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On the Elastic Equilibrium of a Semi-Infinite Solid under given Boundary Conditions, with some Applications.

By

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I. Introduction.

§1. The statical problem concerning an infinite elastic solid bounded by a plane subjected to a given distribution of traction or deformation has attracted the attention of numerous eminent elasticians. The first solution for the case of a purely normal load was given by LAME AND CLAPEYRON[1] by means of FOURIER's theorem, through which an assigned function of two variables is expressed as a quadruple integral. The credit of first improvement on this subject may well be claimed by J. BOUSSINESQ,[2] who introduced several kinds of potentials—direct, inverse and logarithmic with three variables—into the theory of elasticity, and opened a new field of treatment in it. Almost all conceivable cases have been solved by him, especially in relation to what takes place at the boundary surface. Besides Boussinesq, many other authors have touched on this problem, employing the method of integration by Green's functions. Not long ago, Prof. H. LAMB[3] solved a special case of this problem, viz. that in which the boundary condition is a normal pressure symmetrically distributed about a point on the surface, by making use of the integral theorem of Fourier's type concerning Bessel function of the zeroth order; and thus, Lamé and Clapeyron's method, which was considered to

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1) CRELLE's Journal, vol. 7 (1831) p.p. 400–404.
2) Application des Potentiels, Paris, (1885).
3) Lond. Math. Soc. Proc. vol. 34 (1902) p. 276.

be extremely unsuited for obtaining physical results, seems now to have gained practical importance.

§2. The present paper deals with the problem in the case in which the boundary is subjected to any given normal pressure, by generalizing the method adopted by LAMB.[1] In the first two sections the general solution of the problem is obtained in the type of the Bessel-Fourier expansion of a function. The third section discusses several examples in the case of symmetry about an axis normal to the boundary, and forms the main part of this communication. Most of these special examples have been investigated by the authors above cited: the behaviour at the surface especially; and yet it may be worth while to discuss them again more closely, referring especially to the behaviour inside the boundary.

The last section is added as an appendix, supplying the general solutions corresponding to several boundary conditions, excepting that of normal pressure, in the case of symmetry about a normal to the boundary.

§3. The results of these special examples applied to find the limit of rupture of a foundation over which a heavy load is distributed. Strictly speaking, by applying the mathematical theory of elasticity, we can treat of rupture only, for some kinds of brittle solids like cast iron, in which the linear relation of stress and strain holds and, moreover, the strains are so small that their squares are negligible up to the point where rupture takes place. For a ductile material, such as mild steel, and for an imperfectly elastic material, like cement or sandstone, we must bear in mind that the theoretical results indicate only roughly the state of stress when, in the first case, it begins to take permanent set, and, in the second case, when it breaks.

§4. Another application will be found in a problem of geophysics. In his elaborate observations on the lunar deflection of gravity, Dr. O. HECKER has pointed out that the force acting on the pendulum at Potsdam is a large fraction of the moon's force when it acts towards the east or west than when it acts towards the

1) Though the writer had not read his paper until the work was almost finished.

north or south.[1] Various explanations of this anomaly have been proposed, among them one, suggested by Prof. A.E.H. Love,[2] is that a possible cause may perhaps be found in the effect of the tide wave in the North Atlantic. Recently Prof. A.A. MICHELSON[3] has found a similar result in his arduous task of measuring the lunar perturbation of a very long water-level at Chicago. Prof. Sir J. LARMOR kindly suggested to me a query whether the excess-pressure of the tide in the North Atlantic would affect much the measurement of the water-level at Chicago, owing to the depression of the solid earth that it would produce. A calculation is undertaken in order to ascertain to what extent the consideration of the tilting of the ground is important for the explanation of this geodynamical discrepancy; we may in a first estimation neglect the curvature of the earth.[4]

II. Solution of Equation of Equilibrium.

§5. The equation of equilibrium of an isotropic elastic body free from the action of a body force is

$$\operatorname{curl}\operatorname{curl} u = \frac{\lambda+2\mu}{\mu}\operatorname{grad}\Delta, \qquad (1)$$

where u denotes the displacement, λ and μ Lamé's constants which specify the elastic nature of the body concerned, and Δ is the amount of dilatation defined by the equation

$$\Delta = \operatorname{div} u. \qquad (2)$$

Our first object is to find the solution of the equation of equilibrium, which is appropriate for the discussion of the problem concerning a semi-infinite elastic body.

Since div. curl of any vector quantity vanishes, if we perform the operation div on both sides of equation (1) we have simply

$$\operatorname{div}\operatorname{grad}\Delta = 0, \qquad (3)$$

1) A similar result has been found by A. ORLOFF at Dorpat and T. Shida at Kyoto.
2) Some Problems of Geodynamics, Cambridge, (1911) p. 88.
3) The Astrophysical Journal, vol. 39 (1914) p. 105—.
4) The geodynamical application will appear shortly in the Trans. Roy. Soc. London.

which determines the dilatation $\varDelta$. If $\varDelta$ is found from this equation, the displacement u can be determined by solving the equation

$$\operatorname{grad}\operatorname{div} u - \operatorname{curl}\operatorname{curl} u = -\frac{\lambda+\mu}{\mu}\operatorname{grad}\varDelta. \tag{4}$$

The elastic body which we deal with is supposed to be bounded by an infinite (say) horizontal plane in its natural state and to extend without limit both horizontally and downwards. Take the cylindrical coordinates (r, θ, z) such that the axis of z coincides with an inward normal to the boundary and the origin lies on the boundary surface of the body in its natural state, then we have, for $z > o$, the equations

$$\frac{\partial^2\varDelta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varDelta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varDelta}{\partial\theta^2}+\frac{\partial^2\varDelta}{\partial z^2}=0 \tag{5}$$

to determine $\varDelta$, and

$$\left.\begin{aligned}
&\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{\partial z^2}-\frac{1}{r^2}\left(u_r-\frac{\partial^2 u_r}{\partial\theta^2}\right)-\frac{2}{r^2}\cdot\frac{\partial u_\theta}{\partial\theta}=-\frac{\lambda+\mu}{\mu}\cdot\frac{\partial\varDelta}{\partial r},\\
&\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{1}{r^2}\left(u_\theta-\frac{\partial^2 u_\theta}{\partial\theta^2}\right)+\frac{2}{r^2}\cdot\frac{\partial u_r}{\partial\theta}=-\frac{\lambda+\mu}{\mu}\cdot\frac{1}{r}\frac{\partial\varDelta}{\partial\theta},\\
&\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{r^2}\cdot\frac{\partial^2 u_z}{\partial\theta^2}=-\frac{\lambda+\mu}{\mu}\cdot\frac{\partial\varDelta}{\partial z},
\end{aligned}\right\} \tag{6}$$

where u_r, u_θ, u_z are the components of the vector u. The equation (5) follows from (3), and (6) from (4).

§6. To solve these equations, assume

$$\varDelta=\delta(r).\, e^{-kz}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta,$$

where k is a positive constant so that there may be no dilatation at $z=\infty$, and m is an integer, positive or negative, or zero so that the solution may be unique round the origin, then the equation (5) gives

$$\frac{d^2\delta}{dr^2}+\frac{1}{r}\cdot\frac{d\delta}{dr}+\left(k^2-\frac{m^2}{r^2}\right)\delta=0,$$

of which the solution is

$$\delta=C_m J_m(kr)\ ^{1)}$$

1) The second solution is rejected for the reason of its singularity at the origin which we exclude from our present investigation.

where $J_m(r)$ is the Bessel coefficient of order m, and C_m is a constant of integration. Thus we obtain

$$\varDelta = C_m e^{-kz} J_m(kr) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\theta. \tag{7}$$

§7. To find u corresponding to

$$\varDelta = C_m e^{-kz} J_m(kr) \cos m\theta,$$

assume that

$$u_r = U_r \cos m\theta,$$
$$u_\theta = U_\theta \sin m\theta,$$
$$u_z = U_z \cos m\theta,$$

U_r, U_θ, U_z, being functions of r and z. Then equations (6) transform into

$$\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial r} + \frac{\partial^2 U_r}{\partial z^2} - \frac{m^2+1}{r^2} U_r - \frac{2m}{r^2} U_\theta = -\frac{\lambda+\mu}{\mu} k C_m e^{-kz} J'_m(kr), \tag{8}$$

$$\frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial U_\theta}{\partial r} + \frac{\partial^2 U_\theta}{\partial z^2} - \frac{m^2+1}{r^2} U_\theta - \frac{2m}{r^2} U_r = \frac{\lambda+\mu}{\mu} \cdot \frac{m}{r} C_m e^{-kz} J_m(kr), \tag{9}$$

$$\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r} \frac{\partial U_z}{\partial r} + \frac{\partial^2 U_z}{\partial z^2} - \frac{m^2}{r^2} U_z = \frac{\lambda+\mu}{\mu} k C_m e^{-kz} J_m(kr) \tag{10}$$

in which $J_m'(x)$ means $dJ_m(x)/dx$ as usual.

The last equation suggests that U_z has the form $VJ_m(kr)$, where V is a function of z only and satisfies the equation

$$\frac{d^2 V}{dz^2} - k^2 V = \frac{\lambda+\mu}{\mu} k C_m e^{-kz}$$

The solution of this equation is

$$V = \left(-\frac{\lambda+\mu}{2\mu} C_m z + D_m\right) e^{-kz},$$

D_m being an arbitrary constant. Thus

$$u_z = -\left(\frac{\lambda+\mu}{2\mu} C_m z - D_m\right) e^{-kz} J_m(kr) \cos m\theta. \tag{11}$$

To find U_r, and U_θ, write

$$U_r+U_\theta = X,$$
$$U_r-U_\theta = Y,$$

then combining the equations (8) and (9) we have

$$\frac{\partial^2 X}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial X}{\partial r}+\frac{\partial^2 X}{\partial z^2}-\frac{(m+1)^2}{r^2}X = \frac{\lambda+\mu}{\mu}kC_m e^{-kz}J_{m+1}(kr),$$

$$\frac{\partial^2 Y}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial Y}{\partial r}+\frac{\partial^2 Y}{\partial z^2}-\frac{(m-1)^2}{r^2}Y = -\frac{\lambda+\mu}{\mu}kC_m e^{-kz}J_{m-1}(kr).$$

These equations suggest that X and Y vary as $J_{m+1}(kr)$ and $J_{m-1}(kr)$ respectively with regard to r, and they can be solved in a manner similar to the equation (10), The result is

$$X = \left(-\frac{\lambda+\mu}{2\mu}C_m z+E_m\right)e^{-kz}J_{m+1}(kr),$$

$$Y = \left(\frac{\lambda+\mu}{2\mu}C_m z+F_m\right)e^{-kz}J_{m-1}(kr),$$

E_m and F_m being arbitrary constants. Writing

$$E_m+F_m = 2A_m,$$
$$E_m-F_m = 2B_m,$$

and simplifying, we have

$$\left.\begin{aligned} U_r &= \frac{\lambda+\mu}{2\mu}C_m zJ'_m(kr)+\left[A_m\frac{m}{kr}J_m(kr)-B_mJ'_m(kr)\right] \\ U_\theta &= -\frac{\lambda+\mu}{2\mu}C_m z\frac{m}{kr}J_m(kr)+\left[B_m\frac{m}{kr}J_m(kr)-A_mJ'_m(kr)\right] \end{aligned}\right\}e^{-kz}.$$

Thus we have a system of solutions:—

$$\left.\begin{aligned} \Delta &= C_mJ_m(kr)e^{-kz}\cos m\theta, \\ u_r &= \left\{\left[\frac{\lambda+\mu}{2\mu}C_m z-B_m\right]J'_m(kr)+A_m\frac{m}{kr}J_m(kr)\right\}e^{-kz}\cos m\theta, \\ u_\theta &= -\left\{\left[\frac{\lambda+\mu}{2\mu}C_m z-B_m\right]\frac{m}{kr}J_m(kr)+A_mJ'_m(kr)\right\}e^{-kz}\sin m\theta, \\ u_z &= -\left\{\frac{\lambda+\mu}{2\mu}C_m z-D_m\right\}J_m(kr)e^{-kz}\cos m\theta. \end{aligned}\right\} \quad (12)$$

The constants B_m, C_m and D_m are not independent of one another, but are connected by the relation

$$k(B_m - D_m) = \frac{\lambda + 3\mu}{2\mu} C_m, \tag{13}$$

which follows from the definition of $\varDelta$.

§8. Next, the solutions corresponding to $\varDelta$ which has $\sin m\theta$ in its factor can be found in a similar way. They are distinguished from the above by placing a bar on every quantity.

$$\left.\begin{aligned}
\overline{\varDelta} &= \overline{C}_m J_m(kr) e^{-kz} \sin m\theta, \\
\overline{u}_r &= \left\{ \left[\frac{\lambda+\mu}{2\mu} \overline{C}_m z - \overline{B}_m \right] J'_m(kr) + \overline{A}_m \frac{m}{kr} J_m(kr) \right\} e^{-kz} \sin m\theta, \\
\overline{u}_\theta &= \left\{ \left[\frac{\lambda+\mu}{2\mu} \overline{C}_m z - \overline{B}_m \right] \frac{m}{kr} J_m(kr) + \overline{A}_m J'_m(kr) \right\} e^{-kz} \cos m\theta, \\
\overline{u} &= -\left\{ \frac{\lambda+\mu}{2\mu} \overline{C}_m z - \overline{D}_m \right\} J_m(kr) e^{-kz} \sin m\theta \,;
\end{aligned}\right\} \tag{14}$$

with

$$k(\overline{B}_m - \overline{D}_m) = \frac{\lambda + 3\mu}{2\mu} \overline{C}_m. \tag{15}$$

§9. The stress can be calculated by using the formulae

$$\left.\begin{aligned}
\widehat{zz} &= \lambda\varDelta + 2\mu \frac{\partial u_z}{\partial z}, \\
\widehat{zr} &= \mu \left\{ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right\}, \\
\widehat{z\theta} &= \mu \left\{ \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right\}, \\
\widehat{rr} &= \lambda\varDelta + 2\mu \frac{\partial u_r}{\partial r}, \\
\widehat{\theta\theta} &= \lambda\varDelta + 2\mu \left\{ \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right\}, \\
\widehat{r\theta} &= \mu \left\{ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial \theta} \right\}.
\end{aligned}\right\} \tag{16}$$

Thus corresponding to the first solution (12), we obtain

$$
\left.\begin{aligned}
\widehat{zz} &= \left\{ (\lambda+\mu)kC_m z - \mu C_m - 2\mu k D_m \right\} J_m(kr) e^{-kz} \cos m\theta, \\
\widehat{zr} &= -\left\{ \left[(\lambda+\mu)k C_m z - \frac{\lambda+\mu}{2} C_m - \mu k (B_m + D_m) \right] J'_m(kr) \right. \\
&\qquad \left. + \frac{\mu m}{r} A_m J_m'(kr) \right\} e^{-kz} \cos m\theta, \\
\widehat{z\theta} &= \left\{ \left[(\lambda+\mu)kC_m z - \frac{\lambda+\mu}{2} C_m - \mu k (B_m + D_m) \right] \frac{m}{kr} J_m(kr) \right. \\
&\qquad \left. + \mu k A_m J'_m(kr) \right\} e^{-lz} \sin m\theta, \\
&\qquad \text{etc.};
\end{aligned}\right\} \quad (17)
$$

and corresponding to the second solution (14),

$$
\left.\begin{aligned}
\overline{\widehat{zz}} &= \left\{ (\lambda+\mu)k\overline{C}_m z - \mu \overline{C}_m - 2\mu k \overline{D} \right\} J_m(kr) e^{-lz} \sin m\theta, \\
\overline{\widehat{zr}} &= -\left\{ \left[(\lambda+\mu)k\overline{C}_m z - \frac{\lambda+\mu}{2} \overline{C}_m - \mu k (\overline{B} + \overline{D}_m) \right] J'_m(kr) \right. \\
&\qquad \left. + \frac{\mu m}{r} \overline{A}_m J_m(kr) \right\} e^{-lz} \sin m\theta, \\
\overline{\widehat{z\theta}} &= -\left\{ \left[(\lambda+\mu)k\overline{C}_m z - \frac{\lambda+\mu}{2} \overline{C}_m + \mu k (\overline{B}_m + \overline{D}_m) \right] \frac{m}{kr} J_m(kr) \right. \\
&\qquad \left. + \mu k \overline{A}_m J'_m(kr) \right\} e^{-kz} \cos m\theta, \\
&\qquad \text{etc.}
\end{aligned}\right\} \quad (18)
$$

III. Lamé and Clapeyron's Problem.

§10. We will apply the solution obtained in the last article to discuss the effect caused by a given normal pressure applied locally on the boundary. Suppose as a preliminary that a system of stress of the form

$$
\left.\begin{aligned}
\widehat{zz} &= \left\{ Z \cos m\theta + \overline{Z} \sin m\theta \right\} J_m(kr), \\
\widehat{zr} &= 0, \qquad \widehat{z\theta} = 0
\end{aligned}\right\} \quad (19)
$$

is given at the surface $z=0$. Then we must have the following relations between the arbitrary constants:—

$$
\begin{aligned}
-\mu C_m - 2\mu k D_m &= Z, \\
\frac{\lambda+\mu}{2} C_m + \mu k(B_m + D_m) &= 0, \\
A_m &= 0; \\
-\mu \overline{C}_m - 2\mu k \overline{D}_m &= \overline{Z}, \\
\frac{\lambda+\mu}{2} \overline{C}_m + \mu k(\overline{B}_m + \overline{D}_m) &= 0, \\
\overline{A}_m &= 0.
\end{aligned}
$$

From these equations, combining (13) and (15), we have

$$
\begin{aligned}
C_m &= \frac{Z}{\lambda+\mu}, & \overline{C}_m &= \frac{\overline{Z}}{\lambda+\mu}, \\
B_m &= \frac{Z}{2k(\lambda+\mu)}, & \overline{B}_m &= \frac{\overline{Z}}{2k(\lambda+\mu)}, \\
D_m &= -\frac{(\lambda+2\mu)Z}{2\mu k(\lambda+\mu)}, & \overline{D}_m &= -\frac{(\lambda+2\mu)\overline{Z}}{2\mu k(\lambda+\mu)}, \\
A_m &= 0, & \overline{A}_m &= 0,
\end{aligned}
$$

Putting these values in the formulæ (12) and (14) and adding them together, we have an exact solution of the form

$$
\left.
\begin{aligned}
u_r &= \left\{\frac{z}{2\mu} - \frac{1}{2(\lambda+\mu)k}\right\}\left\{Z\cos m\theta + \overline{Z}\sin m\theta\right\} J'_m(kr)e^{-kz}, \\
u_\theta &= \left\{\frac{z}{2\mu} - \frac{1}{2(\lambda+\mu)k}\right\}\left\{\overline{Z}\cos m\theta - Z\sin m\theta\right\} \frac{m}{kr} J_m(kr)e^{-kz}, \\
u_z &= -\left\{\frac{z}{2\mu} + \frac{\lambda+2\mu}{2\mu(\lambda+\mu)k}\right\}\left\{Z\cos m\theta + \overline{Z}\sin m\theta\right\} J_m(kr)e^{-kz}
\end{aligned}
\right\} \quad (20)
$$

corresponding to the boundary conditions (19). The formulæ for stress are

$$
\left.
\begin{aligned}
\widehat{zz} &= (1+kz)\left\{Z\cos m\theta + \overline{Z}\sin m\theta\right\} J_m(kr)e^{-kz}, \\
\widehat{zr} &= -kz\left\{Z\cos m\theta + \overline{Z}\sin m\theta\right\} J'_m(kr)e^{-kz} \\
\widehat{z\theta} &= -kz\left\{\overline{Z}\cos m\theta - Z\sin m\theta\right\} \frac{m}{kr} J_m(kr)e^{-kz}, \\
&\text{etc.}
\end{aligned}
\right\} \quad (21)
$$

§11. To generalize the above so that the solution may be suitable to discuss the effect of any given normal pressure at the boundary surface, suppose that Z and $\overline{Z}$ are functions of k of the form $Z_m(k)dk$ and $\overline{Z}_m(k)dk$ respectively, and superpose the corresponding solutions for all positive values of k, then add them together for all integral values of m. Thus, corresponding to the boundary conditions

$$\left.\begin{aligned} \widehat{zz} &= \sum_{m=0}\int_0^\infty \Big\{Z_m(kr)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}J_m(kr)\,dk,\\ \widehat{zr} &= 0, \qquad \widehat{z\theta} = 0 \end{aligned}\right\} \tag{22}$$

at $z=0$, we have

$$\left.\begin{aligned} u_r &= \sum_{m=0}^\infty \frac{Z}{2\mu}\int_0^\infty \Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J'_m(kr)dk\\ &\quad -\sum_{m=0}^\infty \frac{1}{2(\lambda+\mu)}\int_0^\infty \Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J'_m(kr)\frac{dk}{k},\\ u_\theta &= \sum_{m=0}^\infty \frac{mz}{2\mu r}\int_0^\infty \Big\{\overline{Z}_m(k)\cos m\theta - Z_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)\frac{dk}{k}\\ &\quad -\sum_{m=0}^\infty \frac{m}{2(\lambda+\mu)r}\int_0^\infty \Big\{\overline{Z}_m(k)\cos m\theta - Z_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)\frac{dk}{k^2},\\ u_z &= -\sum_{m=0}^\infty \frac{z}{2\mu}\int_0^\infty \Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)dk\\ &\quad -\sum_{m=0}^\infty \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty \Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)\frac{dk}{k}; \end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} \widehat{zz} &= \sum_{m=0}^\infty \int_0^\infty (1+kz)\Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)dk,\\ \widehat{zr} &= -\sum_{m=0}^\infty z\int_0^\infty k\Big\{Z_m(k)\cos m\theta + \overline{Z}_m(k)\sin m\theta\Big\}e^{-kz}J'_m(kr)dk,\\ \widehat{z\theta} &= -\sum_{m=0}^\infty \frac{mz}{r}\int_0^\infty \Big\{\overline{Z}_m(k)\cos m\theta - Z_m(k)\sin m\theta\Big\}e^{-kz}J_m(kr)dk,\\ &\qquad \text{etc.} \end{aligned}\right\} \tag{24}$$

More generally, if the boundary conditions are given in the form of an arbitrary function, e.g.

$$\left.\begin{aligned} \widehat{zz} &= f(r, \theta), \\ \widehat{zr} &= 0, \qquad \widehat{z\theta} = 0 \end{aligned}\right\} \tag{25}$$

at $z=0$, the general solution can be obtained by making use of the integral theorem

$$F(r) = \int_0^\infty J_m(kr)\, k\, dk \int_0^\infty F(a) J_m(ka)\, a\, da, \tag{26}$$

provided the function $f(r, \theta)$ can be expanded into a trigonometrical series of the form

$$f(r, \theta) = \sum_{m=0}^\infty \left\{ f_m(r) \cos m\theta + \bar{f}_m(r) \sin m\theta \right\}, \tag{27}$$

where $f_m(r)$ and $\bar{f}_m(r)$ are supposed to satisfy the above integral theorem.

Comparing the expansion (27) with $(\widehat{zz})_0$ which follows from the first formula of (24) by putting $z=0$, we see that the functions $Z_m(k)$ and $\bar{Z}_m(k)$ which correspond to the boundary conditions (25) or (27) are the solutions of the integral equations

$$f_m(r) = \int_0^\infty Z_m(k) J_m(kr)\, dk,$$

$$\bar{f}_m(r) = \int_0^\infty \bar{Z}_m(k) J_m(kr)\, dk.$$

Looking at the integral theorem (26), the solutions of these integral equations appear easily to be

$$\left.\begin{aligned} Z_m(k) &= k \int_0^\infty f_m(a) J_m(ka)\, a\, da, \\ \bar{Z}_m(k) &= k \int_0^\infty \bar{f}_m(a) J_m(ka)\, a\, da. \end{aligned}\right\} \tag{28}$$

Thus, substituting these values of $Z_m(k)$ and $\bar{Z}_m(k)$ in the formulæ (23) and (24) we get the solution answering to the boundary condition (25). In his book of differential equations, H. WEBER[1] solves the same problem by using the Cartesian coordinates. But it seems that his mode of using Fourier's theorem was anticipated by Lamé and Clapeyron.

IV. Examples in the Case of Symmetry.

§12. The solution for the case of symmetry round the origin, which is discussed by BOUSSINESQ with numerous examples, has been afterwards obtained by Prof. LAMB in the same way as adopted here. This case is implied, of course, in our solution. Suppose

$$\left.\begin{aligned} \widehat{zz} &= f(r), \\ \widehat{zr} &= 0, \quad \widehat{z\theta} = 0 \end{aligned}\right\} \tag{29}$$

are given at the surface $z=0$. The corresponding solution will then be obtained from (23) and (24), by taking only the first term ($m=0$) in the summation. Thus

$$\left.\begin{aligned} u_r &= -\frac{z}{2\mu}\int_0^\infty Z(k)e^{-kz}J_1(kr)dk \\ &\quad+\frac{1}{2(\lambda+\mu)}\int_0^\infty Z(k)e^{-kz}J_1(kr)\frac{dk}{k}, \\ u_\theta &= 0, \\ u_z &= -\frac{z}{2\mu}\int_0^\infty Z(k)e^{-kz}J_0(kr)dk \\ &\quad-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty Z(k)e^{-kz}J_0(kr)\frac{dk}{k}; \end{aligned}\right\} \tag{30}$$

and

$$\begin{aligned} \widehat{rr} &= -z\int_0^\infty Z(k)e^{-kz}J_0(kr)kdk+\frac{z}{r}\int_0^\infty Z(k)e^{-kz}J_1(kr)dk \\ &\quad+\int_0^\infty Z(k)e^{-kz}J_0(kr)dk-\frac{\mu}{(\lambda+\mu)r}\int_0^\infty Z(k)e^{-kz}J_1(kr)\frac{dk}{k}, \end{aligned}$$

1) Part. Diff. Gleichungen, vol. 2 (1912) §76—.

$$\left.\begin{aligned}
\widehat{\theta\theta} &= -\frac{z}{r}\int_0^\infty Z(k)e^{-kz}J_1(k)dk + \frac{\mu}{(\lambda+\mu)r}\int_0^\infty Z(k)e^{-kz}J_1(kr)\frac{dk}{k} \\
&\quad + \frac{\lambda}{\lambda+\mu}\int_0^\infty Z(k)e^{-kz}J_0(kr)dk, \\
\widehat{zz} &= z\int_0^\infty Z(k)e^{-kz}J_0(kr)kdk + \int_0^\infty Z(k e^{-kz}J_0(kr)dk, \\
\widehat{\theta z} &= 0, \\
\widehat{zr} &= z\int_0^\infty Z(k)e^{-kz}J_1(kr)kdk, \\
\widehat{r\theta} &= 0;
\end{aligned}\right\} \quad (31)$$

where the anxiliary function Z is connected with the prescribed condition by

$$Z(k) = k\int_0^\infty f(a)J_0(ka)ada. \quad (32)$$

We shall now apply the general theorem to a few special examples.

Example I.

§13. The first example, which is discussed by Prof. Lamb, is to find the effect of a given normal pressure concentrated at a point on the boundary, on the supposition that $f(r)$ is zero for all values of r except those in the immediate neighbourhood of the origin, where it becomes infinitely great in such a manner that

$$\int_0^\infty f(r) \,.\, 2\pi r \,.\, dr = -\Pi, \quad (33)$$

in which Π is the total amount of the pressure applied and is finite. The function $Z(k)$ now is

$$Z(k) = -\frac{\Pi}{2\pi}k.$$

Putting this in (30) and (31), and then integrating we get

$$\left.\begin{aligned} u_r &= \frac{\Pi}{4\pi\mu}\left\{\frac{zr}{(r^2+z^2)^{3/2}} - \frac{\mu}{\lambda+\mu}\left[\frac{1}{r} - \frac{z}{r(r^2+z^2)^{1/2}}\right]\right\} \\ u_z &= \frac{\Pi}{4\pi\mu}\left\{\frac{z^2}{(r^2+z^2)^{3/2}} + \frac{\lambda+2\mu}{\lambda+\mu}\cdot\frac{1}{(r^2+z^2)^{1/2}}\right\}; \end{aligned}\right\}^{1)} \qquad (34)$$

and

$$\left.\begin{aligned} \widehat{rr} &= -\frac{\Pi}{2\pi}\left\{\frac{3zr^2}{(r^2+z^2)^{5/2}} - \frac{\mu}{\lambda+\mu}\left[\frac{1}{r^2} - \frac{z}{r^2(r^2+z^2)^{1/2}}\right]\right\}, \\ \widehat{\theta\theta} &= \frac{\Pi}{2\pi}\cdot\frac{\mu}{\lambda+\mu}\left\{\frac{z(2r^2+z^2)}{r^2(r^2+z^2)^{3/2}} - \frac{1}{r^2}\right\}, \\ \widehat{zz} &= -\frac{\Pi}{2\pi}\cdot\frac{3z^3}{(r^2+z^2)^{5/2}}, \\ \widehat{zr} &= -\frac{\Pi}{2\pi}\cdot\frac{3z^2r}{(r^2+z^2)^{5/2}}, \\ \widehat{r\theta} &= 0, \qquad \widehat{\theta z} = 0. \end{aligned}\right\} \qquad (35)$$

At the surface ($z=0$) they reduce simply to

$$\left.\begin{aligned} (u_r)_0 &= -\frac{\Pi}{4\pi(\lambda+\mu)}\cdot\frac{1}{r}, \\ (u_z)_0 &= \frac{\Pi(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\cdot\frac{1}{r}; \end{aligned}\right\} \qquad (36)$$

$$\left.\begin{aligned} (\widehat{rr})_0 &= \frac{\Pi\mu}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2}, \\ (\widehat{\theta\theta})_0 &= -\frac{\Pi\mu}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2}, \\ (\widehat{zz})_0 &= 0. \end{aligned}\right\} \qquad (37)$$

§14. It is interesting to show by graph the state of deformation at different depths from the surface. For the sake of simplicity we assume that the material satisfies the Poisson's relation $\lambda=\mu$, and we take only one component of the displacement u_z for reference, which is now written in the simple form

$$u_z = \frac{\Pi}{4\pi\mu}\left\{\frac{z^2}{(r^2+z^2)^{3/2}} + \frac{3}{2}\cdot\frac{1}{(r^2+z^2)^{1/2}}\right\}.$$

The attached diagram is drawn on the supposition that $\Pi=4\pi\mu$

1) The integrations have been carried out by Lamb, and so it is unnecessary to recapitulate them here.

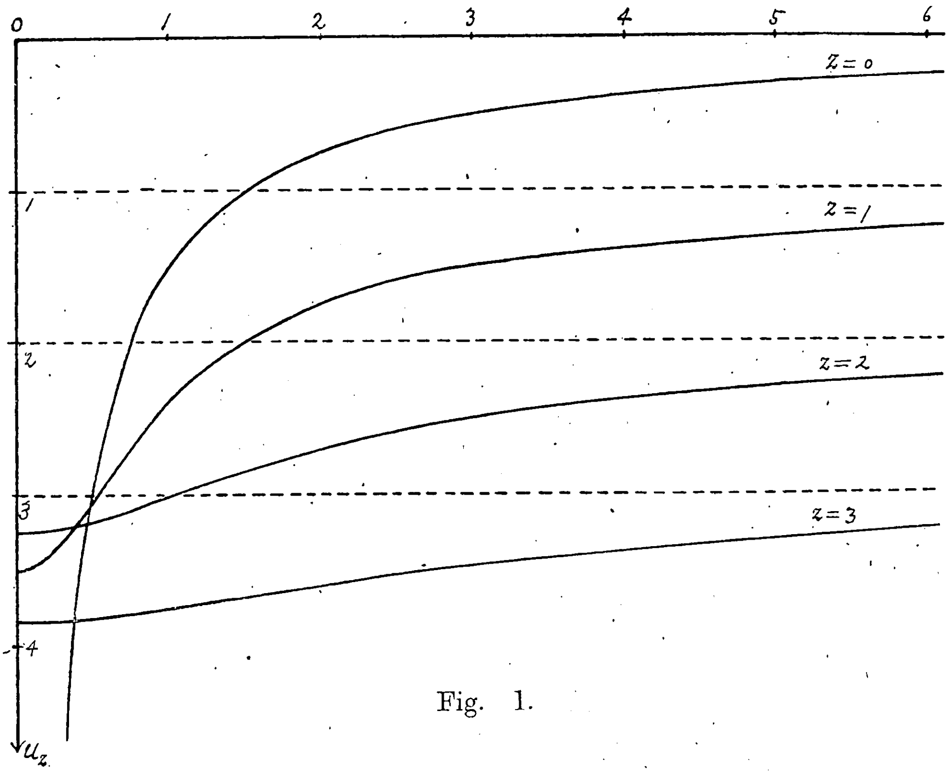

Fig. 1.

As seen from the diagram the state of affairs in the neighbourhood of the origin and even at a finite distance from it is an impossible one, and the mathematical theory of elasticity does not apply to such a case. The above argument must, if possible, be amended by a suitable process of analysis. The general solution found above is restricted to a special class of functions $f(r)$ which satisfy the integral theorem (26). The hypothesis of point concentration of given pressure does not, in a strictly mathematical sense, satisfy this important condition, and the solution deduced from it may not be looked upon as a legitimate one, at least in the vicinity of the origin.[1] What follows from the assumption of point-concentration of given pressure may, however, be considered, except locally, as the limiting case of the effect of a pressure which

1) A quite similar failure of the solution will occur in the problem of the deep-sea water-wave and allied problems which can be solved by the aid of Fourier's Integral Theorem.

acts on a slightly extended area of the boundary, which will be discussed in the next example.

Example II.

§15. Suppose the normal pressure at $z=0$ is given in the form

$$f(r) = -\frac{\Pi}{2\pi}\cdot\frac{a}{(a^2+r^2)^{3/2}}, \quad a>0, \tag{38}$$

Π being its total amount. The function $Z(k)$ is

$$Z(k) = -\frac{\Pi}{2\pi}\cdot k\int_0^\infty \frac{aJ_0(k\alpha)\alpha d\alpha}{(a^2+\alpha^2)^{3/2}} = -\frac{\Pi}{2\pi}ke^{-ak} \tag{39}$$

Thus the solution corresponding to the boundary condition (38) will be given by

$$\left.\begin{aligned} u_r &= \frac{\Pi z}{4\pi\mu}\int_0^\infty e^{-(z+a)k}J_1(kr)kdk \\ &\quad -\frac{\Pi}{4\pi(\lambda+\mu)}\int_0^\infty e^{-(z+a)k}J_1(kr)dk, \\ u_z &= \frac{\Pi z}{4\pi\mu}\int_0^\infty e^{-(z+a)k}J_0(kr)kdk \\ &\quad +\frac{\Pi(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\int_0^\infty e^{-(z+a)k}J_0(kr)dk\,; \end{aligned}\right\} \tag{40}$$

and

$$\left.\begin{aligned} \widehat{rr} &= \frac{\Pi z}{2\pi}\int_0^\infty e^{-(z+a)k}J_0(kr)k^2dk - \frac{\Pi z}{2\pi r}\int_0^\infty e^{-(z+a)k}J_1(kr)kdk \\ &\quad -\frac{\Pi}{2\pi}\int_0^\infty e^{-(z+a)k}J_0(kr)kdk + \frac{\Pi\mu}{2\pi(\lambda+\mu)r}\int_0^\infty e^{-(z+a)k}J_1(kr)dk, \\ \widehat{\theta\theta} &= \frac{\Pi z}{2\pi r}\int_0^\infty e^{-(z+a)k}J_1(kr)kdk - \frac{\Pi\mu}{2\pi(\lambda+\mu)r}\int_0^\infty e^{-(z+a)k}J_1(kr)dk \\ &\quad -\frac{\Pi\lambda}{2\pi(\lambda+\mu)}\int_0^\infty e^{-(z+a)k}J_0(kr)kdk, \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}
\widehat{zz} &= -\frac{\Pi z}{2\pi}\int_0^\infty e^{-(z+a)k}J_0(kr)k^2dk-\frac{\Pi}{2\pi}\int_0^\infty e^{-(z+a)k}J_0(kr)kdk,\\
\widehat{zr} &= -\frac{\Pi z}{2\pi}\int_0^\infty e^{-(z+a)k}J_1(kr)k^2dk,\\
\widehat{r\theta} &= 0, \qquad \widehat{\theta z} = 0.
\end{aligned}\right\}$$

These integrals can be expressed in terms of zonal harmonics and the associated functions. If $P_n(x)$ denotes the zonal harmonic of nth order and

$$P_n^{-m}(x) = (1-x^2)^{-\frac{m}{2}}\int_x^1\int_x^1\ldots\ldots P_n(x)dx^m,$$

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^mP_n(x)}{dx^m},$$

then

$$\left.\begin{aligned}
\int_0^\infty k^n e^{-kz}J_m(kr)dk &= \frac{(n+m)!}{(r^2+z^2)^{n+1/2}}\cdot P_n^{-m}\left(\frac{z}{\sqrt{r^2+z^2}}\right),^{1)}\\
\int_0^\infty k^n e^{-kz}J_m(kr)dk &= \frac{(n-m)!}{(r^2+z^2)^{n+1/2}}\cdot P_n^{m}\left(\frac{z}{\sqrt{r^2+z^2}}\right),\ (m\leqq n).
\end{aligned}\right\}\quad(42)$$

By making use of these formulæ, we have

$$\left.\begin{aligned}
u_r &= \frac{\Pi}{4\pi\mu}\left\{\frac{zr}{(r^2+(z+a)^2)^{3/2}}-\frac{\mu}{\lambda+\mu}\left[\frac{1}{r}-\frac{z+a}{r(r^2+(z+a)^2)^{1/2}}\right]\right\},\\
u_z &= \frac{\Pi}{4\pi\mu}\left\{\frac{z(z+a)}{(r^2+(z+a)^2)^{3/2}}+\frac{\lambda+2\mu}{\lambda+\mu}\cdot\frac{1}{(r^2+(z+a)^2)^{1/2}}\right\};
\end{aligned}\right\}\quad(43)$$

and

$$\left.\begin{aligned}
\widehat{rr} &= -\frac{\Pi}{2\pi}\left\{\frac{a(r^2+(z+a)^2)+3zr^2}{(r^2+(z+a)^2)^{5/2}}-\frac{\mu}{\lambda+\mu}\left[\frac{1}{r^2}-\frac{z+a}{r^2(r^2+(z+a)^2)^{1/2}}\right]\right\},\\
\widehat{\theta\theta} &= \frac{\Pi}{2\pi}\left\{\frac{z}{(r^2+(z+a)^2)^{3/2}}-\frac{\mu}{\lambda+\mu}\left[\frac{1}{r^2}-\frac{z+a}{r^2(r^2+(z+a)^2)^{1/2}}\right]\right.\\
&\qquad\left.-\frac{\lambda}{\lambda+\mu}\cdot\frac{z+a}{(r^2+(z+a)^2)^{3/2}}\right\},\\
\widehat{zz} &= -\frac{\Pi}{2\pi}\cdot\frac{(3z+a)(z+a)^2+ar^2}{(r^2+(z+a)^2)^{5/2}},
\end{aligned}\right\}\quad(44)$$

1) See K. Terazawa, Proc. Roy. Soc. London, A. vol. 92 (1915) p. 57—.

$$\left.\begin{aligned}\widehat{zr} &= -\frac{\Pi}{2\pi}\cdot\frac{3zr(z+a)}{(r^2+(z+a)^2)^{5/2}},\\ \widehat{r\theta} &= 0, \qquad \widehat{\theta z} = 0.\end{aligned}\right\}$$

If we proceed to the limit $a \to 0$, we have the same result as that due to the pressure of point concentration. But this limiting process is, in general, not permissible. A little examination of the value of $\frac{\partial u_z}{\partial z}$ shows that the quantity a has a lower limit, such that

$$27\,a^2 > \frac{\Pi(2\lambda+3\mu)^3}{4\pi\mu(\lambda+\mu)^3}, \tag{45}$$

in order to avoid the impossible state of affairs near the z-axis. At a distance from the origin great compared with a, these solutions reduce, in a first approximation, to those in Ex. I., so that the solution which follows from the assumption of point concentration of given pressure may be valid at a great distance from the origin, though only approximately.

§16. It is desirable here also to see how the displacement varies with the depth. On the same hypothesis as before, that $\lambda=\mu$, the variation of U_z is shown in the attached diagram, in which the upper curve represents the distribution of applied pressure (38) and the lower ones represent u_z on a proper scale. a is put equal to unity for the sake of simplicity.

§17. At the surface, the expressions for displacement and stress become

$$\left.\begin{aligned}(u_r)_0 &= -\frac{\Pi}{4\pi(\lambda+\mu)}\left\{\frac{1}{r}-\frac{a}{r(r^2+a^2)^{1/2}}\right\},\\ (u_z)_0 &= \frac{\Pi(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\cdot\frac{1}{(r^2+a^2)^{1/2}};\end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned}(\widehat{rr})_0 &= -\frac{\Pi}{2\pi}\left\{\frac{a}{(r^2+a^2)^{3/2}}-\frac{\mu}{\lambda+\mu}\left[\frac{1}{r^2}-\frac{a}{r^2(r^2+a^2)^{1/2}}\right]\right\},\\ (\widehat{\theta\theta})_0 &= -\frac{\Pi}{2\pi}\left\{\frac{\mu}{\lambda+\mu}\left[\frac{1}{r^2}-\frac{a}{r^2(r^2+a^2)^{1/2}}\right]+\frac{\lambda}{\lambda+\mu}\cdot\frac{a}{(r^2+a^2)^{3/2}}\right\}\\ (\widehat{zz})_0 &= -\frac{\Pi}{2\pi}\cdot\frac{a}{(r^2+a^2)^{3/2}},\\ (\widehat{zr})_0 &= 0,\end{aligned}\right\} \tag{47}$$

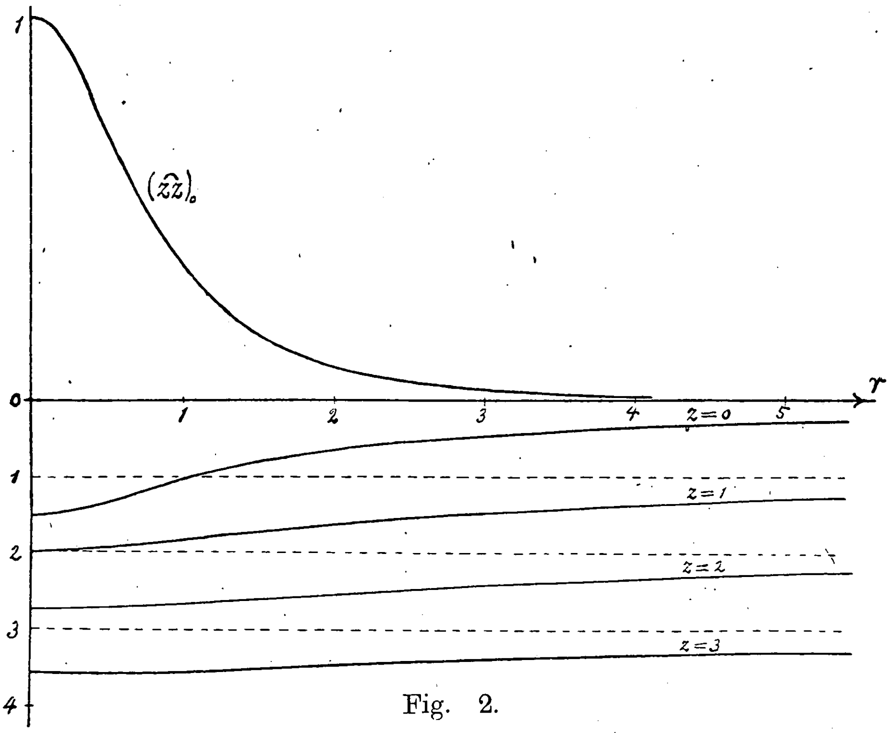

Fig. 2.

By integrating $(u_z)_0$ over the surface, it may be easily seen that the total depression of the surface appears to be infinitely great, though it is caused by a finite normal pressure of total amount Π. This seems again to be paradoxical, but that is not the case; if we calculate the work done by the given pressure, instead of total depression, it will appear to be finite, equal to $\frac{\Pi}{8\pi a}\cdot\frac{\lambda+2\mu}{\mu(\lambda+\mu)}$, inversely as a.

§18. Let us now proceed to apply the results of this example to the theory of rupture of a foundation over which a heavy load is spread. There have been proposed several hypotheses concerning the conditions under which an elastic body is ruptured or nearly so. Among those hypotheses usually adopted there are two in which a limitation on stress is taken as the measure of tendency to rupture: the one which was introduced by Lamé is that the greatest tension should be less than a certain limit which is different for different materials; the other which was recommended by G. H. Darwin asserts that, as mere hydrostatic pressure can hardly affect the case, the maximum difference of the greatest and the least principal stresses should be less than a certain limit.[1] These two hypotheses lead, in general, to different results. Either of them will give warning that danger is being approached, and in any case a certain factor of safety must, in practice, be adopted. Here we shall calculate the limits following from these two hypotheses and compare corresponding results.

For this purpose we have, in the first place, to find the distribution of principal stresses throughout the body concerned. Let N_1, N_2, N_3 denote the values of principal stresses at the point (r, θ, z). Owing to the hypothesis of symmetry round the axis of z, the component $\widehat{\theta\theta}$ is one of the principal stresses, say N_2, as is to be seen from the formulae (31); and any plane passing through the z-axis is one of the principal planes of stress. The other two principal stresses will be found by

$$\left.\begin{aligned} N_1 &= \frac{1}{2}(\widehat{rr}+\widehat{zz})+\frac{1}{2}\sqrt{(\widehat{rr}-\widehat{zz})^2+4\widehat{rz}^2}, \\ N_3 &= \frac{1}{2}(\widehat{rr}+\widehat{zz})-\frac{1}{2}\sqrt{(\widehat{rr}-\widehat{zz})^2+4\widehat{rz}^2}. \end{aligned}\right\} \quad (48)$$

At the surface, since $\widehat{rz}=0$, the stress components $\widehat{rr}, \widehat{\theta\theta}, \widehat{zz}$ themselves are the principal stresses.

§19. Now, to apply these formulae to this example, let us assume[2] that the pressure modulus λ is so great compared with the rigidity μ that the material may be considered to be incompressible. Thus, substituting the values of the stress-components found in (44) in the formulae (48) and making $\lambda=\infty$, we have simply

$$\left.\begin{aligned} N_1 &= -\frac{\Pi}{2\pi}\cdot\frac{3z+a}{(r^2+(z+a)^2)^{3/2}}, \\ N_3 = N_2 &= -\frac{\Pi}{2\pi}\cdot\frac{a}{(r^2+(z+a)^2)^{3/2}}, \end{aligned}\right\} \quad (49)$$

---

1 There is another view often adopted, in which a limitation on strain is taken as the measure.

2) This supposition is not at all necessary, but it makes the calculation extraordinarily simple.

N_1 being greater than N_2 and N_3 which are equal. Thus the stress quadric at any point in the interior is a spheroid, of which the axis of rotation lies in a plane passing through the z-axis.

If we denote by D the difference of the greatest and the least principal stresses, then

$$D = N_1 - N_3 = -\frac{\Pi}{2\pi} \cdot \frac{3z}{(r^2+(z+a)^2)^{3/2}} . \tag{50}$$

D is a maximum at the point $\left(r=0,\ z=\frac{a}{2}\right)$ and its value is

$$D_{max.} = -\frac{2\Pi}{9\pi a^2} , \tag{51}$$

while N_1, the greatest principal stress, is a maximum at the point $(r=0,\ z=0)$ and

$$N_{1max} = -\frac{\Pi}{2\pi a^2} . \tag{52}$$

Thus the latter maximum is greater than the former, and, moreover, the position where the rupture might occur is quite different for the two hypotheses: it is at a certain distance below the surface in the former, while it is at the surface in the latter.

In Fig. 3 it is shown how the greatest principal stress and the greatest difference of principal stresses along the z-axis vary with the depth from the surface.

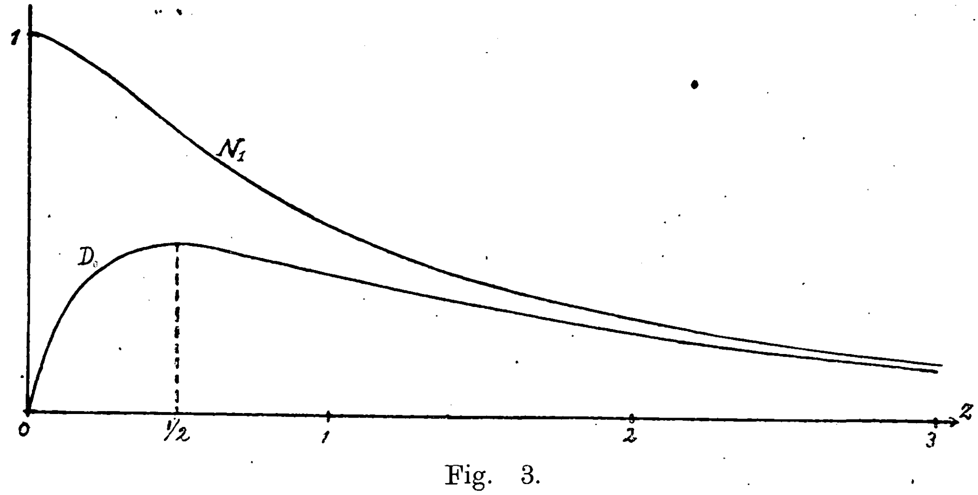

Fig. 3.

§20. To fix matters, suppose an isolated mountain or a high tower of uniform density with a circular base standing on the ground; the surface of the mountain or the other being given by the other by the equation

$$-z = \frac{m^3h^4}{(m^2h^2+(n^{\frac{2}{3}}-1)r^2)^{3/2}} . \qquad (53)$$

This equation is so adjusted that the height at the centre is h and that at the point $r=mh$ is h/n. We shall make the rough assumption that each point on the surface of the ground is pressed normally downwards with a pressure given by the product of the specific gravity and the height of the mountain at that point. The quantity a, used in the above, is now

$$a = \frac{mh}{\sqrt{n^{3/2}-1}} .$$

In the annexed diagram, the upper curves are supposed to represent the profiles of mountains and the inner ones those of columnar buildings such as chimneys or monuments, the height being taken as unity, and m and n chosen properly.

Curve	n	m	$a/2$
C_1	1·837	1	0·7071
C_2	1·837	1/2	0·3536
C_3	2·828	1/3	0·1667
C_4	5·196	1/5	0·0707
C_5	11·180	1/10	0·0250

If we denote the specific gravity of the mountains or other bodies by w, then the total amount of pressure will be

$$\Pi = \frac{2\pi w n^2 h^3}{n^{\frac{2}{3}}-1} .$$

The maximum of the greatest principal stress becomes

$$N_{1\,max} = -wh, \qquad (54)$$

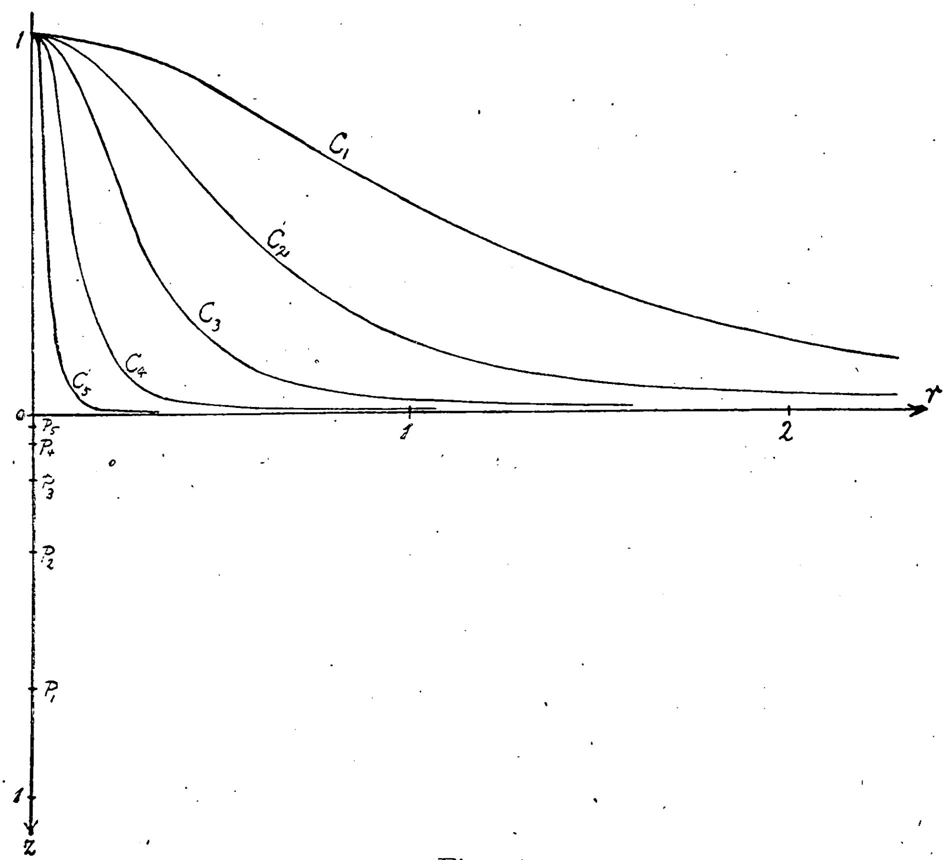

Fig. 4.

acting at the origin; and the maximum of the difference between the greatest and the least principal stress is

$$D_{max} = -\frac{4}{9} wh, \qquad (55)$$

acting at the point $\left(r=0,\ z=\frac{a}{2}\right)$. The values of $a/2$ corresponding to the curves are calculated in the last column of the above table, and the positions of the critical points corresponding to them are shown also in the figure by the points $P_1, P_2, \ldots\ldots$

It will be interesting to find the limiting height of mountains, which stand on bases of several kinds of materials without crushing the latter. In the following table a few examples are given: the third column contains the greatest heights of a mountain given by the formula (54), and in the fourth column are given those calculated by means of the formula (55); the value of w being taken as 3 grammes weight per cubic centimetre.

Material in the base	Strength to resist crushing in kg. per cm.^{2}	Max. height in km. by (54)	Max. height in km. by (55)
Cement	320	1·07	2·40
Strong sand stone	800	2.67	6·00
Strong granite	1600	5·33	12·00
Strong glass	2400	8·00	18·00
Wrought iron	3200	10·67	24.00

Example III.

§21. Next we will consider the case in which a uniform normal pressure acts on the surface within a circular area of radius a, outside of which the surface is free from traction.

Suppose

$$\left.\begin{aligned} f(r) &= -\frac{\Pi}{\pi a^2}, \text{ for } a > r, \\ &= 0 \qquad\quad ,, \;\; a < r \end{aligned}\right\} \tag{56}$$

Π being again the total amount of the given pressure. The function $Z(k)$ will be

$$Z(k) = -\frac{\Pi}{\pi a^2} k \int_0^a J_0(ka)\, a da = -\frac{\Pi}{\pi a} J_1(ka), \tag{57}$$

and the components of displacement can be computed from the formulæ

$$\left.\begin{aligned} u_r = &\frac{\Pi z}{2\pi a\mu} \int_0^\infty e^{-kz} J_1(kr) J_1(ka) dk \\ &-\frac{\Pi}{2\pi a(\lambda+\mu)} \int_0^\infty e^{-kz} J_1(kr) J_1(ka) \frac{dk}{k}, \end{aligned}\right\} \tag{58}$$

$$
\left.\begin{aligned}
u_z = {} & \frac{\Pi z}{2\pi a\mu}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)dk \\
& +\frac{\Pi(\lambda+2\mu)}{2\pi a\mu(\lambda+\mu)}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)\frac{dk}{k};
\end{aligned}\right\}
$$

and those of stress are given by

$$
\left.\begin{aligned}
\widehat{rr} = {} & \frac{\Pi z}{\pi a}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)kdk-\frac{\Pi z}{\pi ar}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)dk \\
& -\frac{\Pi}{\pi a}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)dk+\frac{\Pi\mu}{\pi(\lambda+\mu)ar}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)\frac{dk}{k}, \\
\widehat{\theta\theta} = {} & \frac{\Pi z}{\pi ar}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)dk-\frac{\Pi\mu}{\pi(\lambda+\mu)ar}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)\frac{dk}{k} \\
& -\frac{\Pi\lambda}{\pi(\lambda+\mu)a}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)dk, \\
\widehat{zz} = {} & -\frac{\Pi z}{\pi a}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)kdk-\frac{\Pi}{\pi a}\int_0^\infty e^{-kz}J_0(kr)J_1(ka)dk, \\
\widehat{zr} = {} & -\frac{\Pi z}{\pi a}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)kdk, \\
\widehat{r\theta} = {} & 0, \qquad \widehat{\theta z} = 0.
\end{aligned}\right\} \tag{59}
$$

§22. The integrals required here cannot be evaluated in a very simple way. Some of them are closely connected to the magnetic potential due to a circular current, or to the velocity-potential and stream function of a circular vortex and have been discussed by various authors. In his paper on the inductance of circular coils,[1] Prof. H. NAGAOKA has devised a comparatively simple method which may he applied to evaluate all the integrals needed here. Let us follow his method and describe it briefly.

Put

$$
R = \sqrt{(a^2-2ar\cos\theta+r^2)}, \tag{60}
$$

then we have

$$
\left.\begin{aligned}
J_1(kr)J_1(ka) &= \frac{1}{\pi}\int_0^\pi J_0(kR)\cos\theta\,d\theta, \\
J_0(kr)J_1(ka) &= \frac{1}{\pi}\int_0^\pi J_1(kR)\frac{a-r\cos\theta}{R}d\theta,
\end{aligned}\right\} \tag{61}
$$

1) Phil. Mag. VI. 6 1903, p. 19—.

$$J_1(kr)J_1(ka) = \frac{kar}{\pi}\int_0^\pi J_1(kR)\frac{\sin^2\theta}{R}d\theta \qquad (61)$$

which follow from Neumann's addition theorem for the Bessel function. Making use of these formulæ and on referring to the formulæ (42), we obtain

$$\left.\begin{aligned}
\int_0^\infty e^{-kz}J_1(kr)J_1(ka)dk &= \frac{1}{\pi}\int_0^\pi \frac{\cos\theta}{\sqrt{R^2+z^2}}d\theta,\\
\int_0^\infty e^{-kz}J_0(kr)J_1(ka)dk &= \frac{1}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2}d\theta - \frac{z}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2\sqrt{R^2+z^2}}d\theta,\\
\int_0^\infty e^{-kz}J_1(kr)J_1(ka)\frac{dk}{k} &= \frac{ar}{\pi}\int_0^\pi \frac{\sin^2\theta}{R^2}d\theta - \frac{arz}{\pi}\int_0^\pi \frac{\sin^2\theta}{R^2\sqrt{R^2+z^2}}d\theta,\\
\int_0^\infty e^{-kz}J_0(kr)J_1(ka)\frac{dk}{k} &= \frac{1}{\pi}\int_0^\pi \frac{\sqrt{R^2+z^2}}{R^2}(a-r\cos\theta)d\theta\\
&\quad -\frac{z}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2}d\theta
\end{aligned}\right\} \qquad (62)$$

To find these integrals, put

$$\left.\begin{aligned}
\alpha &= \left(\frac{2}{ar}\right)^{\frac{1}{3}}, \qquad \beta = \frac{a^2+r^2+z^2}{6ar},\\
e_1 &= \frac{2\beta}{\alpha} = \frac{a^2+r^2+z^2}{3ar\alpha},\\
e_2 &= \frac{1-\beta}{\alpha} = -\frac{a^2+r^2+z^2-6ar}{6ar\alpha},\\
e_3 &= -\frac{1+\beta}{\alpha} = -\frac{a^2+r^2+z^2+6ar}{6ar\alpha}
\end{aligned}\right\} \qquad (63)$$

so that

$$e_3 < e_2 < e_1,$$
$$e_1+e_2+e_3 = 0,$$

and change the integration variable from θ to s by putting

$$\cos\theta = \alpha s+\beta,$$

then we have

$$\int_0^\pi \frac{\cos\theta}{\sqrt{R^2+z^2}}d\theta = a^2\int_{e_3}^{e_2}\frac{(s+\frac{1}{2}e_1)ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}},$$

similarly for the others.

Put again

$$s = \wp(u)$$

and denote by ω_1 and ω_3 the real and imaginary half-period respectively and $\omega_2=\omega_1+\omega_3$, then, since s or $\wp(u)$ is real and lies between e_3 and e_2, $s=e_3$ and $s=e_2$ correspond to $u=\omega_3$ and $u=\omega_2$ respectively, if we take the sign of $\wp'(u)$ to be positive.[1] Thus

$$\int_0^\pi \frac{\cos\theta}{\sqrt{R^2+z^2}}d\theta = a^2\int_{\omega_3}^{\omega_2}\left\{\frac{1}{2}e_1+\wp(u)\right\}du$$

$$= a^2\left(\frac{1}{2}e_1\omega_1-\eta_1\right). \tag{64}$$

For the evaluation of the other integral, write

$$\wp(v) = \frac{a^2+r^2-2ar\beta}{2ar\alpha} = \frac{2(a^2+r^2)-z^2}{6ar\alpha}, \tag{65}$$

then we have

$$\int_0^\pi \frac{a-r\cos\theta}{R^2\sqrt{R^2+z^2}}d\theta = \frac{\alpha}{2a}\,\omega_1 - \frac{r^2-a^2}{4a^2r}\int_{\omega_3}^{\omega_2}\frac{du}{\wp(v)-\wp(u)},$$

etc.

Now

$$\left.\begin{aligned}\int_{\omega_3}^{\omega_2}\frac{\wp'(v)du}{\wp(v)-\wp(u)} &= \left[\log\frac{\sigma(u+v)}{\sigma(u-v)}-2u\zeta(v)\right]_{\omega_3}^{\omega_2}\\ &= 2\left\{v\eta_1-\omega_1\zeta(v)+m\pi i\right\}\end{aligned}\right\} \tag{66}$$

in which the term $m\pi i$ enters because of the many-valued property of a logarithm. The actual value of m and $\wp'(v)$ will be determined by the following consideration.

---

1 If we assume $\wp'(u)$ to be negative, then $s=e_3$ and $s=e_2$ correspond to $u=\omega_2$ and $u=2\omega_1+\omega_3$ respectively. But the same result will, as a matter of course, be obtained after integration.

From the definition of $\wp(v)$ and e_1, e_2, e_3, it follows immediately that

$$\wp(v)-e_1 = -\frac{z^2}{2ara},$$

$$\wp(v)-e_2 = -\frac{(a-r)^2}{2ara},$$

$$\wp(v)-e_3 = -\frac{(a+r)^2}{2ara},$$

accordingly

$$e_2 < \wp(v) < e_1.$$

The last inequality shows that the value of v must be one of the following:

$$\left.\begin{array}{ll}\text{(i)} & v = (2n+1)\omega_1+(2n'+\theta)\omega_3\,, \\ \text{(ii)} & v = (2n+1)\omega_1+(2n'+2-\theta)\omega_3\end{array}\right\} \qquad (67)$$

where n and n' denote any integers, positive or negative, or zero and θ a positive number less than unity.

To determine the value of m in the formula (66) for the value of v given in (i) of (67), observe that the integral on the left hand side of (66) and the function $v\eta_1-\omega_1\zeta(v)$ change their values continuously as θ varies from 0 to 1, while m remains unchanged during this variation. In the limit as $\theta\to 0$, the value of the integral is nil and

$$2v\eta_1-2\omega_1\zeta(v) \to 2n'\pi i,$$

and therefore we have

$$\text{(i)} \quad m = -n'.$$

Similarly for the value of v given in (ii) of (67), proceeding to the limit $\theta\to 0$, we find

$$\text{(ii)} \quad m = -(n'+1).$$

The value of $\wp'(v)$ will be obtained from

$$\wp'^2(v) = 4\{\wp(v)-e_1\}\{\wp(v)-e_2\}\{\wp(v)-e_3\}.$$

In the present case, we get

$$\wp'(v) = \pm i \frac{z(a^2-r^2)}{2ar}.$$

It may easily be shown that the value of $\wp'(v)$ is a pure imaginary quantity for the values of v given in (67), with positive sign for (i) and with negative sign for (ii). Therefore we have to take

$$\left.\begin{aligned} \wp'(v) &= +i\left|\frac{z(a^2-r^2)}{2ar}\right|, \\ m &= -n', \\ \text{for} \quad v &= (2n+1)\omega_1+(2n'+\theta)\omega_3 \end{aligned}\right\};$$

and

$$\left.\begin{aligned} \wp'(v) &= -i\left|\frac{z(a^2-r^2)}{2ar}\right|, \\ m &= -(n'+1), \\ \text{for} \quad v &= (2n+1)\omega_1+(2n'+2-\theta)\omega_3 \end{aligned}\right\},$$

in which $0<\theta<1$.

Hereafter we shall, for the sake of simplicity, take into account only the value of v which will be obtained by putting $n=0$, $n'=0$ in (i) of (67) viz.

$$v = \omega_1+\theta\omega_3.$$

By this convention the term $m\pi i$ disappears and the value of $\wp'(v)$ is to be taken as

$$\wp'(v) = +i\left|\frac{z(a^2-r^2)}{2ar}\right|. \tag{68}$$

Thus

$$\int_0^\pi \frac{a-r\cos\theta}{R^2\sqrt{R^2+z^2}}d\theta = \frac{a}{2a}\omega_1 - \frac{r^2-a^2}{2a^2r}\cdot\frac{1}{\wp'(v)}\left\{v\eta_1-\omega_1\zeta(v)\right\}. \tag{69}$$

Similarly we have

$$\int_0^\pi \frac{\sin^2\theta}{R^2\sqrt{R^2+z^2}}d\theta = \frac{a^2}{2ar}\left\{\omega_1\left[\wp(v)+e_1\right]-\eta_1 - \frac{2[\wp(v)-e_2][\wp(v)-e_3]}{\wp'(v)}\left[v\eta_1-\omega_1\zeta(v)\right]\right\}, \tag{70}$$

$$\int_0^{\pi} \frac{\sqrt{R^2+z^2}}{R^2}(a-r\cos\theta)d\theta = \frac{2}{a\alpha}(\eta_1+e_1\omega_1) - \frac{r^2-a^2}{a^2 r \alpha^2}\omega_1$$
$$+\frac{2(r^2-a^2)}{a^2 r\alpha^2}\cdot\frac{\wp(v)-e_1}{\wp'(v)}\left\{v\eta_1-\omega_1\zeta(v)\right\}. \quad (71)$$

The other integrals in (62) are found without the knowledge of elliptic functions.

$$\left.\begin{aligned}\int_0^{\pi}\frac{a-r\cos\theta}{R^2}d\theta &= \frac{\pi}{a} \text{ for } r<a,\\ &= 0 \quad ,, \quad r>a;\end{aligned}\right\} \quad (72)$$

$$\left.\begin{aligned}\int_0^{\pi}\frac{\sin^2\theta}{R^2}d\theta &= \frac{\pi}{2a^2} \quad ,, \quad r<a,\\ &= \frac{\pi}{2r^2} \quad ,, \quad r>a,\end{aligned}\right\} \quad (73)$$

§23. Substituting these values of integrals in (58) we obtain the following expressions for the displacement:

$$u_r = \frac{\Pi a^2 z}{2\alpha\pi^2\mu}\left\{\frac{1}{2}e_1\omega_1-\eta_1\right\}$$
$$-\frac{\Pi}{2\alpha\pi(\lambda+\mu)}\left\{\begin{bmatrix}\frac{r}{2a}(r<a)\\ \frac{a}{2r}(r>a)\end{bmatrix}+\frac{a^2 z}{2\pi}\left[\eta_1-\omega_1(\wp(v)+e_1)\right]\right.$$
$$\left.+\frac{a^2 z}{\pi}\left[\wp(v)-e_2\right]\cdot\left[\wp(v)-e_3\right]\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\right\}, \quad (74)$$

$$u_z = \frac{\Pi z}{2\alpha\pi\mu}\left\{\begin{bmatrix}\frac{1}{a}(r<a)\\ 0\ (r>a)\end{bmatrix}-\frac{az\omega_1}{2\alpha\pi}+\frac{z(r^2-a^2)}{2\alpha^2 r\pi}\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\right\}$$
$$+\frac{\Pi(\lambda+2\mu)}{2\alpha\pi\mu(\lambda+\mu)}\left\{-\begin{bmatrix}\frac{z}{a}(r<a)\\ 0\ (r>a)\end{bmatrix}+\frac{2}{a\alpha\pi}(\eta_1+e_1\omega_1)-\frac{r^2-a^2}{\pi a^2 r\alpha^2}\omega_1\right.$$
$$\left.+\frac{2(r^2-a^2)}{\pi a^2 r\alpha^2}\left[\wp(v)-e_1\right]\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\right\}. \quad (75)$$

§24. For purposes of calculation, it will be very convenient to have the formulæ expressed in terms of JACOBI's q-series. q is defined by

$$q = e^{i\pi\tau}, \qquad \tau = \frac{\omega_3}{\omega_1}.$$

After WEIERSTRASS, if we put

$$l = \frac{(e_1-e_3)^{\frac{1}{4}}-(e_1-e_2)^{\frac{1}{4}}}{(e_1-e_3)^{\frac{1}{4}}+(e_1-e_2)^{\frac{1}{4}}},$$

then q can be computed from

$$q = \frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+150\left(\frac{l}{2}\right)^{13}+O(l^{17}),$$

of which the first two or three terms are usually sufficient. The q-series of the functions needed here are as follows:—

$$\omega_1 = \frac{\pi}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}}\vartheta_2^2(o),$$

$$\eta_1 = -\frac{1}{12\omega_1}\cdot\frac{\vartheta_1'''(o)}{\vartheta_1'(o)},$$

$$\frac{1}{2}e_1\omega_1-\eta_1 = \frac{1}{8\omega_1}\left\{\frac{\vartheta_1'''(o)}{\vartheta_1'(o)}-\frac{\vartheta_2''(o)}{\vartheta_2(o)}\right\},$$

$$\eta_1+e_1\omega_1 = \frac{\pi^2}{\omega_1}\left\{\frac{1}{4}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}\right\};$$

and

$$\vartheta_2(o) = 2q^{\frac{1}{4}}(1+q^2+q^6+q^{12}+\ldots\ldots),$$

$$\vartheta_1'''(o) = -2\pi^3q^{\frac{1}{4}}(1-3^3q^2+5^3q^6-7^3q^{12}+\ldots\ldots),$$

$$\vartheta_1'(o) = 2\pi q^{\frac{1}{4}}(1-3q^2+5q^6-7q^{12}+\ldots\ldots),$$

$$\vartheta_2''(o) = -2\pi^2q^{\frac{1}{4}}(1+3^2q^2+5^2q^6+7^2q^{12}+\ldots\ldots).$$

§25. The calculation of the term $v\eta_1-\omega_1\zeta(v)$ requires a little explanation. By the fermula

$$e^{-\frac{1}{2}\cdot\frac{\eta_1}{\omega_1}v^2}\cdot\sigma(v) = 2\omega_1\frac{\vartheta_1\left(\frac{v}{2\omega_1}\right)}{\vartheta_1'(o)},$$

we have

$$v\eta_1 - \omega_1\zeta(v) = -\frac{1}{2}\cdot\frac{\vartheta_1'\left(\frac{v}{2\omega_1}\right)}{\vartheta_1\left(\frac{v}{2\omega_1}\right)}.$$

Making use of the expansion formula of the right member, we get

$$v\eta_1 - \omega_1\zeta(v) = -\frac{\pi}{2}\left\{\cot\frac{\pi v}{2\omega_1} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\cdot\sin\frac{n\pi v}{\omega_1}\right\}. \qquad (76)$$

The quantity v may be calculated by utilizing the WEIERSTRASS' formula to any degree of accuracy,[1] and therefore the value of the function $v\eta_1 - \omega_1\zeta(v)$. The approximate value of this function can, however, be found in the following way:

Put

$$\frac{1}{2}\cdot\frac{(e_1-e_2)^{\frac{1}{4}}(\wp(v)-e_3)^{\frac{1}{4}} - (e_1-e_3)^{\frac{1}{4}}(\wp(v)-e_2)^{\frac{1}{4}}}{(e_1-e_2)^{\frac{1}{4}}(\wp(v)-e_3)^{\frac{1}{4}} + (e_1-e_3)^{\frac{1}{4}}(\wp(v)-e_2)^{\frac{1}{4}}} = t,$$

and

$$\frac{1}{2}\,\frac{(e_1-e_3)^{\frac{1}{4}}\sigma_2(v) - (e_1-e_2)^{\frac{1}{4}}\sigma_3(v)}{(e_1-e_3)^{\frac{1}{4}}\sigma_2(v) + (e_1-e_2)^{\frac{1}{4}}\sigma_3(v)}$$

$$= \frac{q\cos\frac{v\pi}{\omega_1} + q^9\cos\frac{3v\pi}{\omega_1} + \ldots\ldots}{1 + 2q^4\cos\frac{2v\pi}{\omega_1} + 2q^{16}\cos\frac{4v\pi}{\omega_1} + \ldots\ldots} = Q(v),$$

then, since $e_2 < \wp(v) < e_1$, we shall have

$$Q(v-\omega_1) = t.^{2)}$$

As q^4 is usually a very small quantity, we may take

$$\cos\frac{\pi(v-\omega_1)}{\omega_1} = \frac{t}{q} = s \text{ say}$$

with a close approximation. Since $v-\omega_1 = \theta\omega_3$ purely imaginary, we may put

$$\frac{\pi(v-\omega_1)}{\omega_1} = ix,$$

1) This method has been adopted by Prof. NAGAOKA. l.c.

2) HALPHEN, Traité des Fonctions Elliptiques, I. p. 274.

x being a real quantity, then

$$\cosh x = s,$$

and we have

$$\cot\frac{\pi v}{2\omega_1} = -i\tanh\frac{x}{2} = -i\left(\frac{s-1}{s+1}\right)^{\frac{1}{2}},$$

$$\sin\frac{\pi v}{\omega_1} = -i\sinh x = -i(s^2-1)^{\frac{1}{2}},$$

$$\sin\frac{2\pi v}{\omega_1} = i\sinh 2x = 2is(s^2-1)^{\frac{1}{2}},$$

$$\cdots\cdots\cdots\cdots\cdots$$

Substituting these values in the equation (76), we have

$$v\eta_1-\omega_1\zeta(v) = \frac{i\pi}{2}\left\{\left(\frac{s-1}{s+1}\right)^{\frac{1}{2}}+4q^2(s^2-1)^{\frac{1}{2}}[1-q^2(2s-1)]\right\}.$$

This approximation formula is recommended for the case in which $\theta<\frac{1}{2}$.

For the case $\theta>\frac{1}{2}$, if we put

$$\frac{(e_1-e_2)^{\frac{1}{4}}(e_1-e_3)^{\frac{1}{4}}-(e_1-\wp(v))^{\frac{1}{2}}}{(e_1-e_2)^{\frac{1}{4}}(e_1-e_3)^{\frac{1}{4}}+(e_1-\wp(v))^{\frac{1}{2}}} = t',$$

then we shall have

$$Q(v-\omega_2) = t',$$

and making use of this formula, we shall obtain an approximation formula which is convenient for the case $\theta>\frac{1}{2}$.

§26. The expressions obtained in §23 are rather complicated, and the state of deformation can hardly be grasped at a glance. The formula for the displacement at the surface is, however, comparatively simple and can be calculated with any accuracy.

Putting $z=0$ in the general formulæ, we obtain

$$(u_r)_0 = -\frac{\Pi}{4\pi a(\lambda+\mu)}\left\{\begin{array}{l}\dfrac{r}{a}\,(r<a)\\[2ex] \dfrac{a}{r}\,(r>a)\end{array}\right\} \tag{77}$$

and

$$(u_z)_0 = \frac{\Pi}{\pi a^2} \cdot \frac{\lambda+2\mu}{2(\lambda+\mu)\mu} \cdot \frac{1}{\pi} \left\{ \frac{2}{\alpha} (\eta_1 + e_1 \omega_1) - \frac{r^2 - a^2}{ar\alpha^2} \omega_1 \right\} \quad (78)$$

where

$$\alpha = \left(\frac{2}{ar} \right)^{\frac{1}{3}},$$

$$e_1 = \frac{a^2+r^2}{3ar\alpha}, \qquad e_2 = -\frac{a^2+r^2-6ar}{6ar\alpha}, \qquad e_3 = -\frac{a^2+r^2+6ar}{6ar\alpha}.$$

At the surface the value of l becomes simply

$$l = \frac{(r+a)^{\frac{1}{2}} - |r-a|^{\frac{1}{2}}}{(r+a)^{\frac{1}{2}} + |r-a|^{\frac{1}{2}}},$$

so that the use of the q-series is very advantageous for the calculation, especially at the points near the origin or at a distance from it. For example, at the point $r = 3a$, we have

$$\frac{l}{2} = 0{\cdot}085786, \qquad q = 0{\cdot}085796;$$

thus, even for the value $r = 3a$, we may neglect the terms after l^4 and q^4 in the series. For larger values of r the q-series converge of course very rapidly.[1)]

§27. The formula for $(u_z)_0$ can be transformed into the form which is convenient for the use of Legendre's table of elliptic integrals. If we remember that

$$\eta_1 + e_1 \omega_1 = \sqrt{e_1 - e_3}\, E,$$

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}},$$

K and E being the first and the second complete elliptic integrals with the moduli k and k', and

$$k^2 = \frac{4ar}{(r+a)^2}, \qquad k'^2 = \left(\frac{r-a}{r+a} \right)^2,$$

the expression for $(u_z)_0$ becomes

---

1) For the point near the edge of the circle, the calculation may be undertaken by using similar series, specified by q_1 and l_1. See below.

$$(u_z)_0 = \frac{\Pi}{\pi a^2} \cdot \frac{\lambda+2\mu}{2(\lambda+\mu)\mu} \cdot \frac{1}{\pi}\left\{(r+a)E-(r-a)\,K\right\}. \tag{79}$$

At the centre of the loaded circle, since

$$k = 0, \quad K = E = \frac{\pi}{2},$$

we get

$$(u_z)_0 = \frac{\Pi}{\pi a} \cdot \frac{\lambda+2\mu}{2(\lambda+\mu)\mu},^{1)} \tag{80}$$

and at the periphery of the circle ($r=a$), since

$$k = 1, \quad E = 1, \quad K \to \log\frac{4}{k'}, \quad (r-a)\,K \to 0,$$

we have

$$(u_z)_0 = \frac{\Pi}{\pi a} \cdot \frac{\lambda+2\mu}{2(\lambda+\mu)\mu} \cdot \frac{2}{\pi}. \tag{81}$$

The values of $(u_z)_0$ at the centre and at the periphery of the circle bear the constant ratio $\pi/2$, and this is independent of the elastic constants and radius.

At the centre, as will be seen from the formula (80), the vertical displacement varies inversely as the radius, when the same amount of total pressure is applied to different circular areas, while it varies directly as the radius when the pressure of the same intensity is applied to different areas. The same relation holds at the edge of the circle, with regard to both the components, radial and vertical, of the displacement.

By the aid of the LEGENDRE table of K and E, we can trace by a graph the general march of $(u_z)_0$. The next diagram is drawn in this way, where the radius a is taken as unity and the pressure Π is taken equal to $2\pi a\mu(\lambda+\mu)/(\lambda+2\mu)$

§28. To find the formulæ for stress we need two more integrals which can be also carried out by the same method as before. We have, from (61), that

1) This result may be obtained from (58) directly by using the formula $\int_0^\infty J_1(ka)\frac{dk}{k}=1$. The result (80) and (81) agree with those given by BOUSSINESQ. p. 140, l.c.

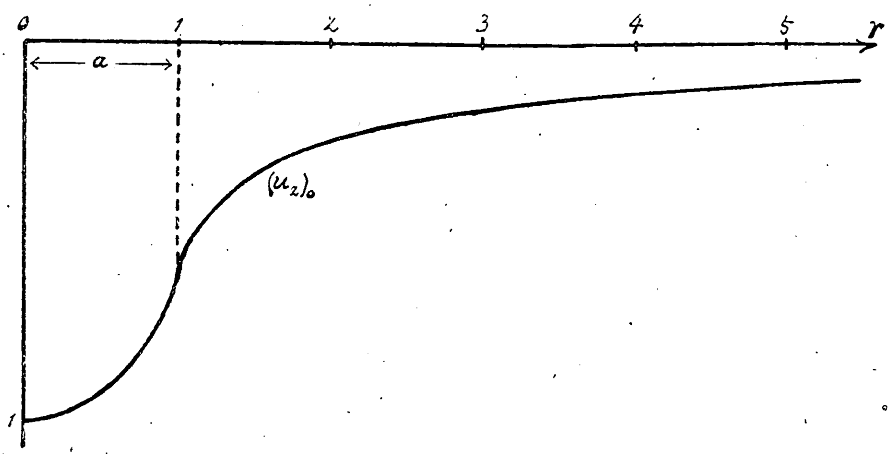

Fig. 5.

$$\left.\begin{aligned}\int_0^\infty e^{-kz}J_1(kr)J_1(ka)kdk &= \frac{z}{\pi}\int_0^\infty \frac{\cos\theta}{(R^2+z^2)^{3/2}}\,d\theta,\\ \int_0^\infty e^{-kz}J_0(kr)J_1(ka)kdk &= \frac{1}{\pi}\int_0^\pi \frac{a-r\cos\theta}{(R^2+z^2)^{3/2}}\,d\theta\end{aligned}\right\} \qquad (82)$$

R being defined by (60). Of course we may find their values by differentiation of the integrals already found with respect to z, but, owing to the complexity of the elliptic functions, it will be seen that the direct method of integration is much easier. By the same transformation of variables as before, we have

$$\int_0^\pi \frac{\cos\theta}{(R^2+z^2)^{3/2}}\,d\theta = \frac{a^4}{4}\int_{\omega_3}^{\omega_2}\frac{3e_1/2}{e_1-\wp(u)}\,du-\frac{a^4}{4}\omega_1,$$

and by the aid of the formula

$$\frac{1}{\wp(u)-e_1}=\frac{\wp(u+\omega_1)-e_1}{(e_1-e_2)(e_1-e_3)},$$

the integral can be found to be

$$\int_0^\pi \frac{\cos\theta}{(R^2+z^2)^{3/2}}\,d\theta = \frac{a}{2ar}\left\{\frac{3e_1(\eta_1+e_1\omega_1)}{2(e_1-e_2)(e_1-e_3)}-\omega_1\right\} \qquad (83)$$

Similarly the second integral in (82) is

$$\int_0^{\pi} \frac{a-r\cos\theta}{(R^2+z^2)^{3/2}}d\theta = \frac{1}{2ar}\left\{\frac{(2a-3e_1ra)(\eta_1+e_1\omega_1)}{2(e_1-e_2)(e_1-e_3)}+ar\omega_1\right\} \tag{84}$$

Substituting these values and those found before in the formulæ (59), we have for the stress

$$\begin{aligned}\widehat{rr} = \frac{\Pi z\alpha}{2\pi^2a^2r}\Bigg\{&\frac{(2a-3are_1)(\eta_1+e_1\omega_1)}{2\alpha(e_1-e_2)(e_1-e_3)}+2r\omega_1-a\alpha(e_1\omega_1-2\eta_1)\\ &-\frac{r^2-a^2}{a\alpha}\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}+\frac{a\alpha\mu}{\lambda+\mu}\Big[\eta_1-e_1\omega_1-\omega_1\wp(v)\\ &+2[\wp(v)-e_2]\cdot[\wp(v)-e_3]\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\Big]\Bigg\}\\ &-\frac{\Pi}{\pi a}\left\{\begin{bmatrix}\frac{1}{a}(r<a)\\ 0\ (r>a)\end{bmatrix}-\frac{\mu}{(\lambda+\mu)r}\begin{bmatrix}\frac{r}{2a}(r<a)\\ \frac{a}{2r}(r>a)\end{bmatrix}\right\},\end{aligned} \tag{85}$$

$$\begin{aligned}\widehat{\theta\theta} = \frac{\Pi z\alpha^2}{2\pi^2ar}\Bigg\{&e_1\omega_1-2\eta_1+\frac{\lambda}{(\lambda+\mu)a\alpha}\left[r\omega_1-\frac{r^2-a^2}{a\alpha}\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\right]\\ &-\frac{\mu}{\lambda+\mu}\left[\eta_1-e_1\omega_1-\omega_1\wp(v)+2[\wp(v)-e_2]\cdot[\wp(v)-e_3]\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp'(v)}\right]\Bigg\}\\ &-\frac{\Pi}{\pi(\lambda+\mu)ar}\left\{\mu\begin{bmatrix}\frac{r}{2a}(r<a)\\ \frac{a}{2r}(r>a)\end{bmatrix}+\lambda\begin{bmatrix}\frac{r}{a}(r<a)\\ 0\ (r>a)\end{bmatrix}\right\},\end{aligned} \tag{86}$$

$$\begin{aligned}\widehat{zz} = -\frac{\Pi z}{2\pi^2a^2r}&\left\{\frac{(2a-3are_1)(\eta_1+e_1\omega_1)}{2(e_1-e_2)(e_1-e_3)}+\frac{r^2-a^2}{a}\cdot\frac{v\eta_1-\omega_1\zeta(v)}{\wp(v)}\right\}\\ &-\frac{\Pi}{\pi a}\begin{Bmatrix}\frac{1}{a}(r<a)\\ 0\ (r>a)\end{Bmatrix},\end{aligned} \tag{87}$$

$$\widehat{zr} = -\frac{\Pi z^2 a}{2\pi^2a^2r}\left\{\frac{3e_1(\eta_1+e_1\omega_1)}{2(e_1-e_2)(e_2-e_3)}-\omega_1\right\}, \tag{88}$$

$$\widehat{r\theta} = 0, \qquad \widehat{\theta z} = 0.$$

§29. At the surface these expressions for the stress reduce to simpler ones.

$$\left.\begin{aligned}
(\widehat{rr})_0 &= -\frac{2\lambda+\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi a^2}(r<a),\\
&= \frac{\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi r^2}(r>a)\,;\\
(\widehat{\theta\theta})_0 &= -\frac{2\lambda+\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi a^2}(r<a)\,;\\
&= -\frac{\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi r^2}(r<a).\\
(\widehat{zz})_0 &= -\frac{\Pi}{\pi a^2} \qquad (r<a),\\
&= 0 \qquad (r>a)\,;\\
(\widehat{zr})_0 &= 0,
\end{aligned}\right\} \tag{89}$$

All the tractions acting on the boundary vanish, as they ought to, under the conditions of our problem, except a uniform pressure on the circle of radius a. The state of stress just below the surface is made up of a simple and beautiful scheme of the pressure system with a radial tension equal to $-\frac{2\lambda+\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi a^2}$ inside the circle, and $\frac{\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi r^2}$ outside it; and a transverse tension equal to $-\frac{2\lambda+\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi a^2}$ inside the circle, and $-\frac{\mu}{2(\lambda+\mu)}\cdot\frac{\Pi}{\pi r^2}$ outside it.

§30. Along the edge of the loaded circle there occurs a singularity of stress. We have seen already that as a rule the component stress $\widehat{zr}$ vanishes at the boundary surface. But this is not always the case. If we put $r=a$ in (88) and then proceed to the limit $z \to 0$, we shall have

$$(\widehat{zr})_0 = -\frac{1}{\pi}\cdot\frac{\Pi}{\pi a^2}\,,\ (r=a). \tag{90}$$

Thus the tangential traction $(\widehat{zr})_0$ does not vanish at the periphery of the loaded area, which is contradictory to our assumed boundary condition. It appears that along the circumference of the loaded circle a radial shearing stress of magnitude equal to the given normal pressure, divided by π, should be applied. This was also pointed out by BOUSSINESQ.[1] But the area on the boundary over which this shearing stress applies is infinitely small, so that it is practically of no account at all. This singularity possibly means that the region in the interior of the body in which the stress component $\widehat{zr}$ exists has a cuspidal edge, which touches the boundary surface at the periphery of the circle. To avoid the above difficulty, BOUSSINESQ supposed that at the edge of the loaded area the pressure decreases more or less rapidly to zero, instead of vanishing abruptly.[2] If, in the actual problem, there were no singularity, this consideration might lead to legitimate results.

§31. In this example, it is not easy to calculate the maximum of the greatest principal stress or that of the difference between the greatest and the least principal stresses, even when the material is incompressible, consequently we shall abandon the general discussion concerning the conditions of rupture. But if we confine our attention only to the condition which determines how much load the body can sustain without breaking at the surface, the problem becomes tractable.

The equation (89) gives

$$(\widehat{rr})_0 = (\widehat{\theta\theta})_0 = -\frac{E-\mu}{2\mu} \cdot \frac{\Pi}{\pi a^2},$$

$$(\widehat{zz})_0 = -\frac{\Pi}{\pi a^2}$$

for $r < a$, in which the elastic constants λ and μ are replaced by Young's modulus E and rigidity μ. Since $3\mu > E$ in ordinary materials the component $(\widehat{zz})_0$ is the greatest. The difference between the greatest and the least principal stresses at the surface is

1) BOUSSINESQ p. 148. l.c.

2) For example, we might take $f(r) = \frac{A}{1+r^{100}}$ or a similar relation. But the analysis might be very complicated.

$$D_0 = -\frac{3\mu - E}{2\mu} \cdot \frac{\Pi}{\pi a^2}.$$

The values of $(\widehat{zz})_0$ and D_0 might give the condition of rupture of the surface.

§32. Now we shall apply this solution to the geophysical phenomena mentioned in the introduction. Dr. C. CHREE[1] followed by Prof. NAGAOKA[2] finds a formula, by using the solution obtained by BOUSSINESQ, to calculate the deviation of the direction of gravity due to the attraction of a material loading on the surface of the earth. The same result will be attained of course from our solution. The expression of the vertical displacement at a point on the surface

$$(u_z)_0 = \frac{\lambda + 2\mu}{2\mu(\lambda+\mu)} \int_0^\infty J_0(kr)\,dk \int_0^a p(r')J_0(kr')r'dr'$$

where $p(r')$ is the pressure produced by the material load, can be transformed into

$$(u_z)_0 = \frac{1}{2\pi} \cdot \frac{\lambda + 2\mu}{2\mu(\lambda+\mu)} \int_0^{2\pi}\int_0^a \frac{p(r')}{R'} r'dr'd\varphi$$

by making use of NEUMANN's addition theorem for the BESSEL function, where R′ stands for

$$R' = \sqrt{(r^2 - 2rr'\cos\varphi + r'^2)}$$

On the other hand, if we denote the attraction constant by γ, and gravity, prior to the application of the load, by g, then the gravitation-potential at a point on the surface due to the load can be expsessed by

$$V = \frac{\gamma}{g} \int_0^{2\pi}\int_0^a \frac{p(r')}{R'} r'dr'd\varphi,$$

provided the height of the material load is negligibly small compared with the distance of the point under consideration from

1) Phil. Mag. (V) vol. 43 (1897) p. 177.

2) Tokyo, Sug. But. Kizi (VI) (1912) p. 208.

any point in the loaded area. Comparing the above two expressions we have

$$V = \frac{2\pi\gamma}{g} \cdot \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(u_z)_0$$

Thus the direction of gravity becomes, in consequence of the attraction of the load, inclined to the vertical at the angle ψ which will be determined by

$$\tan\psi = \frac{2\pi\gamma}{g^2} \cdot \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial u_z}{\partial r}\right)_0, \tag{91}$$

while its tilting effect is expressed by

$$\tan\varphi = \left(\frac{\partial u_z}{\partial r}\right)_0. \tag{92}$$

§33. In the present example, in which a uniform material loading is confined in the circle of radius a, we have, from the formula (58),

$$\left(\frac{\partial u_z}{\partial r}\right)_0 = -\frac{\Pi}{\pi a} \cdot \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\left[\int_0^\infty e^{-kz}J_1(kr)J_1(ka)dk\right]_{z=0}.$$

Referring to the formulæ (61), (62) and (64), we obtain

$$\tan\varphi = -\frac{\Pi}{\pi a^2} \cdot \frac{\lambda+2\mu}{2\mu(\lambda+\mu)\pi} \cdot aa^2\left(\frac{1}{2}e_1\omega_1-\eta_1\right)_{z=0}, \tag{93}$$

$$\tan\psi = -\frac{\Pi}{\pi a^2} \cdot \frac{2\gamma}{g^2} \cdot aa^2\left(\frac{1}{2}e_1\omega_1-\eta_1\right)_{z=0}. \tag{94}$$

The function $\frac{1}{2}e_1\omega_1-\eta_1$ has been discussed already and the expression which is suitable for the calculation of its value at a point not near to the edge of the circle has been established in terms of q. Using the q-series in §24 we shall have

$$aa^2\left(\frac{1}{2}e_1\omega_1-\eta_1\right) = 2\pi\sqrt{\frac{a}{r}} \cdot q^{3/2}(1+3q^4-4q^6-9q^8+22q^{12}+\ldots\ldots). \tag{95}$$

It is equally interesting and important to find the value of φ at the point near the edge of the loaded area. This will be accomplished by using the quantity q_1, instead of q, which is defined by

$$q_1 = e^{i\pi\tau_1}, \qquad \tau_1 = -\frac{\omega_1}{\omega_3} = -\frac{1}{\tau}.$$

Now, by the aid of the relation

$$\frac{\vartheta_1'''(o)}{\vartheta_1'(o)} = \frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}$$

the expression of $\frac{1}{2}e_1\omega_1 - \eta_1$ found in §24 may be transformed into

$$4\pi\sqrt{\frac{a}{2}}\left(\frac{1}{2}e_1\omega_1 - \eta_1\right) = \frac{1}{\vartheta_2^2(o/\tau)}\cdot\left\{\frac{\vartheta_0''(o/\tau)}{\vartheta_0(o/\tau)} + \frac{\vartheta_3''(o/\tau)}{\vartheta_3(o/\tau)}\right\}.$$

Making use of the transformation formulæ of Theta-functions it will be easily shown that

$$\begin{aligned}
\frac{\vartheta_0''(o/\tau)}{\vartheta_0(o/\tau)} &= 2i\pi\tau_1 + \tau_1^2\frac{\vartheta_2''(o/\tau_1)}{\vartheta_2(o/\tau_1)}, \\
\frac{\vartheta_3''(o/\tau)}{\vartheta_3(o/\tau)} &= 2i\pi\tau_1 + \tau_1^2\frac{\vartheta_3''(o/\tau_1)}{\vartheta_3(o/\tau_1)}, \\
\vartheta_2^2(o/\tau) &= -i\tau_1\vartheta_0^2(o/\tau_1),
\end{aligned}$$

consequently we have

$$aa^2\left(\frac{1}{2}e_1\omega_1 - \eta_1\right) = \frac{1}{2}\sqrt{\frac{a}{r}}\cdot\frac{1}{\vartheta_0^2(o/\tau_1)}\left\{-4 + \frac{1}{\pi^2}\log_e q_1\left[\frac{\vartheta_2''(o/\tau_1)}{\vartheta_2(o/\tau_1)} + \frac{\vartheta_3''(o/\tau_1)}{\vartheta_3(o/\tau_1)}\right]\right\}. \tag{96}$$

The q_1-series for the functions needed here are as follows:

$$\begin{aligned}
\vartheta_2'' &= -2\pi^2 q^{\frac{1}{4}}(1 + 3^2q_1^2 + 5^2q_1^6 + 7^2q_1^{12} + \ldots\ldots), \\
\vartheta_2 &= 3q^{\frac{1}{4}}(1 + q_1^2 + q_1^6 + q_1^{12} + \ldots\ldots), \\
\vartheta_3'' &= -8\pi^2(q_1 + 2^2q_1^4 + 3^2q_1^9 + \ldots\ldots), \\
\vartheta_3 &= 1 + 2q_1 + 2q_1^4 + 2q_1^9 + \ldots\ldots, \\
\vartheta_0 &= 1 - 2q_1 + 2q_1^4 - 2q_1^9 + \ldots\ldots.
\end{aligned}$$

The quantity q_1 will be found from

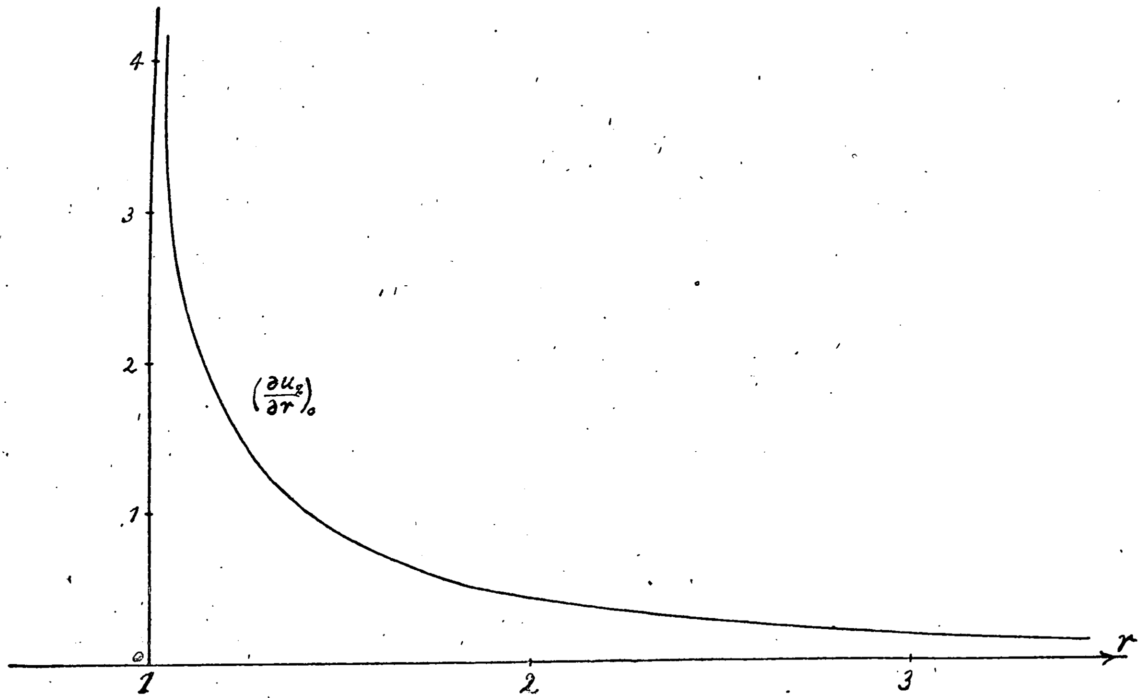

Fig. 6.

$$q_1 = \frac{l_1}{2} + 2\left(\frac{l_1}{2}\right)^5 + 15\left(\frac{l_1}{2}\right)^9 + 150\left(\frac{l_1}{2}\right)^{13} + O(l_1^{17}),$$

$$l_1 = \frac{1-\sqrt{k}}{1+\sqrt{k}} = \frac{\sqrt{(r+a)}-\sqrt{(2\sqrt{ar})}}{\sqrt{(r+a)}+\sqrt{(2\sqrt{ar})}} .$$

Thus the deviation of the direction of gravity at any point can be calculated with any accuracy.

In the next diagram the approximate course of $aa^2(\frac{1}{2}e_1\omega_1 - \eta_1)$ is exhibited as a function of the distance of the point of observation from the centre of the loaded circle, the radius of which is taken as unity.

§34. If we liken the North Atlantic to a circular basin of a large radius and determine the relative position of Potsdam or Chicago referring to the centre of it, the attraction effect of the periodic filling and emptying of tide, which might assist in producing the extra east-west force in observations of the lunar disturbance of gravity, may be computed by our formula. If we suppose the place of the observation not to be very near to the circular basin, the effect, as we see from the above diagram, is of course small, but it increases repidly as the edge is approached.

For the water-level measurement, the effect of a material loading will appear in the form $\varphi+\psi$, instead of ψ only, where ψ is due to the attraction exerted by the material loading and φ to the deformation caused by its weight.

For example, suppose the radius of the North Atlantic basin to be 2000 km, the position of Chicago to be 3000 km from the centre, and the level of the water in this area to be raised *one metre*, then

$$\frac{r}{a} = 1{\cdot}5, \qquad q_1 = 0{\cdot}00255.$$

$$aa^2\left(\frac{e_1\omega_1}{2}-\eta_1\right) = 0{\cdot}8639.$$

Further assume that the density of sea water is unity and in c.g.s.

$$\gamma = 6{\cdot}65\times 10^{-8}, \qquad g = 980,$$

$$\frac{\lambda+2\mu}{2(\lambda+\mu)} = \frac{3}{4}, \qquad \mu = 6\times 10^{11},$$

then we shall have

$$\psi = 1{\cdot}17\times 10^{-8} = 0''{\cdot}0024,$$

$$\varphi = 3{\cdot}37\times 10^{-8} = 0''{\cdot}0069,$$

accordingly the total effect amounts to

$$\psi+\varphi = 4{\cdot}54\times 10^{-8} = 0''{\cdot}009.$$

It will be noticed that the effect of tilting is about three times as great as that of the attraction; so far as the material constants are assumed as above. According to Lord KELVIN,[1] who initiated these investigations, the direct lunar effect on the deviation of a plumbline is a maximum when the moon is at the

1) Natural Philosophy, Part II. p. 333.

altitude 45° and amounts to 0.″017 nearly. The total effect of a tide of amplitude one metre (which is possibly two or three times the actual amount) found here is not small enough to be neglected compared with the direct effect of the moon. As the tilting effect and the attraction effect of the tide wave are directly proportional to the height of the tide, the total effect oscillates in time in accordance with the law which the tide obeys. There is, in general, a difference in phase between the lunar effect and tidal effect, which is worthy of closer investigation. But we must bear in mind that the calculation adopted here is nothing but a rough estimation of order of magnitude, since the north Atlantic is far from circular, the tidal loading in it is never uniform. Nevertheless the above analysis shows that the tidal effect on the water-level measurement, even at a point as far from the coast as Chicago, plays an important role and cannot be regarded as a small correction.

Example IV.

§35. Let us take another example by assuming the normal pressure of the form

$$\left.\begin{aligned} f(r) &= -\frac{3\Pi}{2\pi a^3}\sqrt{a^2-r^2} \quad \text{for } r<a \\ &= 0 \qquad\qquad\qquad\quad ,, \quad r>a \end{aligned}\right\} \tag{97}$$

to be given at $z=0$, Π being its total amount. In this case the function $Z(k)$ becomes

$$\begin{aligned} Z(k) &= -\frac{3\Pi}{2\pi a^3}\,k\int_0^a \sqrt{a^2-\alpha^2}\,J_0(k\alpha)\alpha d\alpha \\ &= -\frac{3\Pi}{2\pi a}\left\{\frac{\sin ka-ka\cos ka}{k^2a^2}\right\}. \end{aligned} \tag{98}$$

Therefore the components of displacement are given by

$$
\left.
\begin{aligned}
u_r &= \frac{3\Pi z}{4\pi a\mu}\int_0^\infty e^{-kz}\left\{\frac{\sin ka - ka\cos ka}{k^2a^2}\right\}J_1(kr)dk \\
&\quad - \frac{3\Pi}{4\pi(\lambda+\mu)}\int_0^\infty e^{-kz}\left\{\frac{\sin ka - ka\cos ka}{k^3a^3}\right\}J_1(kr)dk, \\
u_z &= \frac{3\Pi z}{4\pi a\mu}\int_0^\infty e^{-kz}\left\{\frac{\sin ka - ka\cos ka}{k^2a^2}\right\}J_0(kr)dk \\
&\quad + \frac{3\Pi(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\int_0^\infty e^{-kz}\left\{\frac{\sin ka - ka\cos ka}{k^3a^3}\right\}J_0(kr)dk.
\end{aligned}
\right\} \quad (99)
$$

The integrals contained in the above can be obtained by expanding the trigonometric functions into power series of k and making use of the formulæ (42). In this way we have

$$
\left.
\begin{aligned}
u_r &= \frac{3\Pi}{2\pi a\mu}\cdot\frac{z}{\sqrt{r^2+z^2}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-2)!}{(2n+1)!}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n-1}P^1_{n-1}(\nu) \\
&\quad - \frac{3\Pi}{2\pi(\lambda+\mu)}\cdot\frac{1}{\sqrt{r^2+z^2}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-3)!}{(2n+1)!}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n-2}P^1_{2n-2}(\nu),^{1)} \\
u_z &= \frac{3\Pi}{2\pi a\mu}\cdot\frac{z}{\sqrt{r^2+z^2}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-1)!}{(2n+1)!}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n-1}P_{2n-1}(\nu) \\
&\quad + \frac{3\Pi(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\cdot\frac{1}{\sqrt{r^2+z^2}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-2)!}{(2n+1)!}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n-2}P_{2n-2}(\nu)
\end{aligned}
\right\} \quad (100)
$$

where

$$\nu = \frac{z}{\sqrt{r^2+z^2}}.$$

These series converge for $\sqrt{r^2+z^2} > a$, and are applicable in this region.

At the boundary, we have to put $z=0$ and $\nu=0$. Since

$$P_0^{-1}(0) = 1, \qquad P^1_{2n-2}(0) = 0,$$

$$P_{2n-2}(0) = (-1)^{n-1}\frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots(2n-2)},$$

---

1) For the first term ($n=1$) of the second series we have to take $\frac{1}{3!}P_0^{-1}(\nu)$.

we have

$$\left.\begin{aligned}&\int_0^{\infty}\frac{\sin ka-ka\cos ka}{k^3a^3}J_1(kr)dk=\frac{1}{3r},\\&\int_0^{\infty}\frac{\sin ka-ka\cos ka}{k^3a^3}J_0(kr)dk=\frac{1}{3r}F\left(\frac{1}{2},\frac{1}{2},\frac{5}{2},\frac{a^2}{r^2}\right)\\&\qquad=\frac{1}{2a}\left\{\left(1-\frac{1}{2}\cdot\frac{r^2}{a^2}\right)\sin^{-1}\frac{a}{r}+\frac{r}{2a}\sqrt{1-\frac{a^2}{r^2}}\right)\end{aligned}\right\}\quad(101)$$

for $a \leqq r$. Consequently

$$\left.\begin{aligned}(u_r)_0&=-\frac{\Pi}{4\pi(\lambda+\mu)}\cdot\frac{1}{r},\\(u_z)_0&=\frac{3\Pi(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\cdot\frac{1}{2a}\left\{\left(1-\frac{r^2}{2a^2}\right)\sin^{-1}\frac{a}{r}+\frac{r}{2a}\sqrt{1-\frac{a^2}{r^2}}\right\}\end{aligned}\right\}(102)$$

for $a \leqq r$.

§36. To find the expressions for the displacement within the loaded circle, we proceed as follows:

Making use of the power series of the Bessel function, we have

$$\left.\begin{aligned}&\int_0^{\infty}e^{-kz}\left\{\frac{\sin ka-ka\cos ka}{k^3a^3}\right\}J_1(kr)dk\\&\qquad=\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{r}{2a}\right)^{2n+1}\Omega_{2n+1}\left(\frac{z}{a}\right),\\&\int_0^{\infty}e^{-kz}\left\{\frac{\sin ka-ka\cos ka}{k^3a^3}\right\}J_0(kr)dk\\&\qquad=\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{r}{2a}\right)^{2n}\Omega_{2n}\left(\frac{r}{a}\right)\end{aligned}\right\}\quad(103)$$

where Ω stands for

$$\Omega_m(x)=\int_0^{\infty}e^{-\lambda x}\left\{\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right\}\lambda^m d\lambda.$$

By the aid of the formulæ

$$\int_0^\infty e^{-\lambda x}\frac{\sin\lambda}{\lambda}d\lambda = \tan^{-1}\frac{1}{x},$$

$$\int_0^\infty e^{-\lambda x}\cos\lambda\, d\lambda = \frac{x}{1+x^2}$$

the evaluation of the function $\Omega_m(x)$ can be undertaken. A little calculation will give us

$$\Omega_0(x) = \frac{\pi}{4} - \frac{1}{2}\left\{x + \tan^{-1}x - x^2\tan^{-1}\frac{1}{x}\right\},$$

$$\Omega_1(x) = 1 - x\tan^{-1}\frac{1}{x},$$

$$\Omega_2(x) = \tan^{-1}\frac{1}{x} - \frac{x}{1+x^2},$$

$$\Omega_3(x) = \frac{2}{(1+x^2)^2},$$

and in general

$$\Omega_m(x) = (-1)^{m-1}\frac{d^{m-3}}{dx^{m-3}}\left\{\frac{2}{(1+x^2)^3}\right\}, \quad m > 2.$$

Thus the integrals on the left hand side of (103) can be expanded in accending power series of r/a which probably converge for limited values of r if the value of z is fixed. These series and those found in (100) have a common region in which they are both convergent and therefore they must be congruent to each other in that region. On the proof of this proposition we shall not enter; but we shall find the region of convergency of these latter series at the boundry. Let us take the first series of (103). Expand the function $\Omega_{2n+1}(\frac{z}{a})$ for $n \geqq 1$ into a power series of z/a, supposing z/a to be sufficiently small, then the first term of it will be $(-1)^{n-1}2n(2n-2)!$. Thus if we put $z=0$ in the first series of (103), its general term will then be

$$\frac{2n\,(2n-2)!}{n!\,(n+1)!\,2^{2n+1}}\left(\frac{r}{a}\right)^{2n+1}.$$

The series which has this expression as its general term converges obviously for the value of r smaller than a. Similarly for the second series.

Since, for $z=0$,

$$\Omega_0(o) = \frac{\pi}{4}, \qquad \Omega_1(o) = 1,$$

$$\Omega_2(o) = \frac{\pi}{2}, \qquad \Omega_3(o) = 2,$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$\Omega_{2n}(o) = 0, \qquad \Omega_{2n+1}(o) = (-1)^{n-1}2n(2n-2)\,!, \quad n > 1,$$

we have, after summation,

$$\left.\begin{aligned} \int_0^\infty \frac{\sin ka - ka\cos ka}{k^3a^3} J_1(kr)dk &= \frac{1}{3r}\left\{1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right\}, \\ \int_0^\infty \frac{\sin ka - ka\cos ka}{k^3a^3} J_0(kr)dk &= \frac{\pi}{4a}\left\{1-\frac{r^2}{2a^2}\right\}, \end{aligned}\right\} \qquad (104)$$

for $r \leqq a$.

Consequently we have

$$\left.\begin{aligned} (u_r)_0 &= -\frac{\Pi}{4\pi(\lambda+\mu)}\cdot\frac{1}{r}\left\{1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right\}, \\ (u_z)_0 &= \frac{3(\lambda+2\mu)}{16(\lambda+\mu)\mu}\cdot\frac{1}{a}\left\{1-\frac{1}{2}\cdot\frac{r^2}{a^2}\right\}, \end{aligned}\right\} \qquad (105)$$

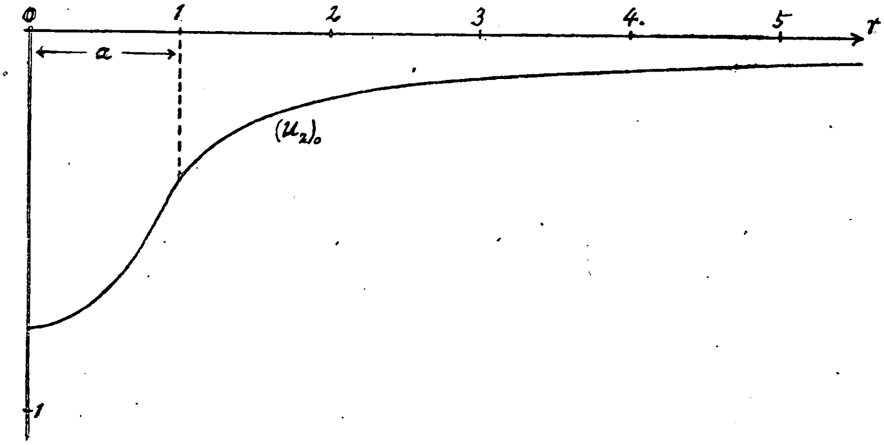

Fig. 7.

for $r \leqq a$. In the annexed diagram the approximate course of the vertical displacement is shown in proper scale.

§37. By a similar process the distribution of stress can be found. Here we shall calculate the stress at the boundary. It may be shown that

$$\left.\begin{aligned} \int_0^{\infty} \frac{\sin ka - ka\cos ka}{k^2a^2} J_0(kr)dk &= 0 \qquad\qquad \text{for } r \geqq a, \\ &= \frac{1}{a}\sqrt{1-\frac{r^2}{a^2}} \quad \text{for } r \geqq a, \end{aligned}\right\} \qquad (106)$$

and, therefore, as the expressions for the stress-components at the surface we have

$$\left.\begin{aligned} (\widehat{rr})_0 &= \frac{\Pi\mu.}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2}, \\ (\widehat{\theta\theta})_0 &= -\frac{\Pi\mu}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2}, \\ (\widehat{zz})_0 &= 0, \qquad (\widehat{zr})_0 = 0 \end{aligned}\right\} \qquad (107)$$

for $r \geqq a$, and

$$\left.\begin{aligned} (\widehat{rr})_0 &= -\frac{3\Pi}{2\pi a^2}\left\{\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{2}} - \frac{\mu}{\lambda+\mu}\cdot\frac{a^2}{3r^2}\left[1-\left(1-\frac{r^2}{a^2}\right)^{\frac{3}{2}}\right]\right\}, \\ (\widehat{\theta\theta})_0 &= -\frac{3\Pi}{2\pi a^2}\left\{\frac{\lambda}{\lambda+\mu}\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{2}} + \frac{\mu}{\lambda+\mu}\cdot\frac{a^2}{3r^2}\left[1-\left(1-\frac{r^2}{a^2}\right)^{\frac{3}{2}}\right]\right\}, \\ (\widehat{zz})_0 &= -\frac{3\Pi}{2\pi a^2}\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{2}}, \\ (\widehat{zr})_0 &= 0, \end{aligned}\right\} \qquad (108)$$

for $r \leqq a$.

The result of this example may be looked upon as a special case of what has been discussed by H. Hertz in his papers[1] concerning the contact of two elastic bodies. He assumed the area on which pressure acts to be an ellipse instead of a circle. If we put $b=a$ in his results, we get exactly the same formulæ for the pressure and for the vertical displacement. And therefore this

1) Gesammelte Werke, I, p. 154—and p. 175.

example may be applied to the discussion of contact of an elastic body upon another with plane-surface.

§38. Another application will be considered here. Suppose we have a material loading of a semi-spheroidal form whose equation is

$$\frac{z^2}{b^2}+\frac{r^2}{a^2}=1, \ (z<0)$$

and of uniform density ρ. This load may be likened to the tidal inequality in the North Atlantic ocean which affects the gravity measurement. In this case Π will be replaced by $\frac{2\pi a^2 bg\rho}{3}$ As before, the deviation of the direction of gravity produced by the attraction of this load is given by

$$\tan\psi=-\frac{3\Pi}{2\pi a^2}\cdot\frac{2\pi\gamma}{g^2}\int_0^\infty\frac{\sin ka-ka\cos ka}{k^2a}J_1(kr)dk,$$

and the level-change due to the deformation of the ground arisen from the load by

$$\tan\varphi=-\frac{3\Pi}{2\pi a^2}\cdot\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty\frac{\sin ka-ka\cos ka}{k^2a}J_1(kr)dk.$$

The evaluation of this integral can be undertaken in a manner similar to those of (99) and will appear to be

$$\left.\begin{aligned}&\int_0^\infty\frac{\sin ka-ka\cos ka}{k^2a}J_1(kr)dk\\&=\frac{r}{2a}\left\{\sin^{-1}\frac{a}{r}-\frac{a}{r}\sqrt{1-\frac{a^2}{r^2}}\right\}, \quad a\leqq r\\&=\frac{\pi r}{4a}, \qquad\qquad r\leqq a\end{aligned}\right\}\tag{109}$$

Thus we have

$$\tan\psi=-\frac{3\Pi}{2\pi a^2}\cdot\frac{\pi\gamma}{g^2}\cdot\frac{r}{a}\left\{\sin^{-1}\frac{a}{r}-\frac{a}{r}\sqrt{1-\frac{a^2}{r^2}}\right\},\tag{110}$$

$$\tan\varphi = -\frac{3\Pi}{2\pi a^2}\cdot\frac{(\lambda+2\mu)}{4\mu(\lambda+\mu)}\cdot\frac{r}{a}\left\{\sin^{-1}\frac{a}{r}-\frac{a}{r}\sqrt{1-\frac{a^2}{r^2}}\right\}, \quad (111)$$

for the point $r > a$.

In the next diagram, the general march of the function $x\sin^{-1}\frac{1}{x}-\sqrt{1-\frac{1}{x^2}}$ is exhibited, where x is the ratio of the distance of the point under consideration from the centre of the loaded circle to its radius. The course of the curve is very similar to that of Fig. 6, except at the point very near to the edge of the loaded area, where, in this case, it remains finite.

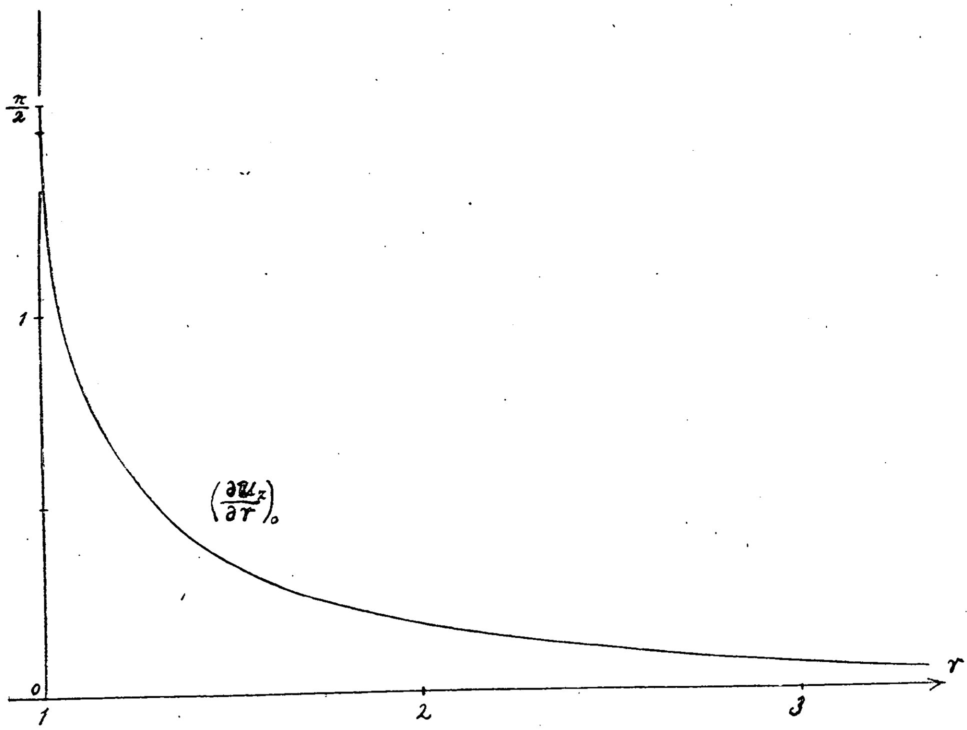

Fig. 8.

For example, with the same assumption regarding the material constants of the earth as in the former example, and supposing that the total amount of the load is the same as before, i.e. the mean height of the tide is one metre, and $a=2\times10^8$, $r=$ $\times 10^8$ cm, we have

$$\psi = 1{\cdot}12 \times 10^{-8} = 0''{\cdot}0023,$$
$$\varphi = 3{\cdot}21 \times 10^{-8} = 0''{\cdot}0066,$$

and

$$\psi + \varphi = 4{\cdot}33 \times 10^{-8} = 0''{\cdot}009$$

nearly the same as the results in the former example.

If we suppose the place of the observation to be very near to the edge of the loaded area, then

$$\psi = 5{\cdot}0 \times 10^{-8} = 0'{\cdot}01,$$
$$\varphi = 14{\cdot}4 \times 10^{-8} = 0''{\cdot}03,$$

and

$$\psi + \varphi = 0''{\cdot}04$$

greater than the maximum of the direct effect of the moon.

Example V.

§39. Lastly, we shall take another example in which the normal pressure of the form

$$f(r) = -\frac{\Pi}{2\pi a} \cdot \frac{1}{\sqrt{a^2 - r^2}} \tag{112}$$

is applied to the boundary within a circle of radius a, which is otherwise left free from traction. This problem has been discussed also by BOUSSINESQ and others, and the expression for the vertical displacement at the boundary has been found. In this case the function $Z(k)$ becomes

$$Z(k) = -\frac{\Pi}{2\pi a} k \int_0^a \frac{J_0(k\alpha)\alpha d\alpha}{\sqrt{a^2 - \alpha^2}} = -\frac{\Pi}{2\pi a} \sin ka. \tag{113}$$

Consequently we have

$$\left.\begin{aligned} u_r = {} & \frac{\Pi z}{4\pi a \mu} \int_0^\infty e^{-kz} \sin ka J_1(kr) dk \\ & - \frac{\Pi}{4\pi a(\lambda + \mu)} \int_0^\infty e^{-kz} \sin ka \, J_1(kr) \frac{dk}{k}, \end{aligned}\right\} \tag{114}$$

$$\left.\begin{aligned} u_z &= \frac{\Pi z}{4\pi a\mu}\int_0^a e^{-kz}\sin ka J_0(kr)dk \\ &+ \frac{\Pi(\lambda+2\mu)}{4\pi a(\lambda+\mu)\mu}\int_0^\infty e^{-kz}\sin ka J_0(kr)\frac{dk}{k}\,. \end{aligned}\right\} \tag{114}$$

The integration can be carried out by expanding $\sin ka$ into a power series and making use of the formulæ (42). Thus we obtain

$$\left.\begin{aligned} u_r &= \frac{\Pi}{4\pi a\mu}\cdot\frac{z}{\sqrt{r^2+z^2}}\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n+1}P_{2n+1}^1(\nu) \\ &- \frac{\Pi}{4\pi a(\lambda+\mu)}\sum_{n=0}^\infty\frac{(-1)^n}{2n(2n+1)}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n+1}P_{2n}^1(\nu),^{1)} \\ u_z &= \frac{\Pi}{4\pi a\mu}\cdot\frac{z}{\sqrt{r^2+z^2}}\sum_{n=0}^\infty(-1)^n\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n+1}P_{2n+1}(\nu) \\ &+ \frac{\Pi(\lambda+2\mu)}{4\pi a(\lambda+\mu)\mu}\cdot\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\left(\frac{a}{\sqrt{r^2+z^2}}\right)^{2n+1}P_{2n}(\nu), \end{aligned}\right\} \tag{115}$$

where

$$\nu = \frac{z}{\sqrt{r^2+z^2}}\,.$$

These series apply for the region $\sqrt{r^2+z^2} > a$.

As in the last example, the expressions which may be applied for small values of r and z can be found by using the power series of the Bessel function and the formula

$$\int_0^\infty k^n e^{-kz}\sin ka\,.\,dk = \frac{n\,!}{(z^2+a^2)^{\frac{n+1}{2}}}\,.\,\sin\left[\,(n+1)\tan^{-1}\frac{a}{z}\,\right]\,.$$

Thus

---

1) For $n=o$ we have to take $\frac{a}{\sqrt{r^2+z^2}}P_0^{-1}(\nu)$.

$$\left.\begin{aligned} u_r &= \frac{\Pi}{4\pi a\mu}\cdot\frac{z}{\sqrt{z^2+a^2}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!}{n!\,(n+1)!\,2^{2n+1}}\left(\frac{r}{\sqrt{z^2+a^2}}\right)^{2n+1}\sin(2n+2)\psi \\ &\quad -\frac{\Pi}{4\pi a(\lambda+\mu)}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{n!\,(n+1)!\,2^{2n+1}}\left(\frac{r}{\sqrt{z^2+a^2}}\right)^{2n+1}\sin(2n+1)\psi, \\ u_z &= \frac{\Pi}{4\pi a\mu}\cdot\frac{z}{\sqrt{z^2+a^2}}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\left(\frac{r}{\sqrt{z^2+a^2}}\right)^{2n}\sin(2n+1)\psi \\ &\quad +\frac{\Pi(\lambda+2\mu)}{4\pi a(\lambda+\mu)\mu}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!}{(n!)^2 2^{2n}}\left(\frac{r}{\sqrt{z^2+a^2}}\right)^{2n}\sin 2n\psi,^{1)} \end{aligned}\right\} \quad (116)$$

where

$$\psi = \tan^{-1}\frac{a}{z}.$$

§40. At the surface ($z=o$) they reduce simply to[2]

$$\left.\begin{aligned} (u_r)_0 &= -\frac{\Pi}{4\pi(\lambda+\mu)a}\cdot\frac{a-\sqrt{a^2-r^2}}{r}, \\ (u_z)_0 &= \frac{\Pi(\lambda+2\mu)}{8(\lambda+\mu)\mu a}, \end{aligned}\right\} \quad (117)$$

for $r \leqq a$; and

$$\left.\begin{aligned} (u_r)_0 &= -\frac{1}{4\pi(\lambda+\mu)}\cdot\frac{1}{r}, \\ (u_z)_0 &= \frac{\Pi(\lambda+2\mu)}{4\pi(\lambda+\mu)\mu a}\cdot\sin^{-1}\frac{a}{r}, \end{aligned}\right\} \quad (118)$$

for $r \geqq a$.

As seen from the formula (117) the vertical displacement is constant over the loaded area, and therefore this solution is applicable when an absolutely rigid body of circular base is pressed normally against an infinite elastic body; the problem has been attacked by various writers from this point of view.

§41. Similarly the expressions for the stresses at the surface will be found to be

---

1) For $n=o$ we have to take ψ instead of zero.

2) If we wish to know the expressions for displacement only at the surface, these can be obtained with less calculation. See LAMB l.c.

$$\left.\begin{aligned}
(\widehat{rr})_0 &= -\frac{\Pi}{2\pi a}\left\{\frac{1}{\sqrt{a^2-r^2}}-\frac{\mu}{\lambda+\mu}\cdot\frac{a-\sqrt{a^2-r^2}}{r^2}\right\},\\
(\widehat{\theta\theta})_0 &= -\frac{\Pi\lambda}{2\pi a(\lambda+\mu)}\left\{\frac{1}{\sqrt{a^2-r^2}}+\frac{\mu}{\lambda}\cdot\frac{a-\sqrt{a^2-r^2}}{r^2}\right\},\\
(\widehat{zz})_0 &= -\frac{\Pi}{2\pi a}\cdot\frac{1}{\sqrt{a^2-r^2}},
\end{aligned}\right\} \qquad (119)$$

for $r < a$; and

$$\left.\begin{aligned}
(\widehat{rr})_0 &= \frac{\Pi\mu}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2},\\
(\widehat{\theta\theta})_0 &= -\frac{\Pi\mu}{2\pi(\lambda+\mu)}\cdot\frac{1}{r^2},\\
(\widehat{zz})_0 &= 0,
\end{aligned}\right\} \qquad (120)$$

for $r > a$.

The stress at the periphery of the loaded area is infinitly great, so that the elastic body would be ruptured at the edge. The present problem may, therefore, throw considerable light upon the explanation of the phenomena of punching.

§42. The expressions for the stress-components in the interior of the body which, so far as I am aware, have not been treated by any one can be found without using any special functions. We shall take here a simple case, for example, in which the material is incompressible.

Since the integral

$$\int_0^\infty e^{-\zeta\lambda}J_0(\xi\lambda)d\lambda = \frac{1}{\sqrt{\zeta^2+\xi^2}}$$

is valid for all values of ζ and ξ, real or complex, provided the real part of ζ is not smaller than the absolute value of the imaginary part of ξ, putting $\zeta = z-ia$, $\xi = r$ in this integral and equating the imaginary parts in both members we have

$$\int_0^\infty e^{-kz}\sin ka.\, J_0(kr)dk = \frac{\sqrt{S^2-(z^2+r^2-a^2)}}{\sqrt{2}S^2} \qquad (121)$$

where S^2 stands for

$$S^2 = \sqrt{(z^2+r^2-a^2)^2+4a^2z^2}. \tag{122}$$

For $z=0$, this formula is still applicable, if we take

$$\left.\begin{aligned} S^2 &= r^2-a^2 \quad \text{for } r > a, \\ &= a^2-r^2 \quad \text{for } r < a. \end{aligned}\right\}$$

Similarly, putting

$$\left.\begin{aligned} P &= \sqrt{S^2-(z^2+r^2-a^2)}, \\ Q &= \sqrt{S^2+(z^2+r^2+a^2)}; \end{aligned}\right\} \tag{123}$$

we have

$$\int_0^\infty e^{-kz}\sin ka \,.\, J_0(kr)kdk = \frac{a(z^2-r^2+a^2)Q+z(z^2+r^2+a^2)P}{\sqrt{2}.\ \ S^6}, \tag{124}$$

$$\int_0^\infty e^{-kz}\sin ka \,.\, J_1(kr)dk = \frac{aQ-zP}{\sqrt{2}.\ rS^2}, \tag{125}$$

$$\int_0^\infty e^{-kz}\sin ka \,.\, J_1(kr)kdk = \frac{r\{(z^2+r^2-a^2)P+2azQ\}}{\sqrt{2}.\ \ S^6}. \tag{126}$$

Thus, for the case of incompressibility, we have

$$\left.\begin{aligned} \widehat{rr} &= -\frac{\Pi}{2\sqrt{2}\pi a}\left\{\frac{P}{S^2}+\frac{z(aQ-zP)}{r^2S^2}-\frac{z[a(z^2-r^2+a^2)Q+z(z^2+r^2+a^2)P]}{S^6}\right\}, \\ \widehat{\theta\theta} &= -\frac{\Pi}{2\sqrt{2}\pi a}\left\{\frac{P}{S^2}-\frac{z(aQ-zP)}{r^2S^2}\right\}, \\ \widehat{zz} &= -\frac{\Pi}{2\sqrt{2}\pi a}\left\{\frac{P}{S^2}+\frac{z[a(z^2-r^2+a^2)Q+z(z^2+r^2+a^2)P]}{S^6}\right\}, \\ \widehat{zr} &= -\frac{\Pi z}{2\sqrt{2}\pi a}\left\{\frac{r(z^2+r^2-a^2)P+2azQ}{S^6}\right\}, \end{aligned}\right\} \tag{127}$$

for the stress components.

V. Boussinesq's Problem.

§43. The problem of LAMÉ and CLAPEYRON is a special case of those known as Boussinesq's, which can be stated as follows:

A limited portion of the surface of a large mass of elastic material is subjected to local stress or to local deformation, it is required to find the strain and stress in the body due to these local disturbances.

In the case of symmetry about an axis perpendicular to the surface of the body, this problem may be discussed, in a general way, by applying our method of analysis. We shall sketch the results here as an addendum.

The typical solution of the equilibrium, in this special case, is

$$\left.\begin{aligned} u_r &= -\left\{\frac{\lambda+\mu}{2\mu}Cz - B\right\}J_1(kr)e^{-kz}, \\ u_\theta &= -\bar{A}J_1(kr)e^{-kz}, \\ u_z &= -\left\{\frac{\lambda+\mu}{2\mu}Cz - D\right\}J_0(kr)e^{-kz}; \end{aligned}\right\} \tag{128}$$

and

$$\left.\begin{aligned} \widehat{zz} &= \{(\lambda+\mu)kCz-\mu C-2\mu kD\}J_0(kr)e^{-kz}, \\ \widehat{z\theta} &= \mu k\bar{A}J_1(kr)e^{-kz}, \\ \widehat{zr} &= \left\{(\lambda+\mu)kCz-\frac{\lambda+\mu}{2}C-\mu k(B+D)\right\}J_1(kr)e^{-kz}, \\ &\text{etc.}; \end{aligned}\right\} \tag{129}$$

with the relation

$$k(B-D) = \frac{\lambda+3\mu}{2\mu}C.$$

in which the components u_θ and $\widehat{z\theta}$ follow from the supposition that Δ is nil. Since these do not give very interesting results, we shall not consider them here.

§44. Case in which all the surface tractions are given.

We suppose first of all that

$$\left.\begin{aligned} \widehat{zz} &= ZJ_0(kr), \\ \widehat{zr} &= RJ_1(kr) \end{aligned}\right\} \tag{130}$$

are given at $z=o$, then we shall have the following values for the arbitrary constants:

$$B = -\frac{\lambda+2\mu}{2\mu(\lambda+\mu)k}R+\frac{1}{2(\lambda+\mu)k}Z,$$

$$D = -\frac{\lambda+2\mu}{2\mu(\lambda+\mu)k}Z+\frac{1}{2(\lambda+\mu)k}R,$$

$$C = -\frac{1}{\lambda+\mu}(Z-R).$$

Putting these in (128) and (129) we have the solution corresponding to the boundary conditions (130).

If the traction over the surface is given in the form

$$\left.\begin{aligned} \widehat{zz} &= p(r), \\ \widehat{zr} &= \tau(r), \end{aligned}\right\} \qquad (131)$$

p and τ being any prescribed functions of r, the corresponding solution can be obtained by making use of the integral theorem (26), on the supposition that the functions $p(r)$ and $\tau(r)$ do not violate that theorem.

Thus

$$\left.\begin{aligned} u_r = &-\int_0^\infty\Big\{\frac{z}{2\mu}[Z(k)-R(k)]+\frac{\lambda+2\mu}{2\mu(\lambda+\mu)k}\cdot R(k) \\ &-\frac{1}{2(\lambda+\mu)k}\cdot Z(k)\Big\}e^{-kz}J_1(kr)dk, \\ u_z = &-\int_0\Big\{\frac{z}{2\mu}[Z(k)-R(k)]+\frac{\lambda+2\mu}{2\mu(\lambda+\mu)k}\cdot Z(k) \\ &-\frac{1}{2(\lambda+\mu)k}\cdot R(k)\Big\}e^{-kz}J_0(kr)dk\,; \end{aligned}\right\} \qquad (132)$$

and

$$\left.\begin{aligned} \widehat{zz} &= \int_0^\infty\{kz[Z(k)-R(k)]+Z(k)\}e^{-kz}J_0(kr)dk, \\ \widehat{zr} &= \int_0^\infty\{kz[Z(k)-R(k)]+R(k)\}e^{-kz}J_1(kr)dk, \\ &\text{etc.}\,; \end{aligned}\right\} \qquad (133)$$

where the functions Z and R are determined by

$$\left.\begin{aligned} Z(k) &= k\int_0^\infty p(a)J_0(ka)a\,da, \\ R(k) &= k\int_0^\infty \tau(a)J_1(ka)a\,da. \end{aligned}\right\} \qquad (134)$$

If we put $\tau(r)=0$ in this solution, we get as a matter of course the solution (30) and (31).

§45. Case in which the normal traction and radial displacement at the surface are given.

If

$$\left.\begin{aligned} u_r &= u(r), \\ \widehat{zz} &= p(r) \end{aligned}\right\} \qquad (135)$$

are given at $z=0$, the corresponding solution is:

$$\left.\begin{aligned} u_r &= -\int_0^\infty \left\{ \frac{z(\lambda+\mu)}{2\mu(\lambda+2\mu)}\left[Z(k)+2\mu k U(k)\right] - U(k)\right\} e^{-kz} J_1(kr)\,dk, \\ u_z &= -\int_0^\infty \left\{ \frac{z(\lambda+\mu)}{2\mu(\lambda+2\mu)}\left[Z(k)+2\mu k U(k)\right] + \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)k} Z(k) \right. \\ &\qquad \left. + \frac{\mu}{\lambda+2\mu} U(k) \right\} e^{-kz} J_0(kr)\,dk\,; \end{aligned}\right\} \qquad (136)$$

and

$$\left.\begin{aligned} \widehat{zz} &= \int_0^\infty \left\{ \frac{z(\lambda+\mu)k}{\lambda+2\mu}\left[Z(k)+2\mu k U(k)\right] + Z(k)\right\} e^{-kz} J_0(kr)\,dk, \\ \widehat{zr} &= \int_0^\infty \left\{ \frac{z(\lambda+\mu)k}{\lambda+2\mu}\left[Z(k)+2\mu k U(k)\right] - \frac{\mu}{\lambda+2\mu} Z(k) \right. \\ &\qquad \left. + \frac{2\mu(\lambda+\mu)k}{\lambda+2\mu} U(k)\right\} e^{-kz} J_1(kr)\,dk, \\ &\qquad \text{etc.}\,; \end{aligned}\right\} \qquad (137)$$

in which U and Z are given by

$$\left.\begin{aligned} U(k) &= k\int_0^\infty u(a)J_1(ka)a\,da, \\ Z(z) &= k\int_0^\infty p(a)J_0(ka)a\,da. \end{aligned}\right\} \qquad (138)$$

46. Case in which the tangential traction and normal displacement at the surface are given.

If

$$\left.\begin{aligned} \widehat{zr} &= \tau(r), \\ u_r &= w(r) \end{aligned}\right\} \qquad (139)$$

are given at $z=o$, then we obtain

$$\left.\begin{aligned} u_r &= -\int_0^\infty \Big\{ -\frac{z(\lambda+\mu)}{2\mu(\lambda+2\mu)}\,[R(k)+2\mu k W(k)] + \frac{\mu}{\lambda+2\mu}\,W(k) \\ &\qquad + \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)k}\,R(k)\Big\} e^{-kz}J_1(kr)\,dk, \\ u_z &= -\int_0^\infty \Big\{ \frac{z(\lambda+\mu)}{2\mu(\lambda+2\mu)}\,[R(k)+2\mu k W(k)] \\ &\qquad + W(k)\Big\} e^{-kz}J_0(kr)\,dk\,; \end{aligned}\right\} \qquad (140)$$

and

$$\left.\begin{aligned} \widehat{zz} &= -\int_0^\infty \Big\{ \frac{z(\lambda+\mu)k}{\lambda+2\mu}\,[R(k)2\mu k\, W(k)] + \frac{2\mu(\lambda+\mu)k}{\lambda+2\mu}\,W(k) \\ &\qquad - \frac{\mu}{\lambda+2\mu}\,R(k)\Big\} e^{-kz}J_0(kr)\,dk, \\ \widehat{zr} &= -\int_0^\infty \Big\{ -\frac{z(\lambda+\mu)k}{\lambda+2\mu}\,[R(k)+2\mu k\, W(k)] \\ &\qquad + R(k)\Big\} e^{-kz}J_1(kr)\,dk, \\ &\text{etc.}\,; \end{aligned}\right\} \qquad (141)$$

where

$$\left.\begin{aligned} R(k) &= k\int_0^\infty \tau(a)J_1(ka)a\,da, \\ W(k) &= k\int_0^\infty w(a)J_0(ka)a\,da. \end{aligned}\right\} \tag{142}$$

§47. Case in which both the displacement components at the boundary are given.

If

$$\left.\begin{aligned} u_r &= u(r), \\ u_z &= w(r) \end{aligned}\right\} \tag{143}$$

are given at $z=o$, we have

$$\left.\begin{aligned} u_r &= \int_0^\infty \left\{ \frac{z(\lambda+\mu)k}{\lambda+3\mu}[W(k)-U(k)] + U(k) \right\} e^{-kz} J_1(kr)\,dk, \\ u_z &= \int_0^\infty \left\{ \frac{2(\lambda+\mu)k}{\lambda+3\mu}[W(k)-U(k)] + W(k) \right\} e^{-kz} J_0(kr)\,dk\,; \end{aligned}\right\} \tag{144}$$

and

$$\left.\begin{aligned} \widehat{zz} &= \int_0^\infty \left\{ \frac{2z\mu(\lambda+\mu)k^2}{\lambda+3\mu}[U(k)-W(k)] - \frac{2\mu^2 k}{\lambda+3\mu}U(k) \right. \\ &\qquad \left. - \frac{2\mu(\lambda+2\mu)k}{\lambda+3\mu}W(k) \right\} e^{-kz} J_0(kr)\,dk, \\ \widehat{rz} &= \int_0^\infty \left\{ \frac{2z\mu(\lambda+\mu)k^2}{\lambda+3\mu}[U(k)-W(z)] - \frac{2\mu^2 k}{\lambda+3\mu}W(k) \right. \\ &\qquad \left. - \frac{2\mu(\lambda+2\mu)k}{\lambda+3\mu}U(k) \right\} e^{-kz} J_1(kr)\,dk, \\ &\text{etc.}\,; \end{aligned}\right\} \tag{145}$$

in which

$$\left.\begin{aligned} U(k) &= k\int_0^\infty u(a)J_1(ka)a\,da, \\ W(k) &= k\int_0^\infty w(a)J_0(ka)a\,da, \end{aligned}\right\} \tag{146}$$

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