C(\lambda)$, it will be seen that, in the two cases (A) and (B), it always tends to zero as $\lambda \to \infty$, when convergent, and, in Case (C), its behaviour is very similar to that of the sine-integral. I have found for it asymptotic formulae whose range of validity is the same as that of the formulae for $S(\lambda)$.

That the formulae should cease to hold when the order of the integrals sinks as low as $\frac{1}{\lambda}$ is to be expected. For, it is easy to see that the parts of the integrals away from $x=0$ are in general of the order $\frac{1}{\lambda}$, so that in such cases the behaviour of $S(\lambda)$ or $C(\lambda)$ is no longer dominated by the parts near $x=0$.

3. The principal results arrived at are as follows: Writing

$$\rho(x) = x^{-a}\theta(x), \qquad x^{\delta} \prec \theta \prec (1/x)^{\delta},$$

in Case (A),

$$S(\lambda) \sim -\Gamma(-a)\sin\left(\tfrac{1}{2}a\pi\right)\rho(1/\lambda)e^{i\phi(1/\lambda)} \qquad (-1<a<0),$$

$$S(\lambda) \sim \tfrac{1}{2}\pi\rho(1/\lambda)e^{i\phi(1/\lambda)} \qquad (a=0,\ \rho\prec 1).$$

Combining these results with Theorem A, we obtain the theorem:

If $1 \prec \sigma \prec l(1/x)$, $\rho \prec \sigma'$ *and*

$$\rho = x^{-a}\theta(x), \qquad x^{\delta} \prec \theta \prec (1/x)^{\delta}, \qquad a \leqq 1,$$

then we have, as $\lambda \to \infty$,

$$(3)\quad \begin{cases} S(\lambda) = O(\lambda^{-1+\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a)\sin\left(\tfrac{1}{2}a\pi\right)\rho(1/\lambda)e^{i\phi(1/\lambda)} & (-1<a<1), \\ S(\lambda) \sim \lambda\, T(1/\lambda) & (a=1), \end{cases}$$

where $$T(x) = \int_0^x \rho(t)e^{i\phi(t)}\,dt.$$

In the particular case $a=0$, *the factor* $-\Gamma(-a)\sin\left(\tfrac{1}{2}a\pi\right)$ *is to be replaced by its limiting value* $\tfrac{1}{2}\pi$.

The corresponding formalae for $C(\lambda)$ are as follows:—

$$(4)\begin{cases} C(\lambda) = O(1/\lambda) & (a<-1 \text{ or } a=-1,\ x\theta' \preceq 1,\ \rho\sigma' \preceq 1), \\ C(\lambda) \sim -(\frac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)} & \\ & (a=-1,\ x\theta' \text{ or } \rho\sigma' \succ 1), \\ C(\lambda) \sim \Gamma(-a)\cos(\frac{1}{2}a\pi)\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a<0), \\ C(\lambda) \sim T(1/\lambda) & (a=0), \end{cases}$$

where
$$T(x) = \int_0^\infty \rho(t)e^{i\sigma(t)}\frac{dt}{t}.$$

In Case (B),

$$S(\lambda) \sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)}$$
$$(-1<a<0 \text{ or } a=0,\ \rho \prec 1).$$

Combining this result with Theorem B, we obtain the theorem:

If $\sigma \sim bl(1/x)$ $(b \neq 0)$, $\rho \prec \dfrac{1}{x}$, *and*

$$\rho = x^{-a}\theta(x), \qquad x^{\delta} \prec \theta \prec (1/x)^{\delta}, \qquad a \leqq 1,$$

then, as $\lambda \to \infty$, *we have*

$$(5)\begin{cases} S(\lambda) = O(\lambda^{-1+\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a-bi)\sin\{\frac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a\leqq 1). \end{cases}$$

The corresponding formulae for $C(\lambda)$ are

$$(6)\begin{cases} C(\lambda) = O(1/\lambda) & (a<-1 \text{ or } a=-1,\ \theta \preceq 1), \\ C(\lambda) \sim \Gamma(-a-bi)\cos\{\frac{1}{2}(a+bi)\pi\}\ \rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\ & (-1<a\leqq 0 \text{ or } a=-1,\ \theta \succ 1). \end{cases}$$

In Case (C), Hardy gave formulae which were shewn to be valid when $x\sqrt{\sigma''} \preceq \rho \prec x\sigma'$. I have succeeded in proving that they are valid for

$$x\sqrt{\sigma''}/\sigma' \prec \rho \prec x\sigma',$$

thus extending the range of validity of the formulae considerably. It will be seen that this lower limit of ρ (namely $\rho \asymp x\sqrt{\sigma''}/\sigma'$) corresponds to our natural limit $\dfrac{1}{\lambda}$ of the order of the integrals $S(\lambda)$ and $C(\lambda)$.

Combining these results with Theorem C, we obtain the theorem:—

The integrals $S(\lambda)$ and $C(\lambda)$ are convergent when $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$ and $\rho \prec x\sigma'$. The behaviour of these integrals, as $\lambda \to \infty$, is determined asymptotically as follows :

If $x^{\Delta} \prec \rho \preceq x\sqrt{\sigma''}/\sigma'$,

$$S(\lambda) = O(1/\lambda), \qquad C(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''}/\sigma' \prec \rho \prec x\sigma'$,

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i} \sqrt{\pi}, \tag{7}$$

$$C(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta+\frac{1}{4}\pi)i} \sqrt{\pi}, \tag{8}$$

where

$$\beta = \lambda\theta + \sigma(\theta)$$

and θ is determined as a function of λ by the equation

$$\sigma'(\theta) + \lambda = 0.$$

In the course of the proof of this theorem, we are led to the comparison of the order of magnitude of the functions

$$\varphi(a) = \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}}, \qquad \psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \to \infty$, when $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1 - \bar{\rho}(x)\}$, where $A > 0$, $\bar{\rho} > 0$ and $\bar{\rho} \prec 1$, a and θ being functions of λ determined respectively by the equations

$$\frac{d}{da}\left[\frac{\rho(a)}{a\{\lambda - \sigma'(a)\}}\right] = 0, \qquad \sigma'(\theta) + \lambda = 0.$$

It will be proved that

$$\varphi(a) \prec \psi(\theta)$$

as $\lambda \to \infty$. The proof of this relation plays an important rôle in the discussion of Case (C).

The integral $S(\lambda)$ is still convergent when $x\sigma' \preceq \rho \prec \sigma'$. Hardy did not went into the discussion of this case, his method ceasing to be applicable in this case. I have succeeded in proving that formula (7) holds also in this case generally, the proof being left incomplete only in a few special cases.

4. In Part II, I will give applications of the results, obtained in Part I, to the determination of the behaviour of the coefficients a_n of a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

as $n \to \infty$, whose radius of convergence is unity and which represents a function $f(z)$ having, on the circle of convergence, one singular point only, at $z=1$.

As is well known, this problem was first systematically treated by Darboux*. Particularly he considered the case in which $f(z)$ has a singularity of the type

$$f(z) = \frac{\varphi(z)}{(1-z)^p},$$

$\varphi(z)$ being a function regular for $z=1$ and p denoting any real constant other than zero or a negative integer. His results were extended by Hamy† who considered the case in which $f(z)$ has a singularity of the type

$$f(z) = \frac{1}{(1-z)^p}\left(\log\frac{1}{1-z}\right)^q,$$

where q is a positive integer. These two authors did not attack the case in which $f(z)$ has an essential singularity for $z=1$. This was first done by Fejèr,‡ who considered the case where

$$f(z) = \frac{1}{(1-z)^p} e^{\frac{1}{z-1}},$$

and shewed that

$$a_n \sim \frac{1}{\sqrt{(e\pi)}} n^{-(\frac{3}{4}-\frac{1}{2}p)} \sin \{2\sqrt{n}+(\tfrac{3}{4}-\tfrac{1}{2}p)\pi\},$$

p being any real constant.

I have considered still more general cases in which the function $f(z)$ has a singularity of the following types:

---

* *Journal de Math.* Série 3, *t.* 4 (1878) *pp.* 5—57, 377—417.

† „ Série 6, *t.* 4 (1908) *pp.* 203—283.

‡ *Comptes Rendus*, 30 Nov., 1908; and "*Asymptotikus értékek megatározásáról*" (1909) Budapest.

(i) $f(z)=\dfrac{1}{(1-z)^p}e^{A/(1-z)^q}$,

(ii) $f(z)=\dfrac{1}{(1-z)^p}e^{A/(1-z)^q}\left(\log\dfrac{1}{1-z}\right)^r$,

(iii) $f(z)=\dfrac{1}{(1-z)^p}e^{A/(1-z)^q}\left(l_1\dfrac{1}{1-z}\right)^{r_1}\left(l_2\dfrac{1}{1-z}\right)^{r_2}\cdots\cdots\left(l_h\dfrac{1}{1-z}\right)^{r_h}$,

A being a certain constant of the form $A=ae^{\alpha i}$.

My results in Case (i) are as follows:—

Let $a>0$ and p be any real constant, then the behaviour of a_n, as $n\to\infty$, is determined asymptotically as follows:

If $q=1$, $\alpha=\pi$,

(9) $a_n\sim\dfrac{1}{\sqrt{\pi}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{\frac{1}{2}p-\frac{3}{4}}\sin\{2a^{\frac{1}{2}}n^{\frac{1}{2}}-(\frac{1}{2}p-\frac{3}{4})\pi\}$*;

if $0<q<1$, $\alpha=(1+q)\dfrac{\pi}{2}$,

(10) $a_n\sim\dfrac{1}{\sqrt{\{2(1+q)\pi\}}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\,exp\left[\{kn^{\frac{q}{1+q}}-(\frac{1}{2}p-\frac{1}{4})\pi\}i\right]$,

where $k=(1+q)q^{-\frac{q}{1+q}}a^{\frac{1}{1+q}}$;

if $0<q<1$, $\alpha=(3-q)\dfrac{\pi}{2}$,

(11) $a_n\sim\dfrac{1}{\sqrt{\{2(1+q)\pi\}}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\,exp\left[-\{kn^{\frac{q}{1+q}}-(\frac{1}{2}p-\frac{1}{4})\pi\}i\right]$,

k being the same as that in the above formula.

Similar results were obtained in Case (ii) and Case (iii).

Now it may be remarked that, owing to the restricted applicability of the method, asymptotic formulae were obtained only in the following three cases:

(1°) $q=1$, $\alpha=\pi$,

(2°) $0<q<1$, $\alpha=(1+q)\dfrac{\pi}{2}$,

(3°) $0<q<1$, $\alpha=(3-q)\dfrac{\pi}{2}$.

* Observe that this becomes Fejèr's formula, if we put $a=1$.

While I was working out this paper at Cambridge, Hardy, being struck with this fact, tried to attack the problem with a quite different method and obtained the following result*:

If $$f(z) = \frac{1}{(1-z)^p} e^{a/(1-z)^q}, \qquad a>0,\ 0<q<1,$$

then

$$(12) \quad a_n \sim \frac{1}{\sqrt{\{2(1+q)\pi\}}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{\frac{1}{2}q+p}{1+q}-1} exp\left\{\frac{q+1}{q}(qa)^{\frac{1}{q+1}} n^{\frac{q}{q+1}}\right\}.$$

It is very probable that formula (12) also holds for complex values of a. If, in this formula, we replace a by

$$A = ae^{\alpha i},$$

where

$$\alpha = (1+q)\frac{\pi}{2} \quad \text{or} \quad (3-q)\frac{\pi}{2},$$

then we get (10) and (11). This shows that (12) holds also when a takes these special complex values.

PART I.†

Oscillating Dirichlet's Integrals

I. *Division of the Problem into three Cases.*

5. We have to consider the integrals

$$(1) \qquad S(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x} dx,$$

$$(2) \qquad C(\lambda) = \int_0^{\xi} \rho(x)\, e^{i\sigma(x)} \frac{\cos \lambda x}{x} dx,$$

where $\rho(x)$ and $\sigma(x)$ are L-functions and $\sigma \succ 1$ as $x \to 0$. As was already mentioned, these integrals will be called "the sine-integral" and "the cosine-integral" respectively. It will be supposed that ξ is a positive number so small that the range of integration does not

* *Messenger of Math.* Vol. XLVI (1916) *pp.* 70–73.

† A preliminary notice of this Part appeared in the *Quarterly Journal*, Vol. XLVIII (1918) *pp.* 113–135.

include any point at which the subject of integration possesses any irrelevant discontinuity or other singularity. We distinguish as in Hardy's papers, the following three cases:

(A) $\sigma \prec l(1/x)$,

(B) $\sigma \asymp l(1/x)$,

(C) $\sigma \succ l(1/x)$.

II. *Lemma for Case (A).*

6. The proofs of the theorems A and B of "O. D. I. 1." are principally carried out by means of H-lemma* 29. By examining the proof of this lemma, we can easily extend the range of validity of the formula given there.

In fact, the integrals

$$\int_0^\infty u^{-a\pm\delta-bi}\left(\frac{\sin u}{u}\right)^2 du,$$

there considered, are absolutely convergent also in the case $-1<a\leqq 0$. Hence the argument of "O. D. I. 1." for the case $0<a<1$ of this lemma holds also in the case $-1<a\leqq 0$. Thus we easily obtain the following modification of this lemma.

Lemma 1. *Let*

(13) $$J(\lambda)=\int_0^T x^{-a-bi}\Phi(x)\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2\tfrac{1}{2}\lambda dx,$$

where $a\leqq 1$ *and*

$$\Phi(x)=\theta(x)e^{i\phi(x)},$$

$$x^\delta<\theta<(1/x)^\delta,\qquad \phi(x)\prec l(1/x),$$

* The work of Mr. Hardy is chiefly included in the proofs of a great number of lemmas. Naturally, in my paper, these lemmas will be used freely, being referred as "H-lemma 1", "H-lemma 2", ..., in order to distinguish them from new lemmas which will be established here.

as $x \to 0$. *Then, if* $a \leqq -1$,

$$J(\lambda) = O(\lambda^{-1+\delta});$$

if $-1<a<1$,

$$J(\lambda) \sim \Gamma(-1-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\lambda^{a+bi}\Phi(1/\lambda), \tag{14}$$

except when $a=b=0$, *in which case the right-hand side of (14) is to be replaced by* $\frac{1}{2}\pi\,\Phi(1/\lambda)$; *if* $a=1$, $b \neq 0$, *this result (14) still holds, provided that the integral*

$$T(x) = \int_0^x t^{-1-bi}\,\Phi(t)\,dt$$

is convergent; and if $a=1$, $b=0$, *and* $T(x)$ *is still convergent,*

$$J(\lambda) \sim \tfrac{1}{2}\lambda\,T(1/\lambda).$$

III. *Discussion of Case (A)*: $\sigma(x) \prec l(1/x)$.

7. At first we shall consider the sine-integral

$$S(\lambda) = \int_0^\xi \rho(x)e^{i\sigma(x)}\frac{\sin\lambda x}{x}\,dx.$$

As is given in the paper "O. D. I. 1.", the necessary and sufficient condition for the convergence of this integral is

$$\rho(x) \prec \sigma'(x)$$

as $x \to 0$. As in the same paper, by performing integration by parts, we have

$$S(\lambda) = \frac{\rho(\xi)}{\xi}e^{i\sigma(\xi)}\frac{1-\cos\lambda\xi}{\lambda} + J(\lambda)$$

$$= O(1/\lambda) + J(\lambda), \tag{15}$$

where

$$\begin{cases} J(\lambda) = \displaystyle\int_0^\xi (R_1 - iR_2)e^{i\sigma}\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda\,dx, \\ R_1 = \rho - x\rho', \\ R_2 = x\rho\sigma'; \end{cases}$$

and hence

$$J(\lambda) = J^{(1)}(\lambda) - iJ^{(2)}(\lambda), \tag{16}$$

where

$$\begin{cases} J^{(1)}(\lambda) = \displaystyle\int_0^{\epsilon} R_1 e^{i\sigma} \left(\frac{\sin \frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda\, dx, \\ J^{(2)}(\lambda) = \displaystyle\int_0^{\epsilon} R_2 e^{i\sigma} \left(\frac{\sin \frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda\, dx. \end{cases} \tag{17}$$

Now we can write

$$\rho = x^{-a}\theta(x),$$

where $a \leqq 1$ and $x^{\delta} \prec \theta \prec (1/x)^{\delta}$ as $x \to 0$. Then, if $a \neq -1$,

$$\begin{cases} R_1 = \rho - x\rho' \sim (1+a)\, x^{-a}\theta(x) \quad \text{[by H-lemma 4]}, \\ R_2 = x\rho\sigma' = x^{-a}\,\bar{\theta}(x), \end{cases}$$

where $\bar{\theta}$ is a function of the same type as θ and $\bar{\theta} \prec \theta$ as $x \to 0$*; if $a = -1$,

$$R_1 = -x^2\theta' = -x\theta_1, \qquad R_2 = x\bar{\theta},$$

where $\theta_1 = x\theta'$ is a function of the same type as θ and $\theta_1 \prec \theta$. Hence we have:

(i) Let $a \leqq -1$. Then, applying Lemma 1 to the integrals (17), we obtain

$$J^{(1)}(\lambda) = O(\lambda^{-1+\delta}), \qquad J^{(2)}(\lambda) = O(\lambda^{-1+\delta});$$

hence, by (15) and (16),

$$S(\lambda) = O(\lambda^{-1+\delta}).$$

(ii) Let $-1 < a < 0$. Then, by Lemma 1, we obtain

---

* Since $1 \prec \sigma \prec l(1/x)$, we easily see that σ is a function of the same type as Θ. It can also easily be proved that the function $x\Theta'$ is a function of the same type as Θ and a product of two functions of this type is also of the same type. Hence it follows that $\bar{\Theta}(x) = x\sigma'\Theta$ is a function of the type Θ; and since $x\sigma' \prec 1$, we have $\bar{\Theta} \prec \Theta$.

$$J^{(1)}(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\lambda^a\theta(1/\lambda)\,e^{i\sigma(1/\lambda)},$$
$$J^{(2)}(\lambda) \sim -\Gamma(-1-a)\sin(\tfrac{1}{2}a\pi)\,\lambda^a\,\bar{\theta}(1/\lambda)\,e^{i\sigma(1/\lambda)};$$

and since $\bar{\theta} \prec \theta$, we have

$$J(\lambda) \sim J^{(1)}(\lambda),$$

and hence

$$S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)},$$

for, in this case, $\rho(1/\lambda) \succ \dfrac{1}{\lambda}$ as $\lambda \to \infty$.

(iii) Let $a = 0$ and $\theta \prec 1$. Then

$$R_1 \sim \rho = \theta, \qquad R_2 \prec \rho = \theta.$$

Applying Lemma 1, we have also $J^{(2)} \prec J^{(1)}$ and

$$S(\lambda) \sim \tfrac{1}{2}\pi\rho(1/\lambda)\,e^{i\sigma(1/\lambda)}.$$

Combining these results with Theorem A, we obtain

Theorem I. *The integral*

$$S(\lambda) = \int_0^{\xi} \rho(x)\,e^{i\sigma(x)}\,\frac{\sin\lambda x}{x}\,dx,$$

where $1 \prec \sigma \prec l(1/x)$ and $\rho \prec \sigma'$, *is convergent. If* $\rho = x^{-a}\theta(x)$, *where* $x^{\delta} \prec \theta \prec (1/x)^{\delta}$, *so that* $a \leqq 1$, *the behaviour of* $S(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically by the following formulae:*

$$(3)\quad \begin{cases} S(\lambda) = O(\lambda^{-1+\delta}) & (a \leqq -1), \\ S(\lambda) \sim -\Gamma(-a)\sin(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)} & (-1 < a < 1), \\ S(\lambda) \sim \lambda\, T(1/\lambda) & (a = 1), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t)\,e^{i\sigma(t)}\,dt.$$

In the particular case $a=0$, *the factor* $-\Gamma(-a)$ *sin* $(\tfrac{1}{2}a\pi)$ *is to be replaced by its limiting value* $\tfrac{1}{2}\pi$.

8. Now we pass to the cosine-integral

$$C(\lambda)=\int_0^{\epsilon} \rho(x)e^{i\sigma(x)}\frac{\cos\lambda x}{x}\,dx.$$

The integral

$$\int_0^{\epsilon} \rho(x)e^{i\sigma(x)}\frac{dx}{x}$$

is convergent if, and only if,

$$\rho < x\sigma' \tag{18}$$

as $x\to 0$. This involves $\rho<1$, as $\sigma'<\frac{1}{x}$. As the factor $\cos\lambda x$ of the subject of integration of $C(\lambda)$ is ultimately monotonic, this condition (18) is the necessary and sufficient condition for the convergence of the cosine-integral $C(\lambda)$.

Performing integration by parts, we obtain

$$C(\lambda)=\int_0^{\epsilon}\frac{\rho}{x}e^{i\sigma}\frac{d}{dx}\left(\frac{\sin\lambda x}{\lambda}\right)dx$$

$$=\frac{\rho(\xi)}{\xi}e^{i\sigma(\xi)}\frac{\sin\lambda\xi}{\lambda}+J(\lambda)=O(1/\lambda)+J(\lambda), \tag{19}$$

where

$$\left\{\begin{aligned} J(\lambda)&=\frac{1}{\lambda}\{J^{(1)}(\lambda)-i\,J^{(2)}(\lambda)\},\\ J^{(1)}(\lambda)&=\int_0^{\epsilon}R_1\,e^{i\sigma}\frac{\sin\lambda x}{x}\,dx,\\ J^{(2)}(\lambda)&=\int_0^{\epsilon}R_2\,e^{i\sigma}\frac{\sin\lambda x}{x}\,dx,\\ R_1&=\frac{\rho}{x}-\rho', \qquad R_2=\rho\sigma'. \end{aligned}\right. \tag{20}$$

The integrals $J^{(1)}$ and $J^{(2)}$ are of the type of $S(\lambda)$.

As before, we can write

$$\rho=x^{-\alpha}\,\theta(x),$$

where $a \leqq 0$ and $x^{\delta} < \theta < (1/x)^{\delta}$ as $x \to 0$.*

Then, if $a \leqq 0$ and $a \neq -1$,

$$R_1 = \frac{\rho}{x} - \rho' \sim (a+1)\, x^{-(a+1)}\theta = (a+1)\frac{\rho}{x}\,;$$

if $a = -1$,

$$R_1 = -x\,\theta' = -\theta_1,$$

where θ_1 is a function of the same type as θ and $\theta_1 < \theta$. Hence applying Theorem I, we obtain

$$\begin{cases} J^{(1)}(\lambda) = o(1) & (a < -1), \\ J^{(1)}(\lambda) \sim -(\frac{1}{2}\pi/\lambda)\theta'(1/\lambda)\, e^{i\sigma(1/\lambda)} & (a = -1), \\ J^{(1)}(\lambda) \sim -(a+1)\,\Gamma(-a-1)\sin\{\frac{1}{2}(a+1)\pi\}\lambda\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\ & (-1 < a < 0), \\ J^{(1)}(\lambda) \sim \lambda\, T(1/\lambda) & (a = 0), \end{cases}$$

where

$$T(x) = \int_0^{\infty} \rho(t) e^{i\sigma(t)} \frac{dt}{t}.$$

Observing that $R_2 = \rho\sigma' = x^{-(a+1)}\bar{\theta}(x)$, where $\bar{\theta}$ is a function of the same type as θ and $\bar{\theta} < \theta$, similarly we obtain

$$\begin{cases} J^{(2)}(\lambda) = o(1) & (a < -1), \\ J^{(2)}(\lambda) \sim \frac{1}{2}\pi\rho(1/\lambda)\,\sigma'(1/\lambda)\, e^{i\sigma(1/\lambda)} & (a = -1), \\ J^{(2)}(\lambda) \sim -\Gamma(-a-1)\sin\{\frac{1}{2}(a+1)\pi\}\rho(1/\lambda)\sigma'(1/\lambda)e^{i\sigma(1/\lambda)} & (-1 < a < 0), \\ J^{(2)}(\lambda) \sim \lambda\bar{T}(1/\lambda) & (a = 0), \end{cases}$$

where

$$\bar{T}(x) = \int_0^{\infty} \bar{\theta}(t)\, e^{i\sigma(t)} \frac{dt}{t}.$$

Hence we obtain the following results:

---

* Observe that, when $a = 0$, $x^{\delta} \prec \Theta \prec x\sigma' \prec 1$.

(i) Let $a<-1$. Then, by (20),

$$J(\lambda)=\frac{1}{\lambda}\{o(1)-i\,o(1)\}=o(1/\lambda),$$

and hence, by (19),

$$C(\lambda)=O(1/\lambda).$$

(ii) Let $a=-1$. Then

$$J(\lambda)\sim-(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\,\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)}.$$

Hence, if $x\theta'\preceq 1$ and $\rho\sigma'\preceq 1$,

$$J(\lambda)\preceq\frac{1}{\lambda},$$

and
$$C(\lambda)=O(1/\lambda);$$

if $x\theta'$ or $\rho\sigma'\succ 1$,

$$C(\lambda)\sim J(\lambda)\sim-(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda)+i\rho(1/\lambda)\sigma'(1/\lambda)\}e^{i\sigma(1/\lambda)}.$$

(iii) Let $-1<a<0$. Since $\bar{\theta}\prec\theta$, we have

$$J(\lambda)\sim\frac{1}{\lambda}J^{(1)}(\lambda),$$

and hence

$$C(\lambda)\sim\Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\,e^{i\sigma(1/\lambda)}.$$

(iv) Let $a=0$. Since $\bar{\theta}\prec\theta$, by H-lemma 10, we have

$$\bar{T}(1/\lambda)\prec T(1/\lambda),$$

and
$$J^{(2)}(\lambda)\prec J^{(1)}(\lambda).$$

Hence
$$C(\lambda)\sim T(1/\lambda).$$

We can now state

Theorem II. *The integral*

$$C(\lambda)=\int_0^{\epsilon}\rho(x)e^{i\sigma(x)}\,\frac{\cos\lambda x}{x}\,dx,$$

where $1 \prec \sigma \prec l(1/x)$ *and* $\rho \prec x\sigma'$, *is convergent. If* $\rho = x^{-a}\theta(x)$, *where* $x^{\delta} \prec \theta \prec (1/x)^{\delta}$, *so that* $a \leqq 0$, *the behaviour of* $C(\lambda)$, *as* $\lambda \to \infty$, *is determined asymptotically by the following formulae:**

$$(4)\quad \begin{cases} C(\lambda) = O(1/\lambda) & (a < -1 \text{ or } a = 1,\ x\theta' \preceq 1,\ \rho\sigma' \preceq 1), \\ C(\lambda) \sim -(\tfrac{1}{2}\pi/\lambda)\{(1/\lambda)\theta'(1/\lambda) + i\rho(1/\lambda)\sigma'(1/\lambda)\}\, e^{i\sigma(1/\lambda)} & \\ & (a = -1,\ x\theta' \text{ or } \rho\sigma' \succ 1), \\ C(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\rho(1/\lambda)\, e^{i\sigma(1/\lambda)} & (-1 < a < 0), \\ C(\lambda) \sim T(1/\lambda) & (a = 0), \end{cases}$$

where

$$T(x) = \int_0^x \rho(t) e^{i\sigma(t)} \frac{dt}{t}.$$

IV. *Discussion of Case (B)* : $\sigma(x) \asymp l(1/\lambda)$.

9. In this case, we can write

$$(21)\qquad \sigma(x) = b\, l(1/x) + \bar{\sigma}(x),$$

where $b \neq 0$ and $\bar{\sigma} \prec l(1/x)$. Then

$$\rho e^{i\sigma} = x^{-a-bi}\,\theta(x) e^{i\bar{\sigma}(x)}.$$

Hence, as Hardy remarks in the paper "*O. D. I. 1.*", the treatment of the sine-integral $S(\lambda)$ in Case (B) may be done by precisely the same method as in Case (A), by applying Lemma 1. Thus we can easily see:

If $a \leqq -1$,

$$S(\lambda) = O(\lambda^{-1+\delta});$$

if $- < a < 0$, or if $a = 0$ and $\rho \prec 1$,

$$S(\lambda) \sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\rho(1/\lambda) e^{i\sigma(1/\lambda)}.$$

Combining these results with Theorem B, we obtain

---

* Observe that here always $C(\lambda) = o(1)$ as $\lambda \to \infty$.

Theorem III. *The integral*

$$S(\lambda)=\int_0^{\varepsilon}\rho(x)e^{i\sigma(x)}\frac{\sin\lambda x}{x}\,dx,$$

where $\sigma\sim bl(1/x)$ $(b\neq 0)$ *and* $\rho\prec 1/x$, *is convergent. If* $\rho=x^{-a}\theta(x)$, *where* $x^{\delta}\prec\theta\prec(1/x)^{\delta}$, *so that* $a\leqq 1$, *the behaviour of* $S(\lambda)$, *as* $\lambda\to\infty$, *is determined asymptotically by the following formulae:*

$$(5)\quad\begin{cases} S(\lambda)=O(\lambda^{-1+\delta}) & (a\leqq -1),\\ S(\lambda)\sim -\Gamma(-a-bi)\sin\{\tfrac{1}{2}(a+bi)\pi\}\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & (-1<a\leqq 1).\end{cases}$$

10. We now consider the cosine-integral

$$C(\lambda)=\int_0^{\varepsilon}\rho(x)e^{i\sigma(x)}\frac{\cos\lambda x}{x}\,dx.$$

Since the function σ has the form (21), the condition for the convergence of this integral $C(\lambda)$ is

$$\rho\prec 1$$

as $x\to 0$.

As in Case (A), we have

$$C(\lambda)=O(1/\lambda)+J(\lambda),$$

where

$$\begin{cases} J(\lambda)=\dfrac{1}{\lambda}\displaystyle\int_0^{\varepsilon}(R_1-i\,R_2)e^{i\sigma}\frac{\sin\lambda x}{x}\,dx,\\ R_1=\dfrac{\rho}{x}-\rho',\quad R_2=\rho\sigma'.\end{cases}$$

If we write, as before,

$$\rho=x^{-a}\theta(x),$$

where $\quad a\leqq 0,\quad x^{\delta}\prec\theta\prec(1/x)^{\delta}$,*

we have

$$R_1-i\,R_2\sim(a+1+bi)\,x^{-(a+1)}\,\theta(x).$$

* Observe that, when $a=0$, $x^{\delta}\prec\Theta\prec 1$.

Hence, applying Theorem III, we obtain:
If $a<-1$, or if $a=-1$ and $\theta\preceq 1$,

$$J(\lambda)=O(1/\lambda);$$

if $a=-1$ and $\theta\succ 1$, or if $-1<a\leqq 0$,

$$J(\lambda)\sim -(a+1+bi)\,\Gamma(-a-1-bi)\sin\{\tfrac{1}{2}(a+1+bi)\pi\}\,\rho(1/\lambda)e^{i\sigma(1/\lambda)}.$$

Therefore we have the theorem.

Theorem IV. *The integral*

$$C(\lambda)=\int_0^{\xi}\rho(x)\,e^{i\sigma(x)}\frac{\cos\lambda x}{x}\,dx,$$

where $\sigma\sim bl(1/x)$ $(b\neq 0)$ *and* $\rho\prec 1$, *is convergent. If* $\rho=x^{-a}\theta(x)$, *where* $x^{\delta}\prec\theta\prec(1/x)^{\delta}$, *so that* $a\leqq 0$, *the behaviour of* $C(\lambda)$, *as* $\lambda\to\infty$, *is determined asymptotically by the following formulae :**

$$(6)\begin{cases} C(\lambda)=O(1/\lambda) & (a<-1 \text{ or } a=-1,\ \theta\preceq 1),\\ C(\lambda)\sim\Gamma(-a-bi)\cos\{\tfrac{1}{2}(a+bi)\pi\}\,\rho(1/\lambda)e^{i\sigma(1/\lambda)} & \\ \qquad\qquad (-1<a\leqq 0 \text{ or } a=-1,\ \theta\succ 1). & \end{cases}$$

V. *Examples† of the Cases (A) and (B).*

11. As examples of the two cases (A) and (B), we shall give some discussion about the behaviour of the integral

$$J(\lambda)=\int_0^1 e^{i\lambda x}\,x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx$$

as $\lambda\to\infty$, where

$$R(r)>0,\quad R(s)>0.$$

At first, we consider the case in which

---

* Observe that, here also always $C(\lambda)=o(1)$ as $\lambda\to\infty$.

† In the followings, I have given examples and verifications, quite similar to those given in Hardy's papers, for the purpose of parallelism.

$$(22) \qquad \begin{cases} r = -a, & -1<a<0, \\ s = a+\mathrm{m}i, & 0<a<1, \quad \mathrm{m}\neq 0. \end{cases}$$

Then

$$J(\lambda) = \int_0^1 e^{i\lambda x}\, x^{-a-1}\left(\log\frac{1}{x}\right)^{a-1} e^{\mathrm{m}i\log\log(1/x)}dx$$
$$= I(\lambda)+i\,\bar{I}(\lambda),$$

where

$$I(\lambda) = \int_0^1 x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{\mathrm{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx,$$

$$\bar{I}(\lambda) = \int_0^1 x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{\mathrm{m}i\log\log(1/x)}\frac{\sin\lambda x}{x}\,dx.$$

Now

$$I(\lambda) = \left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1\right)x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{\mathrm{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx$$
$$= I_1(\lambda)+I_2(\lambda)$$

say. Evidently the integral $I_1(\lambda)$ is convergent, if $a<0$. The integral $I_2(\lambda)$ may be written in the form

$$I_2(\lambda) = \int_0^{\frac{1}{2}}(1-x)^{-a-1}\left(\log\frac{1}{1-x}\right)^{a-1} e^{\mathrm{m}i\log\log\{1/(1-x)\}}\cos\lambda(1-x)dx,$$

and, when x is small, we have

$$\log\frac{1}{1-x} = x\{1+O(x)\}.$$

Hence the integral $I_2(\lambda)$ is convergent if $a>0$.

The integral $I_1(\lambda)$ may be divided into the two parts

$$I_1(\lambda) = \left(\int_0^{\xi}+\int_{\xi}^{\frac{1}{2}}\right)x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{\mathrm{m}i\log\log(1/x)}\frac{\cos\lambda x}{x}\,dx$$
$$= I_1'(\lambda)+I_1''(\lambda)$$

say, ξ being a sufficiently small positive number. Then

$$I_1'(\lambda) = \int_0^{\xi}\rho(x)e^{i\phi(x)}\frac{\cos\lambda x}{x}\,dx;$$

where $$\rho = x^{-a}\left(\log\frac{1}{x}\right)^{\alpha-1}, \quad \sigma = \mathrm{m}\log\log(1/x).$$

Applying Theorem II, we obtain

$$I_1'(\lambda) \sim \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\,\lambda^a(\log\lambda)^{\alpha-1}e^{\mathrm{mi}\log\log\lambda} \qquad (-1<a<0).$$

If we put $$f(x)=x^{-a-1}\left(\log\frac{1}{x}\right)^{\alpha-1}e^{\mathrm{mi}\log\log(1/x)},$$

then, by performing integration by parts, we have

$$I_1''(\lambda)=O(1/\lambda)-\frac{1}{\lambda}\int_\varepsilon^{\frac{1}{2}} f'(x)\sin\lambda x\,dx.$$

Evidently $f'(x)$ has no singularity and is absolutely integrable in the interval $(\xi, \frac{1}{2})$. Hence by a well known theorem*

$$\int_\varepsilon^{\frac{1}{2}} f'(x)\sin\lambda x\,dx = o(1)$$

as $\lambda\to\infty$. Hence we have

$$I_1''(\lambda)=O(1/\lambda).$$

Since $-1<a<0$, we have $I_1' > I_1''$ as $\lambda\to\infty$.
Thus we obtain

$$\begin{aligned} I_1(\lambda) &= \Gamma(-a)\cos(\tfrac{1}{2}a\pi)\lambda^a(\log\lambda)^{\alpha-1}e^{\mathrm{mi}\log\log\lambda}(1+\varepsilon_\lambda) \\ &= \Gamma(r)\cos(\tfrac{1}{2}r\pi)\,\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda), \end{aligned}$$

where $$\lim_{\lambda\to\infty}\varepsilon_\lambda = 0.$$

The integral $I_2(\lambda)$ may also be divided into the two parts

$$\begin{aligned} I_2(\lambda) &= \left(\int_0^\varepsilon+\int_\varepsilon^{\frac{1}{2}}\right)(1-x)^{-a-1}\left(\log\frac{1}{1-x}\right)^{\alpha-1}e^{\mathrm{mi}\log\log\{1/(1-x)\}}\cos\lambda(1-x)\,dx \\ &= I_2'(\lambda)+I_2''(\lambda) \end{aligned}$$

say. As in the case of $I_1''(\lambda)$, we easily see that

$$I_2''(\lambda)=O(1/\lambda).$$

---

* Hobson, Theory of Functions of a Real Variable, *p. 672.*

In considering the integral $I_2'(\lambda)$, introduce the relations

$$\log\frac{1}{1-x} = x\{1+O(x)\},$$

$$(1-x)^{-a-1} = 1+O(x),$$

$$e^{\mathrm{mi}\log\{1+O(x)\}} = 1+O(x).$$

Then we have

$$\begin{aligned} I_2'(\lambda) &= \int_0^{\varepsilon} x^a e^{-\mathrm{mi}\log(1/x)}\{1+O(x)\}\frac{\cos\lambda(1-x)}{x}\,dx \\ &= \cos\lambda\int_0^{\varepsilon} x^a e^{-\mathrm{mi}\log(1/x)}\{1+O(x)\}\frac{\cos\lambda x}{x}\,dx \\ &\quad +\sin\lambda\int_0^{\varepsilon} x^a e^{-\mathrm{mi}\log(1/x)}\{1+O(x)\}\frac{\sin\lambda x}{x}\,dx \\ &= j(\lambda)+\bar{j}(\lambda) \end{aligned}$$

say. Then, by Theorems III and IV, we have, for $0<a<1$,

$$\int_0^{\varepsilon} x^a e^{-\mathrm{mi}\log(1/x)}\frac{\cos\lambda x}{x}\,dx \sim \Gamma(a+\mathrm{mi})\cos\{\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}e^{-\mathrm{mi}\log\lambda},$$

$$\int_0^{\varepsilon} x^a e^{-\mathrm{mi}\log(1/x)}\frac{\sin\lambda x}{x}\,dx \sim \Gamma(a+\mathrm{mi})\sin\{\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}e^{-\mathrm{mi}\log\lambda}.$$

Hence, we can easily see that

$$j(\lambda) \sim \cos\lambda\,\Gamma(a+\mathrm{mi})\cos\{\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}\,e^{-\mathrm{mi}\log\lambda},$$

$$\bar{j}(\lambda) \sim \sin\lambda\cdot\Gamma(a+\mathrm{mi})\sin\{\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}e^{-\mathrm{mi}\log\lambda},$$

and

$$I_2'(\lambda) \sim \Gamma(a+\mathrm{mi})\cos\{\lambda-\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}e^{-\mathrm{mi}\log\lambda}.$$

Since $0<a<1$, evidently we have $I_2'>I_2''$ as $\lambda\to\infty$. Thus we obtain

$$\begin{aligned} I_2(\lambda) &= \Gamma(a+\mathrm{mi})\cos\{\lambda-\tfrac{1}{2}(a+\mathrm{mi})\pi\}\,\lambda^{-a}e^{-\mathrm{mi}\log\lambda}(1+\varepsilon_\lambda') \\ &= \Gamma(s)\cos(\lambda-\tfrac{1}{2}s\pi)\,\lambda^{-s}(1+\varepsilon'_\lambda), \end{aligned}$$

where $$\lim_{\lambda\to\infty}\varepsilon_\lambda' = 0.$$

Hence we have

$$I(\lambda) = \Gamma(r)\cos\left(\tfrac{1}{2}r\pi\right)\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda)$$
$$+\Gamma(s)\cos\left(\lambda-\tfrac{1}{2}s\pi\right)\lambda^{-s}(1+\varepsilon_\lambda').$$

Similarly we can prove that

$$\bar{I}(\lambda) = \Gamma(r)\sin\left(\tfrac{1}{2}r\pi\right)\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon_\lambda'')$$
$$+\Gamma(s)\sin\left(\lambda-\tfrac{1}{2}s\pi\right)\lambda^{-s}(1+\varepsilon_\lambda''').$$

Thus we obtain

$$J(\lambda) = \int_0^1 e^{i\lambda x}x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx$$
$$= \Gamma(r)\,e^{\frac{1}{2}r\pi i}\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon)$$
$$+\Gamma(s)\,\lambda^{-s}\,e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where

$$\lim_{\lambda\to\infty}\varepsilon=0,\quad \lim_{\lambda\to\infty}\varepsilon'=0,$$

r and s having the values of (22).

12. This result may be verified as follows.

Hardy proved* that, if $R(r)>0$ and $R(s)>0$, then, for pure imaginary values of t, we have

$$f_{r,s}(t) = \Gamma(s)\sum_{\nu=0}^{\infty}\frac{t^\nu}{(\nu+r)^s\,\nu!} = \int_0^1 e^{tx}x^{r-1}\left(\log\frac{1}{x}\right)^{s-1}dx$$
$$= \Gamma(r)(-t)^{-r}\{\log(-t)\}^{s-1}(1+\varepsilon_t)$$
$$+\Gamma(s)\,t^{-s}\,e^t(1+\varepsilon'_t),$$

where

$$(-t)^{-r} = \exp\{-r\log(-t)\} = \exp[-r\{\log|t|-\tfrac{1}{2}\varepsilon\pi i\}],$$
$$t^{-s} = \exp\{-s\log t\} = \exp[-s\{\log|t|+\tfrac{1}{2}\varepsilon\pi i\}],$$

**Proc. London Math. Soc.* Ser. 2, Vol. 2, *pp. 401 et seq.*

and $\varepsilon = +1$ or $\varepsilon = -1$ according as $\frac{t}{i} > 0$ or $\frac{t}{i} < 0$.

Herein put $\quad t = \lambda i \qquad (\lambda > 0)$,

then $\quad (-t)^{-r} = e^{\frac{1}{2}r\pi i}\lambda^{-r}, \qquad t^{-s} = e^{-\frac{1}{2}s\pi i}\lambda^{-s}$.

Therefore

$$(23) \quad \int_0^1 e^{i\lambda x} x^{r-1}\left(\log\frac{1}{x}\right)^{s-1} dx = \Gamma(r) e^{\frac{1}{2}r\pi i}\lambda^{-r}(\log\lambda)^{s-1}(1+\varepsilon)$$
$$+\Gamma(s)\lambda^{-s}e^{(\lambda-\frac{1}{2}s\pi)i}(1+\varepsilon'),$$

where $\quad \lim_{\lambda\to\infty} \varepsilon = 0, \qquad \lim_{\lambda\to\infty} \varepsilon' = 0.$

This formula quite agrees with our result obtained for the case in which

$$r = -a, \quad -1<a<0; \quad s = a+mi, \quad 0<a<1, \quad m \neq 0.$$

Thus our result is verified.

13. Next consider the case in which

$$\begin{cases} r = -a-bi, & -1<a<0, \quad b \neq 0, \\ s = a, & 0<a<1. \end{cases}$$

In this case

$$J(\lambda) = \int_0^1 e^{i\lambda x} x^{-a-1}\left(\log\frac{1}{x}\right)^{a-1} e^{bi\log(1/x)} dx$$

$$= \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 = J_1(\lambda) + J_2(\lambda)$$

say. Then

$$J_1(\lambda) = \int_0^{\frac{1}{2}} x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{bi\log(1/x)} \frac{\cos\lambda x}{x} dx$$

$$+ i\int_0^{\frac{1}{2}} x^{-a}\left(\log\frac{1}{x}\right)^{a-1} e^{bi\log(1/x)} \frac{\sin\lambda x}{x} dx.$$

Applying Theorems III and IV, we can easily see that

$$\begin{aligned} J_1(\lambda) &= \Gamma(-a-bi)\, e^{-\frac{1}{2}(a+bi)\pi i}\, \lambda^{a+bi} (\log \lambda)^{a-1} (1+\varepsilon) \\ &= \Gamma(r) e^{\frac{1}{2}r\pi i} \lambda^{-r} (\log \lambda)^{s-1} (1+\varepsilon), \end{aligned}$$

where $\lim\limits_{\lambda\to\infty} \varepsilon = 0.$

The integral $J_2(\lambda)$ may be written in the form

$$J_2(\lambda) = \int_0^{\frac{1}{2}} e^{i\lambda(1-x)} (1-x)^{-a-1} \left(\log \frac{1}{1-x}\right)^{a-1} e^{bi\log\{1/(1-x)\}} dx.$$

If we make use of the equations

$$\begin{cases} \log \dfrac{1}{1-x} = x\{1+O(x)\}, \quad (1-x)^{-a-1} = 1+O(x), \\ e^{bi\log\{1/(1-x)\}} = 1+O(x), \\ \displaystyle\int_0^\infty x^{a-1} \cos \lambda x \, dx = \Gamma(a) \lambda^{-a} \cos \tfrac{1}{2} a\pi \\ \displaystyle\int_0^\infty x^{a-1} \sin \lambda x \, dx = \Gamma(a) \lambda^{-a} \sin \tfrac{1}{2} a\pi \end{cases} \qquad (0<a<1),$$

we can easily prove that

$$\begin{aligned} J_2(\lambda) &= \Gamma(a)\, \lambda^{-a}\, e^{(\lambda-\frac{1}{2}a\pi)i} (1+\varepsilon') \\ &= \Gamma(s)\, \lambda^{-s} e^{(\lambda-\frac{1}{2}s\pi)i} (1+\varepsilon'), \end{aligned}$$

where $\lim\limits_{\lambda\to\infty} \varepsilon' = 0.$

Thus we obtain

$$\begin{aligned} J(\lambda) = J_1(\lambda)+J_2(\lambda) &= \Gamma(r) e^{\frac{1}{2}r\pi i} \lambda^{-r} (\log \lambda)^{s-1} (1+\varepsilon) \\ &\quad + \Gamma(s)\, \lambda^{-s}\, e^{(\lambda-\frac{1}{2}s\pi)i} (1+\varepsilon'), \end{aligned}$$

where $\lim\limits_{\lambda\to\infty} \varepsilon = 0, \qquad \lim\limits_{\lambda\to\infty} \varepsilon' = 0.$

Here again we have obtained the result which quite agrees with the formula (23) for $f_{r,s}(\lambda i)$.

14. Finally consider the case in which

$$(24) \qquad \begin{cases} r = -a-bi, & -1<a<0, \quad b\neq 0, \\ s = a+mi, & 0<a<1, \quad m\neq 0. \end{cases}$$

In this case we have

$$J(\lambda) = \int_0^1 e^{i\lambda x} x^{-a-1} \left(\log\frac{1}{x}\right)^{a-1} e^{\{b \log (1/x) + m \log\log (1/x)\} i}\, dx$$

$$= \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 = J_1(\lambda) + J_2(\lambda)$$

say. Then, proceeding as before, we easily see that the discussion of the integral $J_1(\lambda)$ may be carried out by means of the integrals

$$\int_0^{\varepsilon} x^{-a} \left(\log\frac{1}{x}\right)^{a-1} e^{\{b \log (1/x) + m \log\log (1/x)\} i}\, \frac{\sin \lambda x}{x} dx,$$

$$\int_0^{\varepsilon} x^{-a} \left(\log\frac{1}{x}\right)^{a-1} e^{\{b \log (1/x) + m \log\log (1/x)\} i}\, \frac{\cos \lambda x}{x} dx,$$

and that of $J_2(\lambda)$ by means of the integrals

$$\int_0^{\varepsilon} x^a e^{-mi \log (1/x)} \frac{\sin \lambda x}{x} dx,$$

$$\int_0^{\varepsilon} x^a e^{-mi \log (1/x)} \frac{\cos \lambda x}{x} dx.$$

Thus by another application of Theorems III and IV, we obtain

$$J(\lambda) = \Gamma(r) e^{\frac{1}{2} r\pi i} \lambda^{-r} (\log \lambda)^{s-1} (1+\varepsilon)$$

$$+ \Gamma(s) \lambda^{-s} e^{(\lambda - \frac{1}{2} s\pi) i} (1+\varepsilon'),$$

where $\qquad \lim\limits_{\lambda\to\infty} \varepsilon = 0, \qquad \lim\limits_{\lambda\to\infty} \varepsilon' = 0,$

r and s having the values of (24).

This result is nothing but the formula (23) for the case (24).

IV. *Lemmas for Case (C).*

15. Among the lemmas given in the paper "*O. D. I. 2.*", the most important ones are H-lemmas 32 and 33. They give important properties concerning the variations of the functions

$$\frac{\rho(x)}{x\{\lambda-\sigma'(x)\}}, \quad \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\ .$$

for sufficiently large values of λ, provided that

$$\sigma \succ l(1/x),$$

and $$x \prec \rho \prec x\sigma'.^{*}$$

I will give two more lemmas of a similar nature, concerning the variations of these two functions, for the case in which

$$\sigma \succ l(1/x),$$

and $$x^{\Delta} \prec \rho \preceq x.$$

16. Lemma 2. *Let* $\sigma \succ l(1/x)$.

(i) *If* $\rho \prec x$, *or if* $\rho = Ax\{1+\bar{\rho}(x)\}$,

where A *is a positive constant and*

$$\bar{\rho} \geqq 0, \quad \bar{\rho} \prec 1,$$

then the function

$$\varphi = \frac{\rho}{x(\lambda-\sigma')}$$

is a steadily increasing function of x *throughout the interval* $0<x<\xi$.

(ii) *If* $$\rho = Ax\{1-\bar{\rho}(x)\},$$

where $A>0$, $\bar{\rho}>0$ *and* $\bar{\rho} \prec 1$, *then the function* φ *has, for sufficiently large fixed values of* λ, *one and only one stationary value in the range* $0<x<\xi$, *which is a maximum and tends to zero as* $\lambda \to \infty$.

Proof. In the case (i), $\frac{\rho}{x}$ is evidently a steadily increasing function of x (or a constant when $\bar{\rho}=0$) in the interval $(0, \xi)$ and so also is the function

* In the following investigation of Case (C), we shall assume that

$$\rho > 0, \qquad \sigma > 0.$$

We can easily see that, by this assumption, no loss of generality will be introduced.

$$\frac{1}{\lambda-\sigma'(x)},$$

since $\sigma'<0$, $\sigma'\succ 1$. Hence the first part of the lemma follows immediately.

Now consider the second case (ii) of the lemma, in which

$$\rho = Ax\,\{1-\bar{\rho}(x)\},$$

where $$A>0, \quad \bar{\rho}>0, \quad \bar{\rho}\prec 1.$$

Then we have

$$\varphi = \frac{A(1-\bar{\rho})}{\lambda-\sigma'},$$

and $\dfrac{d\varphi}{dx}=0$ gives

(25) $$\lambda = \frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'}+\sigma'.$$

Let us write $$\bar{\rho} = \sigma'\gamma,$$

so that $$\gamma<0, \quad \sigma'\gamma\prec 1.$$

Then $$\frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'}+\sigma' = \frac{\sigma''+\sigma'^2\gamma'}{\bar{\rho}'};$$

and from the relation $\sigma'\gamma\prec 1$, we obtain $\gamma\prec\dfrac{1}{\sigma'}$ and, by differentiation,

$$\sigma'^2\gamma'\prec\sigma''.$$

Hence (25) becomes

(26) $$\lambda = \frac{\sigma''+\sigma'^2\gamma'}{\bar{\rho}'} \sim \frac{\sigma''}{\bar{\rho}'}.$$

Here we have

$$\sigma''\succ\frac{1}{x^2},$$

since $\sigma\succ l(1/x)$; and, since $\bar{\rho}\prec 1$, we have $x\bar{\rho}\prec x$ and, by differentiation,

$$x\bar{\rho}'+\bar{\rho} \prec 1.$$

But $$\bar{\rho}>0, \qquad \bar{\rho}<1, \qquad \bar{\rho}'>0,$$

and hence $$x\bar{\rho}'<1.$$

Therefore $$\frac{\sigma''}{\bar{\rho}'}>\frac{1}{x^2\bar{\rho}'}>\frac{1}{x}>1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (25) has one, and only one, root. Thus the function φ has one stationary value; and as φ is positive and $\varphi<1$, this value is plainly a maximum.

If the root of the equation (25) is $x=a$, then the value of $\varphi(a)$ is given by

$$\varphi(a)=\frac{A\bar{\rho}'(a)}{\sigma''(a)}=\frac{\rho(a)-a\rho'(a)}{a^2\sigma''(a)}. \tag{27}$$

For $$\varphi(a)=\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}=\frac{A\{1-\bar{\rho}(a)\}}{\lambda-\sigma'(a)},$$

and, by (25), $$\lambda-\sigma'(a)=\frac{\sigma''(a)\{1-\bar{\rho}(a)\}}{\bar{\rho}'(a)}.$$

Hence $$\varphi(a)=\frac{A\bar{\rho}'(a)}{\sigma''(a)};$$

and $$A\bar{\rho}'(x)=-\frac{d}{dx}\left(\frac{\rho}{x}\right)=\frac{\rho-x\rho'}{x^2},$$

which proves the equation (27).

Since $\dfrac{\sigma''}{\bar{\rho}'}>1$, we see that $\varphi(a)$ tends to zero as $\lambda\to\infty$

The proof of the lemma is thus completed.

17. Lemma 3. *Let* $\sigma>l(1/x)$.

(i) *If* $\rho<x$ *or if* $\rho=Ax\{1+\bar{\rho}(x)\}$,

where A is a positive constant and

$$\bar{\rho}>0, \qquad \bar{\rho}<1,$$

then the function

$$\varphi=\frac{\rho}{x(\lambda+\sigma')}$$

has, for sufficiently large fixed values of λ, *one and only one stationary value in the range* $0<x<\xi$, *which is a minimum and tends to zero as* $\lambda\to\infty$. *The value* a *of* x *corresponding to this minimum is greater than the value* θ *of* x *for which the function* φ *becomes infinite,* θ *being given by the equation* $-\sigma'(\theta)=\lambda$; *so that the function* φ *is continuous, except for this value* $x=\theta$, *and is a steadily decreasing function of* x *in the interval* $0<x<a$ *and a steadily increasing function in the interval* $a<x<\xi$.

(ii) *If*
$$\rho = Ax\{1-\bar{\rho}(x)\},$$
where $A>0$, $\bar{\rho}\geqq 0$ *and* $\bar{\rho}<1$, *then the function* φ *has no stationary value in the interval* $0<x<\xi$. *It becomes infinite for one value* θ *of* x, *given by* $-\sigma'(\theta)=\lambda$, *and otherwise it is continuous and is a steadily decreasing function of* x *throughout the interval* $0<x<\xi$.

Proof. We observe that
$$\sigma'<0,$$
and hence
$$\varphi>0,$$
if $x>\theta$, θ being the root of the equation $\lambda+\sigma'(\theta) = 0$.

(i) At first, consider the case in which
$$\rho < x,$$
and write
$$\rho = x\gamma,$$
so that
$$\gamma > 0, \quad \gamma < 1.$$
Then
$$\varphi = \frac{\gamma}{\lambda+\sigma'},$$
and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma''\gamma}{\gamma'}-\sigma'. \tag{28}$$

Now
$$\sigma'<0, \quad \sigma''>0, \quad \gamma>0, \quad \gamma'>0, \quad \frac{\sigma''\gamma}{\gamma'}>0,$$
so that we have

$$\frac{\sigma''\gamma}{\gamma'}-\sigma'>-\sigma'>1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (28) has one, and only one, root α. Thus the function φ has one stationary value.

By (28), we have

$$\lambda+\sigma'(\alpha)=\frac{\sigma''(\alpha)\gamma(\alpha)}{\gamma'(\alpha)}>0,$$

which proves that $\alpha>\theta.$

Also we have

$$\varphi(\alpha)=\frac{\gamma'(\alpha)}{\sigma''(\alpha)}=-\frac{\rho(\alpha)-\alpha\,\rho'(\alpha)}{\alpha'^2\,\sigma''(\alpha)}. \tag{29}$$

Since $\gamma<1$, we have $x\gamma'<1,$

and, from the property of σ, $\sigma''>\frac{1}{x^2},$

whence it follows that $\frac{\gamma'(x)}{\sigma''(x)}<x^2\gamma'(x)<x<1,$

so that $\varphi(\alpha)$ tends to zero as $\lambda\to\infty$.

Next consider the case in which

$$\rho=Ax\{1+\bar\rho(x)\},$$

where $A>0,\quad \bar\rho>0,\quad \bar\rho<1.$

Then $$\varphi=\frac{A(1+\bar\rho)}{\lambda+\sigma'},$$

and $\frac{d\varphi}{dx}=0$ gives

$$\lambda=\frac{\sigma''(1+\bar\rho)}{\bar\rho'}-\sigma'. \tag{30}$$

Now $\sigma'<0,\quad \sigma''>0,\quad \bar\rho>0,\quad \bar\rho'>0,$

so that we have

$$\frac{\sigma''(1+\bar{\rho})}{\bar{\rho}'} - \sigma' > -\sigma' > 1,$$

whence it follows that, for sufficiently large fixed values of λ, the equation (30) has one, and only one, root α. Thus the function φ has one stationary value.

By (30), we have

$$\lambda + \sigma'(\alpha) = \frac{\sigma''(\alpha)\{1+\bar{\rho}(\alpha)\}}{\bar{\rho}'(\alpha)} > 0,$$

which proves that $\alpha > \theta.$

Also we have

$$(31) \qquad \varphi(\alpha) = \frac{A\bar{\rho}'(\alpha)}{\sigma''(\alpha)} = -\frac{\rho(\alpha) - \alpha\rho'(\alpha)}{\alpha^2\sigma''(\alpha)}.$$

Easily we see that

$$\frac{\bar{\rho}'}{\sigma''} < 1,$$

so that $\varphi(\alpha)$ tends to zero as $\lambda \to \infty$.

(ii) Let $\rho = Ax\{1 - \bar{\rho}(x)\},$

where $A > 0, \quad \bar{\rho} > 0, \quad \bar{\rho} < 1.$

Then $$\varphi = \frac{A(1-\bar{\rho})}{\lambda + \sigma'},$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = -\frac{\sigma''(1-\bar{\rho}) + \sigma'\bar{\rho}'}{\bar{\rho}'}.$$

If we write $\bar{\rho} = \sigma'\gamma,$

so that $\gamma < 0, \quad \sigma'\gamma < 1,$

then we have

$$\lambda = -\frac{\sigma'' + \sigma'^2\gamma'}{\bar{\rho}'}. \tag{32}$$

From the relation $\sigma'\gamma < 1$, we obtain

$$\sigma'' \succ \sigma'^2\gamma'.$$

Hence $$\frac{\sigma'' + \sigma'^2\gamma'}{\bar{\rho}'} \sim \frac{\sigma''}{\bar{\rho}'} > 0,$$

since $\sigma'' > 0$ and $\bar{\rho}' > 0$. Therefore the right-hand side of (32) is ultimately negative and hence there is no stationary value of φ.

If $\bar{\rho} = 0$, then $$\varphi = \frac{A}{\lambda + \sigma'},$$

and evidently there is no stationary value of φ.

We easily see that, as $x \to 0$, φ tends to zero by negative values, and that, as $x \to \theta$ from below, $\varphi \to -\infty$ and, as $x \to \theta$ from above, $\varphi \to +\infty$. Thus in the case (i), the function φ is a steadily decreasing function of x in the interval $0 < x < a$, except for the value $x = \theta$, and it is a steadily increasing function of x in the interval $a < x < \xi$, the stationary value for $x = a$ being plainly a minimum. In the case (ii), φ is a steadily decreasing function of x throughout the interval $0 < x < \xi$, except for the value $x = \theta$.

Evidently φ is continuous throughout the interval $0 < x < \xi$, except for $x = \theta$.

The proof of the lemma is thus completed.

18. I will give other lemmas of a different type.

Lemma 4. *Let $f(y)$ and $f_1(y)$ be L-functions such that*

$$y^\delta \succ f(y) \succ (1/y)^\delta, \quad y^\delta \succ f_1(y) \succ (1/y)^\delta,$$

$$f_1(y) \sim f(y) > 0$$

as $y \to \infty$. If $y = \theta$ and $y = \theta_1$ are respectively the roots of the equations

$$yf(y) = \lambda, \quad yf_1(y) = c\lambda,$$

for large values of λ, c being a positive constant independent of λ, then

$$\theta_1 \sim c\,\theta$$

as $\lambda \to \infty$.

Proof. Evidently $yf(y)$ and $yf_1(y)$ are ultimately monotonic and tend to infinity as $y \to \infty$. Hence each of the equations

$$yf(y) = \lambda, \quad yf_1(y) = c\,\lambda$$

has, for sufficiently large values of λ, one and only one root which tends to infinity as $\lambda \to \infty$.

By hypothesis, we have

$$\theta f(\theta) = \lambda, \quad \theta_1 f_1(\theta_1) = c\,\lambda;$$

and, since $f_1(y) \sim f(y)$, we have

$$f_1(\theta_1) = f(\theta_1)(1+\varepsilon),$$

where $\varepsilon \to 0$ as $\theta_1 \to \infty$ or $\lambda \to \infty$. Hence we obtain

$$c\,\theta f(\theta) = \theta_1 f(\theta_1)(1+\varepsilon).$$

Let η be a function of λ such that

$$\theta_1 = \theta\,\eta.$$

Then we have

$$cf(\theta) = \eta f(\theta_1)(1+\varepsilon), \tag{33}$$

where $\varepsilon \to 0$ as $\lambda \to \infty$.

We have to prove that

$$\eta \sim c$$

as $\lambda \to \infty$.

Evidently η is positive and continuous for all sufficiently large values of λ, and it might tend to infinity or zero, or might oscillate finitely or infinitely as $\lambda \to \infty$.

If we suppose that $\eta \succ 1$ or η oscillates in an infinite range of values, then, corresponding to any prescribed positive number P, however great, there will exist a sequence (E) of values of λ tending to infinity, namely,

$$(E)\ :\ \lambda_1,\ \lambda_2, \ldots\ldots,\ \lambda_n, \ldots\ldots \quad (\lim_{n\to\infty} \lambda_n = \infty),$$

such that, for every λ_n, we have

$$\eta > P,$$

all values of λ in (E) being greater than a certain positive number λ_0 which can be determined corresponding to each given P.

Since $\theta = \dfrac{\theta_1}{\eta} > 1$ as λ tends to infinity, taking the values of the sequence (E), we can always choose a number a such that

$$1 < a < \eta < \theta_1/a.$$

Easily we can see that H-lemma 24 is available in our case. Hence we have

$$f(\theta) = f(\theta_1/\eta) < f(\theta_1) \qquad (f > 1),$$
$$< Kf(\theta_1)/f(\eta) \quad (f < 1).$$

Therefore, by (33), we have

$$\eta f(\theta_1)(1+\varepsilon) < cf(\theta_1) \qquad (f > 1),$$
$$< cKf(\theta_1)/f(\eta) \quad (f < 1),$$

or

$$(34) \qquad \begin{cases} \eta(1+\varepsilon) < c & (f > 1), \\ \eta f(\eta)(1+\varepsilon) < cK & (f < 1). \end{cases}$$

But $\qquad \eta > P, \quad \eta f(\eta) > \eta^{1-\delta} > P^{1-\delta};$

and the value of P may be chosen as large as we please. Hence neither of the inequalities of (34) can be true.

Thus η cannot take values which become indefinitely great as $\lambda \to \infty$.

Next, if we suppose that $\eta < 1$, or η oscillates in such a manner that it takes indefinitely small values as $\lambda \to \infty$, then, corresponding to any prescribed positive number p, however small, there will exist a sequence (E) of values of λ tending to infinity such that, for every value of λ of this sequence, we have

$$\eta < p.$$

In this case, if we write η_1 for $\frac{1}{\eta}$, then

$$\eta_1 > 1/p = P$$

for every value of λ in the sequence (E). Hence we can proceed quite similarly as in the above case, observing that

$$\frac{\theta}{\eta_1} = \theta_1 > 1$$

as $\lambda \to \infty$. Thus we see that η cannot take values which become indefinitely small as $\lambda \to \infty$.

Therefore there must exist two certain positive constants p and P such that

$$p < \eta < P.$$

But in this case we have

$$f(\theta\eta) \sim f(\theta)$$

as $\theta \to \infty$ (or $\lambda \to \infty$), since $y^\delta \succ f(y) \succ (1/y)^\delta$ as $y \to \infty$.*

Hence, by (33), we obtain

$$cf(\theta) \sim \eta f(\theta),$$

whence it follows that

$$\eta \sim c$$

as $\lambda \to \infty$. Thus the lemma is proved.

Let $f(y)$, $f_1(y)$ and c be the same as in our Lemma 4, then we have the following corollary, n denoting any positive constant.

Corollary. *If* $y = \theta$, $y = \theta_1$ *are respectively the roots of*

$$y^n f(y) = \lambda, \quad y^n f_1(y) = c\lambda$$

for large values of λ, *then*

* This can be easily shown by means of H-lemma 18.

$$\theta_1 \sim c^{\frac{1}{n}} \theta$$

as $\lambda \to \infty$.

The truth of this corollary can be inferred immediately by writing our equations in the forms

$$y\{f(y)\}^{\frac{1}{n}} = \lambda^{\frac{1}{n}}, \quad y\{f_1(y)\}^{\frac{1}{n}} = c^{\frac{1}{n}} \lambda^{\frac{1}{n}}.$$

19. Lemma 5. *Let $f(x)$ and $f_1(x)$ be L-functions such that*

$$f > 0, \qquad f_1 > 0, \qquad f_1 \succ f \succ 1$$

as $x \to 0$. If $x = \theta$, $x = \theta_1$ are respectively the roots of the equations

$$f(x) = \lambda, \qquad f_1(x) = \lambda$$

for large values of λ, then

$$\theta \geq \theta_1$$

for every sufficiently large values of λ.

If we notice that f and f_1 are ultimately monotonic and $f < f_1$ for every sufficiently small value of x, then our lemma follows immediately.

VII *Discussion of Case (C)* : $\sigma(x) \succ l(1/x)$.

20. We now pass to the discussion of the behaviour of the integrals*

$$C(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\cos \lambda x}{x} \, dx,$$

$$S(\lambda) = \int_0^{\xi} \rho(x) e^{i\sigma(x)} \frac{\sin \lambda x}{x} \, dx,$$

as $\lambda \to \infty$, when $l(1/x) \prec \sigma \prec (1/x)^{\Delta}$. It will in this case be convenient to separate the real and imaginary parts of the integrals. Thus we have to consider

* Although the sine-integral $S(\lambda)$ has already been treated by Hardy, we shall discuss it again, reproducing briefly his analysis, because, for the purpose of this paper, it is necessary to modify his argument and to extend it to the case $x^{\Delta} \prec \rho \preceq x$, while the same argument applies to the discussion of the cosine-integral $C(\lambda)$.

$$\begin{cases} I_1(\lambda) = \int_0^{\varepsilon} \rho(x) \cos \sigma(x) \dfrac{\cos \lambda x}{x} dx, & I_2(\lambda) = \int_0^{\varepsilon} \rho(x) \sin \sigma(x) \dfrac{\cos \lambda x}{x} dx, \\ I_3(\lambda) = \int_0^{\varepsilon} \rho(x) \cos \sigma(x) \dfrac{\sin \lambda x}{x} dx, & I_4(\lambda) = \int_0^{\varepsilon} \rho(x) \sin \sigma(x) \dfrac{\sin \lambda x}{x} dx. \end{cases}$$

All these integrals are convergent if

$$(35) \qquad x^{\Delta} \prec \rho(x) \prec x\sigma'(x).$$

Hence we shall suppose that this is satisfied. Then, if we put

$$\begin{cases} J_1(\lambda) = \int_0^{\varepsilon} \dfrac{\rho(x)}{x} \cos \{\lambda x + \sigma(x)\} dx, & J_2(\lambda) = \int_0^{\varepsilon} \dfrac{\rho(x)}{x} \cos \{\lambda x - \sigma(x)\} dx, \\ J_3(\lambda) = \int_0^{\varepsilon} \dfrac{\rho(x)}{x} \sin \{\lambda x + \sigma(x)\} dx, & J_4(\lambda) = \int_0^{\varepsilon} \dfrac{\rho(x)}{x} \sin \{\lambda x - \sigma(x)\} dx, \end{cases}$$

we have

$$\begin{cases} I_1(\lambda) = \frac{1}{2}\{J_1(\lambda) + J_2(\lambda)\}, & I_2(\lambda) = \frac{1}{2}\{J_3(\lambda) - J_4(\lambda)\}, \\ I_3(\lambda) = \frac{1}{2}\{J_3(\lambda) + J_4(\lambda)\}, & I_4(\lambda) = \frac{1}{2}\{-J_1(\lambda) + J_2(\lambda)\}, \end{cases}$$

and

$$(36) \qquad \begin{cases} C(\lambda) = \frac{1}{2}[J_1(\lambda) + J_2(\lambda) + i\{J_3(\lambda) - J_4(\lambda)\}], \\ S(\lambda) = \frac{1}{2}[J_3(\lambda) + J_4(\lambda) - i\{J_1(\lambda) - J_2(\lambda)\}]. \end{cases}$$

21. *Integrals J_2 and J_4.* At first we shall consider the integral

$$J_2(\lambda) = \int_0^{\varepsilon} \frac{\rho(x)}{x} \cos y \, dx,$$

where $y = \lambda x - \sigma(x)$. As x increases from 0 to ξ, y increases from $-\infty$ to $\eta = \lambda\xi - \sigma(\xi)$, and η tends to infinity with λ. Also

$$J_2(\lambda) = \int_{-\infty}^{\eta} \frac{\rho(x)}{x\{\lambda - \sigma'(x)\}} \cos y \, dy.$$

Let
$$\varphi = \frac{\rho(x)}{x\{\lambda - \sigma'(x)\}},$$

then, by H-lemma 32 and Lemma 2, we have to separate the following cases.

(i) Let $\sigma \prec x$ or $\rho = Ax(1+\bar{\rho})$, where $A>0$, $\bar{\rho}\geqq 0$, $\bar{\rho}\prec 1$. Then φ is a steadily increasing function of x throughout the interval $0<x<\xi$. Hence, by the Second Mean Value Theorem,

$$J_2(\lambda) = \frac{\rho(\xi)}{\xi\{\lambda-\sigma'(\xi)\}}\int_{\eta_1}^{\eta}\cos y\,dy \quad (-\infty<\eta_1<\eta).$$

Therefore $$J_2(\lambda) = O(1/\lambda).$$

(ii) Let $x\prec\rho\prec x\sigma'$ or $\rho=Ax(1-\bar{\rho})$, where $A>0$, $\bar{\rho}>0$, [redacted]$\prec 1$. Then φ has one stationary value in the interval $0<x<\xi$, which is a maximum given by $x=\alpha$, α being the root of the equation $\dfrac{d\varphi}{dx}=0$.

We now write

$$J_2(\lambda) = \left(\int_{-\infty}^{\beta}+\int_{\beta}^{\eta}\right)\frac{\rho(x)}{x\{\lambda-\sigma'(x)\}}\cos y\,dy = J_2'+J_2''$$

say, where $\beta=\lambda\alpha-\sigma(\alpha)$. Then, by another application of the Second Mean Value Theorem,

$$J_2' = \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\int_{\beta_1}^{\beta}\cos y\,dy \quad (-\infty<\beta_1<\beta).$$

Therefore $$J_2' = \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1) = O\{\varphi(\alpha)\}.$$

Similarly we obtain

$$J_2'' = \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1) = O\{\varphi(\alpha)\}.$$

Hence we have

$$J_2(\lambda) = \frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1) = O\{\varphi(\alpha)\}.$$

The same argument applies to the integral $J_4(\lambda)$. Hence, if $\rho\prec x$ or $\rho=Ax(1+\bar{\rho})$, where $A>0$, $\bar{\rho}\geqq 0$, $\bar{\rho}\prec 1$,

$$J_4(\lambda) = O(1/\lambda);$$

if $x < \rho < x\sigma'$ or $\rho = Ax(1-\bar{\rho})$, where $A > 0$, $\bar{\rho} > 0$, $\bar{\rho} < 1$,

$$J_4(\lambda) = \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}} \cdot O(1) = O\{\varphi(a)\}.$$

22. *Integrals J_1 and J_3.* The same methods apply to both of the integrals J_1, J_3, so that we shall consider the integral

$$(37) \qquad J_1(\lambda) = \int_0^\varepsilon \frac{\rho(x)}{x} \cos y \, dx,$$

where $y = \lambda x + \sigma(x)$. This function y has one stationary value, which is a minimum given by

$$\lambda + \sigma'(x) = 0,$$

or, say,
$$x = \theta(\lambda) = \theta,$$

θ being a positive function of λ which tends steadily to zero as $\lambda \to \infty$.

As in the paper "*O. D. I. 2.*", in the following discussion, we shall use an auxiliary function ε of λ such that

$$(38) \qquad \varepsilon > 0, \quad \varepsilon < \theta, \quad \rho(\theta) < \varepsilon\sigma'(\theta), \quad \varepsilon^2\sigma''(\theta) > 1.$$

Since $x\sigma'' \sim A\sigma'$ in our case (C), the last condition of (38) is equivalent to

$$\varepsilon^2 > \theta/\sigma'(\theta).$$

Let $\sigma'(x) = \dfrac{\mu(x)}{x}$, so that $\mu(x) > 1$, and let $\rho(x) = x\sigma'(x)\nu(x)$, so that $\nu(x) < 1$. Then the above conditions (38) for ε are equivalent to

$$(38') \qquad \varepsilon > 0, \quad \varepsilon < \theta, \quad \varepsilon > \theta\nu(\theta), \quad \varepsilon > \frac{\theta}{\sqrt{\mu(\theta)}},$$

and evidently such a choice of the function ε is always possible.

It is convenient to divide the discussion into the following two cases (i) and (ii).

(i) The case in which $x \prec \rho \prec x\sigma'$, or $\rho = Ax(1-\bar{\rho})$, where $A>0$, $\bar{\rho}\geqq 0$, $\bar{\rho}\prec 1$.

In this case, by H-lemma 33 and Lemma 3, the function

$$\varphi = \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}} \tag{39}$$

is a steadily decreasing function of x, except for the value $x = \theta$, throughout the interval $0 < x < \xi$; and we divide the integral (37) into the three parts

$$J_1(\lambda) = \int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\theta+\varepsilon}^{\xi} = J_1^{(1)} + J_1^{(2)} + J_1^{(3)}. \tag{40}$$

(ii) The case in which $x^{\Delta} \prec \rho \prec x$, or $\rho = Ax(1+\bar{\rho})$, where $A>0$, $\bar{\rho}>0$, $\bar{\rho}\prec 1$.

In this case, by Lemma 3, the function φ has one stationary value, which is a minimum given by $x = \alpha$, α being the root of the equation $\frac{d\varphi}{dx} = 0$. Hereby α is greater than θ, so that the function φ is a steadily decreasing function of x in the interval $0 < x < \alpha$, except only for the value $x = \theta$, and it is a steadily increasing function of x in the interval $\alpha < x < \xi$. As it will be proved presently, we have

$$\theta + \varepsilon < \alpha.$$

Hence we divide the integral J_1 into the four parts

$$J_1(\lambda) = \int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\theta+\varepsilon}^{\alpha} + \int_{\alpha}^{\xi} = J_1^{(1)} + J_1^{(2)} + J_1^{(3)} + J_1^{(4)}. \tag{41}$$

23. *Integrals* $J_1^{(1)}$ and $J_1^{(3)}$. As x increases from 0 to $\theta-\varepsilon$, the function $y = \lambda x + \sigma(x)$ decreases from ∞ to $\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)$, which is large and positive when θ is small and ε smaller. Also

$$J_1^{(1)} = \int_{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)}^{\infty} \left[-\frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\right] \cos y \, dy.$$

The factor $-\frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}$ which multiplies $\cos y$ is positive and monotonic, as we have already seen. Hence, by the Second Mean Value Theorem, we obtain

$$|J_1^{(1)}| < K\left[-\frac{\rho(\theta-\varepsilon)}{(\theta-\varepsilon)\{\lambda+\sigma'(\theta-\varepsilon)\}}\right]$$

$$< K\left[\frac{\rho(\theta-\varepsilon)}{(\theta-\varepsilon)\,.\,\varepsilon\sigma''(\theta-\varepsilon_1)}\right],$$

where $$0 < \varepsilon_1 < \varepsilon.$$

Now ρ, σ and all their derivatives satisfy the condition

$$x^{\Delta} \prec f \prec (1/x)^{\Delta};$$

and so each of them satisfies the relation

$$f(\theta \pm \varphi) \sim f(\theta),$$

if $\varphi \prec \theta$, in virtue of H-lemma 11.
Hence we have

$$J_1^{(1)} = \frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)}\,.\,O(1).$$

Similarly, in both of the cases (i) and (ii), we obtain

$$J_1^{(3)} = \frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)}\,.\,O(1).$$

Now $$\frac{\rho(\theta)}{\varepsilon\,\theta\,\sigma''(\theta)} = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\cdot\frac{\sqrt{2}}{\varepsilon\sqrt{\sigma''(\theta)}} \prec \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}},$$

since, by (38), $$\varepsilon^2\sigma''(\theta) \succ 1.$$

Therefore, in both of the cases (i) and (ii), we have

$$\begin{cases} J_1^{(1)} \prec \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}, \\[2ex] J_1^{(3)} \prec \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}. \end{cases}$$

24. In the case (ii), we have assumed the relation

$$\theta + \varepsilon < a,$$

which may be proved as follows.

When $\rho < x$, $x = \alpha$ is the positive root of the equation

$$f(x) = \frac{\sigma''(x)\,\gamma(x)}{\gamma'(x)} - \sigma'(x) - \lambda = 0,$$

where $$\rho = x\,\gamma(x).$$

This equation has, for sufficiently large fixed values of λ, one and only one positive root α. Hence the function $f(x)$ changes its sign when x passes through the value α.

Now, since $\sigma''(\theta) > 0$, $\gamma(\theta) > 0$, $\gamma'(\theta) > 0$, $\sigma'(\theta) + \lambda = 0$, we have

$$f(\theta) = \frac{\sigma''(\theta)\,\gamma(\theta)}{\gamma'(\theta)} > 0.$$

And
$$\begin{aligned} f(\theta+\varepsilon) &= \frac{\sigma''(\theta+\varepsilon)\,\gamma(\theta+\varepsilon)}{\gamma'(\theta+\varepsilon)} - \sigma'(\theta+\varepsilon) - \lambda \\ &= \frac{\sigma''(\theta+\varepsilon)\,\gamma(\theta+\varepsilon)}{\gamma'(\theta+\varepsilon)} - \varepsilon\,\sigma''(\theta+\varepsilon_1) \qquad (0 < \varepsilon_1 < \varepsilon) \\ &\sim \frac{\sigma''(\theta)\,\gamma(\theta)}{\gamma'(\theta)} - \varepsilon\,\sigma''(\theta) \\ &= \frac{\sigma''(\theta)\{\gamma(\theta) - \varepsilon\,\gamma'(\theta)\}}{\gamma'(\theta)}. \end{aligned}$$

Since $\gamma < 1$, we have

$$\theta\,\gamma'(\theta) \leq \gamma(\theta),$$

and, since $\varepsilon < \theta$, $$\varepsilon\,\gamma'(\theta) < \gamma(\theta).$$

Hence ultimately $$\gamma(\theta) - \varepsilon\,\gamma'(\theta) > 0,$$

and therefore $$f(\theta+\varepsilon) > 0.$$

Thus $f(\theta)$ and $f(\theta+\varepsilon)$ have ultimately the same sign.

Therefore it follows that $\theta < \theta + \varepsilon < \alpha$.

Next, when $\rho = Ax(1-\bar{\rho})$, $x = \alpha$ is the positive root of the equation

$$f(x) = \{1 + \bar{\rho}(x)\}\frac{\sigma''(x)}{\bar{\rho}'(x)} - \sigma'(x) - \lambda = 0.$$

Now $$f(\theta) = \{1+\bar{\rho}(\theta)\}\frac{\sigma''(\theta)}{\bar{\rho}'(\theta)} > 0,$$

since $$\sigma''(\theta) > 0,\quad \bar{\rho}'(\theta) > 0,\quad \bar{\rho}(\theta) < 1;\quad \sigma'(\theta) + \lambda = 0.$$

And $$f(\theta+\varepsilon) = \{1+\bar{\rho}(\theta+\varepsilon)\}\frac{\sigma''(\theta+\varepsilon)}{\bar{\rho}'(\theta+\varepsilon)} - \sigma'(\theta+\varepsilon) - \lambda$$

$$= \{1+\bar{\rho}(\theta+\varepsilon)\}\frac{\sigma''(\theta+\varepsilon)}{\bar{\rho}'(\theta+\varepsilon)} - \varepsilon\,\sigma''(\theta+\varepsilon_1) \quad (0<\varepsilon_1<\varepsilon)$$

$$\sim \frac{\sigma''(\theta)}{\bar{\rho}'(\theta)} + \sigma''(\theta)\cdot\frac{\bar{\rho}(\theta)-\varepsilon\bar{\rho}'(\theta)}{\bar{\rho}'(\theta)} > 0,$$

since $$\sigma''(\theta) > 0,\quad \bar{\rho}'(\theta) > 0,\quad \varepsilon\bar{\rho}'(\theta) < \bar{\rho}(\theta).$$

Therefore it follows that $\theta < \theta + \varepsilon < a$.

25. *Integral* $J_1^{(2)}$. We now consider the integral

$$J_1^{(2)} = \int_{\theta-\varepsilon}^{\theta+\varepsilon} \frac{\rho(x)}{x}\cos y\,dx = J_1' + J_1'',$$

where $$\begin{cases} J_1' = \displaystyle\int_\beta^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \left\{\frac{\rho(x)}{x}\cdot\frac{\cos y}{\lambda+\sigma'(x)}\right\}dy, \\ J_1'' = \displaystyle\int_\beta^{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)} \left\{-\frac{\rho(x)}{x}\cdot\frac{\cos y}{\lambda+\sigma'(x)}\right\}dy, \end{cases}$$

$$\beta = \lambda\theta + \sigma(\theta).$$

Now let us consider the following difference of integrals

$$(42)\qquad j' = \int_\beta^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\cos y\,dy - \int_\beta^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)(y-\beta)\}}}\cos y\,dy$$

which may be written in the form

$$(43)\qquad j' = \int_{x=\theta}^{x=\theta+\varepsilon} \frac{r(x)-r(\theta)}{\lambda+\sigma'(x)}\cos y\,dy + \frac{\rho(\theta)}{\theta}\int_{x=\theta}^{x=\theta+\varepsilon}\chi(y)\cos y\,dy,$$

where $$r(x) = \frac{\rho(x)}{x},\quad \chi(y) = \frac{1}{\lambda+\sigma'(x)} - \frac{1}{\sqrt{\{2\sigma''(\theta)(y-\beta)\}}}.$$

By the analysis of §§ 33—35 of "*O. D. I. 2.*", we see that

$$\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon_2} \chi(y)\cos y\,dy < \frac{1}{\sqrt{\sigma''(\theta)}},$$

$$\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon_2} \frac{\cos y\,dy}{\lambda+\sigma'(x)} < \frac{K}{\sqrt{\sigma''(\theta)}},$$

where $$0 \leqq \varepsilon_1 \leqq \varepsilon_2 \leqq \varepsilon.$$

Hence we have

$$\frac{\rho(\theta)}{\theta}\int_{x=\theta}^{x=\theta+\varepsilon} \chi(y)\cos y\,dy < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}. \tag{44}$$

If we put $$k' = \int_{x=\theta}^{x=\theta+\varepsilon} \frac{r(x)-r(\theta)}{\lambda+\sigma'(x)}\cos y\,dy,$$

then as in "*O. D. I. 2.*", we obtain

$$k' = \{r(\theta+\varepsilon)-r(\theta)\}\int_{x=\theta+\varepsilon_1}^{x=\theta+\varepsilon} \frac{\cos y}{\lambda+\sigma'(x)}\,dy \qquad (0<\varepsilon_1<\varepsilon),$$

and hence $$|k'| < \frac{K\,\varepsilon\, r'(\theta)}{\sqrt{\sigma''(\theta)}};$$

and, if $x < \rho < x\sigma'$,

$$\varepsilon\, r'(\theta) < r(\theta).$$

If $x^4 < \rho \leq x$, we have $r \leq 1$ and we easily see that $xr' \leq r$, whence it also follows that

$$\varepsilon\, r'(\theta) < r(\theta).$$

Therefore, in both of the cases (i) and (ii), we have

$$|k'| < \frac{r(\theta)}{\sqrt{\sigma''(\theta)}},$$

or $$k' < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Introducing (44) and this result into (43), we obtain

$$j' < \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Hence, by (42), we have

$$J_1' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\left\{\int_\beta^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1)\right\}.$$

Similarly we obtain

$$J_1'' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\left\{\int_\beta^{\lambda(\theta-\varepsilon)+\sigma(\theta-\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1)\right\}.$$

Now $\qquad \lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon) = \beta+\frac{1}{2}\varepsilon^2\sigma''(\theta+\varepsilon_1),$

where $0<\varepsilon_1<\varepsilon$, so that $\qquad \sigma''(\theta+\varepsilon_1) \sim \sigma''(\theta)$;

and, by (38), $\qquad \varepsilon^2\sigma''(\theta) \succ 1.$

Hence we have

$$\int_\beta^{\lambda(\theta+\varepsilon)+\sigma(\theta+\varepsilon)} \frac{\cos y}{\sqrt{(y-\beta)}}\,dy = \int_\beta^\infty \frac{\cos y}{\sqrt{(y-\beta)}}\,dy + o(1).$$

But

$$\int_\beta^\infty \frac{\cos y}{\sqrt{(y-\beta)}}\,dy = \cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}.$$

Therefore we obtain

$$J_1' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

Similarly

$$J_1'' = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

Hence, in both of the cases (i) and (ii), we obtain

$$J_1^{(2)} = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}.$$

26. *Integral* $J_1^{(4)}$. Finally we consider the integral

$$J_1^{(4)} = \int_a^\xi \frac{\rho(x)}{x}\cos y\,dx = \int_{x=a}^{x=\xi} \frac{\rho(x)}{x\{\lambda+\sigma'(x)\}}\cos y\,dy.$$

In this integral, $\dfrac{\rho(x)}{x\{\lambda+\sigma'(x)\}}$ is a steadily increasing function of x in the interval $a<x<\xi$. Hence, by the Second Mean Value Theorem, we obtain

$$J_1^{(4)} = \frac{\rho(\xi)}{\xi\{\lambda+\sigma'(\xi)\}} \cdot O(1) = O(1/\lambda).$$

27. Introducing the results of §§ 23, 25 and 26 into (40) and (41), we obtain:

In the case (i)

$$J_1(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\};$$

in the case (ii)

$$J_1(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''\,\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda).$$

Similarly we obtain:

In the case (i)

$$J_3(\lambda) = \frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\};$$

in the case (ii)

$$J_3(\lambda) = \frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda).$$

Now we can state

Theorem V. *If* $x^4<\rho<x$ *or* $\rho=Ax\{1+\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho}>0$ *and* $\bar{\rho}<1$, *then*

$$J_1(\lambda)=\frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda),\quad J_2(\lambda)=O(1/\lambda),$$

$$J_3(\lambda)=\frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}+O(1/\lambda),\quad J_4(\lambda)=O(1/\lambda);$$

if $x<\rho<x\sigma'$ *or* $\rho=Ax\{1-\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho}>0$ *and* $\bar{\rho}<1$, *then*

$$J_1(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\},\quad J_2(\lambda)=\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}\cdot O(1),$$

$$J_3(\lambda)=\frac{2\,\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\},\quad J_4(\lambda)=\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}\cdot O(1);$$

finally, if $\rho = Ax$, *where* $A > 0$, *then*

$$J_1(\lambda) = \frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}, \quad J_2(\lambda) = O(1/\lambda),$$

$$J_3(\lambda) = \frac{2\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}+o(1)\}, \quad J_4(\lambda) = O(1/\lambda);$$

and, in these formulae, θ *and* α *are functions of* λ *determined respectively by the equations*

$$\sigma'(\theta) + \lambda = 0, \quad \frac{d}{d\alpha}\left[\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\right] = 0,$$

and
$$\beta = \lambda\theta + \sigma(\theta).$$

Corollary 1. *If* $x^4 \prec \rho \prec x$ *or* $\rho = Ax\{1+\bar{\rho}(x)\}$, *where* $A > 0$, $\bar{\rho} \geqq 0$ *and* $\bar{\rho} \prec 1$, *then*

$$\begin{cases} C(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+O(1/\lambda), \\[2ex] S(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+O(1/\lambda); \end{cases}$$

if $x \prec \rho \prec x\sigma'$ *or* $\rho = Ax\{1-\bar{\rho}(x)\}$, *where* $A > 0$, $\bar{\rho} > 0$ *and* $\bar{\rho} \prec 1$, *then*

$$\begin{cases} C(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+\dfrac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1), \\[2ex] S(\lambda) = \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\{e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}+o(1)\}+\dfrac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}}\cdot O(1); \end{cases}$$

θ, α, β *being the same as in the theorem.*

By H-lemma 32 and Lemma 2, we know that

$$\frac{\rho(\alpha)}{\alpha\{\lambda-\sigma'(\alpha)\}} \prec 1$$

as $\lambda \to \infty$, and evidently $\dfrac{1}{\lambda} \to 0$ as $\lambda \to \infty$. But $\dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$ never tends to zero as $\lambda \to \infty$, if $\rho \succeq x\sqrt{\sigma''}$. Hence we have

Corollary 2. *If* $x\sqrt{\sigma''} \preceq \rho \prec x\sigma'$, *then*

$$(45)\qquad \begin{cases} C(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}, \\[2ex] S(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \end{cases}$$

θ, β having the meanings defined in the theorem.

This formula for $S(\lambda)$ is nothing but the one obtained in "*O. D. I. 2*". In the followings we are going to prove that the formulae (45) hold also when $\rho \prec x\sqrt{\sigma''}$ so long as $C(\lambda) \succ 1/\lambda$ and $S(\lambda) \succ 1/\lambda$ as $\lambda \to \infty$.

28. For the purpose of this paper, it is necessary to compare the order of magnitude of

$$\frac{1}{\lambda}, \quad \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}}, \quad \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \to \infty$, when $\rho \prec x\sqrt{\sigma''}$.

At first, we consider the first and the last of these functions. Now, since $\lambda + \sigma'(\theta) = 0$, we have

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \Big/ \frac{1}{\lambda} = -\frac{\rho(\theta)\,\sigma'(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}.$$

Hence, if $\rho \preceq x\sqrt{\sigma''}/\sigma'$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \preceq \frac{1}{\lambda};$$

if $\rho \succ x\sqrt{\sigma''}/\sigma'$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \succ \frac{1}{\lambda}.$$

We know that, if $\sigma \succ l(1/x)$, then

$$\sigma' \succ \sqrt{\sigma''}. \qquad \text{[H-lemma 31]}$$

Hence, if $\rho \succeq x$, then

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \succ \frac{1}{\lambda}.$$

Therefore we obtain :

If $\rho \preceq x\sqrt{\sigma''}/\sigma'$, *then*

$$C(\lambda) = O(1/\lambda), \qquad S(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''}/\sigma' \prec \rho \prec x$ *or if* $\rho = Ax\{1+\bar{\rho}(x)\}$, *where* $A>0$, $\bar{\rho} \geqq 0$ *and* $\bar{\rho} \prec 1$, *then the formulae* (45) *hold, i.e.*,

$$\begin{cases} C(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi}, \\ S(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\, e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \end{cases} \qquad (\lambda\to\infty).$$

29. It now remains to compare the order of magnitude of

$$\varphi(a) = \frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}, \qquad \psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda\to\infty$, when $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A>0$, $\bar{\rho}>0$ and $\bar{\rho} \prec 1$.

When $x \prec \rho \prec x\sqrt{\sigma''}$, we write, as before,

(46) $$\rho(x) = x^{-a}\,\theta(x),$$

where $a \geqq -1$ and $x^{\delta} \prec \theta \prec (1/x)^{\delta}$ as $x\to 0$. We observe that $\theta \succ 1$, if $a = -1$.

Under the supposition that $l(1/x) \prec \sigma \prec (1/x)^{A}$, we can write

(47) $$\sigma'(x) = -\,x^{-1-b}\,\theta_1(x),$$

where $b \geqq 0$ and $x^{\delta} \prec \theta_1 \prec (1/x)^{\delta}$ as $x\to 0$. We observe that $\theta_1 \succ 1$, if $b = 0$, since $\sigma' \succ 1/x$.

From the condition $\rho \prec x\sqrt{\sigma''}$, we obtain

$$a < \tfrac{1}{2}b,$$

or

$$a = \tfrac{1}{2}b, \quad \theta \prec \theta_1^{\frac{1}{2}}.$$

We have to separate the discussion into several cases as follows.

30. (i) Let $a > -1$.

At first we consider the case in which

$$b > 0 \qquad \text{or} \qquad b = 0, \quad -1 < a < 0.$$

Then we have $$a < b,$$
since $a \leqq \frac{1}{2}b$.

Let us write $$\frac{\rho}{x} = \sigma'\gamma(x),$$
so that $$\gamma = -x^{b-a}\,\theta/\theta_1.^*$$
Then $$\gamma < 0, \quad \gamma \prec 1$$
since $a < b$. Now

$$\varphi(x) = \frac{\rho}{x(\lambda-\sigma')} = \frac{\sigma'\gamma}{\lambda-\sigma'},$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma'^2\gamma'}{\sigma'\gamma'+\sigma''\gamma}. \tag{48}$$

From (47) and the above expression for γ, we obtain

$$x\gamma' \sim (b-a)\gamma, \qquad x\sigma'' \sim -(1+b)\sigma',$$

and $$\frac{\sigma'^2\gamma'}{\sigma'\gamma'+\sigma''\gamma} \sim -\frac{b-a}{a+1}\sigma'.$$

Hence (48) becomes $$\lambda \sim -\frac{b-a}{a+1}\sigma',$$

or $$-\sigma'\{1+e(x)\} = \frac{a+1}{b-a}\lambda,$$

where $e \prec 1$ as $x \to 0$. Now $x = \alpha$ is the root of this equation, while $x = \theta$ is that of the equation

$$-\sigma' = \lambda.$$

Therefore, by the corollary to Lemma 4, we obtain

$$\alpha \sim c\theta, \qquad c = \left(\frac{b-a}{a+1}\right)^{\frac{1}{1+b}}$$

as $\lambda \to \infty$.

Now $$\varphi(\alpha) = \frac{\sigma'(\alpha)\gamma(\alpha)}{\lambda-\sigma'(\alpha)} \sim -\frac{a+1}{1+b}\gamma(\alpha),$$

---

* Observe that $\Theta > 0$ and $\Theta_1 > 0$, which follow from the assumption $\rho > 0$ and $\sigma > 0$.

and
$$\gamma(a) = -a^{b-a}\frac{\theta(a)}{\theta_1(a)}$$
$$\sim -c^{b-a}\,\theta^{b-a}\frac{\theta(\theta)}{\theta_1(\theta)} = c^{b-a}\gamma(\theta),$$

since
$$\begin{cases} a^{b-a} \sim (c\theta)^{b-a} \\ \theta(a) \sim \theta(c\theta) \sim \theta(\theta), \\ \theta_1(a) \sim \theta_1(c\theta) \sim \theta_1(\theta). \end{cases}$$

Hence we have
$$\varphi(a) \sim -\frac{a+1}{b+1}c^{b-a}\gamma(\theta) = K\frac{\rho(\theta)}{\theta\,\sigma'(\theta)}.$$

Therefore
$$\frac{\varphi(a)}{\psi(\theta)} \sim K\frac{\sqrt{\{2\sigma''(\theta)\}}}{\sigma'(\theta)} \prec 1$$

since $\sqrt{\sigma''} \prec \sigma'$. Thus we obtain
$$\varphi(a) \prec \psi(\theta)$$

as $\lambda \to \infty$.

Next consider the case in which
$$b = 0; \quad a = 0.$$

Then we have
$$\begin{cases} \rho = \theta(x), & \sigma' = -x^{-1}\theta_1(x), \\ \theta \prec \sqrt{\theta_1}, & \theta_1 \succ 1. \end{cases}$$

In this case
$$\varphi = \frac{\theta}{x(\lambda-\sigma')} = \frac{\sigma'\gamma}{\lambda-\sigma'};$$

$$\gamma = -\frac{\theta}{\theta_1}, \quad x\gamma' \prec \gamma, \quad \sigma'' \sim -x^{-1}\sigma'.$$

Hence
$$\frac{\sigma'^2\gamma'}{\sigma'\gamma'+\sigma''\gamma} \sim \sigma'\frac{x\gamma'}{\gamma} \prec \sigma',$$

and the equation (48) takes the form

$$-\sigma'(x)\, t(x) = \lambda,$$

where
$$t \sim \frac{x\gamma'}{\gamma} \prec 1$$

as $x \to 0$.

As $x = a$ is the root of this equation, we have

$$\frac{\lambda}{\sigma'(a)} = -\, t(a) \prec 1,$$

whence

$$\varphi(a) = \frac{\gamma(a)}{\lambda/\sigma'(a) - 1} \sim -\,\gamma(a) = \theta(a)/\theta_1(a);$$

and

$$a < \theta$$

by Lemma 5.

Now
$$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sim \frac{\theta(\theta)}{\sqrt{\{2\,\theta_1(\theta)\}}},$$

and hence we have

$$\frac{\varphi(a)}{\psi(\theta)} \sim \sqrt{2}\cdot\frac{\theta(a)}{\theta_1(a)}\cdot\frac{\sqrt{\theta_1(\theta)}}{\theta(\theta)}.$$

We may write

$$\frac{\theta(a)}{\theta_1(a)}\cdot\frac{\sqrt{\theta_1(\theta)}}{\theta(\theta)} = \frac{\theta(a)}{\theta(\theta)}\cdot\frac{\theta_1(\theta)}{\theta_1(a)}\cdot\frac{1}{\sqrt{\theta_1(\theta)}}$$

$$= \frac{\theta(a)}{\sqrt{\theta_1(a)}}\cdot\left\{\frac{\theta_1(\theta)}{\theta_1(a)}\right\}^{\frac{1}{2}}\cdot\frac{1}{\theta(\theta)};$$

and we have

$$\frac{\theta(a)}{\sqrt{\theta_1(a)}} \prec 1, \qquad \frac{1}{\sqrt{\theta_1(\theta)}} \prec 1, \qquad \frac{\theta_1(\theta)}{\theta_1(a)} < 1.$$

For $\theta \prec \sqrt{\theta_1}$, $\theta_1 > 1$ and $a < \theta$; and

if $\theta \prec 1$,
$$\frac{\theta(a)}{\theta(\theta)} < 1;$$

if $\theta \sim A$, $$\frac{\theta(a)}{\theta(\theta)} \sim 1;$$

if $\theta > 1$, $$\frac{1}{\theta(\theta)} < 1.$$

Hence it follows that in all cases we have

$$\psi\ \frac{\varphi(a)}{\cancel{\chi}(\theta)} < 1.$$

Thus we obtain $$\varphi(a) < \psi(\theta)$$

as $\lambda \to \infty$.

31. (ii) Let $a = -1$ and $\theta > 1$.

In this case we have

$$\rho = x\theta, \qquad \varphi = \frac{\theta}{\lambda - \sigma'};$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = -\frac{\sigma''\theta}{\theta'} + \sigma'.$$

Now $$-\frac{\sigma''\theta}{\theta'} \sim (1+b)\,\sigma'.\,\frac{\theta}{x\theta'} > \sigma'$$

since $\theta > x\theta'$. Hence the equation for $x = a$ takes the form

$$-\sigma'(x)\,t\,(x) = \lambda,$$

where $t \sim -(1+b)\dfrac{\theta}{x\theta'} > 1$. Hence we have

$$\frac{-\sigma'(a)}{\lambda} = \frac{1}{t(a)} < 1,$$

whence

$$\varphi(a) = \frac{\theta(a)}{\lambda - \sigma'(a)} \sim \frac{\theta(a)}{\lambda};$$

and $$a > \theta$$

by Lemma 5.

Now $$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \frac{\theta(\theta)}{\sqrt{\{2\sigma''(\theta)\}}},$$

and hence

$$\frac{\varphi(\alpha)}{\psi(\theta)} \sim \frac{\Theta(\alpha)}{\Theta(\theta)} \cdot \frac{\sqrt{\{2\sigma''(\theta)\}}}{\lambda}$$

$$= -\frac{\Theta(\alpha)}{\Theta(\theta)} \cdot \frac{\sqrt{\{2\sigma''(\theta)\}}}{\sigma'(\theta)} \prec 1,$$

since $\sqrt{\sigma''} \prec \sigma'$ and $\dfrac{\Theta(\alpha)}{\Theta(\theta)} < 1$, as $\theta \succ 1$ and $\alpha > \theta$.

Thus we obtain $$\varphi(\alpha) \prec \psi(\theta)$$

as $\lambda \to \infty$.

32. (iii) Let $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A > 0$, $\bar{\rho} > 0$ and $\bar{\rho} \prec 1$.

In this case

$$\varphi = \frac{A(1-\bar{\rho})}{\lambda-\sigma'},$$

and $\dfrac{d\varphi}{dx} = 0$ gives

$$\lambda = \frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'} + \sigma'.$$

In § 16, we have seen that

$$\frac{\sigma''(1-\bar{\rho})}{\bar{\rho}'} + \sigma' \sim \frac{\sigma''}{\bar{\rho}'}, \qquad x\bar{\rho}' \prec 1.$$

Hence, observing that $x\sigma'' \sim -(1+b)\sigma'$, we have

$$\frac{\sigma''}{\bar{\rho}'} \sim -(1+b)\sigma' . \frac{1}{x\bar{\rho}'} \succ \sigma'.$$

Therefore the equation for $x = \alpha$ takes the form

$$-\sigma'(x)t(x) = \lambda,$$

where $t \sim \dfrac{1+b}{x\bar{\rho}'} \succ 1$; and hence

$$\frac{-\sigma'(\alpha)}{\lambda} = \frac{1}{t(\alpha)} \prec 1,$$

whence $$\varphi(\alpha) = \frac{A\{1-\bar{\rho}(\alpha)\}}{\lambda-\sigma'(\alpha)} \sim \frac{A}{\lambda}.$$

Now $$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \sim \frac{A}{\sqrt{\{2\sigma''(\theta)\}}},$$

and hence, observing that $\lambda = -\sigma'(\theta)$, we have

$$\frac{\varphi(a)}{\psi(\theta)} \sim \frac{\sqrt{\{2\sigma''(\theta)\}}}{\lambda} = \frac{\sqrt{\{2\sigma''(\theta)\}}}{-\sigma'(\theta)} \prec 1$$

since $\sqrt{\sigma''} \prec \sigma'$. Thus we obtain

$$\varphi(a) \prec \psi(\theta)$$

as $\lambda \to \infty$.

Thus we have completely proved the following proposition.

Let $x \prec \rho \prec x\sqrt{\sigma''}$ *or* $\rho = Ax\{1-\bar{\rho}(x)\}$, *where* $A > 0$, $\bar{\rho} > 0$ *and* $\bar{\rho} \prec 1$. *Then*

$$\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}} \prec \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}$$

as $\lambda \to \infty$, a *and* θ *being functions of* λ *determined respectively by the equations*

$$\frac{d}{da}\left[\frac{\rho(a)}{a\{\lambda-\sigma'(a)\}}\right] = 0, \quad \sigma'(\theta)+\lambda = 0.$$

33. In the above arguments, except in a few special cases, the whole thing depends on the fundamental Lemma 4. We can also prove the same proposition, with exception of a few special cases, by a more direct method without recourse to this lemma. The principal object of the method is to find such asymptotic expressions in terms of λ for the functions $\varphi(a)$ and $\psi(\theta)$, which are of convenient forms for the purpose of comparing their order of magnitude as $\lambda \to \infty$. The analysis is not very difficult and I content myself with giving only the following results.

Du Bois-Reymond proved* that, *if* y *be the root of the equation*

---

* *Math. Annalen* Bd. VIII (1875), *pp. 394 et seq.* Du Bois-Reymond does not state clearly the conditions to which his functions are subjected.

$$y f(y) = \lambda,$$

where $f(y)$ is an L-function such that $y^\delta \succ f(y) \succ (1/y)^\delta$ as $y \to \infty$, then, for large values of λ, we have

$$y = \lambda \{f(\lambda)\}^{-1+\nu}$$

where ν is a certain function of λ, tending to zero as $\lambda \to \infty$.

Easily we can prove that

$$\lambda^\delta \succ \{f(\lambda)\}^{-1+\nu} \succ (1/\lambda)^\delta$$

as $\lambda \to \infty$. Hence we have

Lemma 6. *Let $f(y)$ be an L-function such that*

$$y^\delta \succ f(y) \succ (1/y)^\delta$$

as $y \to \infty$. If $y = \theta$ is the root of the equation

$$y f(y) = \lambda$$

for large values of λ, then θ can be expressed in the form

$$\theta = \lambda g(\lambda),$$

where g is a certain function of the same type as f, namely,

$$y^\delta \succ g(y) \succ (1/y)^\delta$$

as $y \to \infty$.

As before, write

$$\rho = x^{-a}\, \Theta(x), \qquad \sigma' = -\, x^{-(1+b)}\, \Theta_1(x).$$

Then, applying Lemma 6, we arrive at the following result.

Let $x \prec \rho \prec x\sqrt{\sigma''}$ or $\rho = Ax\{1-\bar{\rho}(x)\}$, where $A > 0$, $\bar{\rho} > 0$ and $\bar{\rho} \prec 1$. Then, if $b > 0$, we have, for large values of λ,

$$\varphi(a) = \frac{\rho(a)}{a\{\lambda - \sigma'(a)\}} = \lambda_1^{a-b}\, g(\lambda_1),$$

$$\psi(\theta) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \lambda_1^{a-\frac{1}{2}b}\, h(\lambda_1),$$

where $\lambda_1 = \lambda^{\frac{1}{1+b}}$, *and* g, h *are certain functions satisfying the condition*

$$y^\delta > f(y) > (1/y)^\delta$$

as $y \to \infty$.

Hence we have

$$\frac{\varphi(a)}{\psi(\theta)} = \lambda_1^{-\frac{1}{2}b} \frac{g(\lambda_1)}{h(\lambda_1)} < 1$$

as $\lambda_1 \to \infty$, since $b > 0$. Thus we obtain

$$\varphi(a) < \psi(\theta)$$

as $\lambda \to \infty$.

34. We can now state

Theorem VI. *The integrals*

$$S(\lambda) = \int_0^\xi \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

$$C(\lambda) = \int_0^\xi \rho(x)\, e^{i\sigma(x)} \frac{\cos \lambda x}{x}\, dx,$$

where $l(1/x) < \sigma < (1/x)^\Delta$ *and* $\rho < x\sigma'$, *are convergent. The behaviour o these integrals, as* $\lambda \to \infty$, *is determined asymptotically as follows.*

If $x^\Delta < \rho \leq x\sqrt{\sigma''/\sigma'}$, *then*

$$S(\lambda) = O(1/\lambda), \qquad C(\lambda) = O(1/\lambda);$$

if $x\sqrt{\sigma''/\sigma'} < \rho < x\sigma'$, *then*

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi}, \tag{7}$$

$$C(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta + \frac{1}{4}\pi)i} \sqrt{\pi}, \tag{8}$$

where
$$\beta = \lambda\theta + \sigma(\theta),$$

and θ *is determined as a function of* λ *by the equation*

$$\sigma'(\theta) + \lambda = 0.$$

Corollary. *If*

$$\begin{cases} I_1(\lambda) = \int_0^\varepsilon \rho(x)\cos\sigma(x)\dfrac{\cos\lambda x}{x}\,dx, & I_2(\lambda) = \int_0^\varepsilon \rho(x)\sin\sigma(x)\dfrac{\cos\lambda x}{x}\,dx, \\ I_3(\lambda) = \int_0^\varepsilon \rho(x)\cos\sigma(x)\dfrac{\sin\lambda x}{x}\,dx, & I_4(\lambda) = \int_0^\varepsilon \rho(x)\sin\sigma(x)\dfrac{\sin\lambda x}{x}\,dx, \end{cases}$$

then

$$\begin{cases} I_1(\lambda) = O(1/\lambda), & I_2(\lambda) = O(1/\lambda), \\ I_3(\lambda) = O(1/\lambda), & I_4(\lambda) = O(1/\lambda), \end{cases}$$

or

$$\begin{cases} I_1(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_2(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_3(\lambda) \sim \dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\sin(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}, \\ I_4(\lambda) \sim -\dfrac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}\cos(\beta+\tfrac{1}{4}\pi)\sqrt{\pi}, \end{cases} \tag{49}$$

under conditions the same as those of Theorem VI.

It may be remarked that all the integrals $S(\lambda)$, $C(\lambda)$, $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ and $I_4(\lambda)$ tend to zero as $\lambda\to\infty$, when $\rho \prec x\sqrt{\sigma''}$.

35. In Case (*C*), the sine-integral $S(\lambda)$ is still convergent when

$$x\sigma' \preceq \rho \prec \sigma'$$

as $x\to 0$. Hardy has not proceeded to the discussion in this case, his method ceasing to be applicable, as he remarks.* I have succeeded in proving that the formula (7) of Theorem VI holds also in this case generally.

The following proof is not complete, having some inaccuracy in a few special cases. At first it will be shown that the proof may be carried out in its full generality if we make an assumption

* "*O. D. I. 2*" *p. 260.*

which appears to be quite probable ; and after that, a rigorous proof will be given, with exception of a few special cases.

Let

$$(50)\qquad \begin{cases} S(\lambda) = \int_0^{\varepsilon} \rho(x)\, e^{i\sigma(x)} \dfrac{\sin \lambda x}{x}\, dx, \\ \bar{S}(\lambda) = \int_0^{\varepsilon} \bar{\rho}(x)\, e^{i\sigma(x)} \dfrac{\sin \lambda x}{x}\, dx, \end{cases}$$

where σ, ρ, $\bar{\rho}$ are L-functions such that

$$(51)\qquad l(1/x) \prec \sigma \prec (1/x)^{\Delta}, \quad x\sigma' \preceq \rho \prec \sigma', \quad \bar{\rho} \prec \rho$$

as $x \to 0$.

Now we shall assume that, for all sufficiently large values of λ,

$$(52)\qquad \left| \frac{\bar{S}(\lambda)}{S(\lambda)} \right| < K,$$

K being a certain positive constant, independent of λ.

We observe that this relation (52) evidently holds when σ, ρ are the functions treated in Theorems I, III and VI; namely when they belong to each one of the cases

(i) $\quad \sigma \prec l(1/x), \quad \rho \prec \sigma', \quad S(\lambda) \succ 1/\lambda;$

(ii) $\quad \sigma \sim A\, l(1/x), \quad \rho \prec \sigma', \quad S(\lambda) \succ 1/\lambda;$

(iii) $\quad \sigma \succ l(1/x), \quad x\sqrt{\sigma''}/\sigma' \prec \rho \prec x\sigma'.$

In the followings it will be seen that the same relation holds also in our case (51), except in a few special cases. Hence the above assumption seems very likely to be admissible.

36. With the above assumption, we can prove the lemma.

Lemma 7. *If* $S(\lambda)$, $\bar{S}(\lambda)$ *are the integrals of (50), then*

$$\bar{S}(\lambda) \prec S(\lambda)$$

as $\lambda \to \infty$.

Proof. It is convenient to separate our integrals into the real and imaginary parts; the same methods apply to both parts. Thus we consider

$$I(\lambda)=\int_0^{\xi}\rho(x)\cos\sigma(x)\frac{\sin\lambda x}{x}\,dx,$$

$$\bar{I}(\lambda)=\int_0^{\xi}\bar{\rho}(x)\cos\sigma(x)\frac{\sin\lambda x}{x}\,dx.$$

These integrals are convergent, if $\rho \prec \sigma'$.

Put
$$\varepsilon(x)=\frac{\bar{\rho}(x)}{\rho(x)}.$$

Then $\varepsilon(x)$ is ultimately monotonic and tends to zero as $x\to 0$; and we assume that ξ is chosen sufficiently small to ensure that $\varepsilon(x)$ is monotonic in the interval $0<x<\xi$.

We may write

$$\bar{I}(\lambda)=\int_0^{\xi}\varepsilon(x)\,\rho(x)\cos\sigma(x)\frac{\sin\lambda x}{x}\,dx$$

$$=\int_0^{\xi}\varepsilon(x)f(x)\sin\lambda x\,dx,$$

where
$$f(x)=\frac{\rho(x)\cos\sigma(x)}{x},$$

so that
$$I(\lambda)=\int_0^{\xi}f(x)\sin\lambda x\,dx.$$

Now, corresponding to any prescribed positive number δ, however small, there always exists a positive number ξ', independent of λ, such that

$$0<\varepsilon(\xi')<\delta, \qquad (0<\xi'<\xi).$$

We have

$$\bar{I}(\lambda)=\left(\int_0^{\xi'}+\int_{\xi'}^{\xi}\right)\varepsilon(x)f(x)\sin\lambda x\,dx$$

$$=J^{(1)}(\lambda)+J^{(2)}(\lambda)$$

say. Then, in the integral

$$J^{(2)}(\lambda)=\int_{\xi'}^{\xi}\varepsilon(x)f(x)\sin\lambda x\,dx,$$

the coefficient of $\sin \lambda x$ in the subject of integration is absolutely integrable in the range of integration $\xi' \leqq x \leqq \xi$. Hence, by a well-known theorem,* we have

$$J^{(2)}(\lambda) = o(1)$$

as $\lambda \to \infty$.

In the integral

$$J^{(1)}(\lambda) = \int_0^{\xi'} \epsilon(x) f(x) \sin \lambda x \, dx,$$

$\epsilon(x)$ is monotonic and, being an L-function, has a differential coefficient with a constant sign in the range of integration. Hence, by the Second Mean Value Theorem, we have

$$\begin{aligned} J^{(1)}(\lambda) &= \epsilon(\xi') \int_{\xi_1}^{\xi'} f(x) \sin \lambda x \, dx \quad (0 < \xi_1 < \xi') \\ &= \epsilon(\xi') \left(\int_0^{\xi} - \int_0^{\xi_1} \right) f(x) \sin \lambda x \, dx \\ &= \epsilon(\xi') \{ j(\lambda) - j'(\lambda) \} \end{aligned}$$

say. Then

$$j(\lambda) = \int_0^{\xi'} f(x) \sin \lambda x \, dx = I(\lambda) - \int_{\xi'}^{\xi} f(x) \sin \lambda x \, dx,$$

and

$$\int_{\xi'}^{\xi} f(x) \sin \lambda x \, dx = o(1)$$

as $\lambda \to \infty$, by the same reason as in the case of $J^{(2)}(\lambda)$. Hence

$$j(\lambda) = I(\lambda) + o(1).$$

As to the integral

$$j'(\lambda) = \int_0^{\xi_1} f(x) \sin \lambda x \, dx \qquad (0 < \xi_1 < \xi' < \xi),$$

we observe that the upper limit ξ_1 of integration is a function of λ, and it may be inferred that

$$| j'(\lambda) | < K \, | I(\lambda) |$$

---

* Hobson, *Theory of Functions of a Real Variable*, p. 672.

for all sufficiently large values of λ, K being a constant independent of λ. For, if not, corresponding to any given K, there would exist some large values of λ for which

$$|j'(\lambda)| > K|I(\lambda)|,$$

and

$$|J^{(1)}(\lambda)| = \left|\varepsilon(\xi')\{j(\lambda)-j'(\lambda)\}\right| > \varepsilon(\xi')\Big|(K-1)|I(\lambda)| - |o(1)|\Big|.$$

Hence $$|J^{(1)}(\lambda)| > \Big|K|I(\lambda)| - o(1)\Big|,^*$$

and we obtain

$$|\bar{I}(\lambda)| > K|I(\lambda)|.$$

Therefore it follows that

$$|\bar{S}(\lambda)| > K|S(\lambda)|,$$

contrary to our assumption (52).

Thus we have

$$|j'(\lambda)| < K|I(\lambda)|$$

for all sufficiently large values of λ. Hence we have

$$|J^{(1)}(\lambda)| < \delta K|I(\lambda)| + o(1),$$

and

$$|\bar{I}(\lambda)| < \delta K|I(\lambda)| + o(1),$$

whence it follows that

$$|\bar{S}(\lambda)| < \delta K|S(\lambda)| + o(1).$$

As will be seen presently,† $S(\lambda)$ does not tend to zero as $\lambda\to\infty$. Therefore

$$\left|\frac{\bar{S}(\lambda)}{S(\lambda)}\right| < \delta K + o(1) < (K+1)\delta$$

by choosing λ sufficiently large. Now K is a constant independent of λ and δ may be chosen as small as we please. Hence

---

* Here K is written for $\varepsilon(\xi')(K-1)$ and, as $\varepsilon(\xi')$ is a constant independent of λ, K in this expression may take any large value as we please.

† See the next paragraph 37.

$K+1)\delta$ may be made as small as we please. Hence it follows that

$$\overline{S}(\lambda) \prec S(\lambda)$$

as $\lambda \to \infty$.

37. Now we consider the integral

$$S(\lambda) = \int_0^{\varepsilon} \rho(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx.$$

Performing integration by parts, we have

$$S(\lambda) = \left| -i \frac{\rho}{\sigma'} e^{i\sigma} \frac{\sin \lambda x}{x} \right|_0^{\varepsilon} + i \int_0^{\varepsilon} e^{i\sigma} \frac{d}{dx} \left\{ \frac{\rho}{\sigma'}\, \frac{\sin \lambda x}{x} \right\} dx.$$

Since $\rho \prec \sigma'$, we obtain

$$S(\lambda) = O(1) + C_1(\lambda) + i\, S_1(\lambda), \tag{53}$$

where

$$\left\{ \begin{aligned} C_1(\lambda) &= i\lambda \int_0^{\varepsilon} \frac{\rho}{\sigma'} e^{i\sigma} \frac{\cos \lambda x}{x}\, dx, \\ S_1(\lambda) &= \int_0^{\varepsilon} \rho_1\, e^{i\sigma} \frac{\sin \lambda x}{x}\, dx, \\ \rho_1 &= x \frac{d}{dx} \left(\frac{\rho}{x\sigma'} \right) = -\frac{\rho}{x\sigma'} + \frac{\rho'}{\sigma'} - \frac{\rho\sigma''}{(\sigma')^2}. \end{aligned} \right. \tag{54}$$

Then, in the integral $C_1(\lambda)$, we have

$$x \preceq \frac{\rho}{\sigma'} \prec x\sigma',$$

since σ, ρ satisfy the first two conditions of (51). Hence Theorem VI may be applied to this integral. Thus we have

$$C_1(\lambda) = i\lambda \frac{\rho(\theta)}{-\theta\sigma'(\theta)\sqrt{\{2\sigma''(\theta)\}}} \{ e^{(\beta + \frac{1}{4}\pi)i} \sqrt{\pi} + o(1) \},$$

where $\sigma'(\theta) + \lambda = 0, \quad \beta = \lambda\theta + \sigma(\theta).$

Hence we obtain

$$C_1(\lambda) = \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \{ e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi} + o(1) \}. \tag{55}$$

We observe that $$C_1(\lambda) \succ 1$$

as $\lambda \to \infty$, since $x\sigma' \preceq \rho$.

Now take the integral

$$S_1(\lambda) = \int_0^\varepsilon \rho_1 e^{i\sigma} \frac{\sin \lambda x}{x} \, dx,$$

where

$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{\rho'}{\sigma'} - \frac{\rho\sigma''}{(\sigma')^2}.$$

If we write

$$(56) \qquad \begin{cases} \rho = x^{-a}\,\theta, & a \geqq 0, \quad x^\delta \prec \theta \prec (1/x)^\delta, \\ \sigma' = -\,x^{-(1+b)}\,\theta_1, & b \geqq 0, \quad x^\delta \prec \theta_1 \prec (1/x)^\delta, \end{cases}$$

we have, by the condition $x\sigma' \preceq \rho \prec \sigma'$,

$$(57) \qquad b \leqq a \leqq 1 + b,$$

and, if $a = b$, $$\theta_1 \preceq \theta;$$

if $a = 1 + b$, $$\theta_1 \succ \theta.$$

And we observe that, if $b = 0$,

$$\theta_1 \succ 1,$$

since $\sigma \succ l(1/x)$.

From (56), we have $$x\sigma'' \sim -(1+b)\sigma',$$

and

$$\begin{cases} x\rho' \sim -a\rho & (a > 0), \\ x\rho' \prec \rho & (a = 0). \end{cases}$$

Hence

$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{x\rho'}{x\sigma'} - \frac{\rho}{x\sigma'} \cdot \frac{x\sigma''}{\sigma'}$$

$$\sim -(a-b)\frac{\rho}{x\sigma'} \prec \rho$$

since $x\sigma' \succ 1$.

Now $S(\lambda)$ cannot tend to zero as $\lambda\to\infty$. For, if $S(\lambda)\prec 1$, then, by the relation (52),* we have

$$S_1(\lambda)\prec 1,$$

and, by (53), $$C_1(\lambda)+O(1)\prec 1,$$

contradictory to the above result $C_1(\lambda)\succ 1$. Thus $S(\lambda)$ does not tend to zero as $\lambda\to\infty$. Hence, by Lemma 7, we have

$$S_1(\lambda)\prec S(\lambda) \tag{58}$$

as $\lambda\to\infty$. Hence, from (53) and (55), we obtain

$$S(\lambda)\sim\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

which is nothing but the formula (7) in our case. Thus we may state, by combining this result with theorem VI,

Theorem VII. *The integral*

$$S(\lambda)=\int_0^{\varepsilon}\rho(x)\,e^{i\sigma(x)}\frac{\sin\lambda x}{x}\,dx,$$

where $l(1/x)\prec\sigma\prec(1/x)^4$ and $\rho\prec\sigma'$, is convergent. The behaviour of $S(\lambda)$, as $\lambda\to\infty$, is determined asymptotically as follows:

If $x^4\prec\rho\preceq x\sqrt{\sigma''}/\sigma'$, then

$$S(\lambda)=O(1/\lambda);$$

if $x\sqrt{\sigma''}/\sigma'\prec\rho\prec\sigma'$, then

$$S(\lambda)\sim\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}}e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}, \tag{7}$$

where $$\beta=\lambda\theta+\sigma(\theta),$$

and θ is determined as a function of λ by the equation

$$\sigma'(\theta)+\lambda=0.$$

* Here $S_1(\lambda)$ is replaced for $\bar{S}(\lambda)$.

38. We now pass to another proof which is quite independent of the assumption (52).

We have to prove the relation (58), or

$$S_1(\lambda) \prec C_1(\lambda)$$

as $\lambda \to \infty$.

At first we consider the case in which

$$b > 0.$$

Now in the integral

$$S_1(\lambda) = \int_0^{\xi} \rho_1(x)\, e^{i\sigma(x)} \frac{\sin \lambda x}{x}\, dx,$$

we have

$$\begin{cases} \rho_1 \sim -(a-b)\dfrac{\rho}{x\sigma'} & (a > b), \\[2ex] \rho_1 \prec \dfrac{\rho}{x\sigma'} & (a = b). \end{cases}$$

By (56), $\quad x\sigma' = -x^{-b}\theta_1, \qquad \dfrac{\rho}{x\sigma'} = -x^{-(a-b)}\dfrac{\theta}{\theta_1}.$

Hence, if $a-b<b$, we have

$$\frac{x\sqrt{\sigma''}}{\sigma'} \prec \rho_1 \precsim \frac{\rho}{x\sigma'} \prec x\sigma';$$

and Theorem VI may be applied to the integral $S_1(\lambda)$. Thus we have

$$S_1(\lambda) \sim \frac{\rho_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}\,;$$

and, as $\rho_1 \prec \rho$, we obtain

$$S_1(\lambda) \prec C_1(\lambda).$$

If $a-b \geqq b$, then, by performing integration by parts, we obtain

$$S_1(\lambda) = O(1) + C_2(\lambda) + iS_2(\lambda),$$

where

$$\begin{cases} C_2(\lambda) = i\lambda \int_0^{\varepsilon} \frac{\rho_1}{\sigma'} e^{i\sigma} \frac{\cos \lambda x}{x}\, dx \sim \frac{\rho_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i} \sqrt{\pi}, \\ S_2(\lambda) = \int_0^{\varepsilon} \rho_2\, e^{i\sigma} \frac{\sin \lambda x}{x}\, dx, \\ \rho_2 = -\frac{\rho_1}{x\sigma'} + \frac{\rho_1'}{\sigma'} - \frac{\rho_1 \sigma''}{(\sigma')^2}. \end{cases}$$

Now

$$\begin{cases} \rho_2 \sim -(a-2b)\frac{\rho_1}{x\sigma'} & (a > 2b), \\ \rho_2 \prec \frac{\rho_1}{x\sigma'} & (a = 2b), \end{cases}$$

and

$$\frac{\rho_1}{x\sigma'} \sim (a-b)\, x^{-(a-2b)} \frac{\theta}{\theta_1^2}.$$

If $a-2b < b$, then

$$\rho_2 \preceq \frac{\rho_1}{x\sigma'} \prec x\sigma',$$

and, by another application of Theorem VI, we obtain

$$S_1(\lambda) \prec C_1(\lambda),$$

since $S_2(\lambda) \prec C_2(\lambda) \prec C_1(\lambda)$ as $\lambda \to \infty$.

If $a-2b \geqq b$, then repeat a similar process.

Since $b > 0$ and $b \leqq a \leqq 1+b$ by (57), there exists a positive integer n such that

$$(n-1)\, b \leqq a < nb.$$

Hence, after repeating n times the above process, we are led to the equation

$$S_{n-1}(\lambda) = O(1) + C_n(\lambda) + i\, S_n(\lambda),$$

where

$$\begin{cases} C_n(\lambda) \sim \frac{\rho_{n-1}(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i} \sqrt{\pi}, \\ S_n(\lambda) = \int_0^{\varepsilon} \rho_n\, e^{i\sigma} \frac{\sin \lambda x}{x}\, dx, \\ \rho_n \preceq \frac{\rho_{n-1}}{x\sigma'} \sim -(a-b)(a-2b)\cdots(a-\overline{n-1}b)\, x^{-(a-nb)} \frac{\theta}{\theta_1^{\,n}}. \end{cases}$$

Since $a < nb$, Theorem VI may be applied to the integral $S_n(\lambda)$, and we obtain

$$S_n(\lambda) \prec C_n(\lambda) \sim S_{n-1}(\lambda).$$

Since $\rho \succ \rho_1 \succ \cdots \succ \rho_{n-1}$, we have

$$C_1(\lambda) \succ C_2(\lambda) \succ \cdots \succ C_n(\lambda),$$

and hence we obtain

$$S_1(\lambda) \prec C_1(\lambda).$$

Thus, in the case $b > 0$, always we have

$$S(\lambda) \sim C_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i} \sqrt{\pi}.$$

Thus the proof is completed for the case $b > 0$.

39. Next we consider the case in which

$$b = 0.$$

Thus $$\sigma' = -x^{-1}\theta_1, \qquad \theta_1 \succ 1,$$

$$\rho = x^{-a}\theta.$$

We observe that, if $a = 1$, $\quad \theta \prec \theta_1$;

if $a = 0$, $\quad \theta \succeq \theta_1$.

In this case $\quad x\sigma' = -\theta_1$.

(i) If $a > 0$, we have

$$\rho_1 = -\frac{\rho}{x\sigma'} + \frac{\rho'}{\sigma'} - \frac{\rho\sigma''}{(\sigma')^2} \sim -a\, x^{-a}\frac{\theta}{\theta_1},$$

and, for any positive integer n,

$$\rho_n \sim (-a)^n x^{-a} \frac{\theta}{\theta_1^{\,n}} \succ -\theta_1 = x\sigma'.$$

Hence, if $a > 0$, then the method of the last paragraph fails.

(ii) Now consider the case in which

$$a = 0.$$

First, let $$\theta \sim A\theta_1.$$

Then $$\theta = A\theta_1\{1+\varepsilon(x)\},$$

where $\varepsilon(x)$ is an L-function such that $\varepsilon(x) \prec 1$ as $x \to 0$; and we have

$$S(\lambda) = A\int_0^{\xi} \theta_1 e^{i\sigma} \frac{\sin \lambda x}{x}\, dx + A\int_0^{\xi} \varepsilon\theta_1 e^{i\sigma} \frac{\sin \lambda x}{x}\, dx$$

$$= A\{I_1(\lambda) + I_2(\lambda)\}$$

say. Then, performing integration by parts, we have

$$I_1(\lambda) = O(1) - i\lambda \int_0^{\xi} e^{i\sigma} \cos \lambda x \, dx$$

$$= O(1) - i\lambda \frac{1}{\sqrt{\{2\sigma''(\theta)\}}} \{e^{(\beta+\frac{1}{4}\pi)i}\sqrt{\pi} + o(1)\} \quad \text{[by Theorem VI]},$$

$$\sim \frac{\theta_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi}.$$

In the integral $I_2(\lambda)$, we have

$$\varepsilon\theta_1 \prec \theta_1 = -x\sigma'.$$

Hence, by Theorem VI,

$$I_2(\lambda) \sim \frac{\varepsilon(\theta)\,\theta_1(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

or $$I_2(\lambda) = O(1/\lambda);$$

and we have

$$I_2(\lambda) \prec I_1(\lambda).$$

Therefore

$$S(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta-\frac{1}{4}\pi)i}\sqrt{\pi},$$

since $\rho \sim A\theta_1$.

Thus, in the case when $a = 0$ and $\theta \sim A\theta_1$, the truth of the formula (7) is proved.

Next let $$\theta \succ \theta_1.$$

Then in the integral

$$S_1(\lambda) = \int_0^\xi e^{i\sigma} \frac{d}{dx}\left(\frac{\rho}{x\sigma'}\right) \sin \lambda x \, dx$$

we have
$$\frac{\rho}{x\sigma'} = -\frac{\theta}{\theta_1} \succ 1.$$

Let us write

$$\frac{\theta(x, 1)}{\theta_1} = -x \frac{d}{dx}\left(\frac{\theta}{\theta_1}\right),$$

so that
$$\theta(x, 1) = \theta \left\{ \frac{x\theta_1'}{\theta_1} - \frac{x\theta'}{\theta} \right\} \prec \theta$$

since $x\theta' \prec \theta$ and $x\theta_1' \prec \theta_1$. We observe that, since $\dfrac{\theta}{\theta_1} \succ 1$, we have $\theta(x, 1) > 0$, and $\theta(x, 1)$ is a function of the same type as θ, namely

$$x^\delta \prec \theta(x, 1) \prec (1/x)^\delta.$$

Thus we have

$$S_1(\lambda) = \int_0^\xi \rho_1 e^{i\sigma} \frac{\sin \lambda x}{x} \, dx,$$

where
$$\rho_1 = \frac{\theta(x, 1)}{\theta_1}.$$

If $\theta(x, 1) \prec \theta_1^2$, then $\rho_1 \prec x\sigma'$ and, as before, applying Theorem VI, we see that

$$S_1(\lambda) \prec C_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} e^{(\beta - \frac{1}{4}\pi)i} \sqrt{\pi}.$$

If $\theta(x, 1) \sim A\theta_1^2$, then, by proceeding as in the case $\theta \sim A\theta_1$, we easily arrive at the same result.

If $\theta(x, 1) \succ \theta_1^2$, then repeat a similar process.

Thus we have to consider successively the functions $\theta(x, 1)$, $\theta(x, 2)$, ..., $\theta(x, n)$ defined by the equations

$$(59)\quad \left\{\begin{array}{l} \dfrac{\theta(x,1)}{\theta_1} = -x\dfrac{d}{dx}\left\{\dfrac{\theta}{\theta_1}\right\}, \\[2ex] \dfrac{\theta(x,2)}{\theta_1^2} = -x\dfrac{d}{dx}\left\{\dfrac{\theta(x,1)}{\theta_1^2}\right\}, \\ \dotfill \\ \dfrac{\theta(x,n)}{\theta_1^n} = -x\dfrac{d}{dx}\left\{\dfrac{\theta(x,n-1)}{\theta_1^n}\right\}, \end{array}\right.$$

where

$$\theta(x, n-1) \succ \theta_1^n.$$

We easily see that

$$\theta \succ \theta(x,1) \succ \theta(x,2) \succ \cdots\cdots \succ \theta(x,n).$$

There are two different cases.

(a) For a certain integer n, we have

$$\theta(x,n) \preceq \theta_1^{n+1}.$$

In this case, applying Theorem VI, we obtain

$$S_n(\lambda) \prec S_{n-1}(\lambda) \prec \cdots\cdots \prec S_1(\lambda) \prec C_1(\lambda).$$

Thus, in this case, the formula (7) holds also.

(b) For any integer n, however great, we have always

$$\theta(x,n) \succ \theta_1^{n+1}.$$

In this case the above method again fails.

We have thus proved that the formula (7) holds also when $x\sigma' \preceq \rho \prec \sigma'$, with the exception of the following special cases.

(i) $\qquad b = 0, \qquad 0 < a \leqq 1;$

(ii) $\qquad b = 0, \qquad a = 0, \qquad \theta(x,n) \succ \theta_1^4,$

for any integer n, $\theta(x,n)$ being the function defined by the equations (59).

I have already got a certain proof for some of these special cases, but not yet completed it. Perhaps I may return to this problem on another occasion.

40. Here I will give another lemma which will be useful in Part II.

Lemma 8. *Let $\rho(x)$ and $\sigma(x)$ be the L-functions which are treated in Theorem VI, and let $\varpi(x)$ be a real function, not necessarily an L-function, but continuous and differentiable in the interval $(0, \xi)$ save for $x = 0$, satisfying the relation $\varpi(x) \sim \rho(x)$ in such a way that*

$$\varpi(x) = \rho(x)\{1 + \varepsilon(x)\},$$

where $\varepsilon(x)$ is ultimately monotonic and tends to zero as $x \to 0$. If there exists an L-function $\gamma(x)$ such that

$$\varepsilon(x) \sim \gamma(x),$$

then, under the conditions the same as those of Theorem VI, the same asymptotic formulae (7), (8) and (49) hold respectively for the integrals obtained by replacing $\varpi(x)$ for $\rho(x)$ in $S(\lambda)$, $C(\lambda)$, $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ and $I_4(\lambda)$.

Proof. If $\varepsilon(x)$ be an L-function, then the lemma follows immediately from Theorem VI and its corollary.

If $\varepsilon(x)$ is not an L-function, still it behaves like an L-function under our hypothesis and hence the truth of the lemma may be conjectured from Theorem VI.

Take the integral

$$J(\lambda) = \int_0^\xi \varpi(x) \cos \sigma(x) \frac{\cos \lambda x}{x} dx$$

$$= \int_0^\xi \rho(x) \cos \sigma(x) \frac{\cos \lambda x}{x} dx + \int_0^\xi \varepsilon(x)\, \rho(x) \cos \sigma(x) \frac{\cos \lambda x}{x} dx.$$

Let $\bar{\gamma}(x)$ be an L-function such that

$$\gamma(x) \prec \bar{\gamma}(x) \prec 1^*$$

as $x \to 0$; and write

$$\bar{\rho}(x) = \rho(x)\,\bar{\gamma}(x), \qquad \bar{\varepsilon}(x) = \frac{\varepsilon(x)}{\bar{\gamma}(x)},$$

so that $$\bar{\rho} \prec \rho, \qquad \bar{\varepsilon} \prec 1,$$

since $\varepsilon \sim \gamma$. Then we have

* For instance, we may take $\bar{\gamma} = \{\gamma(x)\}^{\frac{1}{2}}$.

$$J(\lambda) = I_1(\lambda) + \int_0^{\xi} \bar{\varepsilon}(x)\, \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x}\, dx,$$

where, by the corollary to Theorem VI, we have

$$I_1(\lambda) = O(1/\lambda) \qquad (\rho \preceq x\sqrt{\sigma''}/\sigma'),$$

$$I_1(\lambda) \sim \frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi} \qquad (x\sqrt{\sigma''}/\sigma' \prec \rho \prec x\sigma'),$$

θ, β being the same as those in Theorem VI.

Since $\bar{\rho} \prec \rho$, the integral

$$\int_0^{\xi} \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x}\, dx$$

is evidently convergent when $\rho \prec x\sigma'$; and since, by hypothesis, ϖ is differentiable, $\bar{\varepsilon}$ is also differentiable and $\frac{d\bar{\varepsilon}}{dx}$ has a constant sign in the interval $(0, \xi)$, ξ being chosen sufficiently small. Hence by the Second Mean Value Theorem, we obtain

$$\begin{aligned} \bar{J}(\lambda) &= \int_0^{\xi} \bar{\varepsilon}(x)\, \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x}\, dx \\ &= \bar{\varepsilon}(\xi) \int_{\xi_1}^{\xi} \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x}\, dx \qquad (0 < \xi_1 < \xi) \\ &= \bar{\varepsilon}(\xi) \left(\int_0^{\xi} - \int_0^{\xi_1} \right) \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x}\, dx \\ &= \bar{\varepsilon}(\xi)\, \{ j(\lambda) - j'(\lambda) \} \end{aligned}$$

say. Then, by the corollary to Theorem VI, we have

$$j(\lambda) = O(1/\lambda),$$

if $\bar{\rho} \preceq x\sqrt{\sigma''}/\sigma'$; and, if $x\sqrt{\sigma''}/\sigma' \prec \bar{\rho} \prec x\sigma'$,

$$j(\lambda) \sim \frac{\bar{\rho}(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} \cos(\beta + \tfrac{1}{4}\pi)\sqrt{\pi},$$

θ, β being the same as those in the above formula for $I_1(\lambda)$.

Therefore, from the relation $\bar{\rho} \prec \rho$, it follows that

$$j(\lambda) \prec I_1(\lambda)$$

as $\lambda \to \infty$, when $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$.

In the integral

$$j'(\lambda) = \int_0^{\xi_1} \bar{\rho}(x) \cos \sigma(x) \frac{\cos \lambda x}{x} \, dx \qquad (0 < \xi_1 < \xi),$$

we observe that ξ_1 varies with λ; still, if we examine the proof of Theorem VI, we can see without difficulty that

$$j'(\lambda) \preceq K\, j(\lambda).$$

Hence

$$\bar{J}(\lambda) = \bar{\varepsilon}(\xi)\{j(\lambda) - j'(\lambda)\} \prec I_1(\lambda),$$

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$. Therefore we have:

If $\rho \preceq x\sqrt{\sigma''/\sigma'}$, then $\qquad J(\lambda) = O(1/\lambda)$;

if $x\sqrt{\sigma''/\sigma'} \prec \rho \prec x\sigma'$, then $\qquad J(\lambda) \sim I_1(\lambda)$.

The same argument applies to the other integrals.
Thus the lemma is completely proved.

VIII *Examples of Case (C)*

41. Let us consider the case in which

$$\rho = x^{-a}, \qquad \sigma = \frac{m}{x},$$

where m is positive, so that

$$I_1(\lambda) = \int_0^{\xi} x^{-a} \cos\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x} \, dx, \quad I_2(\lambda) = \int_0^{\xi} x^{-a} \sin\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x} \, dx,$$

$$I_3(\lambda) = \int_0^{\xi} x^{-a} \cos\left(\frac{m}{x}\right) \frac{\sin \lambda x}{x} \, dx, \quad I_4(\lambda) = \int_0^{\xi} x^{-a} \sin\left(\frac{m}{x}\right) \frac{\sin \lambda x}{x} \, dx,$$

$$S(\lambda) = \int_0^{\xi} x^{-a} e^{(m/x)i} \frac{\sin \lambda x}{x} \, dx, \qquad C(\lambda) = \int_0^{\xi} x^{-a} e^{(m/x)i} \frac{\cos \lambda x}{x} \, dx.$$

Since $\sigma = \dfrac{m}{x} \succ l(1/x)$, Theorems VI and VII are applicable.
Now

$$\sigma' = -\frac{m}{x^2}, \qquad \sigma'' = \frac{2m}{x^3}.$$

The conditions $\rho \lesssim x\sqrt{\sigma''}/\sigma'$, $\rho < x\sigma'$, $\rho < \sigma'$ give respectively

$$a \leqq -\tfrac{3}{2}, \quad a < 1, \quad a < 2.$$

The equation $\lambda + \sigma'(\theta) = 0$ gives

$$\theta = \left(\frac{m}{\lambda}\right)^{\frac{1}{2}},$$

and we have

$$\beta = \lambda\theta + \sigma(\theta) = 2(m\lambda)^{\frac{1}{2}},$$

$$\frac{\rho(\theta)}{\theta\sqrt{\{2\sigma''(\theta)\}}} = \frac{\theta^{\frac{1}{2}-a}}{2\sqrt{m}} = \tfrac{1}{2}m^{-\frac{1}{2}a-\frac{1}{4}}\,\lambda^{\frac{1}{2}a-\frac{1}{4}}.$$

Hence we have:

If $-\frac{3}{2} < a < 1$, then

$$\begin{cases} I_1(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\cos\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\tfrac{1}{4}\pi\right), \\ I_2(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\sin\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\tfrac{1}{4}\pi\right), \\ C(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\exp\left\{\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\tfrac{1}{4}\pi\right)i\right\}; \end{cases}$$

if $-\frac{3}{2} < a < 2$, then

$$\begin{cases} I_3(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\cos\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\tfrac{1}{4}\pi\right), \\ I_4(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\sin\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\tfrac{1}{4}\pi\right), \\ S(\lambda) \sim \tfrac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\exp\left\{\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}-\tfrac{1}{4}\pi\right)i\right\}. \end{cases}$$

We observe that all these integrals tend to zero as $\lambda \to \infty$ if $a < \frac{1}{2}$, and they oscillate if $a \geqq \frac{1}{2}$.

42. These results may be verified as follows.

Hardy proved* that

$$\int_0^\infty \cos u \cos\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}$$

$$= \frac{\pi\beta^\nu}{4\sin\frac{1}{2}\nu\pi}\left\{-J^\nu(2\beta) + J^{-\nu}(2\beta) - e^{-\frac{1}{2}\nu\pi i}J^\nu(2i\beta) + e^{\frac{1}{2}\nu\pi i}J^{-\nu}(2i\beta)\right\}.$$

---

* *Messenger of mathematics* Vol. XL, *pp. 44 et seq.*

From this we can easily deduce

$$\int_0^\infty \cos u \sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}$$

$$= \frac{\pi\beta^\nu}{4\cos\frac{1}{2}\nu\pi}\{J^\nu(2\beta)+J^{-\nu}(2\beta)+e^{-\frac{1}{2}\nu\pi i}J^\nu(2i\beta)-e^{\frac{1}{2}\nu\pi i}J^{-\nu}(2i\beta)\}.$$

These formulae hold respectively for

$$-1<\nu<1,$$

and for $$-1<\nu<2,$$

it being understood that, in certain special cases, the expression of the right-hand side must be replaced by its limits. For $\nu=\frac{1}{2}$ they assume the forms

$$\int_0^\infty \cos u \cos\left(\frac{\beta^2}{u}\right)\frac{du}{\sqrt{u}} = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(-\sin 2\beta+\cos 2\beta+e^{-2\beta}),$$

$$\int_0^\infty \cos u \sin\left(\frac{\beta^2}{u}\right)\frac{du}{\sqrt{u}} = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\sin 2\beta+\cos 2\beta-e^{-2\beta})\,;$$

and their values are expressible in terms of elementary functions also when

$$\nu=-\tfrac{1}{2},\ \tfrac{3}{2}$$

(the last value, of course, only in the second integral).

Now write λm for β^2, and put $u=\lambda x$ in the integrals. We obtain

$$\int_0^\infty \cos\lambda x\cos\left(\frac{m}{x}\right)\frac{dx}{x^{1-\nu}} = \frac{\pi}{4\sin\frac{1}{2}\nu\pi}\left(\frac{m}{\lambda}\right)^{\frac{1}{2}\nu}\Big[-J^\nu\{2\sqrt{(m\lambda)}\}$$

$$+J^{-\nu}\{2\sqrt{(m\lambda)}\}-e^{-\frac{1}{2}\nu\pi i}J^\nu\{2i\sqrt{(m\lambda)}\}+e^{\frac{1}{2}\nu\pi i}J^{-\nu}\{2i\sqrt{(m\lambda)}\}\Big],$$

$$\int_0^\infty \cos\lambda x\sin\left(\frac{m}{x}\right)\frac{dx}{x^{1-\nu}} = \frac{\pi}{4\cos\frac{1}{2}\nu\pi}\left(\frac{m}{\lambda}\right)^{\frac{1}{2}\nu}\Big[J^\nu\{2\sqrt{(m\lambda)}\}$$

$$+J^{-\nu}\{2\sqrt{(m\lambda)}\}+e^{-\frac{1}{2}\nu\pi i}J^\nu\{2i\sqrt{(m\lambda)}\}-e^{\frac{1}{2}\nu\pi i}J^{-\nu}\{2i\sqrt{(m\lambda)}\}\Big].$$

Now, when β is large,

$$J^{\nu}(2\beta) = \frac{1+\epsilon_{\beta}}{\sqrt{(\pi\beta)}} \cos\{2\beta - \tfrac{1}{4}(1+2\nu)\pi\},$$

where $$|\epsilon_{\beta}| < \frac{K}{\beta}.$$

Hence, when β is large, $-J^{\nu}(2\beta) + J^{-\nu}(2\beta)$ and $J^{\nu}(2\beta) + J^{-\nu}(2\beta)$ behave respectively like

$$-\frac{2 \sin \tfrac{1}{2}\nu\pi}{\sqrt{(\pi\beta)}} \sin(2\beta - \tfrac{1}{4}\pi),$$

and $$\frac{2\cos\tfrac{1}{2}\nu\pi}{\sqrt{(\pi\beta)}} \cos(2\beta - \tfrac{1}{4}\pi).$$

On the other hand

$$e^{-\frac{1}{2}\nu\pi i} J^{\nu}(2i\beta) - e^{\frac{1}{2}\nu\pi i} J^{-\nu}(2i\beta) = \frac{i}{\sin \nu\pi} e^{\frac{1}{2}\nu\pi i} H_1^{\nu}(2i\beta)$$

tends *exponentially* to zero as $\beta \to \infty$ i.e., as $\lambda \to \infty$, so that this term is negligible in comparing with the remaining terms. Hence we obtain the results:—

The integral

$$A(\lambda) = \int_0^{\infty} x^{\nu} \cos\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x}\, dx$$

is convergent if $-1 < \nu < 1$, and, as $\lambda \to \infty$, we have

$$A(\lambda) = -\tfrac{1}{2}\pi^{\frac{1}{2}} m^{\frac{1}{2}\nu - \frac{1}{4}} \lambda^{-\frac{1}{2}\nu - \frac{1}{4}} \{\sin(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \tfrac{1}{4}\pi) + o(1)\}.$$

The integral

$$B(\lambda) = \int_0^{\infty} x^{\nu} \sin\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x}\, dx$$

is convergent if $-1 < \nu < 2$, and, as $\lambda \to \infty$, we have

$$B(\lambda) = \tfrac{1}{2}\pi^{\frac{1}{2}} m^{\frac{1}{2}\nu - \frac{1}{4}} \lambda^{-\frac{1}{2}\nu - \frac{1}{4}} \{\cos(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \tfrac{1}{4}\pi) + o(1)\}.$$

43. Now put $\nu = -a$ and write

$$A(\lambda) = \left(\int_0^{\epsilon} + \int_{\epsilon}^{\infty}\right) x^{-a} \cos\left(\frac{m}{x}\right) \frac{\cos \lambda x}{x}\, dx \qquad (-1 < a < 1),$$

$$= I_1(\lambda) + J_1(\lambda)$$

say. Then, by writing $f(x) = x^{-1-a}\cos\left(\frac{m}{x}\right)$, we have

$$J_1(\lambda) = -\frac{f(\xi)\sin\lambda\xi}{\lambda} - \frac{1}{\lambda}\int_\xi^\infty f'(x)\sin\lambda x\,dx$$

and $$f'(x) = -(1+a)\,x^{-(a+2)}\cos\left(\frac{m}{x}\right) + mx^{-(3+a)}\sin\left(\frac{m}{x}\right),$$

whence

$$\left|\int_\xi^\infty f'(x)\sin\lambda x\,dx\right| \leqq |a+1|\int_\xi^\infty x^{-(2+a)}\,dx + m\int_\xi^\infty x^{-(3+a)}\,dx$$
$$< K,$$

since $a > -1$. Therefore we obtain

$$J_1(\lambda) = O(1/\lambda),$$

and evidently

$$J_1(\lambda) \prec \lambda^{\frac{1}{2}a-\frac{1}{4}} \qquad (-1 < a < 1),$$

whence it follows that

$$A(\lambda) \sim I_1(\lambda) \qquad \left(\begin{array}{c}\nu = -a,\\ -1 < a < 1\end{array}\right).$$

Similarly

$$B(\lambda) \sim I_2(\lambda) \qquad \left(\begin{array}{c}\nu = -a\\ -1 < a < 1\end{array}\right).$$

Therefore we obtain

$$\left\{\begin{array}{ll} I_1(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\cos(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi) & (-1 < a < 1), \\ I_2(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\sin(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi) & (-1 < a < 1), \\ C(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}}m^{-\frac{1}{2}a-\frac{1}{4}}\lambda^{\frac{1}{2}a-\frac{1}{4}}\exp\{(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}}+\frac{1}{4}\pi)i\} & (-1 < a < 1), \end{array}\right.$$

which agree with the results obtained from our general Theorem VI, only the difference being that the lower limit of a is -1 instead of $-\frac{3}{2}$, this limitation being introduced from the condition for convergence of the integral $A(\lambda)$.

As Hardy gives in "*O. D. I. 2.*", from the values of the integrals

$$\int_0^\infty \sin u\cos\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}, \qquad \int_0^\infty \sin u\sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}},$$

we can infer that

$$\begin{cases} I_3(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \cos\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi\right) & (-1 < a < 2), \\ I_4(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \sin\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi\right) & (-1 < a < 2), \\ S(\lambda) \sim \frac{1}{2}\pi^{\frac{1}{2}} m^{-\frac{1}{2}a-\frac{1}{4}} \lambda^{\frac{1}{2}a-\frac{1}{4}} \exp\left\{\left(2m^{\frac{1}{2}}\lambda^{\frac{1}{2}} - \frac{1}{4}\pi\right)i\right\} & (-1 < a < 2), \end{cases}$$

which also agree with the results obtained from Theorem VII, only the difference being that the lower limit of a is -1 instead of $-\frac{3}{2}$.

Thus our theorems VI and VII are verified.

PART II

Coefficients of Power Series

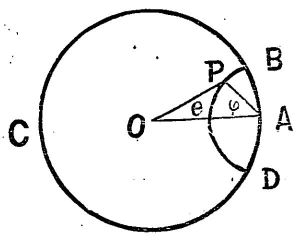
I. *Preliminaries.*

44. Consider a power series

$$\sum_{n=0}^{\infty} a_n z^n, \tag{60}$$

whose radius of convergence is unity, representing a function $f(z)$ which has, on the circle of convergence, one singular point only at $z = 1$, being regular at every other point on it.

Let $ABCD$ be the circle of convergence of the series (60), A being the point $z = 1$ and O the centre. Draw a circular arc BPD inside the circle of convergence, cutting it at the points B and D, with the centre at A and the radius $r_1 < 1$.



Let $I(r_1)$ denote the integral

$$I(r_1) = \frac{1}{2\pi i}\int_{\widehat{DPB}} \frac{f(z)}{z^{n+1}}\, dz, \tag{61}$$

where the path of integration is the arc DPB, starting at the point D, turning round the point A in the clock-wise direction along this arc and ending at the point B.

Lemma 9. *If*

$$\lim_{r_1 \to 0} I(r_1) = 0, \tag{62}$$

then
$$a_n = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z^{n+1}}\, dz,$$

provided that this integral is convergent, the contour (C) *of integration being the circle of convergence.*

Proof. Since the function $f(z)$ is regular at every point on the circle of convergence except only at the point A, the integral

$$\int_{\widehat{BCD}} \frac{f(z)}{z^{n+1}}\, dz,$$

where the path of integration is the arc BCD, is convergent and so also is the integral $I(r_1)$ for every r_1 such that $0 < r_1 < 1$. Hence, by means of Cauchy's Theorem, we have

$$a_n = \frac{1}{2\pi i}\int_{BCDPB} \frac{f(z)}{z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i}\int_{\widehat{BCD}} \frac{f(z)}{z^{n+1}}\, dz + I(r_1).$$

Now let r_1 tend to zero. Then the arc BCD tends to the whole circle of convergence and we have

$$a_n = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z^{n+1}}\, dz + \lim_{r_1 \to 0} I(r_1) = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z^{n+1}}\, dz$$

since, by hypothesis, $\lim_{r_1 \to 0} I(r_1) = 0$ and the last integral is convergent.

45. Lemma 10. *Let* ξ *be a small positive constant and*

$$I(n) = \frac{1}{2\pi}\left(\int_0^{\xi} + \int_{2\pi-\xi}^{2\pi}\right) f(e^{\theta i})\, e^{-n\theta i}\, d\theta \qquad (0 < \xi < \pi). \tag{63}$$

Then, if the conditions of Lemma 9 are satisfied, the behaviour of the coefficient a_n *of the power series (60), as* $n \to \infty$, *is asymptotically determined as follows :*

If $I(n) \leq \frac{1}{n}$, *then* $a_n = O(1/n)$;

if $I(n) \succ \frac{1}{n}$, *then* $a_n \sim I(n)$.

Proof. By Lemma 9, we have

$$a_n = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$= \frac{1}{2\pi}\left(\int_0^{\xi} + \int_{2\pi-\xi}^{2\pi}\right) f(e^{\theta i})\,e^{-n\theta i}\,d\theta + \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$(0 < \xi < \pi)$$

$$= I(n) + I'(n)$$

say. Then

$$I'(n) = \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} f(e^{\theta i})\,e^{-n\theta i}\,d\theta$$

$$= \frac{1}{2\pi}\int_{\xi}^{2\pi-\xi} (U+iV)\,(\cos n\theta - i\sin n\theta)\,d\theta,$$

where $$f(e^{\theta i}) = U+iV,$$

U and V denoting real functions of θ. Since $f(z)$ is regular on the circle of convergence, except at $z = 1$, the functions

$$U,\quad V,\quad \frac{dU}{d\theta},\quad \frac{dV}{d\theta}$$

have no singularities and are integrable in the interval $(\xi, 2\pi-\xi)$. Hence

$$\int_{\xi}^{2\pi-\xi} U\cos n\theta\,d\theta = \left|\frac{U\sin n\theta}{n}\right|_{\xi}^{2\pi-\xi} - \frac{1}{n}\int_{\xi}^{2\pi-\xi}\frac{dU}{d\theta}\sin n\theta\,d\theta$$

$$= O(1/n).$$

Similarly the integrals

$$\int_{\xi}^{2\pi-\xi} U\sin n\theta\,d\theta,\quad \int_{\xi}^{2\pi-\xi} V\,{\cos \atop \sin}\,n\theta\,d\theta$$

have values of the same type. Therefore we obtain

$$I'(n) = O(1/n).$$

Thus we have

$$a_n = I(n) + O(1/n).$$

Hence, if $I(n) \leq \dfrac{1}{n}$, then

$$a_n = O(1/n);$$

if $I(n) > \dfrac{1}{n}$, then

$$a_n \sim I(n).$$

46. Now we may write

$$I(n) = J(n) + \bar{J}(n), \tag{64}$$

where

$$\left\{\begin{aligned} J(n) &= \frac{1}{2\pi}\int_0^{\varepsilon} f(e^{\theta i})\, e^{-n\theta i}\, d\theta, \\ \bar{J}(n) &= \frac{1}{2\pi}\int_{2\pi-\varepsilon}^{2\pi} f(e^{\theta i})\, e^{-n\theta i}\, d\theta = \frac{1}{2\pi}\int_0^{\varepsilon} f\{e^{(2\pi-\theta)i}\}\, e^{n\theta i}\, d\theta, \end{aligned}\right. \tag{65}$$

n being a positive integer.

If, in the neighbourhood of $\theta = 0$, both of the functions $f(e^{\theta i})$ and $f\{e^{(2\pi-\theta)i}\}$ take the form

$$\varphi(\theta)\, e^{i\psi(\theta)},$$

where $\varphi(x)$ and $\psi(x)$ are real functions of x such that

$$\varphi(x) \sim \frac{\rho(x)}{x}, \quad \psi(x) \sim \sigma(x)$$

as $x \to 0$, ρ and σ denoting certain L-functions, then the behaviour of $J(n)$ and $\bar{J}(n)$, as $n \to \infty$, may be determined by applying the results of Part I of this paper.

II *Case in which the Singularity is of the Type* $\frac{1}{(1-z)^p}e^{A/(1-z)^q}$.

47. Let us consider the case in which the function $f(z)$ has a singularity of the type

$$f(z)=\frac{1}{(1-z)^p}e^{A/(1-z)^q},$$

where
$$A = a\,e^{\alpha i},$$

and p, q, a, α are real constants such that

$$p >, =, <0, \quad a > 0, \quad q > 0, \quad 0 \leqq \alpha < 2\pi.$$

It is to be understood that, when p and q are not integers, $(1-z)^p$ and $(1-z)^q$ assume respectively the values

$$e^{p\log(1-z)}, \qquad e^{q\log(1-z)},$$

where $\log(1-z)$ assumes its principal value.

At first we consider the integral

$$I(r_1)=\frac{1}{2\pi i}\int_{\widehat{DPB}}\frac{f(z)}{z^{n+1}}\,dz.$$

Let P be any point $z = r\,e^{\theta i}$ on the arc DPB and let φ denote the angle between the radius OA and the straight line AP, namely

$$\varphi = \angle\, OAP.$$

Then
$$1-z = 1-r\cos\theta-ir\sin\theta = r_1\cos\varphi-ir_1\sin\varphi$$
$$= r_1\,e^{-\varphi i},$$

$$\frac{1}{(1-z)^p}=\frac{1}{r_1{}^p}e^{p\varphi i},$$

$$\frac{A}{(1-z)^q}=\frac{a}{r_1{}^q}e^{(\alpha+q\varphi)i}=\frac{a}{r_1{}^q}\{\cos(\alpha+q\varphi)+i\sin(\alpha+q\varphi)\},$$

$$f(z)=\frac{1}{r_1{}^p}e^{a\cos(\alpha+q\varphi)/r_1{}^q}\,.\,e^{\{p\varphi+a\sin(\alpha+q\varphi)/r_1{}^q\}i},$$

$$dz = ir_1\,e^{-\varphi i}\,d\varphi.$$

Hence

(66)

$$I(r_1)=\frac{1}{2\pi r_1^{p-1}}\int_{-(\frac{\pi}{2}-\varepsilon)}^{\frac{\pi}{2}-\varepsilon} e^{ar_1^{-q}\cos(\alpha+q\varphi)}\cdot e^{\{(p-1)\varphi+ar_1^{-q}\sin(\alpha+q\varphi)\}i}\cdot\frac{d\varphi}{(1-r_1e^{-\varphi i})^{n+1}},$$

where ε denotes the difference of the angle OAB and a right-angle, namely

$$\frac{\pi}{2}-\varepsilon=\angle OAB=\angle OAD,$$

and we observe that $$\lim_{r_1\to 0}\varepsilon=0.$$

Now, if $\cos(\alpha+q\varphi)>0$ in any part of the range

$$-(\tfrac{\pi}{2}-\varepsilon)\leqq\varphi\leqq\tfrac{\pi}{2}-\varepsilon,$$

then $e^{ar_1^{-q}\cos(\alpha+q\varphi)}$ tends exponentially to infinity as $r_1\to 0$, so that $I(r_1)$ does not necessarily tend to zero. Hence we shall put aside this case and confine ourselves to the case in which

$$\cos(\alpha+q\varphi)\leqq 0,$$

or

(67) $$(4m+1)\frac{\pi}{2}\leqq\alpha+q\varphi\leqq(4m+3)\frac{\pi}{2}\qquad(-\tfrac{\pi}{2}+\varepsilon\leqq\varphi\leqq\tfrac{\pi}{2}-\varepsilon),$$

m being a positive integer or zero.

If we observe that, by hypothesis,

$$0\leqq\alpha<2\pi,$$

and that the condition (67) is to be satisfied by all values of φ in the interval $-\frac{\pi}{2}+\varepsilon\leqq\varphi\leqq\frac{\pi}{2}-\varepsilon$, where ε takes any value corresponding to r_1 which tends to zero, it can easily be inferred that

$$\begin{cases} 0<q\leqq 1,\\ (1+q)\dfrac{\pi}{2}\leqq\alpha\leqq(3-q)\dfrac{\pi}{2}.\end{cases}$$

Now we can prove the lemma.

Lemma 11. *If* $0<q\leqq 1$, $(1+q)\frac{\pi}{2}\leqq\alpha\leqq(3-q)\frac{\pi}{2}$ *and* $p<1+q$, *then*

$$\lim_{r_1\to 0}I(r_1)=0.$$

Proof. By hypothesis

$$(1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2} \qquad (0 < q \leqq 1).$$

Hence, if we put $\alpha = \frac{\pi}{2} + q\alpha'$,

we obtain

$$\frac{\pi}{2} \leqq \alpha' \leqq \frac{\pi}{q} - \frac{\pi}{2},$$

$$\cos(\alpha + q\varphi) = \cos\{\tfrac{1}{2}\pi + q(\alpha' + \varphi)\} = -\sin q(\alpha' + \varphi),$$

and $$q\varepsilon \leqq q(\alpha' + \varphi) \leqq \pi - q\varepsilon,$$

since $-\frac{\pi}{2} + \varepsilon \leqq \varphi \leqq \frac{\pi}{2} - \varepsilon$. Hence we have

$$\sin q(\alpha' + \varphi) > 0$$

as $\varepsilon > 0$.

Now, by (66),

$$|I(r_1)| < \frac{1}{2\pi r_1^{p-1}} \int_{-\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}-\varepsilon} e^{a r_1^{-q} \sin q(\alpha'+\varphi)} \frac{d\varphi}{(1-r_1)^{n+1}}$$

$$< \frac{1}{2q\pi r_1^{p-1}(1-r_1)^{n+1}} \int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt,$$

and $$\int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt = 2\int_0^{\frac{\pi}{2}} e^{-a r_1^{-q} \sin t}\, dt$$

$$= \frac{2r_1^q}{a} \int_0^{a/r_1^q} \frac{e^{-u}\, du}{\sqrt{\left(1 - \frac{r_1^{2q}}{a^2} u^2\right)}}.$$

The last integral is convergent and tends to

$$\int_0^{\infty} e^{-u}\, du = 1$$

as $r_1 \to 0$. Hence

$$\int_0^{\pi} e^{-a r_1^{-q} \sin t}\, dt < K r_1^q,$$

and we obtain

$$|I(r_1)| < \frac{K\,r_1^q}{2q\,\pi\,r_1^{p-1}(1-r_1)^{n+1}} = \frac{K\,r_1^{1+q-p}}{2q\pi\,(1-r_1)^{n+1}} \to 0$$

as $r_1 \to 0$, provided that $p < 1+q$.

Thus the lemma is proved.

48. We now consider the integral $I(n)$ under the supposition

$$(68) \qquad 0 < q \leqq 1, \quad (1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2}, \quad p < 1+q.$$

We have

$$1-e^{\theta i} = 2\sin\tfrac{1}{2}\theta\, e^{-(\frac{\pi}{2}-\frac{1}{2}\theta)i},$$

$$\frac{1}{(1-e^{\theta i})^p} = \frac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{\frac{1}{2}p(\pi-\theta)i},$$

$$\frac{A}{(1-e^{\theta i})^q} = \frac{a}{(2\sin\frac{1}{2}\theta)^q}\, e^{\{\alpha+\frac{1}{2}q(\pi-\theta)\}i}.$$

Hence we may write

$$(69) \qquad f(e^{\theta i}) = \varphi(\theta)\, e^{i\psi(\theta)},$$

where

$$(70) \quad \begin{cases} \varphi(\theta) = \dfrac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{a\cos\{\alpha+\frac{1}{2}q(\pi-\theta)\}/(2\sin\frac{1}{2}\theta)^q}, \\[2ex] \psi(\theta) = \frac{1}{2}p(\pi-\theta) + \dfrac{a\sin\{\alpha+\frac{1}{2}q(\pi-\theta)\}}{(2\sin\frac{1}{2}\theta)^q}; \end{cases}$$

and

$$(71) \qquad f\{e^{(2\pi-\theta)i}\} = \bar{\varphi}(\theta)\, e^{i\bar{\psi}(\theta)},$$

where

$$(72) \quad \begin{cases} \bar{\varphi}(\theta) = \dfrac{1}{(2\sin\frac{1}{2}\theta)^p}\, e^{a\cos\{\alpha-\frac{1}{2}q(\pi-\theta)\}/(2\sin\frac{1}{2}\theta)^q}, \\[2ex] \bar{\psi}(\theta) = -\frac{1}{2}p(\pi-\theta) + \dfrac{a\sin\{\alpha-\frac{1}{2}q(\pi-\theta)\}}{(2\sin\frac{1}{2}\theta)^q}; \end{cases}$$

so that we have, by (64) and (65),

$$(73) \quad \begin{cases} J(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} f(e^{\theta i})e^{-n\theta i}\,d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} \varphi(\theta)\, e^{\{\psi(\theta)-n\theta\}i}\,d\theta, \\[2ex] \bar{J}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} f\{e^{(2\pi-\theta)i}\}e^{n\theta i}\,d\theta = \dfrac{1}{2\pi}\displaystyle\int_0^{\varepsilon} \bar{\varphi}(\theta)\, e^{\{\bar{\psi}(\theta)+n\theta\}i}\,d\theta, \end{cases}$$

(64) $$I(n) = J(n) + \bar{J}(n).$$

49. *Integral* $J(n)$. First of all, we shall consider the integral $J(n)$.

Observe that, when θ is very small, we have

$$\frac{1}{2\sin\frac{1}{2}\theta} = \frac{1}{\theta}\{1+O(\theta^2)\},$$

$$\begin{aligned}\cos\{\alpha+\tfrac{1}{2}q(\pi-\theta)\} &= \cos(\alpha+\tfrac{1}{2}q\pi)\{1+O(\theta^2)\} \\ &\quad + \sin(\alpha+\tfrac{1}{2}q\pi).\tfrac{1}{2}q\theta\{1+O(\theta^2)\},\\ \sin\{\alpha+\tfrac{1}{2}q(\pi-\theta)\} &= \sin(\alpha+\tfrac{1}{2}q\pi)\{1+O(\theta^2)\} \\ &\quad - \cos(\alpha+\tfrac{1}{2}q\pi).\tfrac{1}{2}q\theta\{1+O(\theta^2)\}.\end{aligned}$$

Hence the equations (70) may be written in the forms

$$(70') \quad \begin{cases} \varphi(\theta) = \dfrac{1}{\theta^p}\{1+O(\theta^2)\}e^{a\theta^{-q}\{\cos(\alpha+\frac{1}{2}q\pi)+\frac{1}{2}q\theta\sin(\alpha+\frac{1}{2}q\pi)\}+O(\theta^{2-q})}, \\ \psi(\theta) = \tfrac{1}{2}p(\pi-\theta) + \dfrac{a}{\theta^q}\{\sin(\alpha+\tfrac{1}{2}q\pi)-\tfrac{1}{2}q\theta\cos(\alpha+\tfrac{1}{2}q\pi)\} + O(\theta^{2-q}). \end{cases}$$

Under the conditions (68), the discussion may be divided into the four cases

(i)	$q=1,$	$\alpha=\pi,$	$p<2,$
(ii)	$0<q<1,$	$\alpha=(1+q)\frac{\pi}{2},$	$p<1+q,$
(iii)	$0<q<1,$	$\alpha=(3-q)\frac{\pi}{2},$	$p<1+q,$
(iv)	$0<q<1,$	$(1+q)\frac{\pi}{2}<\alpha<(3-q)\frac{\pi}{2},$	$p<1+q.$

50. (i) The case in which $q=1$, $\alpha=\pi$, $p<2$. In this case

$$\alpha+\tfrac{1}{2}q\pi = \tfrac{3}{2}\pi.$$

Hence, by (70),

$$\varphi(\theta) = \frac{1}{\theta^p}e^{-\frac{1}{2}a}\{1+O(\theta^2)\},\quad \psi(\theta) = -\frac{a}{\theta}+\tfrac{1}{2}p\pi+O(\theta),$$

and, if we write

$$\rho(x) = x^{-(p-1)}, \quad \sigma(x) = ax^{-1},$$

then

$$J(n) = \frac{1}{2\pi} e^{-\frac{1}{2}a+\frac{1}{2}p\pi i} \int_0^{\varepsilon} \rho(x)\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} e^{-\{nx+\sigma(x)\}i} \frac{dx}{x}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}a+\frac{1}{2}p\pi i} \{J_1(n) - iJ_2(n)\},$$

where

$$J_1(n) = \int_0^{\varepsilon} \frac{\rho(x)}{x}\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} \cos\{nx+\sigma(x)\}\, dx,$$

$$J_2(n) = \int_0^{\varepsilon} \frac{\rho(x)}{x}\{1+\varepsilon_1(x)+i\varepsilon_2(x)\} \sin\{nx+\sigma(x)\}\, dx,$$

$\varepsilon_1(x)$ and $\varepsilon_2(x)$ being real functions such that

$$\varepsilon_1(x) = O(x), \quad \varepsilon_2(x) = O(x).$$

We easily see* that there exist certain L-functions, $\gamma_1(x)$ and $\gamma_2(x)$ such that

$$\varepsilon_1 \sim \gamma_1, \quad \varepsilon_2 \sim \gamma_2$$

as $x \to 0$, and also that $\frac{d\varepsilon_1}{dx}$ and $\frac{d\varepsilon_2}{dx}$ have ultimately constant signs.

---

* From (69) and (70), we see that

$$\varepsilon_1(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p \cos\left(\frac{a}{x} - \frac{1}{2}a\cot\frac{x}{2} - \frac{1}{2}px\right) - 1$$

$$= \left(1+\frac{1}{24}x^2+\cdots\right)^p \cos\left\{\left(\frac{a}{12}-\frac{p}{2}\right)x+\cdots\right\} - 1,$$

$$\varepsilon_2(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p \sin\left(\frac{a}{2} - \frac{1}{2}a\cot\frac{x}{2} - \frac{1}{2}px\right)$$

x

$$= \left(1+\frac{1}{24}x^2+\cdots\right)^p \sin\left\{\left(\frac{a}{12}-\frac{p}{2}\right)x+\cdots\right\}.$$

Hence ε_1 and ε_2 may be expressed as power series of x, which are uniformly convergent for sufficiently small values of x. Thus the first terms of these series may respectively be taken as γ_1 and γ_2, and it immediately follows that $\frac{d\varepsilon_1}{dx}$ and $\frac{d\varepsilon_2}{dx}$ have ultimately constant signs.

The integrals $J_1(n)$, $J_2(n)$ and $J(n)$ are all convergent, if

$$\rho \prec x\sigma'$$

as $x \to 0$. Introducing the above expressins of ρ and σ, this condition becomes

$$p < 2,$$

which is nothing but our hypothesis.

Thus, ε_1 and ε_2 having the above properties, Lemma 8 of Part I may be applied to our integrals, and we obtain

$$J_1(n) \sim \int_0^{\varepsilon} x^{-p} \cos\left(nx + \frac{a}{x}\right) dx,$$

$$J_2(n) \sim \int_0^{\varepsilon} x^{-p} \sin\left(nx + \frac{a}{x}\right) dx$$

as $n \to \infty$, and hence we have

$$J_1(n) \sim \pi^{\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} n^{\frac{1}{2}p-\frac{3}{4}} \cos\left(2a^{\frac{1}{2}}n^{\frac{1}{2}} + \tfrac{1}{4}\pi\right) \qquad \left(-\tfrac{1}{2} < p < 2\right),$$

$$J_2(n) \sim \pi^{\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} n^{\frac{1}{2}p-\frac{3}{4}} \sin\left(2a^{\frac{1}{2}}n^{\frac{1}{2}} + \tfrac{1}{4}\pi\right) \qquad \left(-\tfrac{1}{2} < p < 2\right).$$

Hence we have

$$(74) \quad \left\{ \begin{aligned} & J(n) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}} n^{\frac{1}{2}p-\frac{3}{4}} \exp\left\{-\left(2a^{\frac{1}{2}}n^{\frac{1}{2}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi\right)i\right\}, \\ & \qquad\qquad \left(-\tfrac{1}{2} < p < 2\right). \end{aligned} \right.$$

51. (ii) The case in which $0 < q < 1$, $\alpha = (1+q)\dfrac{\pi}{2}$, $p < 1+q$. In this case

$$\alpha + \tfrac{1}{2}q\pi = \tfrac{1}{2}\pi + q\pi,$$

and this lies between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, since $0 < q < 1$. By (70), we have:

If $q \neq \frac{1}{2}$,

$$\left\{ \begin{aligned} \varphi(\theta) &= \frac{1}{\theta^p} e^{-a\theta^{-q} \sin q\pi} \{1 + O(\theta^{1-q})\}, \\ \psi(\theta) &= \frac{a}{\theta^q} \cos q\pi + \tfrac{1}{2}p\pi + O(\theta^{1-q}); \end{aligned} \right.$$

if $q=\frac{1}{2}$,

$$\begin{cases} \varphi(\theta) = \dfrac{1}{\theta^p} e^{-a\theta^{-q}}\{1+O(\theta^{2-q})\}, \\ \psi(\theta) = \frac{1}{2}p\pi + O(\theta^{\frac{1}{2}}). \end{cases}$$

Hence we have

$$J(n) = \frac{1}{2\pi} e^{\frac{1}{2}p\pi i} \int_0^{\xi} \varpi(x)\, e^{-\{nx+\sigma(x)\}i}\, dx,$$

where

$$\begin{cases} \sigma(x) = ax^{-q}\cos q\pi, \\ \varpi(x) = x^{-p} e^{-ax^{-q}\sin q\pi}\{1+\epsilon_1(x)+i\epsilon_2(x)\}, \end{cases}$$

ϵ_1 and ϵ_2 being real, continuous and differentiable functions of x in the interval $(0, \xi)$, such that

$$\lim_{x\to 0} \epsilon_1(x) = 0, \qquad \lim_{x\to 0} \epsilon_2(x) = 0.$$

Since $\sin q\pi > 0$, $\varpi(x)$ tends exponentially to zero as $x\to 0$. Therefore the integral $J(n)$ is absolutely convergent. Now

$$(75)\quad \int_0^{\xi} \varpi(x)\, e^{-\{nx+\sigma(x)\}i}\, dx = \int_0^{\xi} \varpi(x)\cos\sigma(x)\cos nx\, dx$$

$$- \int_0 \varpi(x)\sin\sigma(x)\sin nx\, dx$$

$$-i\int_0 \varpi(x)\cos\sigma(x)\sin nx\, dx$$

$$-i\int_0^{\xi} \varpi(x)\sin\sigma(x)\cos nx\, dx.$$

First we consider the integral

$$\int_0^{\xi} \varpi(x)\cos\sigma(x)\cos nx\, dx$$
$$= \frac{\varpi(\xi)\cos\sigma(\xi)\sin n\xi}{n} - \frac{1}{n}\int_0^{\xi} \frac{d}{dx}\{\varpi(x)\cos\sigma(x)\}\sin nx\, dx,$$

and $\left|\int_0^{\xi} \dfrac{d}{dx}\{\varpi(x)\cos\sigma(x)\}\sin nx\, dx\right| \leqq \int_0^{\xi} \{|\varpi'(x)| + |\varpi(x)\sigma'(x)|\}dx.$

But $|\varpi'(x)|+|\varpi(x)\sigma'(x)|$ tends exponentially to zero as $x\to 0$. Hence the last integral is convergent. Therefore we have

$$\int_0^\epsilon \varpi(x)\cos\sigma(x)\cos nx\,dx = O(1/n).$$

Similarly the other three integrals on the right-hand side of (75) assume values of the same type.

Thus we obtain

$$J(n) = O(1/n). \tag{76}$$

We observe that, in this case, p may take any value, positive or negative.

52. (iii) The case in which $0<q<1$, $a=(3-q)\frac{\pi}{2}$, $p<1+q$. In this case

$$a+\tfrac{1}{2}q\pi = \tfrac{3}{2}\pi.$$

Hence, by (70′),

$$\varphi(\theta) = \frac{1}{\theta^p}\{1+O(\theta^{1-q})\}, \qquad \psi(\theta) = -\frac{a}{\theta^q}+\tfrac{1}{2}p\pi+O(\theta),$$

and, if we write

$$\rho(x) = x^{-(p-1)}, \qquad \sigma(x) = ax^{-q};$$

then

$$J(n) = \frac{1}{2\pi}e^{\frac{1}{2}p\pi i}\int_0 \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}\,e^{-\{nx+\sigma(x)\}i}\,dx$$

$$= \frac{1}{2\pi}e^{\frac{1}{2}p\pi i}\{J_1(n)-iJ_2(n)\},$$

where

$$J_1(n) = \int_0^\epsilon \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}\cos\{nx+\sigma(x)\}\,dx,$$

$$J_2(n) = \int_0^\epsilon \frac{\rho(x)}{x}\{1+\epsilon_1(x)+i\epsilon_2(x)\}\sin\{nx+\sigma(x)\}\,dx,$$

$\epsilon_1(x)$ and $\epsilon_2(x)$ being real functions such that

$$\epsilon_1(x) = O(x^{1-q}), \quad \epsilon_2(x) = O(x^{1-q}).$$

We easily see* that there exist certain L-functions $\gamma_1(x)$ and $\gamma_2(x)$ such that

$$\varepsilon_1 \sim \gamma_1, \quad \varepsilon_2 \sim \gamma_2$$

as $x \to 0$, and also that $\frac{d\varepsilon_1}{dx}$ and $\frac{d\varepsilon_2}{dx}$ have ultimately constant signs.

By hypothesis $\qquad p < 1+q,$

hence the condition $\qquad \rho \prec x\sigma'$

is satisfied, so that the integrals $J_1(n)$, $J_2(n)$ and $J(n)$ are convergent.

Thus, applying Lemma 8 of Part I, we obtain

$$J_1(n) \sim \int_0^{\varepsilon} x^{-p} \cos\{nx + ax^{-q}\}\, dx,$$

$$J_2(n) \sim \int_0^{\varepsilon} x^{-p} \sin\{nx + ax^{-q}\}\, dx$$

as $n \to \infty$. Hence, we have

$$\begin{cases} J_1(n) \sim \left(\dfrac{2\pi}{1+q}\right)^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \cos(k n^{\frac{q}{1+q}} + \tfrac{1}{4}\pi), \\ J_2(n) \sim \left(\dfrac{2\pi}{1+q}\right)^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \sin(k n^{\frac{q}{1+q}} + \tfrac{1}{4}\pi), \end{cases}$$

and hence

$$(77)\quad \begin{cases} J(n) \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\{-(k n^{\frac{q}{1+q}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ \qquad k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}, \qquad (-\tfrac{1}{2}q < p < 1+q). \end{cases}$$

---

* The exact forms of ε_1 and ε_2 are

$$\varepsilon_1(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p e^{-a\sin\frac{1}{2}qx/(2\sin\frac{1}{2}x)^q} \cos\left\{\frac{a}{x^q} - \frac{a\cos\frac{1}{2}qx}{(2\sin\frac{1}{2}x)^q} - \tfrac{1}{2}px\right\} - 1,$$

$$\varepsilon_2(x) = \left(\frac{x}{2\sin\frac{1}{2}x}\right)^p e^{-a\sin\frac{1}{2}qx/(2\sin\frac{1}{2}x)^q} \sin\left\{\frac{a}{x^q} - \frac{a\cos\frac{1}{2}qx}{(2\sin\frac{1}{2}x)^q} - \tfrac{1}{2}px\right\}.$$

These functions are continuous and differentiable when x is sufficiently small, including the value $x=0$. And we can easily obtain the said result.

53. (iv) The case in which $0<q<1$, $(1+q)\frac{\pi}{2}<\alpha<(3-q)\frac{\pi}{2}$, $p<1+q$. In this case

$$\tfrac{1}{2}\pi+q\pi<\alpha+\tfrac{1}{2}q\pi<\tfrac{3}{2}\pi.$$

Hence
$$\cos(\alpha+\tfrac{1}{2}q\pi)<0,$$

and therefore, by (70′), we see that $\varphi(\theta)$ tends exponentially to zero as $\theta\to 0$, so that, by proceeding as in the case (ii), we easily obtain

$$J(n)=O(1/n). \tag{78}$$

Thus we have established the following results.

(i) *If* $q=1$, $\alpha=\pi$, $p<2$, *then*

$$\begin{cases} J(n)\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}a}n^{\frac{1}{2}p-\frac{3}{4}}\exp\{-(2a^{\frac{1}{2}}n^{\frac{1}{2}}-\tfrac{1}{2}p\pi+\tfrac{1}{4}\pi)i\}, \\ \qquad (-\tfrac{1}{2}<p<2); \end{cases}$$

(ii) *if* $0<q<1$, $\alpha=(1+q)\frac{\pi}{2}$, $p<1+q$, *then*

$$J(n)=O(1/n);$$

(iii) *if* $0<q<1$, $\alpha=(3-q)\frac{\pi}{2}$, $p<1+q$, *then*

$$\begin{cases} J(n)\sim\left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}(qa)^{-\frac{p-\frac{1}{2}}{1+q}}n^{\frac{p-1-\frac{1}{2}q}{1+q}}\exp\{-(kn^{\frac{q}{1+q}}-\tfrac{1}{2}p\pi+\tfrac{1}{4}\pi)i\}, \\ k=(1+q)\,q^{-\frac{q}{1+q}}a^{\frac{1}{1+q}}, \qquad (-\tfrac{1}{2}q<p<1+q); \end{cases}$$

(iv) *if* $0<q<1$, $(1+q)\frac{\pi}{2}<\alpha<(3-q)\frac{\pi}{2}$, $p<1+q$, *then*

$$J(n)=O(1/n).$$

54. *Integral* $\bar{J}(n)$. The discussion is quite similar as in the case of $J(n)$.

We have

$$\bar{\varphi}(\theta)=\frac{1}{\theta^p}\{1+O(\theta^2)\}e^{a\theta^{-q}\{\cos(\alpha-\frac{1}{2}q\pi)-\frac{1}{2}q\theta\sin(\alpha-\frac{1}{2}q\pi)\}+O(\theta^{2-q})},$$

$$\bar{\psi}(\theta) = -\tfrac{1}{2}p(\pi-\theta) + \frac{a}{\theta^q}\{\sin(\alpha-\tfrac{1}{2}q\pi) + \tfrac{1}{2}q\theta\cos(\alpha-\tfrac{1}{2}q\pi)\} + O(\theta^{2-q})\,;$$

and, without difficulty, we obtain the following results.

(i) *If* $q=1$, $\alpha=\pi$, $p<2$, *then*

$$\begin{cases} \bar{J}(n) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} \exp\{(2a^{\frac{1}{2}}n^{\frac{1}{2}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ \qquad (-\tfrac{1}{2} < p < 2)\,; \end{cases}$$

(ii) *if* $0<q<1$, $\alpha=(1+q)\dfrac{\pi}{2}$, $p<1+q$, *then*

$$\begin{cases} \bar{J}(n) \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\{(k\,n^{\frac{q}{1+q}} - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi)i\}, \\ k = (1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}} \qquad (-\tfrac{1}{2}q < p < 1+q)\,; \end{cases}$$

(iii) *if* $0<q<1$, $\alpha=(3-q)\dfrac{\pi}{2}$, $p<1+q$, *then*

$$\bar{J}(n) = O(1/n);$$

(iv) *if* $0<q<1$, $(1+q)\dfrac{\pi}{2} < \alpha < (3-q)\dfrac{\pi}{2}$, $p<1+q$, *then*

$$\bar{J}(n) = O(1/n)\,.$$

55. Hence, by (64) and Lemma 10, we obtain :

If $q=1$, $\alpha=\pi$, *and* $-\tfrac{1}{2}<p<2$, *then*

$$a_n \sim \pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{-\frac{1}{2}p+\frac{3}{4}} \sin\{2a^{\frac{1}{2}}n^{\frac{1}{2}} - (\tfrac{1}{2}p-\tfrac{3}{4})\pi\}\,; \tag{79}$$

if $0<q<1$, $\alpha=(1+q)\dfrac{\pi}{2}$, *and* $-\tfrac{1}{2}q<p<1+q$, *then*

$$\begin{cases} a_n \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\left[\{kn^{\frac{q}{1+q}} - (\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\right], \\ k = (1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}}\,; \end{cases} \tag{80}$$

if $0<q<1$, $\alpha=(3-q)\dfrac{\pi}{2}$, *and* $-\tfrac{1}{2}q<p<1+q$, *then*

$$\begin{cases} a_n \sim \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\left[-\{kn^{\frac{q}{1+q}} - (\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\right], \\ k = (1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}}\,; \end{cases} \tag{81}$$

$0<q<1$, $(1+q)\frac{\pi}{2}<\alpha<(3-q)\frac{\pi}{2}$, *and* $p<1+q$, *then*

$$(82) \qquad a_n = O(1/n).$$

56. We have thus obtained asymptotic formulae for a_n, as $n\to\infty$, in the three cases*

(i) $\quad q=1, \quad \alpha=\pi,$

(ii) $\quad 0<q<1, \quad \alpha=(1+q)\frac{\pi}{2},$

(iii) $\quad 0<q<1, \quad \alpha=(3-q)\frac{\pi}{2},$

always with the condition

$$-\tfrac{1}{2}q<p<1+q,$$

which is introduced from the conditions that the integral $I(n)$ is convergent and $I(n)\succ 1/n$ as $n\to\infty$. But, by proceeding as follows, it will be seen that this restriction about the value of p may be removed.

Now, in a certain region near the point $z=1$ and interior to the circle of convergence, we may put

$$\frac{1}{(1-z)^p}e^{A/(1-z)^q}=\sum_{n=0}^{\infty}a_n z^n,$$

and we may differentiate this equation with respect to z, since our series $\sum a_n z^n$ is uniformly convergent in the said region. Thus we obtain

$$\frac{p}{(1-z)^{p+1}}e^{A/(1-z)^q}+\frac{qA}{(1-z)^{p+q+1}}e^{A/(1-z)^q}=\sum_{n=0}^{\infty}(n+1)\,a_{n+1}z^n.$$

Observing that a_n is a function of n and p, we write

$$a_n=a(n,p).$$

Then we have

$$p\sum_{n=0}^{\infty}a(n,p+1)z^n+qA\sum_{n=0}^{\infty}a(n,p+q+1)z^n=\sum_{n=0}^{\infty}(n+1)a(n+1,p)\,z^n,$$

* Our method fails to determine the asymptotic formulae in other cases.

whence

$$(83) \qquad p\,a(n, p+1)+qAa(n, p+q+1) = (n+1)\,a(n+1, p)$$
$$(n = 0, 1, 2, \cdots\cdots).$$

Hence

$$(84) \qquad a(n, p+q+1) = \frac{n+1}{qA}\,a(n+1, p) - \frac{p}{qA}\,a(n, p+1).$$

For instance, take the case

$$0 < q < 1, \qquad a = (1+q)\frac{\pi}{2}.$$

Then, by (80), we have

$$\begin{cases} a(n, p) \sim C(qa)^{-(p-\frac{1}{2})/(1+q)} n^{(p-1-\frac{1}{2}q)/(1+q)} \exp\{(kn^{q/(1+q)} - \frac{1}{2}p\pi + \frac{1}{4}\pi)i\}, \\ \qquad (-\frac{1}{2}q < p < 1+q), \end{cases}$$

where $\qquad C = \left\{\dfrac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}}, \quad k = (1+q)\,q^{-\frac{q}{1+q}}\,a^{\frac{1}{1+q}}$

are independent of n and p.

Let us suppose that

$$p < q,$$

so that $\qquad 1+p < 1+q,$

and we have

$$a(n, p+1) \sim C(qa)^{-(p+\frac{1}{2})/(1+q)} n^{(p-\frac{1}{2}q)/(1+q)} \exp[\{kn^{q/(1+q)} - \tfrac{1}{2}(p+1)\pi + \tfrac{1}{4}\pi\}i],$$

whence

$$|\,a(n, p+1)\,| = O\left\{n^{(p-\frac{1}{2}q)/(1+q)}\right\}.$$

We have also

$$|\,(n+1)\,a(n+1, p)\,| = O\left\{(n+1)^{(p+\frac{1}{2}q)/(1+q)}\right\} = O\left\{n^{(p+\frac{1}{2}q)/(1+q)}\right\}.$$

Hence we have

$$(n+1)\,a(n+1, p) \succ a(n, p+1).$$

Thus we obtain, by (84),

$$a(n, p+q+1) \sim \frac{1}{qA}(n+1)\,a(n+1, p).$$

Now

$$qA = qa\,e^{(1+q)\frac{\pi}{2}i},$$

$$a(n+1,p) \sim C(qa)^{-(p-\frac{1}{2})/(1+q)}(n+1)^{(p-1-\frac{1}{2}q)/(1+q)}\exp\left[\{k(n+1)^{q/(1+q)}-\tfrac{1}{2}p\pi+\tfrac{1}{4}\pi\}i\right],$$

and

$$e^{ik(n+1)^{q/(1+q)}} = \exp\left[i\,k\left\{n^{\frac{q}{1+q}}+O(n^{-\frac{1}{1+q}})\right\}\right]$$

$$= e^{ik\,n^{q/(1+q)}}\left\{1+O(n^{-\frac{1}{1+q}})\right\}$$

$$\sim e^{ik\,n^{q/(1+q)}}.$$

Hence, writing $p_1 = p+q+1$, we obtain

$$a(n, p_1) \sim C(qa)^{-(p_1-\frac{1}{2})/(1+q)}n^{(p_1-1-\frac{1}{2}q)/(1+q)}\exp\left[\{kn^{q/(1+q)}-\tfrac{1}{2}p_1\pi+\tfrac{1}{4}\pi\}i\right],$$

where $$p_1 < 1+2q.$$

Thus, in the formula (80), the upper limit of p is increased by q. By repeating this process m times, the upper limit of p in (80) may be increased by mq. Thus we see that, in the formula (80), p may take any positive value, whatever.

Next, by (83), we have

$$a(n+1,p) = \frac{p}{n+1}\,a(n, p+1)+\frac{qA}{n+1}\,a(n, p+q+1).$$

By repeated applications of this formula, the lower limit of p in the formula (80), may be decreased as much as we please. Thus we see that, in the formula (80), p may take any negative value, whatever.

Therefore the formula (80) holds for all real values of p.

The same argument applies to the other two formulae. Hence we can state

Theorem VIII. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, whose radius of convergence is unity, representing a function $f(z)$ which has on the circle of convergence, one singular point only at $z=1$, being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p}\,e^{A/(1-z)^q},\quad A = a\,e^{\alpha i},$$

where p is any real constant, $a>0$ and q, α are certain constants, then the behaviour of the coefficient a_n, as $n \to \infty$, is determined asymptotically as follows :

(i) *If $q=1$, $\alpha=\pi$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{a/(z-1)} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0 \end{pmatrix},$$

then

$$(9) \qquad a_n \sim \frac{1}{\sqrt{\pi}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} \sin\{2a^{\frac{1}{2}}n^{\frac{1}{2}}-(\tfrac{1}{2}p-\tfrac{3}{4})\pi\}$$

(ii) *If $0<q<1$, $\alpha=(1+q)\frac{\pi}{2}$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{-a(\sin\frac{1}{2}q\pi - i\cos\frac{1}{2}q\pi)/(1-z)^q} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0,\,0<q<1 \end{pmatrix}.$$

then

$$(10) \quad a_n \sim \left\{\frac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\left[\,\{k\,n^{\frac{q}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\,\right],$$

where $$k=(1+q)\,q^{-\frac{q}{1+q}}a^{\frac{1}{1+q}}.$$

(iii) *If $0<q<1$, $\alpha=(3-q)\frac{\pi}{2}$, or the singularity is of the type*

$$f(z)=\frac{1}{(1-z)^p}e^{-a(\sin\frac{1}{2}q\pi + i\cos\frac{1}{2}q\pi)/(1-z)^q} \qquad \begin{pmatrix} p>,=,<0\,; \\ a>0,\,0<q<1 \end{pmatrix},$$

then

$$(11) \quad a_n \sim \left\{\frac{1}{2(1+q)\pi}\right\}^{\frac{1}{2}} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} \exp\left[\,-\{k\,n^{\frac{q}{1+q}}-(\tfrac{1}{2}p-\tfrac{1}{4})\pi\}i\,\right],$$

k being the same as in the case (ii).

III *Case in which the Singularity is of the Type* $\dfrac{1}{(1-z)^p}e^{A/(1-z)^q}\left(\log\dfrac{1}{1-z}\right)^r$.

57. We now pass to the case in which $f(z)$ has a singularity of the type

$$f(z)=\frac{1}{(1-z)^p}e^{A/(1-z)^q}\left(\log\frac{1}{1-z}\right)^r,$$

where $$A = a\, e^{\alpha i},\ a>0,\ 0 \leqq \alpha < 2\pi,\ q>0,$$

and p, r denote arbitrary real constants.

First of all, in considering the integral $I(r_1)$, we observe that

$$\log \frac{1}{1-z} = \log\left(\frac{1}{r_1} e^{\varphi i}\right) = \log \frac{1}{r_1} + \varphi i.$$

Hence, without difficulty, we can prove that Lemma 11 holds also in this case; namely, if $0<q\leqq 1$, $(1+q)\frac{\pi}{2} \leqq \alpha \leqq (3-q)\frac{\pi}{2}$ and $p<1+q$, then

$$\lim_{r_1 \to 0} I(r_1) = 0.$$

Next, when θ is small, we have

$$\begin{aligned}\log \frac{1}{1-e^{\theta i}} &= \log \frac{1}{2 \sin \frac{1}{2}\theta} + (\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) i \\ &= \log \frac{1}{\theta} + \tfrac{1}{2}\pi i + O(\theta) \\ &= \log \frac{1}{\theta}\left\{1 + O\left(1\Big/\log \frac{1}{\theta}\right)\right\},\end{aligned}$$

whence

$$\left(\log \frac{1}{1-e^{\theta i}}\right)^r = \left(\log \frac{1}{\theta}\right)^r \left\{1 + O\left(1\Big/\log \frac{1}{\theta}\right)\right\}.$$

Similarly

$$\left(\log \frac{1}{1-e^{(2\pi-\theta) i}}\right)^r = \left(\log \frac{1}{\theta}\right)^r \left\{1 + O\left(1\Big/\log \frac{1}{\theta}\right)\right\}.$$

Hence the dicussion may be carried out quite similarly as in the preceding case, the presence of the logarithmic factor producing no great change in the analysis, and we content ourselves with giving only the following results.

Theorem IX. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, whose radius of convergence is unity, representing a function $f(z)$ which has, on the circle of convergence, one singular point only at $z=1$, being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r, \quad A = a\, e^{\alpha i},$$

where $a > 0$, q *and* α *denote certain constants, and* p, r *arbitrary real constants, then the behaviour of the coefficient* a_n, *as* $n \to \infty$, *is determined asymptotically as follows.*

(i) *If* $q = 1$, $\alpha = \pi$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{a/(z-1)} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim 2^{-r} \pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} (\log n)^r \sin\left\{2a^{\frac{1}{2}} n^{\frac{1}{2}} - (\tfrac{1}{2}p - \tfrac{3}{4})\pi\right\}.$$

(ii) *If* $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi - i\cos\frac{1}{2}q\pi)/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} (\log n)^r \exp\left[\left\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\right\} i\right],$$

where
$$k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}.$$

(iii) *If* $0 < q < 1$, $\alpha = (3-q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi + i\cos\frac{1}{2}q\pi)/(1-z)^q} \left(\log \frac{1}{1-z}\right)^r,$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} (\log n)^r \exp\left[-\left\{k\, n^{\frac{q}{1+q}} - (\tfrac{1}{2}p - \tfrac{1}{4})\pi\right\} i\right],$$

k being the same as in the case (ii).

IV *Case in which the Singularity is of the Type*

$$\frac{1}{(1-z)^p} e^{A/(1-z)^q} \left(l_1 \frac{1}{1-z}\right)^{r_1} \left(l_2 \frac{1}{1-z}\right)^{r_2} \cdots \left(l_h \frac{1}{1-z}\right)^{r_h}.$$

58. More generally we obtain the following theorem, the argument being quite similar as in the preceding case.

Theorem X. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be a power series, whose radius of convergence is unity, representing a function* $f(z)$ *which has, on the* *circle of convergence, one singular point only at* $z = 1$, *being regular at every other point on it. If the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{A/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

where $g(x) = \{l_1(x)\}^{r_1} \{l_2(x)\}^{r_2} \cdots\cdots \{l_h(x)\}^{r_h}$, $A = a\, e^{\alpha i}$, $a > 0$, q *and* α *denote certain constants, and* p *and all* r'*s arbitrary real constants, then the behaviour of the coefficient* a_n, *as* $n \to \infty$, *is determined asymptotically as follows.*

(i) *If* $q = 1$, $\alpha = \pi$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{a/(z-1)} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim 2^{-r_1} \pi^{-\frac{1}{2}} a^{-\frac{1}{2}p+\frac{1}{4}} e^{-\frac{1}{2}a} n^{\frac{1}{2}p-\frac{3}{4}} g(n) \sin\left\{2a^{\frac{1}{2}} n^{\frac{1}{2}} - \left(\tfrac{1}{2}p - \tfrac{3}{4}\right)\pi\right\}.$$

(ii) *If* $0 < q < 1$, $\alpha = (1+q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi - i\cos\frac{1}{2}q\pi)/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r_1+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} g(n) \exp\left[\left\{k\, n^{\frac{q}{1+q}} - \left(\tfrac{1}{2}p - \tfrac{1}{4}\right)\pi\right\} i\right],$$

where

$$k = (1+q)\, q^{-\frac{q}{1+q}} a^{\frac{1}{1+q}}.$$

(iii) *If* $0 < q < 1$, $\alpha = (3-q)\frac{\pi}{2}$, *or the singularity is of the type*

$$f(z) = \frac{1}{(1-z)^p} e^{-a(\sin\frac{1}{2}q\pi + i\cos\frac{1}{2}q\pi)/(1-z)^q} g\left(\frac{1}{1-z}\right),$$

then

$$a_n \sim \frac{1}{\sqrt{(2\pi)}} (1+q)^{-(r_1+\frac{1}{2})} (qa)^{-\frac{p-\frac{1}{2}}{1+q}} n^{\frac{p-1-\frac{1}{2}q}{1+q}} g(n) \exp\left[-\left\{k\, n^{\frac{q}{1+q}} - \left(\tfrac{1}{2}p - \tfrac{1}{4}\right)\pi\right\} i\right],$$

k *being the same as in the case (ii).*

Published April 20, 1919.