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Theory of the Rainbow due to a Circular Source of light.

By

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With 5 plates.

1. Introduction.

From the time of Descartes, many theories of the rainbow have been proposed, but Airy[1] was the first to establish a satisfactory theory on the undulatory theory of light. Airy's equation of the meridian section of the emergent wave-surface immediately after leaving the surface of the raindrop ($y=ax^3$), was, however, formed with reference to the tangent and normal to the curve at the point of inflexion, based on geometrical optics. This point in Airy's theory was afterwards developed in detail by Boitel,[2] Larmor,[3] and especially by Mascart[4] and L. Lorenz.[5] But all the above investigations were based on the assumption of a point source of light, and for the actual case

(1) Trans. Camb. Phil. Soc. VI. p. 379 (1838); VIII. p. 595 (1848).
(2) Compt. Rend. May 28, 1888; Phil. Mag. XXVI. p. 239 (1888).
(3) Proc. Camb. Phil. Soc. VI. p. 283 (1888).
(4) Traité d'Optique, I. p. 382 (1889); III. p. 430 (1893).
(5) Œuvres Scientifiques, I. p. 405 (Copenhagen, 1898).

of the rainbow which is due to a source of finite dimensions, the result holds only approximately. Recently, the colours of the rainbow have been minutely investigated by Pernter[1] with the use of Maxwell's theory of compound colours. In that paper Pernter also calculated the colours of the rainbow as due to a circular source of light, by a numerical addition of the results, due to seven point-sources in a straight line, each differing by 5′. This method of calculation is not exact, and the result only holds as a rough approximation (see ante).

It is to be remarked that Pernter's values[2] of Airy's integral $f^2(z)$, on which the whole calculation is based, are sometimes discrepant from those originally given by Airy. On comparing them with Airy's values, we found three mistakes at $z=1.8$, 2.2 and 3.6, and Pernter himself, in his second paper,[3] remarked that these mistakes came from Mascart's table and that they did not affect the final result of his calculation. For $z>8$, on drawing the curve representing Pernter's values, we found considerable irregularity. It seemed therefore advisable to repeat the calculation, using Stokes's semiconvergent series (see ante). The results of our calculations were always greater than Pernter's values, excepting the maxima and minima values. Some numerical examples are given in the following table:

z	f^2 : Pernter's	f^2 : our's
8.8	0.189	0.223
9.4	.100	.125
10.0	.240	.268
10.6	.022	.033
11.0	.170	.189

(1) Wien Sitz. Ber. CVI. 2a, p. 135 (1897); Neues über den Regenbogen (Wien, 1898).
(2) Loc. cit., p. 140.
(3) Wien. Sitz. Ber. CXIV. 2a, p. 1 (1905).

In our calculation we did not take the same numerical values for z that Pernter took. Thus in our values given above which were found by interpolation, there are probably small errors in the last figures.

So far as we are aware, the various calculations are as yet limited to cases which, strictly speaking, hold only for a point source of light. These considerations have led us to undertake the following investigation. It may therefore, be looked upon as an extension of Airy's theory to the case in which the source of light is circular, namely, when the apparent diameter of the sun is taken into account.

Next, let us consider the experimental side. Miller(1) and Pulfrich(2) verified Airy's theory in the special case of two dimensions with a cylindrical stream of water (or glass rod), and a straight slit as the source of light. But a question suggests itself in connexion with the problem of the circular source of light—if we take account of the breadth of the slit, assuming its length to be infinite, what difference will occur? This question must be answered.

In the following, we shall start by briefly stating Airy's theory, and then proceeding to find differences when the source of light is replaced by a small circular disk; and after some additional notes on the two-dimensional case, the colours of the rainbow due to the sun are calculated in two cases, which may be taken as illustrations of the difference between the point and the circular source; and lastly experimental results will be discussed.

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(1) Trans. Camb. Phil. Soc. VII. p. 277 (1841).
(2) Wied. Ann. XXXIII. p. 194 (1888).

2. Airy's theory.

It will be necessary, in the first place, to state Airy's theory in a form convenient for use in subsequent investigations. First, let us neglect the visual angle of the drop, i.e. the radius of the drop compared with the distance of the observer from the drop (in the case of table experiment, we have to consider the observer's distance as infinity, the telescope being so focussed). Describe a unit sphere having the centre c coinciding with that of the drop, and let the points o and s on the sphere be the directions of observer and point-source of light respectively, seen from c, and cm be the direction of the ray of minimum deviation in the plane sco. The position of the observer with respect to the sun is specified by the angle sco, or by the angle mco.

Put

$$\theta = \angle mco = D - \angle sco \qquad (1)$$

where $D = \pi$ − angle of minimum deviation,

r = radius of the drop,

n = index of refraction,

$p - 1$ = number of internal reflections,

and $$h = \frac{(p^2-1)^2}{p^2(n^2-1)}\sqrt{\frac{p^2-n^2}{n^2-1}} .$$

Then the emergent wave-surface, being the surface of rotation with the axis sc, is specified by the curve of the intersection with the plane sco. Taking the coordinate origin at c, y−axis in cm and x−axis perpendicular to it, we have the equation of the curve

$$y = -\frac{h}{3a^2}x^3 , \qquad (2)$$

when we confine our attention to small values of θ; and then the intensity of light in the direction o is given by

$$i(\theta) = \text{const. } A f^2(\varkappa\theta) \qquad (3)$$

where

$$A = \left(\frac{r^7}{h^2\lambda}\right)^{\frac{1}{3}}, \qquad (3_a)$$

$$\varkappa = 2\left(\frac{6}{h}\right)^{\frac{1}{3}}\left(\frac{r}{\lambda}\right)^{\frac{2}{3}}, \qquad (3_b)$$

$$f(\varkappa\theta) = \int_0^\infty \cos\frac{\pi}{2}(u^3 - \varkappa\theta u)du. \qquad (3_c)$$

But, if we do not neglect the visual angle of the drop, the definition of θ must be slightly changed. In this case the ray of minimum deviation does not pass c, but meets the surface of the drop at a point say c'. Thus c' must be taken as the coordinate origin and $c'o'$ the direction of the observer; then θ is defined by

$$\theta = \angle o'c'm. \qquad (1')$$

Using this value of θ, and neglecting $r\theta$ compared with the observer's distance, we may state the same formula as the above. For different wave lengths of light, the point c' is slightly displaced, but the amount of the displacement being negligibly small, we may take one position of c' as the coordinate origin for all the wave lengths of a visible ray.

Airy expanded $f(\varkappa\theta)$ as a power series of $\varkappa\theta$, which is not convenient for a practical calculation of values for $\varkappa\theta > 3$, though it always remains convergent. On the other hand, especially for large values of θ, the following semiconvergent series taken from Stokes(1) can be employed with advantage:—

$$f(\varkappa\theta) = 2^{\frac{1}{2}}3^{-\frac{1}{4}}(\varkappa\theta)^{-\frac{1}{4}}\, M \cos\left(\mu - \frac{\pi}{4} - \delta\right),$$

where

$$\mu = \pi\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}},$$

(1) Collected Papers, II. p. 329 (London, 1883).

$$M=1-0.0347\mu^{-2}, \qquad \tan\delta=0.0694\mu^{-1};$$

or, approximately,

$$f(\varkappa\theta)=2^{\frac{1}{2}}3^{-\frac{1}{4}}(\varkappa\theta)^{-\frac{1}{4}}\cos\pi\left\{\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}}-\frac{1}{4}\right\}$$

It must be remarked that both expansions only represent $f(\varkappa\theta)$ for $\theta>0$. But for $\theta<0$, $f(\varkappa\theta)$ being a function having no characteristic property, it is at once seen that no important difference appears between a point and a circular source. In the following discussion the places where $\theta<0$ are therefore excluded.

3. Remarks on Airy's theory.

In his paper, Airy confined the value of θ within 2°, but he did not exactly determine the limit of approximation in his theory. Also he did not discuss the dependency of the intensity of the rainbow on the wave length of light. Pernter applied[1] Airy's theory for $\theta=16°$ not only to determine the positions of maxima and minima, but also to calculate the amount of intensity, and he said[2] that θ might be 20° or 30°. Mascart took[3] as the value of A of (3_a) at first

$$A=\left(\frac{r^4\lambda^2}{\cos^2\theta h^2}\right)^{\frac{1}{3}},$$

and then modified[4] it to the form of (3_a) taking account of dimension.

We shall have to examine these points in detail, before proceeding to our discussion. First, in the equation (2), there was neglected the term of x^4 compared with x^3. Differentiate (2) with respect of x and

(1) Wien. Sitz. Ber. CVI. 2a, Tab. I, II (1897).
(2) „ „ „ CXIV. 2a, p. 6 (1905).
(3) Traité d'Optiques I. p. 394 (Copenhagen, 1889).
(4) „ „ III. p. 437 („ 1893).

$$\frac{dy}{dx} = -\frac{h}{a^2}x^2 ,$$

put
$$\frac{dy}{dx} = \tan \varepsilon ,$$

then ε, being the angle between the tangent to the wave-front and the x-axis or the angle between the wave normal and the y-axis, is of the order of θ. It follows that x is of the order $\theta^{\frac{1}{2}}$. This shows that in (2) $\theta^{\frac{3}{2}}$ was neglected as compared with θ.

Secondly, the phase difference at a point x, y, z on the wave-surface is easily calculated from the equation (2), the z-axis being perpendicular to the x and y axes. Represent the position of the observer by ξ, η, o, then the phase difference is

$$\delta = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2} - \sqrt{\xi^2 + \eta^2} ,$$

or
$$\delta = \delta_1 + \delta_2 ,$$

where
$$\delta_1 = \frac{h}{3a^2}x^3 - \frac{\xi}{\eta}x , \qquad \delta_2 = \frac{z^2}{2\eta} ,$$

x, z, ξ, being small compared with η. Then the intensity, being proportional to the square of the amplitude, is given by

$$i(\theta) = V_c^2 + V_s^2 ,$$

where
$$V_c = \frac{1}{\lambda\eta}\int a \cos\frac{2\pi}{\lambda}\delta \, d\sigma , \qquad V_s = \frac{1}{\lambda\eta}\int a \sin\frac{2\pi}{\lambda}\delta \, d\sigma$$

$d\sigma$ being the surface element of the wave surface, and a the amplitude of the wave. In the case of a spherical drop, we have to put

$$a = cr \sin \mathrm{I} ,$$

where I is the angle of incidence of the ray which has passed through $d\sigma$, and c depends on I, n, p, representing the effect of polarisation

or
$$a = cr \sin (\mathrm{I}_0 + \gamma)$$

where I_0 is the angle of incidence of a ray of minimum deviation and $\gamma = \mathrm{I} - \mathrm{I}_0$. But in the position of minimum deviation

$$\frac{d\varepsilon}{d\gamma}=0 \ , \qquad \frac{d^2\varepsilon}{d\gamma^2}=\text{finite},$$

i.e. ε is of the order γ^2, or γ is of the order of $\theta^{\frac{1}{2}}$.
Thus, if we again neglect $\theta^{\frac{1}{2}}$ as compared with unity, i.e. $\theta^{\frac{3}{2}}$ compared with θ, we have

$$a = \text{const.}\ r$$

In the case of the circular cylinder, we have simply

$$a = \text{const.}$$

Thus, putting $d\sigma = dx\,dz$ and taking the limit of the integration from $-\infty$ to $+\infty$

$$V_c=\frac{\text{const.}}{\lambda\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\frac{2\pi}{\lambda}\delta\,dx\,dz, \qquad V_s=\frac{\text{const.}}{\lambda\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin\frac{2\pi}{\lambda}\delta\,dx\,dz$$

where
$$\cos\frac{2\pi}{\lambda}\delta=\cos\frac{2\pi}{\lambda}(\delta_1+\delta_2)$$
$$=\cos\frac{2\pi}{\lambda}\delta_1\cos\frac{2\pi}{\lambda}\delta_2-\sin\frac{2\pi}{\lambda}\delta_1\sin\frac{2\pi}{\lambda}\delta_2 \ .$$

Therefore

$$V_c=\frac{\text{const.}}{\lambda\eta}\int_{-\infty}^{+\infty}\cos\frac{2\pi}{\lambda}\delta_1\,dx\int_{-\infty}^{+\infty}\cos\frac{2\pi}{\lambda}\delta_2 dz \ .$$

In the first part of these integrals put

$$x=\left(\frac{6\lambda a^2}{h}\right)^{\frac{1}{3}}u \ , \qquad 2\left(\frac{6a^2}{\lambda^2 h}\right)^{\frac{1}{3}}=\varkappa \ ,$$

then
$$\int_{-\infty}^{+\infty}\cos\frac{2\pi}{\lambda}\delta_1\,dx=\left(\frac{6\lambda r^2}{h}\right)^{\frac{1}{3}}\int_0^{\infty}\cos\frac{\pi}{2}(u^3-\varkappa\theta u)du \ ,$$

and in the second part $\qquad z^2=\dfrac{\lambda\eta}{2}v^2$

then
$$\int_{-\infty}^{+\infty}\cos\frac{2\pi}{\lambda}\delta_2 dz=\sqrt{\frac{\lambda\eta}{2}}\int_{-\infty}^{+\infty}\cos\frac{\pi}{2}v^2 dv=\sqrt{\frac{\lambda\eta}{2}} \ .$$

Hence
$$V_c=\frac{\text{const.}}{\sqrt{\lambda}}\left(\frac{\lambda r^2}{h}\right)^{\frac{1}{3}}\int_0^{\infty}\cos\frac{\pi}{2}(u^3-\varkappa\theta u)du \ ,$$

and similarly for V_s

$$V_s = \frac{\text{const.}}{\lambda\eta} \int_{-\infty}^{+\infty} \cos \frac{2\pi}{\lambda}\delta_1 \, dx \int_{-\infty}^{+\infty} \sin \frac{2\pi}{\lambda}\delta_2 \, dz = V_c \ . \ .$$

Thus, for a spherical drop

$$i(\theta) = \text{const.}\left(\frac{r^7}{\lambda h^2}\right)^{\frac{1}{3}} f^2(\varkappa\theta)$$

and for a circular cylinder

$$i(\theta) = \text{const.}\left(\frac{r^4}{\lambda h^2}\right)^{\frac{1}{3}} f^2(\varkappa\theta) \ .$$

Of course, if we leave λ out of consideration, we may take as the expression for V_c and V_s

$$V_c = \text{const.} \int_{-\infty}^{+\infty} \cos \frac{2\pi}{\lambda}\delta_1 dx \ , \quad V_s = \text{const.} \int_{-\infty}^{+\infty} \sin \frac{2\pi}{\lambda}\delta_1 dx = 0 \ ,$$

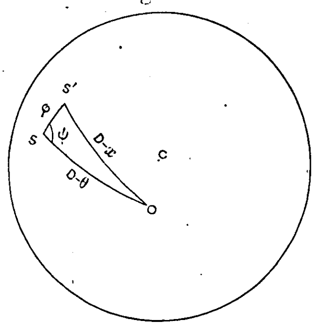
as Airy, δ_2 being small compared with δ_1.

Airy's theory holds good only when $\theta^{\frac{1}{2}}$ is negligibly small, in other cases the theory must be essentially modified. If γ is not negligible, then we have to take into account the dependency of the intensity of light on the angles of reflections and refractions, namely the effect of polarization as indicated by Mascart and Lorenz; on the other hand the form of the wave front must be modified by adding a term of x^4. Thus some of Pernter's calculation for large values of θ can not be regarded as exact.

4. Extension of Airy's theory to that of a circular source.

Passing now to the case of a circular source of uniform intensity, the apparent diameter of the source 2Φ will be supposed so small that we can neglect Φ^2 compared with Φ, and confine our attention to the neighborhood of the minimum deviation, so that Airy's theory applies. It is most convenient, in this case, to neglect the visual angle of the drop. Take the elementary area of the projection s' of the source on the unit sphere, s being that of the centre of the circular source, and denote the angle between $s's$ and so by ψ, the angular distance between s' and s by φ, the angle sco by $D-\theta$ and the angle $s'co$ by $D-x$. Then, in the spherical triangle $ss'o$, we have the relation

Fig. 1.



$$\cos(D-x)=\cos\varphi\cos(D-\theta)+\sin\varphi\sin(D-\theta)\cos\psi\ ,$$

which reduces to the form

$$x=\theta+\varphi\cos\psi,$$

since x, θ, φ are small.

The intensity of light at o due to the elementary area s' which is equal to $\varphi\, d\varphi\, d\psi$, is expressed by

$$i_{\varphi\, d\varphi\, d\psi}(\theta)=\text{const. } Af^2(\varkappa x)$$

$$=\text{const.}\left(\frac{r^7}{h^2\lambda}\right)^{\frac{1}{3}} f^2\left\{\varkappa(\theta+\varphi\cos\psi)\right\}\ ;$$

from which it follows at once as the expression for the total intensity in the direction co, that

$$I(\theta,\Phi)=\text{const. } AF(\varkappa,\Phi,\varkappa\theta) \qquad (4)$$

where
$$F(\varkappa,\Phi,\varkappa\theta)=\frac{1}{\pi\Phi^2}\int_0^{\Phi}\int_0^{2\pi}\varphi\, d\varphi\, d\psi\, f^2\left\{\varkappa(\theta+\varphi\cos\psi)\right\}\ , \qquad (4_a)$$

$$A=\left(\frac{r^7}{h^2\lambda}\right)^{\frac{1}{3}}\ ,$$

$$f^2\{\varkappa(\theta+\varphi\cos\psi)\}=\int_0^{\infty}\cos\frac{\pi}{2}\left\{u^3-\varkappa(\theta+\varphi\cos\psi)u\right\}du\ .$$

Thus the function f^2 in Airy's theory is replaced by a more general function F. From the form of the function f^2

and F, the difference between a point and a circular source is to be found. $f^2(\varkappa\theta)$ does not change for different values of $\varkappa$, if we reduce the scale of θ properly, because it is a function of $\varkappa\theta$, but not a function of $\varkappa$ and θ taken separately; this however, does not hold for F, which is a function of $\varkappa$ as well as of $\varkappa\theta$. Thus it is necessary to consider F more in detail, though its evaluation as a function of $\varkappa$, Φ, and $\varkappa\theta$ is by no means easy.

If we try to expand $f^2\left\{\varkappa(\theta+\varphi\cos\psi)\right\}$ in a power series of $\varkappa\varphi\cos\psi$, then its coefficients gradually increase with $\varkappa\theta$, and are very inconvenient for values of $\varkappa\theta>1$. If we change $f^2\left\{\varkappa(\theta+\varphi\cos\psi)\right\}$ to a double integral

$$f^2\left\{\varkappa(\theta+\varphi\cos\psi)\right\}$$

$$=\int_0^\infty \cos\frac{\pi}{2}\left\{x^3-\varkappa(\theta+\varphi\cos\psi)x\right\}dx\int_0^\infty \cos\frac{\pi}{2}\left\{y^3-\varkappa(\theta+\varphi\cos\psi)y\right\}dy$$

$$=\frac{1}{2}\int_0^\infty dx\int_0^\infty dy\Big[\cos\frac{\pi}{2}\left\{x^3+y^3-\varkappa\theta(x+y)\right\}\cos\frac{\pi}{2}\left\{\varkappa\varphi\cos\psi(x+y)\right\}$$

$$-\sin\frac{\pi}{2}\left\{x^3+y^3-\varkappa\theta(x+y)\right\}\sin\frac{\pi}{2}\left\{\varkappa\varphi\cos\psi(x+y)\right\}$$

$$+\cos\frac{\pi}{2}\left\{x^3-y^3-\varkappa\theta(x-y)\right\}\cos\frac{\pi}{2}\left\{\varkappa\varphi\cos\psi(x-y)\right\}$$

$$-\sin\frac{\pi}{2}\left\{x^3-y^3-\varkappa\theta(x-y)\right\}\sin\frac{\pi}{2}\left\{\varkappa\varphi\cos\psi(x-y)\right\}\Big],$$

and integrate with respect to φ and ψ, by using the relations of Bessel's function

$$J_0(w)=\int_0^\pi \cos(w\cos\mu)d\mu\ ,$$

$$o=\int_0^\pi \sin(w\cos\mu)d\mu\ ,$$

and

$$\frac{d}{dw}\left\{w^{\frac{1}{2}}J_1(\sqrt{w})\right\}=\frac{1}{2}J_0(\sqrt{w})\ ,$$

then the final form is

$$\int_0^\infty\int_0^\infty dxdy\Big[\cos\frac{\pi}{2}\Big\{x^3+y^3-\varkappa\theta(x+y)\Big\}\frac{J_1\Big\{\frac{\pi}{2}\varkappa\Phi(x+y)\Big\}}{\frac{\pi}{2}\varkappa\Phi(x+y)}$$

$$+\cos\frac{\pi}{2}\Big\{x^3-y^3-\varkappa\theta(x-y)\Big\}\frac{J_1\Big\{\frac{\pi}{2}\varkappa\Phi(x-y)\Big\}}{\frac{\pi}{2}\varkappa\Phi(x-y)}\Big].$$

Putting $\quad x=x'\left(\frac{2}{\pi}\right)^{\frac{1}{3}} \quad y=y'\left(\frac{2}{\pi}\right)^{\frac{1}{3}} \quad \varkappa=\varkappa'\left(\frac{2}{\pi}\right)^{\frac{2}{3}}$

and $\quad y'=-y' \quad$ in the latter half,

then applying the well-known sequence equation of the Bessel's function

$$\frac{J_1(w)}{w}=\frac{1}{2}\Big\{J_2(w)+J_0(w)\Big\},$$

this can be reduced to the form

$$\int_0^\infty\int_{-\infty}^{+\infty}dx'dy'\cos\Big\{x'^3+y'^3-\varkappa'\theta(x'+y')\Big\}\Big[J_2\Big\{\varkappa'(x'+y')\Big\}+J_0\Big\{\varkappa'(x'+y')\Big\}\Big],$$

which is almost intractable for practical calculation.

If we transform the variables φ, ψ to x, y which are given by $x=\varphi\cos\psi$, $y=\varphi\sin\psi$, and then integrate with respect to y, we arrive at the expression

$$F(\varkappa,\Phi,\varkappa\theta)=\frac{2}{\pi\Phi^2\varkappa^2}\int_{-\varkappa\Phi}^{+\varkappa\Phi}d(\varkappa x)\sqrt{(\varkappa\varphi)^2-(\varkappa x)^2}f^2(\varkappa\theta+\varkappa x). \qquad (4')$$

This form is most advantageous for the numerical calculation of F by means of mechanical quadrature.

5. Case where the visual angle of the drop is not negligible.

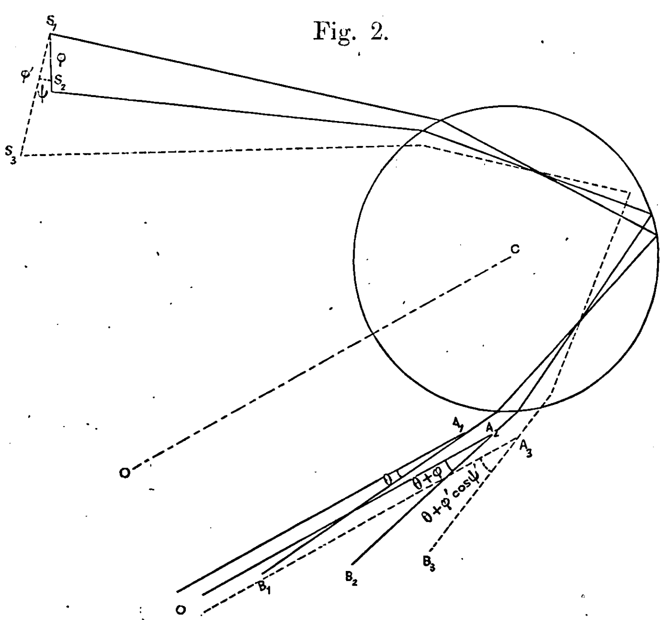
In the preceding article we neglected the visual angle of the drop, i.e. r as compared with the observers distance, but when this is not the case, the consideration is more complicated. Firstly, let us consider the case where the source of light consists of three points s_1, s_2, s_3 lying in a plane perpendicular to s_1c at a great distance. In the plane $s_1c\ o$, the intensity of the rainbow due to s_1, is

$$i_{s_1}(\theta)=c_1\, A f^2(\varkappa\theta)$$

where θ represents the angle $B_1 A_1 O$, $B_1 A_1$ being the direction of the minimum deviation due to s_1. Suppose that s_2 lies in the plane s_1co, and its angular distance φ from s_1, is small; then the intensity due to s_2 in the direction A_1O or A_2O is

$$i_{s_2}(\theta)=c_2\, A\, f^2\left\{\varkappa(\theta+\varphi)\right\}.$$

Fig. 2.



Next suppose that s_3 lies in another plane s_3co, and its position is represented by φ' and ψ, where φ' is the small angular distance of s_3 from s_1 and ψ the angle subtended by $s_1\, s_2$ and $s_1\, s_3$. Now then, in the plane s_3co there is no direction parallel to

A_1O, but there exists a direction which makes the angle θ with $A_1 B_1$, and the intensity in this direction is given by

$$i_{s_3}(\theta) = c_3 A f^2 \left\{ \varkappa(\theta + \varphi' \cos\psi) \right\} .$$

Thus, in the plane s_1co we have as the total intensity

$$i(\theta) = i_{s_1}(\theta) + i_{s_2}(\theta) ,$$

and in the plane s_3co

$$i'(\theta) = i_{s_3}(\theta) .$$

In this case, therefore though $i(\theta)$ and $i'(\theta)$ do not exist in one direction nor in one plane, they exist in the same arc of the rainbow which is specified by θ. Hence, the distribution of the intensity of the rainbow can not be considered as uniform along the arc of the rainbow.

The above method is directly applicable to the case of a circular source of light. To determine the position of a point in the source of light, take the centre s_0 of the source as the origin from which φ is measured, the diameter D_0 of the source which lies in the plane s_0co as the axis from which ψ is measured, and the direction of the minimum deviation due to s_0 as the y-axis from which θ is measured. Then, in the plane s_0co, the intensity of the rainbow is given by

$$I(\theta) = \text{const. } A \int_{-\Phi}^{+\Phi} f^2 \left\{ \varkappa(\theta + \varphi) \right\} d\varphi , \qquad (a)$$

Φ being the angular radius of the circle. Pernter took this integral as the general expression of the intensity of the rainbow due to a circular source of light and condemned our result, but this holds only in the plane s_0co. Let us consider another plane which cuts the source in another line L and contains c and o.

Then, the line L does not pass the centre of the source, but is approximately parallel to D_0. Representing the angular distance of L from D_0 by y, and the angular length measured from the middle point of L along L by x, we have as the intensity due to L in the direction θ in the plane Lco,

$$I(\theta) = \text{const. } A \int_{-\sqrt{\Phi^2 - y^2}}^{+\sqrt{\Phi^2 - y^2}} f^2\left\{\varkappa(\theta + x)\right\} dx \quad . \qquad (b)$$

This value of $I(\theta)$ being a function of y, holds for any plane which contains a part of the source and c, o. For a particular value $y = o$, (b) reduces to (a), and when $y = \Phi$, $I(\theta)$ becomes zero.

The above discussion shows that, in general, when both the angular diameters of the source and of the drop are not negligible, the intensity of the rainbow can not be considered as uniform along the arc of the rainbow; and exact investigation is almost impossible unless we are informed of the distribution of the drops. Again, if we consider the drop so large that the distance of the lines A_1O, A_2O in Fig. (2) is greater than the pupil of the observer's eye, the result must be considerably changed.

If we take the mean value of intensity $I(\theta)$ along the arc of the rainbow which is specified by θ as

$$I_m(\theta) = \text{const. } A \int_0^{\Phi} \int_{-\sqrt{\Phi^2 - y^2}}^{+\sqrt{\Phi^2 - y^2}} f^2\left\{\varkappa(\theta + x)\right\} dy\, dx \ ,$$

changing the order of integration, we have

$$I_m(\theta) = \text{const. } A \int_0^{\Phi} dx \sqrt{\Phi^2 - x^2}\, f^2\left\{\varkappa(\theta + x)\right\}$$

or
$$I_m(\theta)=\text{const. } A\int_{-\varkappa\Phi}^{+\varkappa\Phi} d(\varkappa x) f^2\left\{\varkappa(\theta+x)\right\}\sqrt{(\varkappa\Phi)^2-(\varkappa x)^2}\ .$$

This coincides with (4′), i.e. the result when we neglect the angular diameter of the drop.

6. General nature of the intensity curve.

It will be advantageous to consider first the maxima and minima of F as compared with those of f^2, and then to discuss the general character, and finally proceed to the numerical calculation of F. In the expression (4′) in § 4 put $\varkappa x=z$, then

$$F(\varkappa,\Phi,\varkappa\theta)=\frac{2}{\pi\Phi^2\varkappa^2}\int_{-\varkappa\Phi}^{+\varkappa\Phi} dz\sqrt{(\varkappa\varphi)^2-z^2}\,f^2(\varkappa\theta+z)\ .$$

Thus the maxima and minima of F are given by

$$\int_{-\varkappa\Phi}^{+\varkappa\Phi} dz\sqrt{(\varkappa\varphi)^2-z^2}\,\frac{\partial}{\partial\theta}\left\{f^2(\varkappa\theta+z)\right\}=0\ ;$$

$$\int_{-\varkappa\Phi}^{+\varkappa\Phi} dz\sqrt{(\varkappa\varphi)^2-z^2}\,\frac{\partial}{\partial z}\left\{f^2(\varkappa\theta+z)\right\}=0\ .$$

Or, putting the mean value of $\sqrt{(\varkappa\varphi)^2-z^2}$ in the integral, we arrive at the approximate relation

$$f^2(\varkappa\theta-\varkappa\Phi)=f^2(\varkappa\theta+\varkappa\Phi)\ .$$

For smaller values of θ, especially at the first maximum, f^2 has no symmetry on both sides of the maxima and minima (see Pl. I.); so that the first maximum of F receives a small displacement towards $\theta=0$ as compared with f^2. This displacement becomes smaller and smaller for other maxima and minima. For larger values of θ, as f^2 is nearly symmetrical on both sides, the maxima and minima approximately coincide with those of

f^2; nevertheless it does not follow that the maxima for f^2 always remain as maxima for F.

For the consideration of the general character of F, we shall begin with that of f^2. By Stokes's expression

$$f^2(\varkappa\theta)=\frac{2}{\sqrt{3}}\,\frac{1}{\sqrt{\varkappa\theta}}\cos^2\pi\left\{\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}}-\frac{1}{4}\right\}=\frac{1}{\sqrt{3}\sqrt{\varkappa\theta}}-\frac{\sin 2\pi\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}}}{\sqrt{3}\sqrt{\varkappa\theta}}$$

f^2 is composed of two terms: the mean term $\frac{1}{\sqrt{3}\sqrt{\varkappa\theta}}$, and the oscillating term, whose amplitude is limited by the same numerical factor $\frac{1}{\sqrt{3}\sqrt{\varkappa\theta}}$; therefore at maxima f^2 increases to $2\times$ (mean term) and at minima decreases to zero. But the character of F is slightly different, since

$$F(\varkappa,\Phi,\varkappa\theta)=\frac{2}{\pi\Phi^2\varkappa^2}\int_{-\varkappa\Phi}^{+\varkappa\Phi}dz\sqrt{(\varkappa\varphi)^2-z^2}\left[\frac{1}{\sqrt{3}\sqrt{\varkappa\theta+z}}-\frac{\sin 2\pi\left(\frac{\varkappa\theta+z}{3}\right)^{\frac{3}{2}}}{\sqrt{3}\sqrt{\varkappa\theta+z}}\right].$$

The first term is equal to

$$\frac{2}{\pi\Phi^2\varkappa^2}\int_{-\varkappa\Phi}^{+\varkappa\Phi}dz\sqrt{(\varkappa\varphi)^2-z^2}\;\frac{1}{\sqrt{3}\sqrt{\varkappa\theta}}\left(1-\frac{z}{2\varkappa\theta}+\ \ldots\right),$$

and leads to the same mean term so far as the first order of $\frac{z}{\varkappa\theta}$ is concerned. But the second term, after putting $\frac{\pi\varkappa\Phi}{4}\frac{1}{\varkappa\theta}$ as the mean value of $\frac{\sqrt{(\varkappa\varphi)^2-z^2}}{\varkappa\theta+z}$, and integrating with respect to a new variable $2\pi\left(\frac{\varkappa\theta+z}{3}\right)^{\frac{3}{2}}$, becomes

$$\frac{\cos 2\pi\left(\frac{\varkappa\theta-\varkappa\Phi}{3}\right)^{\frac{3}{2}}-\cos 2\pi\left(\frac{\varkappa\theta+\varkappa\Phi}{3}\right)^{\frac{3}{2}}}{6\pi\varkappa\Phi\varkappa\theta}\doteqdot\frac{\sin 2\pi\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}}\sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})}{3\pi\varkappa\Phi\varkappa\theta}.$$

Thus the amplitude of the oscillating term is limited by $\frac{\sin(\sqrt{3}\pi\varkappa\Phi\sqrt{\varkappa\theta})}{3\pi\varkappa\Phi\varkappa\theta}$ which is much smaller than $\frac{1}{\sqrt{3}\sqrt{\varkappa\theta}}$; and, moreover, this may be positive or negative according to the sign of $\sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})$. For the smaller value of Φ, $\frac{1}{3\pi\varkappa\Phi\varkappa\theta}$ becomes larger and the period of $\sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})$ is prolonged; in the limiting case $\Phi=0$, $\dfrac{\sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})}{3\pi\varkappa\Phi\varkappa\theta}$ becomes equal to $\dfrac{1}{\sqrt{3}\sqrt{\varkappa\theta}}$, as expected. Also for smaller values of $\varkappa$, i.e. of r, the same reasoning will hold true. Thus for large values of Φ and r, the difference of the two cases becomes manifest.

From this approximate expression,

$$F \doteq \frac{1}{\sqrt{3}\sqrt{\varkappa\theta}} - \frac{\sin 2\pi\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}} \sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})}{3\pi\varkappa\Phi\varkappa\theta}\,.$$

it follows that F does not increase at maxima to $2\times$(mean term), but only to $\left(\text{mean term}+\frac{1}{3\pi\varkappa\Phi\varkappa\theta}\right)$; and at minima it does not diminish to zero, but only to $\left(\text{mean term}-\frac{1}{3\pi\varkappa\Phi\varkappa\theta}\right)$; moreover, for values of θ for which $\sin(\sqrt{3}\,\pi\varkappa\Phi\sqrt{\varkappa\theta})<0$, the maxima of $\sin 2\pi\left(\frac{\varkappa\theta}{3}\right)^{\frac{3}{2}}$ changes to minima and the minima to maxima. Finally, the expression for the intensity being

$$i(\theta)=\text{const.}\left(\frac{r^7}{h^2\lambda}\right)^{\frac{1}{3}} f^2(\varkappa\theta) \qquad \text{for point source,}$$

$$I(\theta,\Phi)=\text{const.}\left(\frac{r^7}{h^2\lambda}\right)^{\frac{1}{3}} F(\varkappa,\Phi,\varkappa\theta) \qquad \text{for circular source;}$$

it follows, first, that for larger values of Φ, the difference of $i(\theta)$ and $I(\theta,\Phi)$ becomes larger; secondly, that for larger values of r, the difference of maximum and minimum values of $i(\theta)$ becomes larger in virtue of $r^{\frac{7}{3}}$, but for $I(\theta,\Phi)$ at the same time it is diminished by the presence of F.

7. Case of the cylinder and slit.

We shall now treat the case which has often been tested by experiment with the glass rod, and straight slit as the source of light. In this case, if we neglect the breadth of the slit, Airy's theory applies as well as in the case of the spherical drop, when we change the value of A

$$i(\theta)=\text{const. } A' f^2(\varkappa\theta)$$

where $$A'=\left(\frac{r^4}{h^2\lambda}\right)^{\frac{1}{3}}.$$

If we take into account the breadth of the slit, there is no difficulty in applying reasoning similar to the above, to arrive at the expression

$$I(\theta, \Phi)=\text{const.}\left(\frac{r^4}{h^2\lambda}\right)^{\frac{1}{3}} F_1(\varkappa, \Phi, \varkappa\theta)$$

$$F_1(\varkappa, \Phi, \varkappa\theta)=\int_{-\varkappa\Phi}^{\varkappa\Phi} f^2(\varkappa\theta+z)dz$$

where $2\Phi=$ the angular breadth of the slit as viewed from the centre of the glass rod. Or putting $z=\varkappa\varphi$,

$$F_1(\varkappa, \Phi, \varkappa\theta)=\int_{-\Phi}^{+\Phi} f^2\left\{\varkappa(\theta+\varphi)\right\} d\varphi .$$

This coincides with the integral at (*a*) on which Pernter's calculation was based. Hence, we see that Pernter's integral holds good for the case of slit and cylinder, but not for the case of circle and sphere.

The difference of the expression of A in the two cases of sphere and cylinder was not discussed by Airy and others; but the existence of the difference is evident from the geometrical theory of the rainbow, in which the intensity is proportional to r^2 in the case of the sphere and to r in the case of the cylinder.

In § 6 we always substituted the mean value of $\sqrt{(\varkappa\varphi)^2-z^2}$ before integration, so that the expression for F becomes only roughly approximate; but in the present case, there being no such term as $\sqrt{(\varkappa\varphi)^2-z^2}$, this expression for F must be taken as nearly true.

Thus the maxima and minima of F_1 are given by

$$f^2(\varkappa\theta+\varkappa\Phi)-f^2(\varkappa\theta-\varkappa\Phi)=0\ ,$$

and to determine whether they correspond to maxima or minima, we have to consider the sign of

$$\frac{\partial}{\partial\theta}\left\{f^2(\varkappa\theta+\varkappa\Phi)-f^2(\varkappa\theta-\varkappa\Phi)\right\}\ .$$

This being given by the directions of the tangents to the curve of f^2 (Pl. I) at the points $\varkappa\theta+\varkappa\Phi$, and $\varkappa\theta-\varkappa\Phi$, we see at once that when the intervals of the consecutive maxima or minima are greater than $\varkappa\Phi$, maxima of f^2 correspond to maxima of F, but when $\varkappa\Phi$ exceeds the intervals of the consecutive maxima or minima, the maxima of f^2 correspond to minima of F.

The maxima and minima not only interchange places at certain points, but the interval between the maxima and minima slightly increases, as a consequence that the first maximum being displaced towards $\theta=0$, while the higher maxima, showing the same tendency, are displaced by smaller amounts. In Pulfrich's experiment, where he takes the third maximum as standard, the first and second are displaced slightly toward $\theta=0$, and the other to the opposite side, as compared with Airy's values of f^2. This displacement may be partly due to the breadth of the slit.

8. Numerical calculation.

For numerical calculation we must have recourse to mechanical quadrature. The method of procedure is as follows:—Draw a circle having the centre at $\varkappa\theta$ and radius equal to $\varkappa\Phi$, then plot a curve whose ordinate is equal to the product of the value of f^2 into the corresponding ordinate of this circle; the area of the new curve divided by the area of this circle is the value of F at $\varkappa\theta$. The following table has been made according to this method.

2Φ=apparent diameter of the sun=32′

$\varkappa\theta$: (except 0) max. and mini. of Airy's value.

In Table I

F_1 : $r=0.025$ cm. $\lambda=5893\times 10^{-8}$ cm. $\varkappa=120.95$

F_2 : $r=0.05$ cm. $\lambda=5893\times 10^{-8}$ cm. $\varkappa=192.00$

TABLE I.

$\varkappa\theta$	f^2	F_1	F_2
0	0.443	0.447	0.465
1.084	1.008	0.930	0.840
2.495	0.000	0.103	0.228
3.467	0.617	0.497	0.362
4.363	0.000	0.124	0.237
5.145	0.510	0.378	0.260
5.892	0.000	0.133	0.235
6.578	0.450	0.300	0.206
7.244	0.000	0.140	0.223
7.868	0.404	0.254	0.177
8.479	0.384	0.230	0.164
9.060	0.000	0.148	0.199
10.177	0.362	0.207	0.155
10.716	0.000	0.150	0.195

The graphical representation of this table is given at the end of the paper, where f^2 is represented by the dotted, F_1 by broken and F_2 by solid lines (Pl. I).

In Table II

F_s : $r=0.025$ cm. $\lambda=6302\times10^{-8}$ cm.

F_g : „ $\lambda=5211\times10^{-8}$ cm.

F_b : „ $\lambda=4659\times10^{-8}$ cm.

TABLE II.

$x\theta$	F_s	F_g	F_b
0	0.449	0.456	0.456
1.084	938	927	912
2.495	097	121	139
3.467	492	473	451
4.363	119	148	166
5.145	380	351	333
5.892	126	150	173
6.578	306	281	265
7.244	133	154	175
7.868	250	230	216
8.499	135	155	174
9.060	232	207	196
9.630	137	159	176
10.177	209	190	176
10.716	138	162	177

In Table III

F_s : $r=0.05$ cm. $\lambda=6302\times10^{-8}$ cm.

F_g : „ $\lambda=5211\times10^{-8}$ cm.

F_b : „ $\lambda=4659\times10^{-8}$ cm.

TABLE III.

$x\theta$	F_s	F_g	F_b
0	0.482	0.484	0.484
1.084	869	837	814
2.495	215	262	290

TABLE III. (*Continued*).

$x\theta$	f_s	F_g	F_b
3.467	0.394	0.353	0.326
4.363	231	267	293
5.145	263	237	230
5.892	222	253	262
6.578	216	199	189
7.244	223	237	237
7.868	166	169	181
8.479	208	209	204
9.060	166	165	162
9.630	202	204	204
10.177	165	155	156
10.716	187	196	197

In the case of slit and cylinder, the method of mechanical quadrature is simpler; the mean value of f^2 in the interval $x\theta - x\Phi$ and $x\theta + x\Phi$ representing the value of F_1 at $x\theta$. Pernter's method of calculation which we have described in § 1 is applicable in this case only. He calculated the case where $r=0.025$ cm., and compared[1] with our result, shows that there was no great difference in both results. This fact shows that there is no great difference in the two cases, of slit-cylinder and of circle-sphere. Thus, fortunately, Pernter's method of calculation applies as a rough approximation for the case of circle-sphere, though his reasoning was not exact. It seems rather curious that in his calculation for $r=0.05$ cm., we can not find the general nature in § 7; when the interval of consecutive maxima or minima of F is less than $x\Phi$, the maxima and minima interchange.

(1) Wien Sitz. Ber. CXIV. 2a, p. 13 (1905).

9. Colours of the rainbow.

Pernter calculated the colours of the rainbow due to the sun, but his calculation was not sufficient to establish the above-mentioned results. We repeated the calculation for two cases. The theory of compound colours being the subject of much dispute, there is as yet no settled opinion. But we can admit that, excluding the physiological and psychological points of view, there are three primary colours, as Maxwell's experiment[1] shows. For the discussion of the colours of the rainbow, we may conveniently take only these three primary colours, in such a ratio as to produce white, and proceed in the manner indicated by Maxwell.

We take the primary colours

$\lambda = 6302 \times 10^{-8}$ cm.	5211×10^{-8} cm.	4659×10^{-8} cm.
(Scarlet)	(Green)	(Blue)

corresponding to Maxwell's

[24]	[46]	[64]

in the ratio 1 : 1.62 : 1.60, so as to produce white,

and	$n=1.332$	1.335	1.339
whence	D=42°22	41°80	41°21
and for $r=0.025$ cm.			
	$x=115.3$	131.6	142.9
for $r=0.05$ cm.			
	$x=183.0$	209.0	226.9

Thus we obtain the values of F given by Table II and Table III in the preceding article. The results of compound colours represented by Pl. II......Pl. V, Pl. II and Pl. IV

(1) Scientific Papers, I. p. 410 (Cambridge, 1890).

corresponding to the case of the point source, and Pl. III and Pl. V to the circular source ($2\Phi=32'$ mean angular diameter of the sun), r in Pl. II and Pl. III being 0.025 cm., and in Pl. IV, Pl. V 0.05 cm. The intensity of the scarlet ray is given by dotted; green by broken; blue by solid lines, and the sum of the three intensities, i.e. the total intensity, by curve (1), which is compounded of a portion of white and a portion of the two primary colours.

For example, in Pl. II:—

I at 41°5	consists of	27	percent.	scarlet,	48	percent.	green,	25	percent.	white,
40°5	„	14	„	blue,	14	„	scarlet,	72	„	white,
38°	„	23	„	green,	77	„	blue,	0	„	white.

In Pl. III:—

I at 41°5	consists of	24	percent.	scarlet,	43	percent.	green,	33	percent.	white,
40°5	„	21	„	blue,	9	„	scarlet,	70	„	white,
38°	„	9	„	green,	19	„	blue,	72	„	white.

where the angles correspond to $s_0co=D-\theta$ in § 4.

The above calculation shows that in the colours of the supernumerary bows due to the sun, white predominates, and we can not distinguish many numbers of the supernumerary bows. This explains the fact *that the rainbow in nature is accompanied by only a small number of supernumerary bows, while according to Airy's theory the rainbow ought to be accompanied by numerous bows.*

According to § 6, we notice that the difference between the maximum and minimum values of intensity increases with the size of the drop for a point source; but for a circular source the intensity depends on two factors, one of which enjoys the same property as for a point source, but the other produces a contrary effect. Montigny[1] says that supernumerary bows are numerous when the drops are small. This holds for the case of a circular source and supports our view, but he considers this as the result of Airy's theory, i.e. of a point source, which we cannot understand.

10. Summary.

From the above discussion, we obtain the following result, where (1) represents the case of a point source, (2) a circular source:—

(*a*) The positions of the maxima and minima of (2) approximately coincide with those of (1). Strictly speaking, the first maximum of (2) is displaced by a small amount towards $\theta=0$ as compared with (1), and for other maxima and minima this displacement becomes smaller and smaller. But the maxima of (2) may correspond to the minima of (1), and the minima to the maxima.

(*b*) The value of (2), which corresponds to the maximum of (1), is smaller than that of (1), and the value of (2), which corresponds to the minimum of (1) is greater than that of (1). This difference between (1) and (2) increases with the value of Φ (i.e. with the increased diameter of the source).

(*c*) As the value of θ increases, the maximum value of (1) and (2) gradually decreases. While the minimum value of (1) always remains 0, the minimum value of (2) gradually increases until it becomes equal to the maximum value and assumes a stationary value, then the maxima and minima interchange, the difference of the maximum and minimum values at first increases and then decreases, then again assumes a stationary value, and

(1) Phil. Mag. IX. p. 389 (1880).

so on. If in this interval between the two stationary values the maxima of (2) correspond to those of (1), then in the next interval the maxima of (2) correspond to the minima of (1).

(*d*) For larger values of r (radius of the drop) the intensity of (1) and (2) increases by $r^{\frac{7}{3}}$. But at the same time for (2), the difference between the maximum and minimum values is diminished by another factor F.

(*e*) The above is more manifestly shown in the case of the laboratory experiment with a cylindrical glass rod and a straight slit as the source of light. The stationary points of (2) at which the maximum value coincides with the minimum are easily found by

$\varkappa\Phi = m \times$ interval of the maximum and minimum of (1),

where m represents an integer.

(*f*) According to Airy's theory, the law of the distribution of the colours of the rainbow is independent of the magnitude of the drop. But in the case of the finite source, the colour distributions are changed by the magnitude of the drop, especially in the supernumerary bows.

(*g*) The supernumerary bows almost lose their colour as the consequence of the finiteness of the source. This effect is more remarkable when the drop becomes larger.

11. Note on the experimental side.

To show the above-mentioned results, we repeated rough experiments with glass rods and a straight slit as the source of light.

Using homogeneous light, we see that when the slit is very narrow the phenomena nearly coincide with Airy's theory, and that when the breadth of the slit is increased, the positions of the bows (or fringes in this case) change very little. As another effect of the increase of the breadth of the slit, the bows become indistinct, especially the supernumerary bows. This effect is remarkable when the diameter of the rod is large. We could not observe the turning point, at which the maxima and minima interchange, as the difference of the intensities is very small. But we can roughly say that the point at which the bows become almost indistinguishable corresponds to the position at which the angular breadth of the bow coincides with that of the slit.

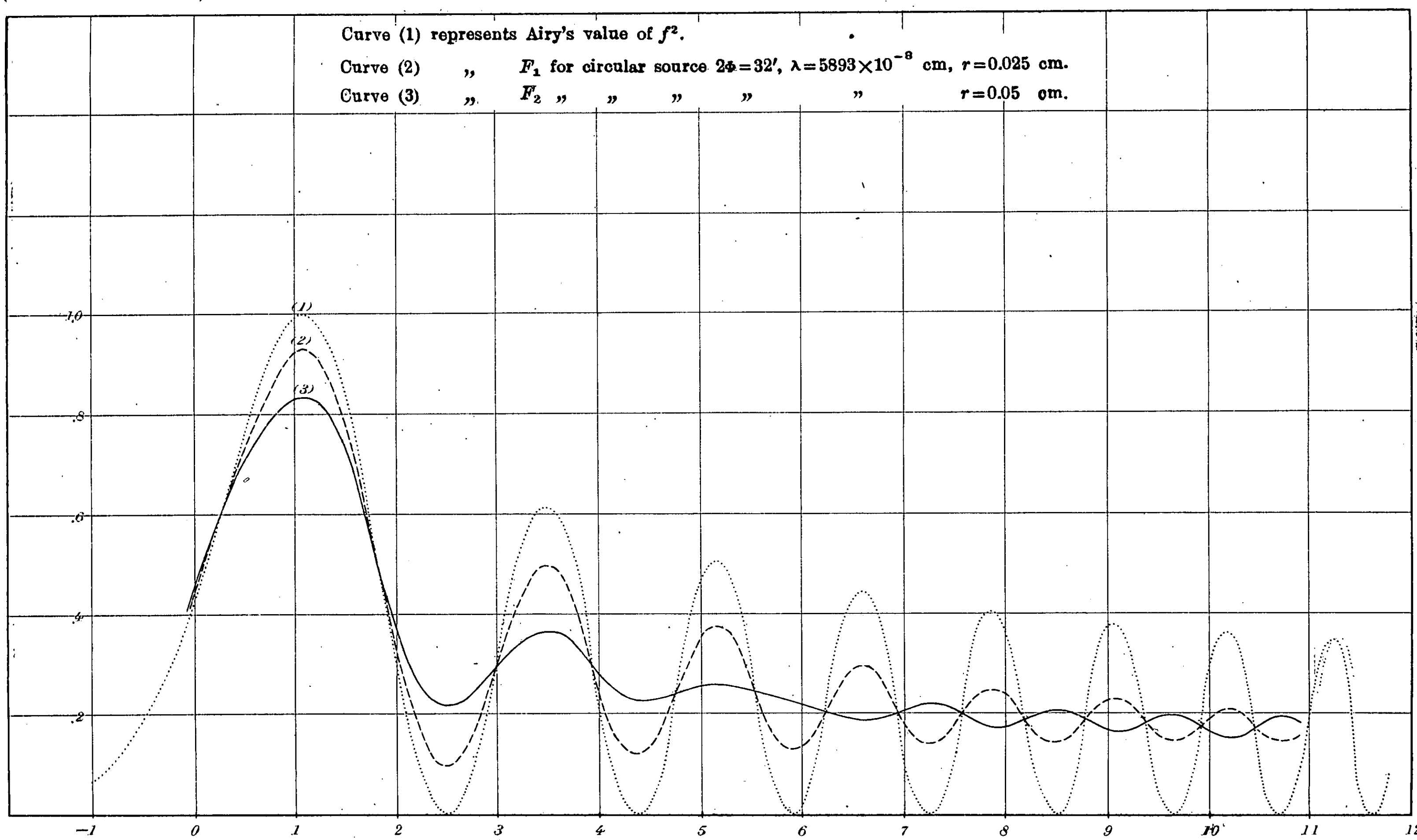
Again, using white light, it is easy to see that the colours of the supernumerary bows change when the magnitude of the rod is changed, and that the supernumerary bows almost lose colour and become indistinct when the breadth of the slit is increased.

In the table experiment, we always observed that, when the breadth of the slit is not too large, the supernumerary bows are numerous for a cylinder with a large radius, but fewer for a small cylinder. This fact may be explained by the presence of the factor $r^{\frac{4}{3}}$. In the case of natural rainbows where, as Pernter indicated, the supernumerary bows were observed when the drops were large, we can not take into account the factor $r^{\frac{7}{3}}$ directly, because the intensity depends, at the same time, on the number of drops which are contained in a unit volume of space, and it is probable that when the radius of the drop is large the number of the drops is small. For instance, let us take the cases $r=0.025$ cm. and $r=0.05$ cm., and suppose that the quantities of the drops per unit volume is equal in the two cases, so that the ratio of the numbers of the drops is $2^3:1$, and that the intensity of the rainbow is proportional to the $\frac{2}{3}$th power of the number of the drops. Then, the numbers in the curves in Pl. III and Pl. V are increased by $2^2(0.025)^{\frac{7}{3}}$ and $(0.05)^{\frac{7}{3}}$, or 1 and 1.26 respectively, and we see that the intensity curves of the former are sharper than those of the latter. Thus, the above result of observation is explained only by saying that there was a comparatively large quantity of drops. But, we have another cause on which the above observation must depend; namely, the imperfectness of Airy's theory for large values of θ. In the strict sense, we can not compare the corresponding supernumerary bows due to two drops of different sizes, because the value of θ being different, the approximation of Airy's and consequently of our theory is not the same in both cases. So far as Airy's theory holds good, we can say that the supernumerary bow due to large drops is less distinct; leaving out of consideration both the factor $r^{\frac{7}{3}}$ and the number of the drops.

The above discussion only holds for supernumerary bows; on the contrary, the principal bow is more distinct for large drops, as Pl. III and Pl. V show. Thus a white rainbow is probably caused by small drops, or rather mixed drops of different sizes. In fact, in many cases in nature, it is absolutely important to consider the inequality of the size of the drops, though actual discussion of this point is almost impossible.

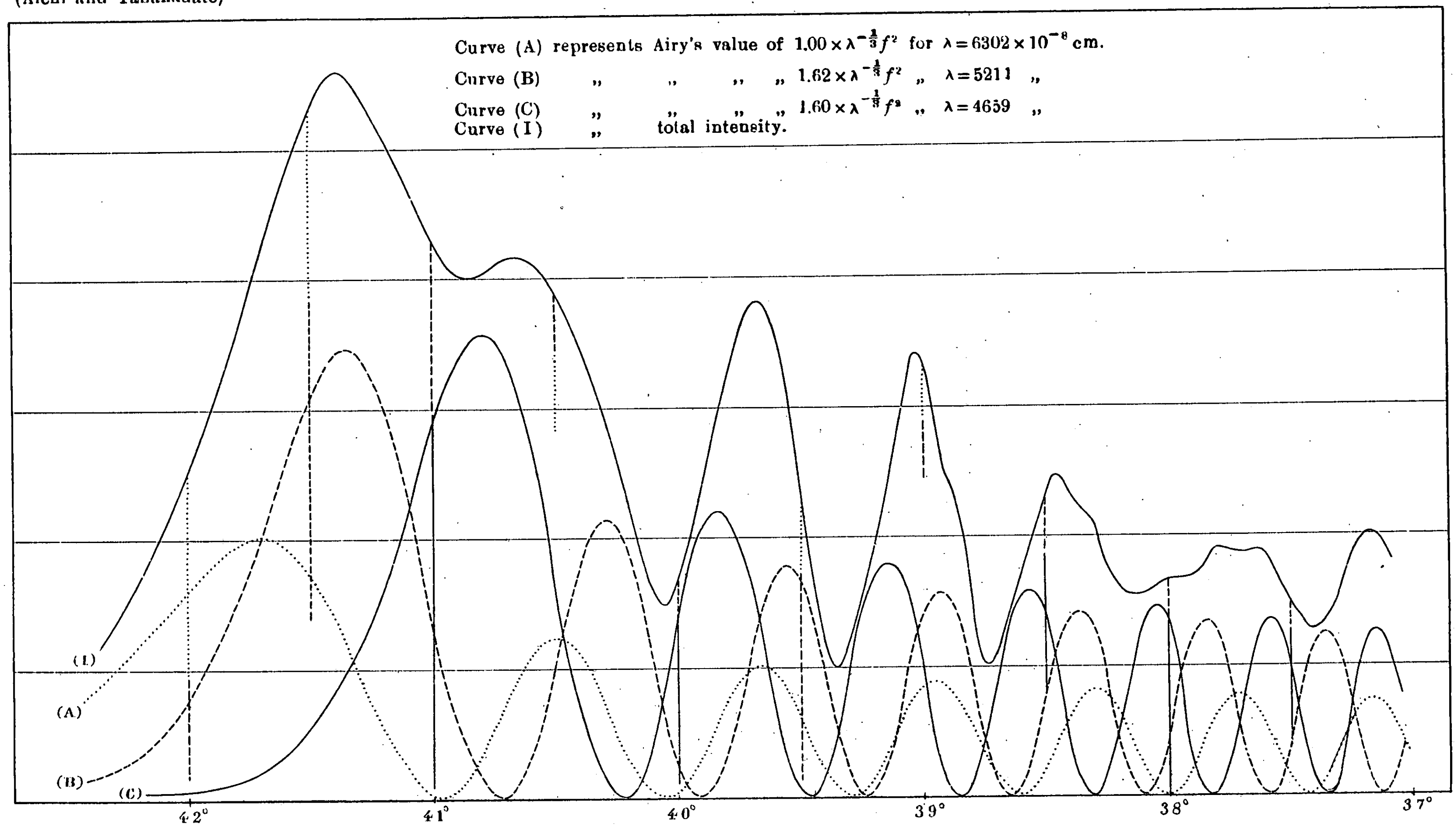
In conclusion, we have to thank Prof. Nagaoka for suggesting the problem and for giving kind advice during the course of our investigations.

(Aichi and Tanakadate)



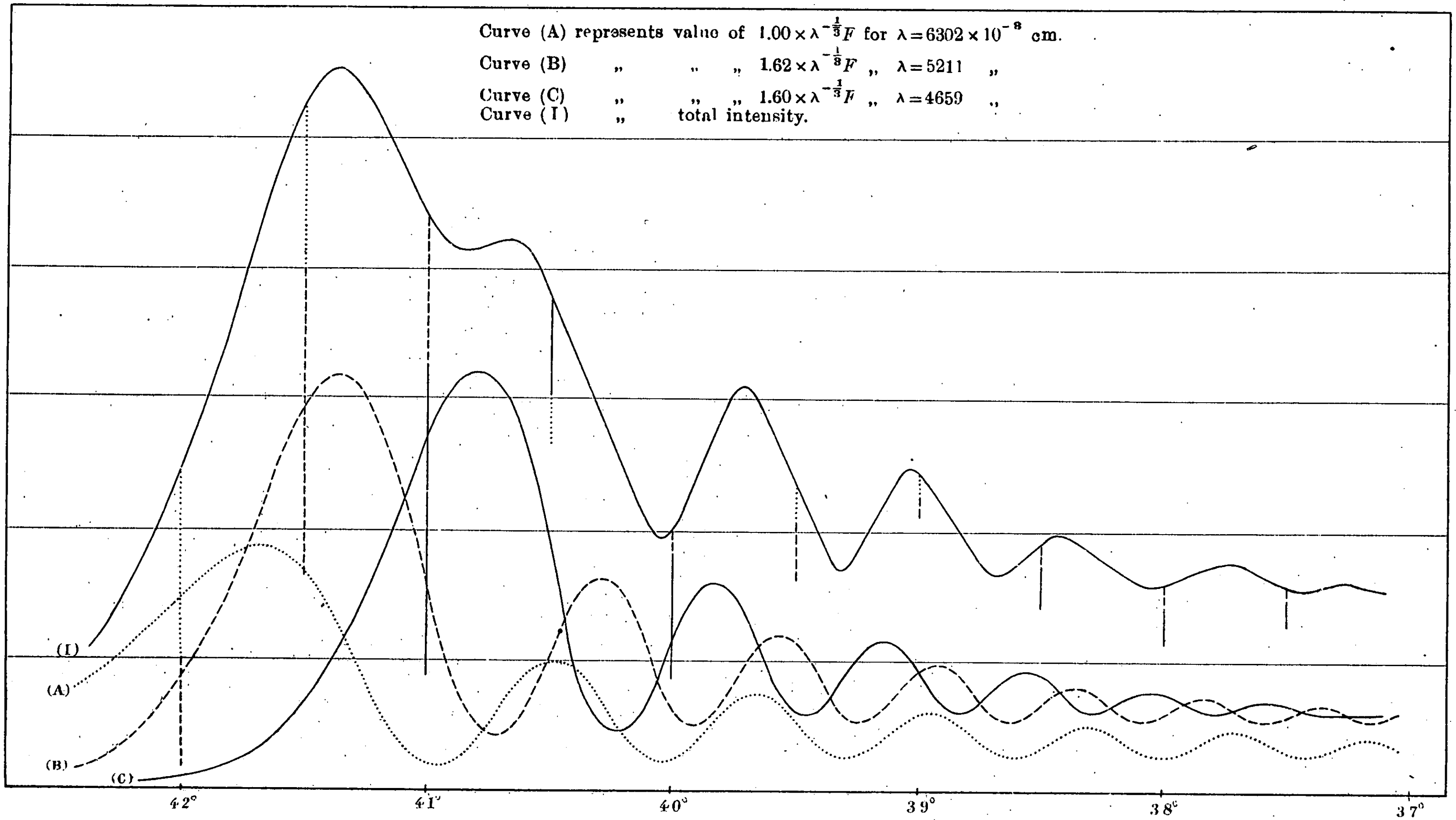
Intensity Curve.

(Aichi and Tanakadate)



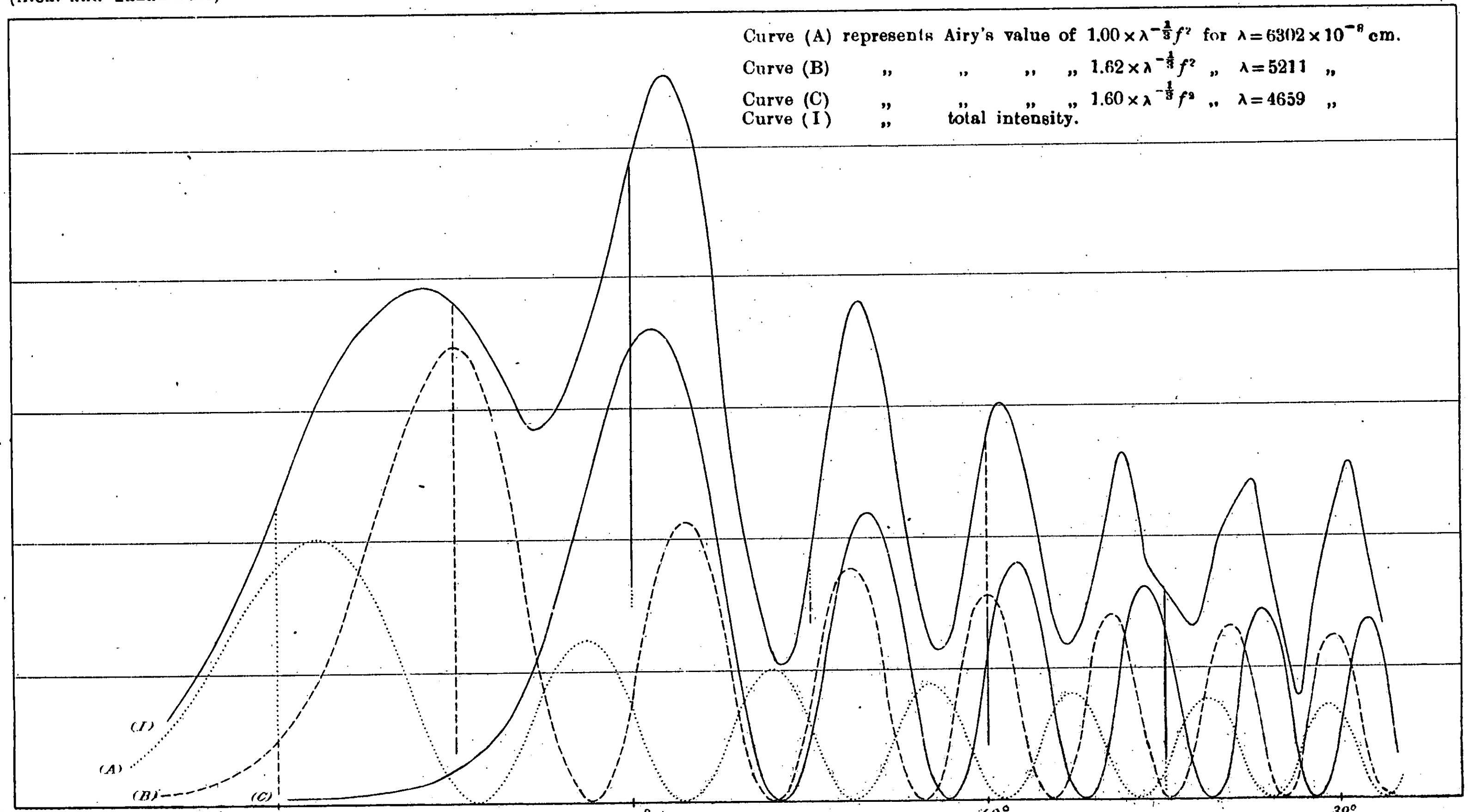
Intensity Curves for Primary Colours (Point Source, $r = 0.025$ cm).

(Aichi and Tanakadate)



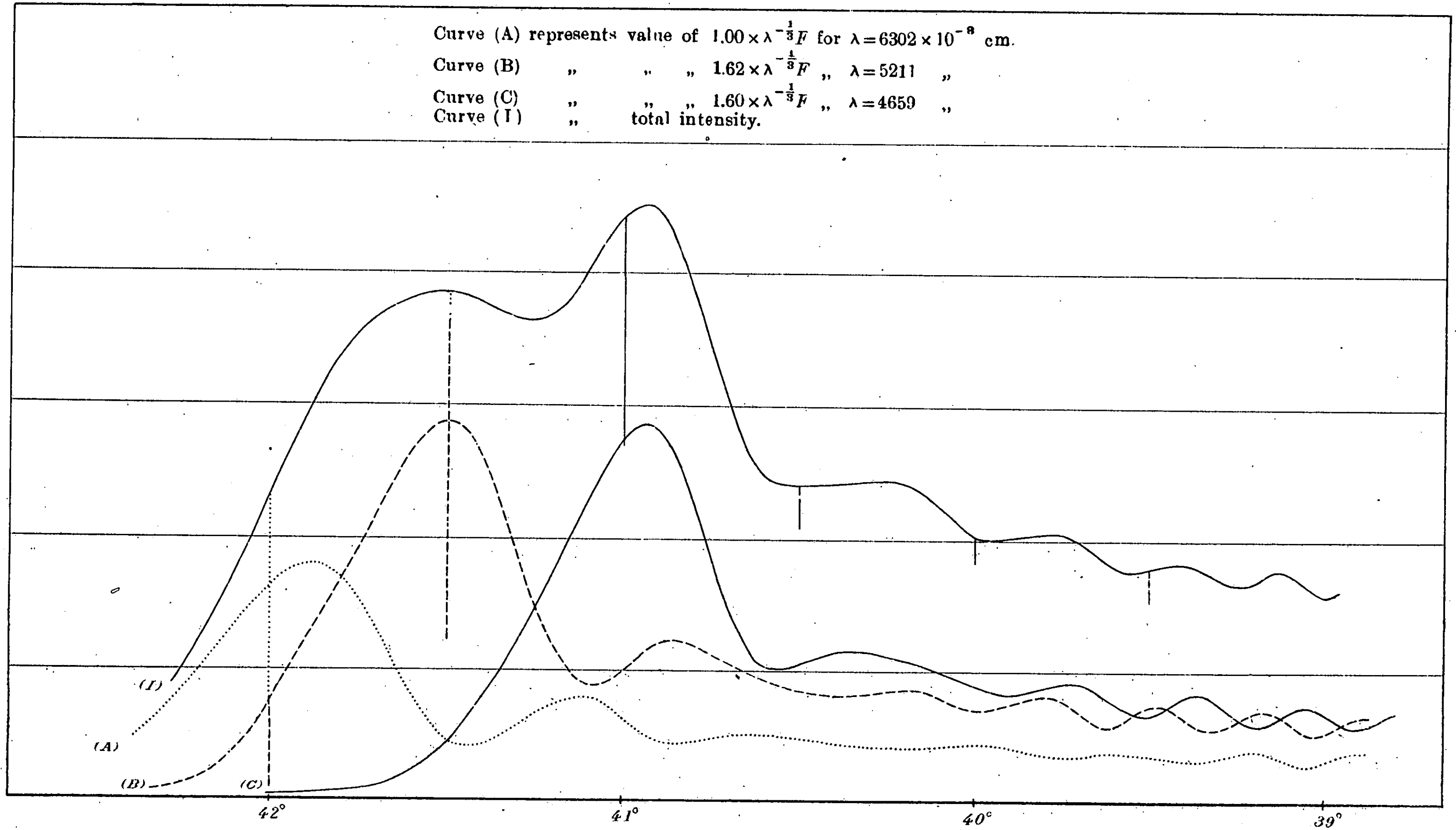
Intensity Curves for Primary Colours (Circular Source $2\Phi = 32'$, $r = 0.025$ cm).

(Aichi and Tanakadate)



Intensity Curves for Primary Colours (Point Source, r=0.05 cm).

(Aichi and Tanakadate)



Intensity Curves for Primary Colours (Circular Source $2\Phi \doteq 32'$, $r = 0.05$ cm).