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The Inductance Coefficients of Solenoids.

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§ 1. The practical importance of having an accurate standard of inductance has lately been felt in electrical measurements relating to alternate currents and electric waves, and different formulæ have been deduced for expressing the inductance coefficients of solenoids. Among the numerous expressions which have from time to time been given by different physicists for calculating the mutual inductance of coaxal circles or coils, we may mention those bearing the names of Maxwell, Weinstein, Heaviside, Roiti, Searle and Airey, Himstedt, Jones, Lorenz, Gray, Cohen, Russell, and Rosa. These formulæ can be conveniently divided into two classes; the first makes use of Legendre's tables of elliptic integrals, while the second utilizes the expansion in series, which takes different forms according to the method of expansion. They have been examined and criticised by Rosa[1] and Cohen in the Bulletin of the Bureau of Standards, Washington.

A few years ago, I gave a simple series for calculating the mutual inductance of two coaxal circles,[2] which is remarkable

(1) Rosa, Bull. Bur. Standards, 4, 301,1907; Rosa and Cohen, 5, 1, 1908.
(2) H. Nagaoka, Journ. Coll. Sci., 16, Art. 15, 1903; Phil. Mag., [6] 6, p. 19,1904.

for its rapid convergence; the object of the present communication is to shew that the same method may be conveniently employed for expressing the mutual inductance of solenoids of any length.

For the self-inductance of solenoids, the formula can be put in a form convenient for practical calculation, and by tabulating a certain factor $\mathfrak{K}$ as a function of the ratio of diameter to length, we can dispense with the rather intricate formula, that has hitherto been employed for the same purpose.

§ 2. The mutual inductance for two circuits is given by

$$M=\int\int\frac{\cos\varepsilon\; ds\; ds'}{r} \qquad (1)$$

where ε is the angle between the elements ds and ds', and r the distance between the two. In the case of two coaxal solenoids of the radii a and A, length $2l$, $2l'$, placed in such a position that the distance between the centres is d,

$$r^2=a^2+A^2+d^2-2aA\cos(\varphi-\varphi'),$$

$$\varepsilon=\varphi-\varphi';\quad ds=ad\varphi,\quad ds'=A\,d\varphi'.$$

For two circles, whose planes are at the distances z, z' from the centre of the coil (a)

$$M_0=\int_0^{2\pi}\int_0^{2\pi}\frac{Aa\cos(\varphi-\varphi')\,d\varphi\,d\varphi'}{\sqrt{A^2+a^2+(z-z')^2-2Aa\cos(\varphi-\varphi^1)}} \qquad (2)$$

Consequently the mutual inductance M of two solenoids with the number of turns per unit length n and n',

$$M=nn'\int_{d-l'}^{d+l'}\int_{-l}^{+l}M_0\,dz\,dz' \qquad (3)$$

I. Mutual Inductance of Coaxal Circles.

§ 3. In the paper already cited, I showed that mutual inductance for two coaxal circles M_0 at the distance c from each other can be written from (2) in the form

$$\frac{M_0}{4\pi\sqrt{Aa}} = 4\pi q^{\frac{2}{3}}(1+\varepsilon)$$

$$\text{where } \varepsilon=3q^4-4q^6+9q^8-12q^{10}+\cdots \qquad (4)$$

and

$$\frac{M_0}{4\pi\sqrt{Aa}} = \frac{1}{2(1-2q_1)^2}\left\{logn\left(\frac{1}{q_1}\right)(1+8q_1-8q_1^2+\varepsilon_1)-4\right\}$$

$$\text{where } \varepsilon_1=32q_1^3-40q_1^4+48q_1^5-\cdots\cdots \qquad (4')$$

In these expressions, q and q_1 are to be calculated by the well-known expressions

$$q=\frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\cdots\cdots,$$

$$q_1=\frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\cdots\cdots,$$

$$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}}, \qquad l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}}; \qquad (5)$$

$$\text{and } k^2=\frac{e_2-e_3}{e_1-e_3}=\frac{4Aa}{(A+a)^2+c^2}=sin^2\gamma,$$

$$k'^2=\frac{e_1-e_2}{e_1-e_3}=\frac{(A-a)^2+c^2}{(A+a)^2+c^2}=cos^2\gamma.$$

When k or k' is very small, we may calculate l or l_1 by the formulæ

$$l=\frac{k^2}{8}+\frac{k^4}{16}+\frac{21k^6}{512}+\frac{31k^8}{1024}+\cdots\cdots \qquad (5')$$

$$l_1=\frac{k'^2}{8}+\frac{k'^4}{16}+\frac{21k'^6}{512}+\frac{31k'^8}{1024}+\cdots\cdots.$$

It is to be remarked that for most practical purposes, it is generally sufficient to put $q=\frac{l}{2}$ or $q_1=\frac{l_1}{2}$. For $k=sin\ 45°$, $\frac{l}{2}=$ 0.0432139, and $\left(\frac{l}{2}\right)^5=0.00000015$, so that the sixth decimal is not affected by neglecting the second term.

Looking at the table[1] of q, we find that q varies from 0 to 0.15 as γ increases from 0° to 73°.1; similarly the range of q_1 is from 0 to 0.01, as γ diminishes from 90° to 67°.4.

For the calculation of the mutual inductance of coaxal coils, it is therefore convenient to use (4) for values of q from 0 to 0.15, and (4′) for q_1 from 0 to 0.01. When q_1 is very small, we can write (4′) in the form

$$\frac{M_0}{4\pi\sqrt{Aa}}=\frac{1}{2(1-2q_1)^2}\left\{ logn\left(\frac{1}{q_1}\right)(1+8\,q_1+\varepsilon_1')-4\right\}$$

where

$$\varepsilon_1'=-8\,q_1^2+32\,q_1^3-40q_1^4+48q_1^5+\cdots\cdots \quad (4'')$$

§ 4. For facilitating the calculation of mutual inductance, it would be convenient to construct the tables of the following quantities.

I. Table of $q\frac{l}{2}$ from $q=0.02$ to 0.15, and of ε and $log_{10}\,(1+\varepsilon)$ for the same interval.

II. Table of ε_1 and $-\varepsilon_1'=8q_1^2-\varepsilon_1$ for $q=0.01$ to $q_1=0.00$.

These tables cover all values of γ from 10°.2 to 90°, thus enabling us to calculate M for any value of A, a and c, without using the tables of elliptic integrals, which by ordinary methods of calculation must always be resorted to. The special advantage of the formulæ (4), (4′) or (4″) lies in their rapid convergence; we may also dispense with the calculation of γ, which is a great disadvantage of the method usually employed.

The following tables have been calculated by Mr. C. Harada.

(1) See Jacobi, Werke, I, pp. 363–368.

I. Table of $q-\frac{l}{2}$, ε, and $log_{10}(1+\varepsilon)$.

q	$q-\frac{l}{2}$	Δ	ε	$\log_{10}(1+\varepsilon)$	Δ
0.020	0.000 00001	0	0.000 00048	0.000 00021	9
0.022	0.000 00001	1	0.000 00070	0.000 00030	13
0.024	0.000 00002	0	0.000 00099	0.000 00043	16
0.026	0.000 00002	1	0.000 00137	0.000 00059	21
0.028	0.000 00003	2	0.000 00184	0.000 00080	25
0.030	0.000 00005	3	0.000 00243	0.000 00105	31
0.032	0.000 00008	1	0.000 00314	0.000 00136	38
0.034	0.000 00009	3	0.000 00400	0.000 00174	44
0.036	0.000 00012	4	0.000 00503	0.000 00218	53
0.038	0.000 00016	5	0.000 00624	0.000 00271	61
0.040	0.000 00021	5	0.000 00766	0.000 00332	72
0.042	0.000 00026	7	0.000 00931	0.000 00404	83
0.044	0.000 00033	8	0.000 01122	0.000 00487	94
0.046	0.000 00041	10	0.000 01339	0.000 00581	109
0.048	0.000 00051	12	0.000 01588	0.000 00690	122
0.050	0.000 00063	13	0.000 01868	0.000 00812	138
0.052	0.000 00076	16	0.000 02186	0.000 00950	154
0.054	0.000 00092	18	0.000 02541	0.000 01104	172
0.056	0.000 00110	21	0.000 02938	0.000 01276	192
0.058	0.000 00131	25	0.000 03380	0.000 01468	213
0.060	0.000 00156	27	0.000 03870	0.000 01681	234
0.062	0.000 00183	32	0.000 04410	0.000 01915	259
0.064	0.000 00215	36	0.000 05006	0.000 02174	283
0.066	0.000 00251	40	0.000 05660	0.000 02457	312
0.068	0.000 00291	45	0.000 06375	0.000 02769	339
0.070	0.000 00336	51	0.000 07156	0.000 03108	369
0.072	0.000 00387	57	0.000 08007	0.000 03477	401
0.074	0.000 00444	63	0.000 08931	0.000 03878	436
0.076	0.000 00507	70	0.000 09933	0.000 04314	470
0.078	0.000 00577	78	0.000 11016	0.000 04784	509
0.080	0.000 00655	86	0.000 12185	0.000 05293	545
0.082	0.000 00741	95	0.000 13444	0.000 05838	588
0.084	0.000 00836	105	0.000 14798	0.000 06426	631
0.086	0.000 00941	114	0.000 16251	0.000 07057	676
0.088	0.000 01055	126	0.000 17808	0.000 07733	724
0.090	0.000 01081	137	0.000 19474	0.000 08457	772
0.092	0.000 01318	150	0.000 21253	0.000 09229	825
0.094	0.000 01468	162	0.000 23152	0.000 10054	878
0.096	0.000 01630	177	0.000 25174	0.000 10932	937
0.098	0.000 01807	193	0.000 27324	0.000 11869	988
0.100	0.000 02000		0.000 29609	0.000 12857	

I. Table of $q-\frac{l}{2}$, ε, and $log_{10}(1+\varepsilon)$.

q	$q-\frac{l}{2}$	Δ	ε	$\log_{10}(1+\varepsilon)$	Δ
0.100	0.000 02000	102	0.000 29609	0.000 12857	519
0.101	0.000 02102	106	0.000 30803	0.000 13376	533
0.102	0.000 02208	110	0.000 32033	0.000 13909	550
0.103	0.000 02318	115	0.000 33299	0.000 14459	566
0.104	0.000 02433	119	0.000 34602	0.000 15025	582
0.105	0.000 02552	123	0.000 35942	0.000 15607	598
0.106	0.000 02675	129	0.000 37321	0.000 16205	616
0.107	0.000 02804	134	0.000 38731	0.000 16821	632
0.108	0.000 02938	138	0.000 40196	0.000 17453	651
0.109	0.000 03076	144	0.000 41694	0.000 18104	668
0.110	0.000 03220	149	0.000 43233	0.000 18772	686
0.111	0.000 03369	154	0.000 44814	0.000 19458	705
0.112	0.000 03523	160	0.000 46438	0.000 20163	724
0.113	0.000 03683	166	0.000 48105	0.000 20887	742
0.114	0.000 03849	172	0.000 49816	0.000 21629	762
0.115	0.000 04021	178	0.000 51572	0.000 22391	783
0.116	0.000 04199	184	0.000 53374	0 000 23174	802
0.117	0.000 04383	191	0.000 55222	0.000 23976	823
0.118	0.000 04574	196	0.000 57117	0.000 24799	843
0.119	0.000 04770	204	0.000 59060	0.000 25642	865
0.120	0.000 04974	210	0.000 61052	0.000 26507	885
0.121	0.000 05184	218	0.009 63093	0.000 27392	908
0.122	0.000 05402	226	0.000 65184	0.000 28300	930
0.123	0.000 05628	232	0.000 67327	0.000 29230	953
0.124	0.000 05860	240	0.000 69522	0.000 30183	975
0.125	0.000 06100	248	0.000 71769	0.000 31158	998
0.126	0.000 06348	255	0.000 74070	0.000 32156	1022
0.127	0.000 06603	265	0.000 76425	0.000 33178	1046
0.128	0.000 06868	272	0.000 78835	0.000 34224	1071
0.129	0.000 07140	280	0.000 81301	0.000 35295	1094
0.130	0.000 07420	290	0.000 83824	0.000 36389	1120
0.131	0.000 07710	299	0.000 86405	0.000 37509	1145
0.132	0.000 08009	308	0.000 89044	0.000 38654	1171
0.133	0.000 08317	317	0.000 91742	0 000 39825	1196
0.134	0.000 08634	327	0.000 94501	0.000 41021	1224
0.135	0.000 08961	336	0.000 97321	0.000 42245	1251
0.136	0.000 09297	347	0.001 00202	0.000 43496	1277
0.137	0.000 09644	357	0.001 03147	0.000 44773	1305
0.138	0.000 10001	367	0.001 06155	0.000 46078	1333
0.139	0.000 10368	378	0.001 09228	0.000 47411	1362
0.140	0.000 10746	389	0.001 12366	0.000 48773	1389
0.141	0.000 11135	401	0.001 15570	0.000 50162	1420
0.142	0.000 11536	411	0.001 18842	0.000 51582	1448
0.143	0.000 11947	423	0.001 22181	0.000 53030	1479
0.144	0.000 12370	435	0.001 25590	0.000 54509	1509
0.145	0.000 12805	448	0.001 29069	0.000 56018	1539
0.146	0.000 13253	459	0.001 32618	0.000 57557	1571
0.147	0.000 13712	473	0.001 36239	0.000 59128	1602
0.148	0.000 14185	485	0.001 39933	0.000 60730	1634
0.149	0.000 14670	498	0.001 43701	0.000 62364	1666
0.150	0.000 15168		0.001 47543	0.000 64030	

II. Table of ε_1 and $-\varepsilon_1'$.

q_1	ε_1	Δ	$-\varepsilon_1'$	Δ
0.0100	0.000 03160	93	0.000 76840	1499
0.0099	0.000 03067	92	0.000 75341	1484
0.0098	0.006 02975	89	0.000 73857	1471
0.0097	0.000 02886	89	0.000 72386	1455
0.0096	0.000 02797	86	0.000 70931	1442
0.0095	0.000 02711	84	0.000 69489	1428
0.0094	0.000 02627	83	0.000 68061	1413
0.0093	0.000 02544	81	0.000 66648	1399
0.0092	0.000 02463	78	0.000 65249	1386
0.0091	0.000 02385	78	0.000 63863	1370
0.0090	0.000 02307	76	0.000 62493	1356
0.0089	6.000 02231	74	0.000 61137	1342
0.0088	0.000 02157	73	0.000 59795	1327
0.0087	0.000 02084	71	0.000 58468	1313
0.0086	0.000 02013	69	0.000 57155	1299
0.0085	0.050 01944	67	0.000 55856	1285
0.0084	0.000 01877	67	0.000 54571	1269
0.0083	0.000 01810	64	0.000 53302	1256
0.0082	0.000 01746	62	0.000 52046	1242
0.0081	0.000 01684	62	0.000 50804	1226
0.0080	0.000 01622	60	0.000 49578	1212
0.0079	0.000 01562	59	0.000 48366	1197
0.0078	0.000 01503	56	0.000 47169	1184
0.0077	0.000 01447	55	0.000 45985	1169
0.0076	0.000 01392	55	0.000 44816	1153
0.0075	0.000 01337	52	0.000 43663	1140
0.0074	0.000 01285	51	0.000 42523	1125
0.0073	0.000 01234	51	0.000 41398	1109
0.0072	0.000 01183	48	0.000 40289	1096
0.0071	0.000 01135	47	0.000 39193	1081
0.0070	0.000 01088	46	0.000 38112	1066
0.0069	0.000 01042	45	0.000 37046	1051
0.0068	0.000 00997	43	0.000 35995	1037
0.0067	0.000 00954	42	0.000 34958	1022
0.0066	0.000 00912	40	0.000 33936	1008
0.0065	0.000 00872	40	0.000 32928	992
0.0064	0.000 00832	38	0.000 31936	978
0.0063	0.000 00794	37	0.000 30958	963
0.0062	0.000 00757	37	0.000 29995	947
0.0061	0.000 00720	34	0.000 29048	934
0.0060	0.000 00686	34	0.000 28114	918
0.0059	0.000 00652	33	0.000 27196	903
0.0058	0.000 00619	30	0.000 26293	890
0.0057	0.000 00589	31	0.000 25403	873
0.0056	0.000 00558	30	0.000 24530	858
0.0055	0.000 00528	27	0.000 23672	845
0.0054	0.000 00501	28	0.000 22827	828
0.0053	0.000 00473	26	0.000 21999	814
0.0052	0.000 00447	26	0.000 21185	798
0.0051	0.000 00421		0.000 20387	

II. Table of ε_1 and $-\varepsilon_1'$.

q_1	ε_1	Δ	$-\varepsilon_1'$	Δ
0.0050	0.000 00397	24	0.000 19603	784
0.0049	0.000 00374	23	0.000 18834	769
0.0048	0.000 00352	22	0.000 18080	754
0.0047	0.000 00330	22	0.000 17342	738
0.0046	0.000 00309	21	0.000 16619	723
0.0045	0.000 00290	19	0.000 15910	709
0.0044	0.000 00272	18	0.000 15216	694
0.0043	0.000 00253	19	0.000 14539	677
0.0042	0.000 00236	17	0.000 13876	663
0.0041	0.000 00220	16	0.000 13228	648
0.0040	0.000 00204	16	0.000 12596	632
0.0039	0.000 00189	15	0.000 11979	617
0.0038	0.000 00175	14	0.000 11377	602
0.0037	0.000 00161	14	0.000 10791	586
0.0036	0.000 00148	13	0.000 10220	571
0.0035	0.000 00136	12	0.000 09664	556
0.0034	0.000 00125	11	0.000 09123	541
0.0033	0.000 00114	11	0.000 08598	525
0.0032	0.000 00105	9	0.000 08087	511
0.0031	0.000 00095	10	0.000 07593	494
0.0030	0.000 00086	9	0.000 07114	479
0.0029	0.000 00078	8	0.000 06650	464
0.0028	0.000 00070	8	0.000 06202	448
0.0027	0.000 00063	7	0.000 05769	433
0.0026	0.000 00056	7	0.000 05352	417
0.0025	0.000 00050	6	0.000 04950	402
0.0024	0.000 00044	6	0.000 04564	386
0.0023	0.000 00039	5	0.000 04193	371
0.0022	0.000 00034	5	0.000 03838	355
0.0021	0.000 00030	4	0.000 03498	340
0.0020	0.000 00026	4	0.000 03174	324
0.0019	0.000 00022	4	0.000 02866	308
0.0018	0.000 00019	3	0.000 02573	293
0.0017	0.000 00016	3	0.000 02296	277
0.0016	0.000 00013	3	0.000 02035	261
0.0015	0.000 00011	2	0.000 01789	246
0.0014	0.000 00009	2	0.000 01559	230
0.0013	0.000 00007	2	0.000 01345	214
0.0012	0.000 00006	1	0.000 01146	199
0.0011	0.000 00004	2	0.000 00964	182
0.0010	0.000 00003	1	0.000 00797	167
0.0009	0.000 00002	1	0.000 00646	151
0.0008	0.000 00002	0	0.000 00510	136
0.0007	0.000 00001	1	0.000 00391	119
0.0006	0.000 00001	0	0.000 00287	104
0.0005	0.000 00000	1	0.000 00200	87
0.0004	0.000 00000	0	0.000 00128	72
0.0003	0.000 00000	0	0.000 00072	56
0.0002	0.000 00000	0	0.000 00032	40
0.0001	0.000 00000	0	0.000 00008	24

§ 5. The following two examples will serve to illustrate the present method of calculation.

(1) For $k'=0.3420201=\sin 70°$, we find by (5)

$$q=0.1310618$$

Table I gives $\log_{10}(1+\varepsilon)=0.0003758$

Direct calculation gives $\log_{10} 4\pi q^{\frac{3}{2}}=\bar{1}.7754242$

By addition $\log_{10}\dfrac{M_0}{4\pi\sqrt{Aa}}=\bar{1}.7758000.$

which coincides with the value found by Maxwell, who calculated it by using Legendre's tables.

(2) For $k=0.9975641=\sin 86°$, we find by (5)

$$q_1=0.0003048651.$$

Or from (5′)

$$\frac{k'^2}{16}=0.000\ 3041228$$

$$\frac{k'^4}{32}=0.000\ 0007399$$

$$\frac{21k'^6}{1024}=0.000\ 0000024$$

$$q_1\ =0.000\ 3048651$$

which is the same as found by (5)

By direct calculation

$$\log n\left(\frac{1}{q_1}\right)=8.115378$$

Table II gives $-\varepsilon'_1=0.0000007$

Consequently $1+8q_1+\varepsilon'_1=1.0024382$

and $1-2q_1=0.9993903$

By (4″) $\log\dfrac{M_0}{4\pi\sqrt{Aa}}=0.3139097$

as given by Maxwell.

It may be remarked that a slight disadvantage of the present method lies in the fact that q_1 must always be calculated with great accuracy, however small it may be.

§ 6. When two circles are near each other and of nearly equal radii, so that x and y in

$$A=a+x\ ;\quad c=y\ ,\quad r^2=x^2+y^2$$

are small, the mutual inductance can be expressed in a series

$$M_0=4\pi a\ log\ \frac{8a}{r}\left\{1+\frac{1}{2}\frac{x}{a}+\frac{x^2+3y^2}{16a^2}-\frac{x^3+3xy^2}{32a^3}+\cdots\right\}$$

$$+4\pi a\left\{-2-\frac{1}{2}\frac{x}{a}+\frac{3x^2-y^2}{16a^2}-\frac{x^3-6xy^2}{48a^3}+\cdots\right\}\cdots \qquad (6)$$

as shown by Maxwell.

We can easily show that (4′) can be so transformed that it coincides with Maxwell's expression. Evidently

$$q_1=\frac{k'^2}{16}+\frac{k'^4}{32}+\cdots\ ,\qquad \text{where } k'^2=\frac{x^2+y^2}{2a^2\left(1+\frac{x}{a}+\frac{x^2+y^2}{4a^2}\right)}.$$

Consequently

$$\log\frac{1}{q_1}=2\,log\ \frac{8a}{r}+log\left(1+\frac{x}{a}+\frac{x^2+y^2}{4a^2}\right)-log\left(1+\frac{k'^2}{2}\right)$$

$$\sqrt{Aa}\ =\ a\left(1+\frac{x}{2a}-\frac{1}{8}\frac{x^2}{a^2}+\frac{3}{48}\frac{x^3}{a^3}-\cdots\right)\ .$$

To first approximation

$$M_0\ =\ 4\pi\sqrt{Aa}\left\{\frac{1}{2}\ log\frac{1}{q_1}\cdot(1+12q_1)-2(1+4q_1)\right\}\ ,$$

and by expansion

$$\sqrt{Aa}\,(1+12q_1)\ log\ \frac{8a}{r}\ =a\ log\frac{8a}{r}\left(1+\frac{x}{2a}+\frac{x^2+3y^2}{16a^2}-\frac{x^3+3xy^2}{32a^3}\right)$$

$$\sqrt{Aa}\,\frac{(1+12q_1)}{2}\ log\left(1+\frac{x}{a}+\frac{r^2}{4a^2}\right)=a\left(\frac{x}{2a}+\frac{x^2+y^2}{8a}+\frac{x(x^2+3y^2)}{96a^3}\right)$$

$$-\sqrt{Aa}\ \frac{(1+12q_1)}{2}\ log\left(1+\frac{k'^2}{2}\right)=-a\left(\frac{x^2+y^2}{16a^2}-\frac{x\,(x^2+y^2)}{32a^3}\right)$$

$$-\sqrt{Aa}\ .\ 2\,(1+4q_1)\ \ =-a\left(2+\frac{x}{a}-\frac{x^2+y^2}{8a^2}+\frac{x(x^2-y^2)}{16a^3}\right)$$

Adding these four expressions, we obtain the formula (6) as given by Maxwell. Proceeding to the second approximation

$$M_0 = 4\pi a \log\frac{8a}{r}\left\{1+\frac{x}{2a}+\frac{x^2+3y^2}{16a^2}-\frac{x^3+3xy^2}{32a^3}+\frac{17x^4+42x^2y^2-15y^4}{1024a^4}\right\}$$

$$+4\pi a\left\{-2-\frac{x}{2a}+\frac{3x^2-y^2}{16a^2}-\frac{x^3-6xy^2}{48a^3}-\frac{19x^4+534x^2y^2-93y^4}{6144a^4}\right\}\cdots(6')$$

This extension of Maxwell's formula was first given by Rosa and Cohen.[1]

II. Mutual Inductance of Coaxal Solenoids.

§ 7. Returning to formula (3) for the mutual inductance of two coaxal solenoids with the number of turns per unit length n and n', we have

$$M = nn'\int_{d-l'}^{d+l'}\int_{-l}^{+l} M_0\, dz\, dz'. \qquad (3)$$

Substituting for M_0 from (2), the integral can be easily transformed into

$$M = 4\pi\, nn'\int_{d-l'}^{d+l'}\int_{-l}^{+l}\int_0^{\pi}\frac{Aa\cos\psi\, dz\, dz'\, d\psi}{\sqrt{A^2+a^2+(z-z')^2+2Aa\cos\psi}}$$

$$= 4\pi\, nn'\int_{d-l'}^{d+l'}\int_{-l}^{+l'}\int_0^{\pi}\frac{A^2a^2\sin^2\psi\, dzdz'd\psi}{(A^2+a^2+(z-z')^2-2Aa\cos\psi)^{\frac{3}{2}}}$$

Integrating with respect to z and z',

$$M = 4\pi\, nn'\, Aa\,(I_1-I_2-I_3+I_4) \qquad (7)$$

where $$I = Aa\int_0 \frac{\sin^2\psi\,\sqrt{A^2+a^2+c^2-2Aa\cos\psi}}{(A^2+a^2-2Aa\cos\psi)}\, d\psi, \qquad (8)$$

and
$$\begin{aligned} I &= I_1 \quad \text{for} \quad c=d+l+l', \\ &= I_2 \quad ,, \quad c=d+l-l', \\ &= I_3 \quad ,, \quad c=d-l+l', \\ &= I_4 \quad ,, \quad c=d-l-l'. \end{aligned} \qquad (8')$$

(1) Rosa and Cohen, Bulletin Bur. Standards, 2, p. 366, 1906.

I can be written

$$I=\int_0^\pi \frac{\sin^2\psi\ d\psi}{\sqrt{A^2+a^2+c^2-2\,Aa\cos\psi}}$$

$$+c^2\int_0^\pi \frac{\sin^2\psi\ d\psi}{(A^2+a^2-2Aa\cos\psi)\sqrt{A^2+a^2+c^2-2Aa\cos\psi}}\cdots(8'')$$

Putting $\cos\psi = \alpha s + \beta$

$$\left.\begin{array}{l}
\alpha=\left(\dfrac{2}{Aa}\right)^{\frac{1}{3}}, \qquad \beta=\dfrac{A^2+a^2+c^2}{6Aa} \\
e_1=\dfrac{2\beta}{\alpha}; \qquad e_2=\dfrac{1-\beta}{\alpha}, \qquad e_3=-\dfrac{1+\beta}{\alpha}, \\
\text{so that} \quad e_1+e_2+e_3=o \quad \text{and} \quad e_2-e_3=\dfrac{2}{\alpha}, \\
g_2=2\,(e_1^2+e_2^2+e_3^2) \doteq \dfrac{4\,(1+3\beta^2)}{\alpha^2}, \\
g_3=4\,e_1\,e_2\,e_3 \qquad =\dfrac{8\beta\,(\beta^2-1)}{\alpha^3},
\end{array}\right\} \qquad (9)$$

we have

$$\sin\psi\sqrt{A^2+a^2+c^2-2\,Aa\cos\psi}=\sqrt{4(s-e_1)\,(s-e_2)\,(s-e_3)}$$

$$=\sqrt{4s^3-g_2s-g_3}=\sqrt{S}.$$

Let $u=\int_s^\infty \frac{ds}{\sqrt{S}}$ so that $s=pu$,

and let $pv=\dfrac{A^2+a^2-2\,Aa\,\beta}{2\,Aa\,\alpha}=\dfrac{2\,(A^2+a^2)-c^2}{6\,Aa\,\alpha}$;

then

$$\left.\begin{array}{l}
pv-e_1=-\dfrac{c^2}{4Aa}\,(e_2-e_3)=-\dfrac{c^2}{2Aa\,\alpha}, \\
pv-e_2=\dfrac{(A-a)^2}{4Aa}\,(e_2-e_3)=\dfrac{(A-a)^2}{2Aa\,\alpha}, \\
pv-e_3=\dfrac{(A+a)^2}{4Aa}\,(e_2-e_3)=\dfrac{(A+a)^2}{2Aa\,\alpha}; \\
\text{whence} \quad e_1-e_3=\dfrac{(A+a)^2+c^2}{4Aa}\,(e_2-e_3),
\end{array}\right\} \qquad (10)$$

$$e_1-e_2=\frac{(A-a)^2+c^2}{4Aa}(e_2-e_3)\,,$$

$$p'v=\sqrt{4(pv-e_1)(pv-e_2)(pv-e_3)}\,,$$

$$=-i\frac{c\,(A^2-a^2)}{2Aa}\,.$$

By simple substitution

$$\frac{I}{Aa\,a}=\int_{\omega_2}^{\omega_3}(1-\beta^2-2a\beta\,pu-a^2p^2u)\,du-(pv-e_1)\int_{\omega_2}^{\omega_3}\frac{1-\beta^2-2a\beta pu-p^2u}{pv-pu}\,du$$

Since $p^2u=\frac{1}{6}p''u+\frac{1}{12}g_2$, and $p'\omega_2=o$, $p'\omega_3=o$,

$$\frac{I}{Aa\,a}=\left\{1-\beta^2-\frac{a^2g_2}{12}-a\,(pv-e_1)\,(apv+2\beta)\right\}\int_{\omega_2}^{\omega_3}du\;-a^2pv\int_{\omega_2}^{\omega_3}p\,u\,du$$

$$-\frac{(pv-e_1)}{p'v}(1-\beta^2-2a\,\beta\,pv-a^2p^2v)\int_{\omega_2}^{\omega_3}\frac{p'v}{pv-pu}du\,.$$

Again

$$1-\beta^2-\frac{a^2g_2}{12}-a\,(pv-e_1)\,(a\,pv+2\beta)\;=\;a^2\left(\frac{g_2}{6}-p^2v\right).$$

$$1-\beta^2-2a\beta\,pv-a^2p^2v\;=-a^2\,(pv-e_2)\,(pv-e_3)\,.$$

Thus by integration

$$I=2\left\{\left(\frac{g_2}{6}-p^2v\right)\omega_1+pv.\eta_1+\frac{p'v}{2}\left(\eta_1\,.\,v-\omega_1\frac{\sigma'}{\sigma}(v)\right\}\right.\cdots(11)$$

This expression for I is to be substituted for I_1, I_2, I_3 and I_4 in (7), for evaluating the mutual inductance M of coaxal solenoids.

§ 8. Evidently the first two terms of I

$$\left(\frac{g_2}{6}-p^2v\right)\omega_1$$

and $pv.\eta_1$

are easily expressed by means of complete elliptic integrals of the first and second kinds, while the third term corresponds to to an integral of the third kind and forms the chief difficulty in the numerical calculation.

In the refined experiments of the present day, a formula which admits of exact calculation is found essential. The usual formula for M expressed in terms of elliptic integrals of three different kinds is not easy to evaluate. The calculation of the integral of the third kind, which is usually expressed by incomplete integrals of the first and second kinds requires a good deal of labour, even when Legendre's table is accessible, as it is of double entry. When the integral is expanded in powers of k, as in Russell's formula, the convergence is rather slow and the calculation of successive coefficients by the formula of recursion is not easy. In place of the usual method of reduction, the evaluation by means of the q-series, which is rapidly convergent, may be used with great advantage.

§ 9. The expression for I in the above form (11) leads to cumbrously large values of g_2 and p^2v, so that for exact calculation, it is convenient to reduce the above to a simpler form. It can be thrown into various forms suitable for numerical calculation.

Since
$$\eta_1 = -e_2\omega_1 - \frac{1}{4\omega_1}\cdot\frac{\vartheta_3''(o)}{\vartheta_3(o)},$$
$$= -e_3\omega_1 - \frac{1}{4\omega_1}\cdot\frac{\vartheta_0''(o)}{\vartheta_0(o)},$$

we remark that the first two terms in I involve as the coefficient of ω_1 either

$$\frac{g_2}{6} - p^2v - e_2pv$$

or

$$\frac{g_2}{6} - p^2v - e_3pv.$$

But $\frac{g_2}{6}=\frac{2(3\beta^2+1)}{3a^2}$, whence

$$\frac{g_2}{6}-p^2v-e_2pv=\frac{c^2(A^2+a^2)}{4A^2a^2a^2}-\frac{pv}{a}+\frac{2}{3a^2}$$

$$\frac{g_2}{6}-p^2v-e_3pv=\frac{c^2(A^2+a^2)}{4A^2a^2a^2}+\frac{pv}{a}+\frac{2}{3a^2}.$$

Remembering that

$$pv-e_1=-\frac{c^2}{4Aa}(e_2-e_3) \quad =P_1,$$

$$pv-e_2=\frac{(A-a)^2}{4Aa}(e_2-e_3) \quad =P_2,$$

$$pv-e_3=\frac{(A+a)^2}{4Aa}(e_2-e_3) \quad =P_3,$$

we easily find that

$$\frac{c^2(A^2+a^2)}{4A^2a^2a^2}=-\frac{(pv-e_1)\{(pv-e_2)+(pv-e_3)\}}{2}=-\frac{P_1(P_2+P_3)}{2},$$

$$-\frac{pv}{a}+\frac{2}{3a^2}=-\frac{\{(pv-e_1)+2(pv-e_2)\}(e_2-e_3)}{6}=-\frac{(P_1+2P_2)(e_2-e_3)}{6},$$

$$+\frac{pv}{a}+\frac{2}{3a^2}=\frac{\{(pv-e_1)+2(pv-e_3)\}(e_2-e_3)}{6}=\frac{(P_1+2P_3)(e_2-e_3)}{6},$$

Consequently the first two terms of I in the parenthesis are

$$\left(\frac{g_2}{6}-p^2v\right)\omega_1+pv.\eta_1=\left(\frac{g_2}{6}-p^2v-e_2pv\right)\omega_1-\frac{pv}{4\omega_1}\frac{\vartheta_3''(o)}{\vartheta_3(o)}$$

$$=-\left\{\frac{P_1(P_2+P_3)}{2}+\frac{(P_1+2P_2)(e_2-e_3)}{6}\right\}\omega_1-\frac{pv}{4\omega_1}\frac{\vartheta_3''(o)}{\vartheta_3(o)} \tag{12}$$

or
$$=\left(\frac{g_2}{6}-p^2v-e_3pv\right)\omega_1-\frac{pv}{4\omega_1}\frac{\vartheta_0''(o)}{\vartheta_0(o)}$$

$$=-\left\{\frac{P_1(P_2+P_3)}{2}-\frac{(P_1+2P_3)(e_2-e_3)}{6}\right\}\omega_1-\frac{pv}{4\omega_1}\frac{\vartheta_0''(o)}{\vartheta_0(o)} \tag{12'}$$

The quantities within the parenthesis in (12) and (12′) are easily calculated from the known dimensions and position of the coils, and

$$\omega_1 = \frac{2\pi\sqrt{q}}{\sqrt{e_2 - e_3}}(1 + q^2 + q^6 + q^{12} + \ldots) \quad ,$$

or

$$= \frac{\pi}{2\sqrt{e_1 - e_3}}(1 + 2q + 2q^4 + 2q^9 + \ldots)^2 \quad ,$$

or

$$= \frac{\pi}{2\sqrt{e_1 - e_2}}(1 - 2q + 2q^4 - 2q^9 + \ldots)^2 \quad ;$$

and

$$\frac{\vartheta_3''(o)}{\vartheta_3(o)} = -\frac{8\pi^2(q + 4q^4 + 9q^9)}{1 + 2q + 2q^4 + 2q^9} \quad ,$$

$$\frac{\vartheta_0''(o)}{\vartheta_0(o)} = \frac{8\pi^2(q - 4q^4 + 9q^9 + \ldots)}{1 - 2q + 2q^4 - 2q^9 + \ldots} \quad .$$

For long coils, direct calculation of g_2 involves c^4, which is inconveniently large; but in the above form, there only remains c^2A^2 or c^2a^2, and the expression becomes free of one source of error.

Another advantage of the above transformation is that we can check the result of calculation for the terms $\left(\frac{g_2}{6} - p^2v\right)\omega_1 + pv.\eta_1$ by taking either form requires little labour as soon as $pv - e_1$, $pv - e_2$, $pv - e_3$ are once calculated. By summing these quantities and noticing that $e_1 + e_2 + e_3 = o$, we find pv; mutual subtraction gives $e_1 - e_2$, $e_1 - e_3$, $e_2 - e_3$; this again leads to the evaluation of q_1 which is generally very small. These being known, ω_1 and consequently η_1 is calculated by means of the formulæ already given.

It is also worthy of remark that

$$\frac{g_2}{6} - p^2v - e_2pv \quad ,$$

or

$$\frac{g_2}{6} - p^2v - e_3pv \quad ,$$

can be expressed in terms of σ-functions by utilizing the formula

$$pv-e_\lambda=\frac{\sigma_\lambda^2}{\sigma^2}(v) \quad , \qquad \text{where } \lambda=1,2,3.$$

§ 10. The numerical evaluation of the term $p'v\left(\eta_1 v-\omega_1\frac{\sigma'}{\sigma}(v)\right)$ requires little explanation. By means of the formula

$$p'v=-\sqrt{4(pv-e_1)(pv-e_2)(pv-e_3)} \; ,$$

$p'v$ can be calculated from the values of the three quantities under the radical; but it is more accurate to calculate it by the relation (10)

$$p'v=-i\frac{(A^2-a^2)}{2Aa} \; ,$$

Since $$\omega_1\frac{\sigma'}{\sigma}(v)-\eta_1 v=i\frac{\pi}{2}\left\{\frac{z+z^1}{z-z^{-1}}+\sum_n\frac{2q^{2n}z^{-2}}{1-q^{2n}z^{-2}}-\sum_n\frac{2q^{2n}z^2}{1-q^{2n}z^2}\right\}$$

where $$z=e^{\frac{\pi v}{2\omega_1}i}=e^{\pi wi} \; ,$$

it is necessary to calculate z from the known values of pv. Practically the quantity within the parenthesis is nearly equal to 1, so that only for very accurate determinations is it necessary to take the first term of Σ's into account.

In the present case $e_2<pv<e_1$; then

$$\frac{1}{2}\,\frac{\sqrt[4]{e_1-e_3}\,\sqrt{pv-e_2}-\sqrt[4]{e_1-e_2}\,\sqrt{pv-e_3}}{\sqrt[4]{e_1-e_3}\,\sqrt{pv-e_2}+\sqrt[4]{e_1-e_2}\,\sqrt{pv-e_3}}=-\frac{q\cos 2\pi w-q^9\cos 6\pi w}{1+2q^4\cos 4\pi w+\ldots}=s$$

With great approximation

$$\cos 2\pi w=-\frac{s}{q} \; ,$$

since q is generally small; sometimes $q^4\cos 4\pi w$ may enter as a small correction. Knowing s, we can accurately find $\cos 2\pi w=-b$.

Thus $$z^2+z^{-2}=-2b \; ,$$

whence $$z+z^{-1}=i\sqrt{2(b-1)} \; ,$$

$$z - z^{-1} = i\sqrt{2(b+1)} \quad ,$$

and

$$\omega_1 \frac{\sigma'}{\sigma}(v) - \eta_1 . v = i\frac{\pi}{2}\left[\sqrt{\frac{b-1}{b+1}} + 4q^2\sqrt{b^2-1}\{1-q^2(2b-1)\}\right] \quad (13)$$

in which the term involving q^4 is generally negligibly small, and the whole expression is nearly equal to $i\frac{\pi}{2}$ in most practical cases.

Although the expressions (12) and (13) appear somewhat abstruse for numerical calculation, the calculation is not so laborious as in dealing with a formula involving incomplete elliptic integrals, even when Legendre's table is accessible. Taking the case $d=o$, $2l=200$, $2l'=20$, $A=15$, $a=10$, I found $M=4\pi nn' \times 6213.51$,[1] which coincides with the value deduced from Roiti's formula.

It is evident without proof that the formula given by Viriamu Jones for a helix and a circle, and the formula arrived at by Russell for the mutual inductance of a cylindrical current sheet and a coaxal helix can be deduced in a similar manner, and expressed in terms of p-functions.

III. Self-inductance of Solenoids.

§ 11. For the self-inductance of solenoids, several formulæ have been deduced by different physicists. They generally assume different forms according as the solenoid is short or long. Most of them are, however, complicated and not suitable for the use of experimental physicists and engineers. In the following I propose to show that the self-inductance of a solenoid can be easily calculated by tabulating a certain coefficient $\mathfrak{L}$. Evidently the self-inductance of a very long solenoid is given by

(1) For numerical calculation, see Proc. Tokyo Math. Phys. Soc., **4**, 284, 1908.

$$L = 4\pi n^2 \times \text{Area of Cross Section} \times \text{Length}$$

where n is the number of turns per unit length; for solenoids of any length, it will be shown that

$$L = 4\pi n^2 \times \text{Area of Cross Section} \times \text{Length} \times \mathfrak{L}.$$

where $\mathfrak{L}$ can be tabulated once for all as a function of angular aperture or of the ratio of diameter to length of the solenoid. When once the values of $\mathfrak{L}$ are known, the calculation is greatly facilitated, as the rest of the operation is a simple multiplication.

§ 12. It has already been shown in § 7. that the mutual inductance of coaxal solenoids

$$M = 4\pi nn' Aa \; [I_1 - I_2 - I_3 + I_3] \tag{7}$$

where $I = 2\left[\left(\frac{g_2}{6} - p^2 v\right)\omega_1 + pv.\eta_1 + \frac{p'v}{2}\left(\eta_1 v - \omega_1 \frac{\sigma'}{\sigma}(v)\right)\right]$ (11)

The case which deserves special attention is when the radii and the lengths of the solenoids coincide; i.e., when $d = o$, $A = a$, $l = l'$, $n = n'$. In this special case, M is transformed into L

For $\quad c = 2l, \; pv = e_2$, or $v = \omega_2$ and $p'v = o$.

$$e_1 - e_2 = \frac{l^2}{a_2} \cdot (e_2 - e_3), \qquad e_1 - e_3 = \frac{a^2 - l^2}{a^2} \cdot (e_2 - e_3)$$

$$e_2 = \frac{1}{3} \; \frac{a^2 - l^2}{a^2}(e_2 - e_3), \qquad k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{a^2}{a^2 + l^2}$$

For $\quad c = o, \; pv = e_1 = e_2$, or $u = \omega_1 = \omega_2$ and $p'v = o$.

Thus $\quad I_1 = I_4 = 2\left\{\frac{e_1(e_2 - e_3) - e_3(e_1 - e_2)}{3}\,\omega_1 + e_2\eta_1\right\}$

$$I_2 = I_3 = \frac{4a}{3}. \tag{14}$$

Utilizing the relation

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \qquad \eta_1 = \sqrt{e_1 - e_2}E - \frac{e_1}{\sqrt{e_1 - e_3}} K$$

where K and E are complete elliptic integrals of the first and second kinds resp., we find that

$$I_1=\frac{4}{3}\,\frac{\sqrt{a^2+l^2}}{a^2}\,(l^2K+(a^2-l^2)E)\ ,$$

$$I_1-I_2=\frac{4}{3a^2}\left\{\sqrt{a^2+l^2}\,(l^2K+(a^2-l^2)E)-a^3\right\}\ .$$

Consequently

$$L=4\pi n^2\frac{8}{3}\left\{\sqrt{a^2+l^2}\,(l^2K+(a^2-l^2)E)-a^3\right\}\ , \tag{15}$$

which is identical with the formula obtained by Lorenz[1] and Cohen.[2]

Remembering that $k^2=\dfrac{a^2}{a^2+l^2}$, $k'^2=\dfrac{l^2}{a^2+l^2}$, we find

$$I_1-I_2=\frac{4}{3}\,\frac{1}{k'k^2}\left\{k'^2(K-E)+k^2K-k^3\right\}\ .$$

The self inductance of the solenoid is thus

$$L=4\pi n^2.\ \pi a^2.\ 2l.\frac{4}{3\pi}\,\frac{1}{k'}\left\{\frac{k'^2}{k^2}(K-E)+E-k\right\} \tag{16}$$

Putting $$\mathfrak{L}=\frac{4}{3\pi}\,\frac{1}{k'}\left\{\frac{k'^2}{k^2}(K-E)+E-k\right\}\ , \tag{17}$$

we get $$L=4\pi n^2\times\text{Area of Cross Section}\times\text{Length}\times\mathfrak{L}$$

$$=4\pi N^2.\ \frac{\text{Area of Cross Section}}{\text{Length}}\ \mathfrak{L} \tag{18}$$

where N is the total number of windings.

Putting $k=\sin\alpha$, $tg\alpha=\dfrac{a}{l}=\dfrac{k}{k'}$, we see that α is equivalent to a semiangular aperture of the solenoid at the centre. Thus $\mathfrak{L}$ can be generally expressed as a function of angular aperture. In practice, it will be convenient to tabulate $\mathfrak{L}$ as a function

1 Lorenz, Wied. Ann., **7**, p. 170. 1879; Oeuvres, **2**, p. 196.

2 L. Cohen, Bull. Bur. Stand., **3**, p. 303, 190.

of $tg\alpha = \frac{\text{Diameter}}{\text{Length}}$. These tables are given at the end of the paper.

§ 13. The coefficient $\mathfrak{L}$ can be expressed in various ways.

Since $$-k\frac{\partial E}{\partial k} = \frac{k^2}{k'}\frac{\partial E}{\partial k'} = K - E \ ,$$

and $$\frac{k'^2}{k^2}(K-E)+E = -\frac{k'^2}{k^2}\frac{\partial E}{\partial k}+E$$

$$= k'\frac{\partial E}{\partial k'} + E = \frac{\partial (k'E)}{\partial k'}$$

$$\mathfrak{L} = \frac{4}{3\pi}\frac{1}{k'}\left(-\frac{k'^2}{k}\frac{\partial E}{\partial k} + E - k\right) , \qquad (19)$$

$$= \frac{4}{3\pi}\frac{1}{k'}\left(\frac{\partial (k'E)}{\partial k'} - k\right) . \qquad (19')$$

The expression (19) can be conveniently used when the length is very large as compared with the diameter of the solenoid, while the second expression (19′) will be found useful when the said ratio is very small.

By differentiating the power series in k^2 for E

$$E = \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2\frac{k^2}{1} - \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{k^4}{3} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{k^6}{5} - \cdots\cdots\right\} ,$$

we find by (19)

$$\mathfrak{L} = \frac{1}{k'}\left\{1 - \frac{4}{3\pi}k - \left(\frac{1}{2}\right)^2\frac{3}{2}\cdot\frac{k^2}{1} - \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{5}{3}\frac{k^4}{3} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{7}{4}\frac{k^6}{5}\right.$$
$$\left. - \left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)^2\cdot\frac{9}{5}\frac{k^8}{7} - \left(\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\right)^2\frac{11}{6}\frac{k^{10}}{9} - \cdots\cdots\right\} \qquad (20)$$

For convenience in calculating the coefficients, the following table is added

$$\frac{4}{3\pi} = 0.42441318, \qquad log_{10}\frac{4}{3\pi} = \bar{1}.62778887.$$

$$log_{10}[k^2]=\bar{1}.15740313, \qquad log_{10}[k^4]=\bar{2}.8927890,$$
$$log_{10}[k^6]=\bar{2}.5337681, \qquad log_{10}[k^8]=\bar{2}.2838906,$$
$$log_{10}[k^{10}]=\bar{2}.0912009, \qquad log_{10}[k^{12}]=\bar{3}.9280777.$$

The above series is only slowly convergent when the angular aperture is large, so that its application is limited to small values of α.

The second operation (19′) above indicated can be applied to the expression for E under the form given by Schlömilch.

$$E=1+\left(\frac{1}{2}k'^2+\frac{3}{4}\left(\frac{1}{2}\right)^2k'^4+\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{5}{6}k'^6+\ldots\ldots\right)log\frac{4}{k'}$$
$$-\frac{s_2}{2}k'^2-\left(\frac{1}{2}\right)^2\frac{3}{4}(s_2+s_4)\,k'^4-\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{5}{6}(s_4+s_6)\,k'^6-\ldots\ldots$$

where $$s_n=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots-\frac{1}{2n} \quad .$$

Thus

$$\mathfrak{L}=\frac{4}{3\pi}\left\{\left(\frac{3}{2}k'+\left(\frac{1}{2}\right)^2\frac{7}{4}k'^3+\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{11}{6}k'^5+\ldots\ldots\right)log\frac{4}{k'}+\frac{1-k}{k'}\right.$$
$$\left.-\frac{1}{2}\left(1+3s_2\right)k'-\left(\frac{1}{2}\right)^2\cdot\frac{3}{4}\left(1+5(s_2+s_4)\right)k'^3-\ldots\ldots\right\} \qquad (20') \quad .$$

This formula corresponds to that usually given for short solenoids. The above expression can only be used for values of α close to 90°.

§ 20. On account of the simplicity of calculation in series proceeding according to powers of k or k', the two series (20) and (20′) above given will find special favour among practical scientific men, but the limits within which the series may be safely applied is so narrow that it is necessary to deduce other series which can be easily used within a wide range in the value of α. For this purpose, expansions in q-series are specially to be recommended.

Starting from the well-known expressions(1)

$$K(K-E) = 2\pi^2 q(1+2q+4q^2+4q^3+6q^4+8q^5+\cdots\cdots)$$

$$KE = \frac{\pi^2}{4}\{1+8q^2(1-q^2+4q^4-5q^6+\cdots\cdots)\}$$

$$k'K = \frac{\pi}{2}(1-4q+4q^2+4q^4-8q^5+4q^8-\cdots\cdots)\}$$

$$k'K^2 = 4\pi^2(q+4q^3+6q^5+8q^7+\cdots\cdots)$$

we easily find by (I)

$$\mathfrak{L} = 1-\frac{4}{3\pi}\frac{k}{k'}+2q+12q^2+44q^3+116q^4+260q^5+576q^6+\cdots\cdots \quad (21)$$

The limits of α, within which the above series can be conveniently used are wider than the expansions already given; for $\alpha=45°$ $576q^6 = 0.00000375$, so that the values of $\mathfrak{L}$ are right to six decimal places for the above argument. By the way, it may be noted that for the same value of α, the term affected with k^{12} in (20)=0.0001324, and the convergency of (20) is slower than that of (21).

§ 21. In order to arrive at expressions, which would give more exact values of $\mathfrak{L}$, it is necessary to transform formula (17) into ϑ-functions.

Since $\quad \eta_1 = -e_1\omega_1 - \frac{1}{4\omega_1}\frac{\vartheta''_2(o)}{\vartheta_1(o)} \quad = -e_3\omega_1 - \frac{1}{4\omega_1}\frac{\vartheta''_0(o)}{\vartheta_0(o)}$,

$$E = \frac{1}{\sqrt{e_1-e_3}}(\eta_1+e_1\omega_1)$$

$$K-E = \frac{1}{\sqrt{e_1-e_3}}(\eta_1+e_3\omega_1) \quad \text{and} \quad k' = \frac{\sqrt{e_1-e_2}}{\sqrt{e_1-e_3}},$$

we find $\quad \frac{E}{k'} = -\frac{1}{4\omega_1\sqrt{e_1-e_2}}\frac{\vartheta_2''(o)}{\vartheta_2(o)} = -\frac{1}{2\pi\vartheta_0^2(o)}\frac{\vartheta_2''(o)}{\vartheta_2(o)}$

(1) Jacobi, Fundamenta Nova, 105; Werke. **1,** p. 161.

$$\frac{K-E}{k}=\frac{1}{4\omega_1\sqrt{e_1-e_2}}\frac{\vartheta_0''(o)}{\vartheta_0(o)}=\frac{1}{2\pi\vartheta_0^2(o)}\cdot\frac{\vartheta_0''(o)}{\vartheta_0(o)}$$

whence

$$\frac{1}{k'}\left\{(K-E)\frac{k'^2}{k^2}+E\right\}=\frac{1}{2\pi\vartheta_0^2}\left(\frac{\vartheta_0''(o)}{\vartheta_0(o)}\frac{k'^2}{k^2}-\frac{\vartheta_2''(o)}{\vartheta_2(o)}\right).$$

Expressing this in terms of q's, we find for $\mathfrak{L}$

$$\mathfrak{L}=\frac{2}{3(1-2q+2q^4-2q^9)^2}\left\{1+\frac{8(q^2+3q^6+6q^{12}+\cdots)}{1+q^2+q^6+q^{12}}\right.$$

$$\left.+\frac{8(q-4q^4+9q^9-\cdots)}{1-2q+2q^4-2q^9+\cdots}\frac{k'^2}{k^2}\right\}-\frac{4}{3\pi}\frac{k}{k'} \qquad (21')$$

The above expression is applicable within wide limits of α, and is rapidly convergent; although q^{12} is retained, it is generally sufficient for practical calculation from $\alpha=0$ to $\alpha=45°$ to suppress terms beyond q^4. For $\alpha=45°$, $q^4=0.00000349$ and $q^6=0.0000000065$, so that between the said limits

$$\mathfrak{L}=\frac{2}{3(1-2q+2q^4)^2}\left\{1+8(q^2-q^4)+\frac{8(q-4q^4)}{1-2q+2q^4}\frac{k'^2}{k^2}\right\}-\frac{4}{3\pi}\frac{k}{k'} \qquad (21'')$$

will give values accurate to six decimal places, which is superfluous in practice. Since $\frac{k'}{k}=cotg\alpha$ enters into the calculation before finding q, it would be more convenient to retain it in this form than to change it into a q-series.

To find an expression of $\mathfrak{L}$ in terms of q_1, by which the calculation of self-inductance for values of α from 45° to 90° may be easily effected, we have to express E and $K-E$ by means of ω_3 and $\vartheta(o,\tau_1)$'s.

Since
$$E=\frac{1}{2\omega_3\sqrt{e_1-e_3}}\left(\pi i-\frac{\omega_1}{2\omega_3}\frac{\vartheta_0''(o,\tau_1)}{\vartheta_0(o,\tau_1)}\right)$$

$$K-E=-\frac{1}{2\omega_3\sqrt{e_1-e_3}}\left(\pi i-\frac{\omega_1}{2\omega_3}\frac{\vartheta_2''(o,\tau_1)}{\vartheta_2(o,\tau_1)}\right)$$

$$\frac{\omega_1}{\omega_3}=\frac{1}{\pi i}\,log\left(\frac{1}{q_1}\right)\ ;$$

we find by (17)

$$\mathfrak{L}=\frac{4}{3\pi^2 i\vartheta_0^2(o,\tau_1)}\left\{\left(\frac{\vartheta_2''(o,\tau_1)}{\vartheta_2(o,\tau_1)}\frac{k'^2}{k^2}-\frac{\vartheta_0''(o,\tau_1)}{\vartheta_0(o,\tau_1)}\right)\frac{1}{2\pi i}\,log\,\frac{1}{q_1}+\pi i\left(1-\frac{k'^2}{k^2}\right)\right\}$$
$$-\frac{4}{3\pi}\frac{k}{k'}\,.$$

Expressing the ϑ-functions by means of q_1-series, we find

$$\mathfrak{L}=\frac{1}{3\pi\sqrt{q_1}(1+q_1^2+q_1^6+q_1^{12}+\cdots)^2}\left[\left\{\left(1+\frac{8(q_1^2+3q_1^6+6q_1^{12}+\ \ )}{1+q_1^2+q_1^6+q_1^{12}\cdot}\right)\frac{k'^2}{k^2}\right.\right.$$
$$\left.\left.+\frac{8(q_1-4q_1^4+9q_1^9-\cdots)}{1-2q_1+2q_1^4-2q_1^9+\cdots}\right\}\frac{1}{2}\,log\,\frac{1}{q_1}+1-\frac{k'^2}{k_2}\right]-\frac{4}{3\pi}\frac{k}{k'}\,. \tag{22}$$

For practical calculation, it is superfluous to retain terms beyond q_1^4; whence the simplified expression for $\mathfrak{L}$ becomes

$$\mathfrak{L}=\frac{1}{3\pi\sqrt{q_1}\,(1+q_1^2)^2}\left[\left\{\left(1+8(q_1^2-4q_1^4)\right)\frac{k'^2}{k^2}+\frac{8(q_1-4q_1^4)}{1-2q_1+2q_1^4}\right\}\frac{1}{2}\,log\,\frac{1}{q_1}\right.$$
$$\left.+1-\frac{k'^2}{k^2}\right]-\frac{4}{3\pi}\frac{k}{k'}\,. \tag{22'}$$

It is needless to remark that the convergence is extremely rapid. The slight inconvenience which is felt in the evaluation of the above expression is due to the presence of the term $\frac{1}{\sqrt{q_1}}\,log\,\frac{1}{q_1}$. Even for small values of q_1, it must be accurately known; in fact, we shall have to push the calculations for q_1 to several decimal places, which is quite unnecessary for finding the values of ϑ' s, for the simple reason that the expression contains terms multiplied by $\frac{1}{\sqrt{q_1}}\,log\,\frac{1}{q_1}$. It is convenient to calculate q_1 from l_1, where

$$l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}}=\frac{k'^2}{(1+\sqrt{k}\,)^2(1+k)}\,.$$

The two expressions (21) and (22) for $\mathfrak{L}$ in terms of q and q_1 resp. will enable us to calculate the self-inductance L by the formula (18) with any desirable accuracy, while for practical purposes, the simpler expressions (21′) and (22′) will generally suffice.

§ 22. The application of quadric transformation to the elliptic integrals which enter into $\mathfrak{L}$ will lead to an expression, which is expedient for the evaluation of the coefficient, but for the use of physicists and engineers, those already given would be efficient for numerical calculation. It would however not be out of place to notice the different gates, which are open for expressing $\mathfrak{L}$ in a convenient manner.

§ 23. The following numerical examples are given for the sake of comparison of formulæ (20) and (21)

For $\alpha=45°$, formula (20) gives

$$
\begin{aligned}
1 &= 1.00000 \\
-[\ \ ]\,k^2 &= -0.18750 \\
-[\ \ ]\,k^4 &= -0.01953 \\
-[\ \ ]\,k^6 &= -0.00427 \\
-[\ \ ]\,k^8 &= -0.00120 \\
-[\ \ ]\,k^{10} &= -0.00039 \\
-[\ \ ]\,k^{12} &= -0.00013 \\
\hline
1-0.21302 &= 0.78698
\end{aligned}
$$

$$
\begin{aligned}
\frac{0.78698}{k'} &= 1.1129 \\
\frac{4}{3\pi}\cdot\frac{k}{k'} &= 0.4244 \\
\hline
\mathfrak{L} &= 0.6885
\end{aligned}
$$

The fourth figure is slightly in error as formula (21) will show.

For $\alpha = 45°$, $q = 0.04321392$; by formula (21)

$$
\begin{aligned}
1 &= 1.0000000\\
2q &= 0.0864278\\
12q^2 &= 0.0224093\\
44q^3 &= 0.0035509\\
116q^4 &= 0.0004045\\
260q^5 &= 0.0000392\\
576q^6 &= 0.0000038\\
\frac{3760}{3}q_7 &= 0.0000004\\
\hline
& \quad 1.1128359\\
\frac{4}{3\pi}\frac{k}{k'} &= 0.4244132\\
\hline
\mathfrak{L} &= 0.6884227
\end{aligned}
$$

This coincides with the values found from (22) or from (17) by using Legendre's tables of elliptic integrals.

§ 24. Most of the formulæ above deduced admit of easy calculation, but when the values of $\mathfrak{L}$ for different α's are once tabulated, they will have only a theoretical interest. In the eyes of practical men, the result of the various calculations above given for finding $\mathfrak{L}$ will be of little value, when the table is constructed.

The following tables of log_{10} $\mathfrak{L}$ and $\mathfrak{L}$ as functions of α were calculated by Mr. G. Sugimoto from formula (17), by making use of Legendre's tables of elliptic integrals given in 'Exercises' vol. 3. It is given for every degree of α, but in practice it will be more convenient to tabulate $\mathfrak{L}$ as a function of

$$tg\,\alpha = \frac{\text{Diameter}}{\text{Length}} = \frac{2a}{2l}$$

The last table gives $\mathfrak{L}$ with $tg\ \alpha$ as argument. It was obtained from the foregoing table by interpolation.

TABLE OF $LOG_{10}\mathfrak{L}$

α	$Log\ \mathfrak{L}$	Δ_1	Δ_2	α	$Log\ \mathfrak{L}$	Δ_1	Δ_2
0°	10·000 000			30°	9·900 397		−·000 047
		−·003 213				−·003 717	
1	9·996 787		+·000 009	31	9·896 680		−·000 050
		−·003 204				−·003 767	
2	9·993 583		+·000 005	32	9·892 913		−·000 051
		−·003 199				−·003 818	
3	9·990 384		+·000 004	33	9·889 095		−·000 056
		−·003 195				−·003 874	
4	9·987 189		+·000 004	34	9·885 221		−·000 059
		−·003 191				−·003 933	
			+·000 001				−·000 063
5	9·983 998			35	9·881 238		
		−·003 190				−·003 996	
6	9·980 808		−·000 000	36	9·877 292		−·000 065
		−·003 190				−·004 061	
7	9·977 618		−·000 001	37	9·873 231		−·000 070
		−·003 191				−·004 131	
8	9·974 427		−·000 004	38	9·869 100		−·000 074
		−·003 195				−·004 205	
9	9·971 232		−·000 004	39	9·864 895		−·000 078
		−·003 199				−·004 283	
			−·000 007				−·000 083
10	9·968 033			40	9·860 612		
		−·003 206				−·004 366	
11	9·964 827		−·000 008	41	9·856 246		−·000 087
		−·003 214				−·004 453	
12	9·961 613		−·000 009	42	9·851 793		−·000 093
		−·003 223				−·004 546	
13	9·958 390		−·000 012	43	9·847 247		−·000 098
		−·003 235				−·004 644	
14	9·955 155		−·000 012	44	9·842 603		−·000 104
		−·003 247				−·004 748	
			−·000 015				−·000 110
15	9·951 908			45	9·837 855		
		−·003 262				−·004 858	
16	9·948 646		−·000 016	46	9·832 997		−·000 117
		−·003 278				−·004 975	
17	9·945 368		−·000 019	47	9·828 022		−·000 125
		−·003 297				−·005 100	
18	9·942 071		−·000 019	48	9·822 922		−·000 131
		−·003 316				−·005 231	
19	9·938 755		−·000 021	49	9·817 691		−·000 141
		−·003 337				−·005 372	
			−·000 025				−·000 149
20	9·935 418			50	9·812 319		
		−·003 362				−·005 521	
21	9·932 056		−·000 025	51	9·806 798		−·000 158
		−·003 387				−·005 679	
22	9·928 669		−·000 027	52	9·801 119		−·000 170
		−·003 414				−·005 849	
23	9·925 255		−·000 030	53	9·795 270		−·000 181
		−·003 444				−·006 030	
24	9·921 811		−·000 032	54	9·789 240		−·000 193
		−·003 476				−·006 223	
			−·000 034				−·000 207
25	9·918 335			55	9·783 017		
		−·003 510				−·006 430	
26	9·914 825		−·000 037	56	9·776 587		−·000 221
		−·003 547				−·006 651	
27	9·911 278		−·000 038	57	9·769 936		−·000 238
		−·003 585				−·006 889	
28	9·907 693		−·000 041	58	9·763 047		−·000 255
		−·003 626				−·007 144	
29	9·904 067		−·000 044	59	9·755 903		−·000 275
		−·003 670				−·007 419	
			−·000 047				−·000 298
30	9·900 397			60	9·748 484		

a	Log $\mathfrak{L}$	Δ_1	Δ_2	a	Log $\mathfrak{L}$	Δ_1	Δ_2
60	9·743 484	−·007 717	−·000 298	75°	9·580 159	−·018 033	−·001 376
61	9·740 767	−·008 037	−·000 320	76	9·562 126	−·019 625	−·001 592
62	9·732 730	−·008 386	−·000 349	77	9·542 501	−·021 492	−·001 867
63	9·724 344	−·008 763	−·000 377	78	9·521 009	−·023 704	−·002 212
64	9·715 581	−·009 175	−·000 412	79	9·497 305	−·026 362	−·002 658
65	9·706 406	−·009 625	−·000 450	80	9·470 943	−·029 614	−·003 252
66	9·696 781	−·010 118	−·000 493	81	9·441 329	−·033 672	−·004 058
67	9·686 663	−·010 659	−·000 541	82	9·407 657	−·038 868	−·005 196
68	9·676 004	−·011 258	−·000 599	83	9·368 789	−·045 742	−·006 874
69	9·664 746	−·011 920	−·000 662	84	9·323 047	−·055 236	−·009 494
70	9·652 826	−·012 657	−·000 737	85	9·267 811	−·069 154	−·013 918
71	9·640 169	−·013 482	−·000 825	86	9·198 657	−·091 438	−·022 284
72	9·626 687	−·014 410	−·000 928	87	9·107 219	−·132 712	−·041 274
73	9·612 277	−·015 461	−·001 051	88	8·974 507	−·235 451	−·102 739
74	5·596 816	−·016 657	−·001 196	89	8·739 056		
75	9·580 159		−·001 376				

TABLE OF $\mathfrak{L}$

a	$\mathfrak{L}$	Δ_1	Δ_2	a	$\mathfrak{L}$	Δ_1	Δ_2
0°	1·000 000	−·007 370		15°	0·895 175	−·006 699	+·000 019
1	0·992 630	−·007 298	+·000 072	16	0·888 476	−·006 681	+·000 018
2	0·985 332	−·007 231	+·000 067	17	0·881 795	−·006 667	+·000 014
3	0·978 101	−·007 168	+·000 063	18	0·875 128	−·006 657	+·000 010
4	0·970 933	−·007 108	+·000 060	19	0·868 471	−·006 649	+·000 008
5	0·963 825	−·007 054	+·000 054	20	0·861 822	−·006 645	+·000 004
6	0·956 771	−·007 001	+·000 053	21	0·855 177	−·006 643	+·000 002
7	0·949 770	−·006 955	+·000 046	22	0·848·534	−·006 645	−·000 002
8	0·942 815	−·006 909	+·000 046	23	0·841 889	−·006 650	−·000 005
9	0·935 906	−·006 870	+·000 039	24	0·835 239	−·006 659	−·000 009
10	0·929 036	−·006 833	+·000 037	25	0·828 580	−·006 669	−·000 010
11	0·922 203	−·006 799	+·000 034	26	0·821 911	−·006 685	−·000 016
12	0·915 404	−·006 769	+·000 030	27	0·815 226	−·006 702	−·000 017
13	0·908 635	−·006 742	+·000 027	28	0·808 524	−·006 723	−·000 021
14	0·901 893	−·006 718	+·000 024	29	0·801 801	−·006 747	−·000 024
15	0·895 175		+·000 019	30	0·795 054		−·000 028

a	$\mathfrak{L}$	Δ_1	Δ_2	a	$\mathfrak{L}$	Δ_1	Δ_2
30°	0·795 054	−·006 775	−·000 028	**60°**	0·560 382	−·009 869	−·000 214
31	0·788 279	−·006 807	−·000 032	**61**	9·550 513	−·010 095	−·000 226
32	0·781 472	−·006 841	−·000 034	**62**	0·540 418	−·010 334	−·000 239
33	0·774 631	−·006 880	−·000 039	**63**	0·530 084	−·010 590	−·000 256
34	0·767 751	−·006 921	−·000 041	**64**	0·519 494	−·010 860	−·000 270
35	0·760 830	−·006 967	−·000 046	**65**	0·508 634	−·011 148	−·000 288
36	0·753 863	−·007 017	−·000 050	**66**	0·497 486	−·011 456	−·000 308
37	0·746 846	−·007 071	−·000 054	**67**	0·486 030	−·011 784	−·000 328
38	0·739 775	−·007 128	−·000 057	**68**	0·474 246	−·012 135	−·000 351
39	0·732 647	−·007 189	−·000 061	**69**	0·462 111	−·012 511	−·000 376
40	0·725 458	−·007 256	−·000 067	**70**	0·449 600	−·012 914	−·000 403
41	0·718 202	−·007 327	−·000 071	**71**	0·436 686	−·013 349	−·000 435
42	0·710 875	−·007 402	−·000 075	**72**	0·423 337	−·013 816	−·000 467
43	0·703 473	−·007 483	−·000 081	**73**	0·409 521	−·014 322	−·000 506
44	0·695 990	−·007 567	−·000 084	**74**	0·395 199	−·014 871	−·000 549
45	0·688 423	−·007 659	−·000 092	**75**	0·380 328	−·015 468	−·000 597
46	0·680 764	−·007 754	−·000 095	**76**	0·364 860	−·016 121	−·000 653
47	0·673 010	−·007 856	−·000 102	**77**	0·348 739	−·016 838	−·000 717
48	0·665 154	−·007 964	−·000 108	**78**	0·331 901	−·017 629	−·000 791
49	0·657 190	−·008 079	−·000 115	**79**	0·314 272	−·018 510	−·000 881
50	0·649 111	−·008 199	−·000 120	**80**	0·295 762	−·019 495	−·000 985
51	0·640 912	−·008 327	−·000 128	**81**	0·276 267	−·020 611	−·001 116
52	0·632 585	−·008 463	−·000 136	**82**	0·255 656	−·021 886	−·001 275
53	0·624 122	−·008 605	−·000 142	**83**	0·233 770	−·023 370	−·001 484
54	0·615 517	−·008 757	−·000 152	**84**	0·210 400	−·025 128	−·001 758
55	0·606 760	−·008 917	−·000 160	**85**	0·185 272	−·027 272	−·002 144
56	0·597 843	−·009 086	−·000 169	**86**	0·158 000	−·029 997	−·002 725
57	0·588 757	−·009 265	−·000 179	**87**	0·128 003	−·033 704	−·003 707
58	0·579 492	−·009 455	−·000 190	**88**	0·094 299	−·039 464	−·005 760
59	0·570 037	−·009 655	−·000 200	**89**	0·054 835		
60	0·560 382		−·000 214				

TABLE OF $\mathfrak{L}$ AS FUNCTION OF $\frac{\text{Diameter}}{\text{Length}}$.

$\frac{\text{Diameter}}{\text{Length}}$	$\mathfrak{L}$	Δ_1	$\frac{\text{Diameter}}{\text{Length}}$	$\mathfrak{L}$	Δ_1
0·00	1·000 000	−0·004 231	**0·30**	0·883 803	−0·003 498
0·01	0·995 769	−0·004 207	**0·31**	0·880 305	−0·003 476
0·02	0·991 562	−0·004 181	**0·32**	0·876 829	−0·003 452
0·03	0·987 381	−0·004 157	**0·33**	0·873 377	−0·003 429
0·04	0·983 224	−0·004 132	**0·34**	0·869 948	−0·003 406
0·05	0·979 092	−0·004 107	**0·35**	0·866 542	−0·003 384
0·06	0·974 985	−0·004 082	**0·36**	0·863 158	−0·003 359
0·07	0·970 903	−0·004 056	**0·37**	0·859 799	−0·003 338
0·08	0·966 847	−0·004 032	**0·38**	0·856 461	−0·003 315
0·09	0·962 815	−0·004 008	**0·39**	0·853 146	−0·003 293
0·10	0·958 807	−0·003 982	**0·40**	0·849 853	−0·003 270
0·11	0·954 825	−0·003 957	**0·41**	0·846 583	−0·003 248
0·12	0·950 868	−0·003 933	**0·42**	0·843 335	−0·003 225
0·13	0·946 935	−0·003 910	**0·43**	0·840 110	−0·003 204
0·14	0·943 025	−0·003 884	**0·44**	0·836 906	−0·003 183
0·15	0·939 141	−0·003 859	**0·45**	0·833 723	−0·003 160
0·16	0·935 284	−0·003 834	**0·46**	0·830 563	−0·003 139
0·17	0·931 450	−0·003 811	**0·47**	0·827 424	−0·003 117
0·18	0·927 639	−0·003 785	**0·48**	0·824 307	−0·003 096
0·19	0·923 854	−0·003 761	**0·49**	0·821 211	−0·003 075
0·20	0·920 093	−0·003 737	**0·50**	0·818 136	−0·003 054
0·21	0·916 356	−0·003 713	**0·51**	0·815 082	−0·003 033
0·22	0·912 643	−0·003 689	**0·52**	0·812 049	−0·003 012
0·23	0·908 954	−0·003 664	**0·53**	0·809 037	−0·002 991
0·24	0·905 290	−0·003 641	**0·54**	0·806 046	−0·002 971
0·25	0·901 649	−0·003 616	**0·55**	0·803 075	−0·002 950
0·26	0·898 033	−0·003 593	**0·56**	0·800 125	−0·002 930
0·27	0·894 440	−0·003 569	**0·57**	0·797 195	−0·002 910
0·28	0·890 871	−0·003 546	**0·58**	0·794 285	−0·002 890
0·29	0·887 325	−0·003 522	**0·59**	0·791 395	−0·002 870

Diameter / Length	𝔏	Δ_1	Diameter / Length	𝔏	Δ_1
0·60	0·788 525	−0·002 850	**0·95**	0·699 509	−0·002 247
0·61	0·785 675	−0·002 831	**0·96**	0·697 262	−0·002 232
0·62	0·782 844	−0·002 812	**0·97**	0·695 030	−0·002 217
0·63	0·780 032	−0·002 792	**0·98**	0·692 813	−0·002 202
0·64	0·777 240		**0·99**	0·690 611	
		−0·002 773			−0·002 188
0·65	0·774 467	−0·002 754	**1·00**	0·688 423	−0·010 726
0·66	0·771 713	−0·002 735	**1·05**	0·677 697	−0·010 382
0·67	0·768 978	−0·002 716	**1·10**	0·667 315	−0·010 052
0·68	0·766 262	−0·002 697	**1·15**	0·657 263	−0·009 736
0·69	0·763 565		**1·20**	0·647 527	
		−0·002 679			−0·009 432
0·70	0·760 886	−0·002 661	**1·25**	0·638 095	−0·009 145
0·71	0·758 225	−0·002 643	**1·30**	0·628 950	−0·008 864
0·72	0·755 582	−0·002 624	**1·35**	0·620 086	−0·008 599
0·73	0·752 958	−0·002 607	**1·40**	0·611 487	−0·008 343
0·74	0·750 351		**1·45**	0·603 144	
		−0·002 589			−0·008 099
0·75	0·747 762	−0·092 571	**1·50**	0·595 045	−0·007 863
0·76	0·745 191	−0·002 554	**1·55**	0·587 182	−0·007 639
0·77	0·742 637	−0·002 537	**1·60**	0·579 543	−0·007 424
0·78	0·740 100	−0·002 519	**1·65**	0·572 119	−0·007 216
0·79	0·737 581		**1·70**	0·564 903	
		−0·002 502			−0·007 018
0·80	0·735 079	−0·002 486	**1·75**	0·557 885	−0·006 828
0·81	0·732 593	−0·002 467	**1·80**	0·551 057	−0·006 644
0·82	0·730 126	−0·002 451	**1·85**	0·544 413	−0·006 468
0·83	0·727 675	−0·002 435	**1·90**	0·537 945	−0·006 298
0·84	0·725 240		**1·95**	0·531 647	
		−0·002 419			−0·006 137
0·85	0·722 821	−0·002 402	**2·00**	0·525 510	−0·011 809
0·86	0·720 419	−0·002 386	**2·10**	0·513 701	−0·011 229
0·87	0·718 033	−0·002 370	**2·20**	0·502 472	−0·010 690
0·88	0·715 663	−0·002 355	**2·30**	0·491 782	−0·010 191
0·89	0·713 308		**2·40**	0·481 591	
		−0·002 339			−0·009 726
0·90	0·710 969	−0·002 322	**2·50**	0·471 865	−0·009 292
0·91	0·708 647	−0·002 308	**2·60**	0·462 573	−0·008 887
0·92	0·706 339	−0·002 292	**2·70**	0·453 686	−0·008 509
0·93	0·704 047	−0·002 277	**2·80**	0·445 177	−0·008 154
0·94	0·701 770		**2·90**	0·437 023	
		−0·002 261			−0·007 824

Diameter / Length	$\mathfrak{Q}$	Δ_1	Diameter / Length	$\mathfrak{Q}$	Δ_1
3·00	0·429 199	−0·007 512	**4·50**	0·340 898	−0·004 467
3·10	0·421 687	−0·007 219	**4·60**	0·336 431	−0·004 333
3·20	0·414 468	−0·006 944	**4·70**	0·332 098	−0·004 208
3·30	0·407 524	−0·006 684	**4·80**	0·327 890	−0·004 090
3·40	0·400 840	−0·006 439	**4·90**	0·323 800	−0·003 975
3·50	0·394 401	−0·006 209	**5·00**	0·319 825	−0·018 323
3·60	0·388 192	−0·005 989	**5·50**	0·301 502	−0·016 096
3·70	0·382 203	−0·005 782	**6·00**	0·285 406	−0·014 262
3·80	0·376 421	−0·005 587	**6·50**	0·271 144	−0·012 737
3·90	0·370 834	−0·005 401	**7·00**	0·258 407	−0·011 425
4·00	0·365 438	−0·005 227	**7·50**	0·246 982	−0·010 401
4·10	0·360 206	−0·005 059	**8·00**	0·236 581	−0·009 434
4·20	0·355 147	−0·004 898	**8·50**	0·227 147	−0·008 619
4·30	0·350 249	−0·004 746	**9·00**	0·218 528	−0·007 911
4·40	0·345 503	−0·004 605	**9·50**	0·210 617	−0·007 302
			10·00	0·203 315	

§ 25. The formulæ above deduced do not apply to solenoids wound with wires or strips, inasmuch as the calculation is based on the supposition that the solenoid forms a cylindrical current sheet. In the practical problem, we have to take into account the thickness of the wire and of the insulation, for which it will be necessary to add small corrections to the numbers which are obtained from the tables already given. Important formulæ as well as tables to meet such practical problems have been calculated by Rosa[(1)] and Cohen[(2)].

(1) Rosa, Bull. Bur. Standards, 2, 161, 1906; 4, 369, 1908.
(2) Cohen, Bull. Bur. Standards, 4, 183, 1908.