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Modulus of Rigidity of Rocks *

AND

Hysteresis Function.

BY

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With Plates I—XXII, containing 53 Figures.

CONTENTS.

INTRODUCTION.

Since Galileo Galilei(1) published his inquiries concerning the rupture and strength of beams in 1638, the elastic nature of substances

* Detailed descriptions of the experiments are to be found in "The Publications of the Earthquake Investigation Committee in Foreign Languages," No. 14, Tokyo, 1903.

(1) Galileo Galilei. Thomas Salisbury's Mathematical Collections and Transactions; London, 1655. Vol. II.

has from time to time been investigated by many scores of distinguished physicists and engineers. Indeed, the questions of elasticity, having close relation, on one hand, to the strength of materials, and, on the other to almost every branch of physics constitute a problem whose solution has long been hoped for, but not yet solved, either from its theoretical or its practical side.

A law expressed by Hooke[1] with Latin terseness in the words "*Ut tensio, sic vis*" is the foundation of the mathematical theory of elasticity taken in the wide sense. The result of experimental researches is that Hooke's law is nearly fulfilled, for all hard solids, each through the whole range within its limits of elasticity.

Coulomb first gave his theory of torsion for hairs and silk threads[2] and then extended it to metal threads.[3] He brings out clearly that the set-slide,—i.e. non-elastic permanent strain—produced by torsion is at first proportional to the total slide and then to the elastic slide, and that the slide-modulus (réaction de torsion) remains almost the same after any slide-set, that the elastic limit (at least in the case of torsion) can be extended by giving the material a set. Savart[4] endeavoured to extend experimentally the result which Coulomb had obtained for the torsion of a wire, and concluded thus :—" Quelque soit le contour de la section transversale der verges, les arcs de torsion sont directement proportionnels au moment de la force et à la longeur."

The first formula for the torsion of square and rectangular prisms, which was given by Eaton Hodgkinson,[5] has been proved to

(1) Hooke. De potentiâ restitutiva; London, 1678.
(2) Coulomb. Mémoires des savants étrangers. Tom. IX. 1777.
(3) Coulomb. Histoire de l'Academie des Sciences, année 1784. Paris, 1787.
(4) Savart. Annales de chemie et de physique. Tom. 41, 1829.
(5) E. Hodgkinson. Experimental researches on the strength and other properties of cast iron, 1846.

be inexact by the later researches of Saint-Venant.[1] Vicat[2] probably was the first elastician who published a well-considered experiment on yielding. In a paper "De fili Bombycini vi elastica," Göttingen 1841, Weber[3] wrote "......deuplex tensionis effectus...... alter primarius seu momentaneus ac subitus, alter secundarius seu subsequens et continuatus......" It is then noted that this "prolongatio continuata" is not permanent. An experimental discussion of set und yielding is found in a memoir entitled "Sur la torsion des fils métaliques et......" written by Ignace Giulio.[4] His conclusion is instructive: "L'altération de forme produite par l'action d'une force extérieure sur un corps elastique se compose de deux parties: l'une indépendente de la durée de cette action et sensiblement proportionnelle à son intensité; l'autre, croissant plus rapidement que la force qui la produit, et suivant une fonction de la durée de son action." Then he says that when the external force ceases to act, the first part of the deformation disappears instantaneously, but the second part persists, diminishing, however, continuously with the lapse of time.

G. Wiedemann,[5] in his paper "Ueber die Torsion," concludes that the temporary torsions of a wire twisted for the first time by increasing loads increase more rapidly than these loads, and that the torsional sets or torts of the wire increase still more rapidly. In a paper entitled "Memoire sur la torsion," G. Wertheim[6] also divides

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(1) Saint-Venant. Memoire sur la torsion des prismes; Comptes Rendus, Tom. XXXVI. 1853.

(2) Vicat. Annales des ponts et chaussés, 1er semestre, 1834.

(3) W. Weber. W. Weber's Werke, herausgegeben v. d. König. Gesel. d. Wiss. zu Göttingen. Bd. I. pp. 447–474.

(4) Ignace Giulio. Memorie della reale Accademie delle Scienze di Torino. Serie II. Tomo. IV. 1842.

(5) G. Wiedemann. Pogg. Annalen. Bd. CVI. 1859.

(6) G. Wertheim. Annales de chemie et de physique. Tom. 50. 1857.

the angle of torsion into two parts, i.e. the elastic part and the set part; he disregards, however, the elastic yielding.

The experiments on torsion thus far referred to were made on specimens, which were almost wholly metallic substances. Bevan[1] in "Experiments on the Modulus of Torsion" has given a table of the modulus of torsion for different kinds of wood. The strength of stone and mortar was investigated by Bevan[2] with glue and by Rondelt[3] with mortar: but they rather belong under the category of adherence.

Several distinguished elasticians have mathematically treated the elastic nature of rocks—or rather, of crystals—, but the result of their subtle analysis is of little use to the study of geological phenomena. Quite recently, Professor H. Nagaoka[4] has published an essay, containing a valuable table of the rigidity modulus of the various rocks, which compose the outer coating of our planet.

The present experiments were undertaken under the professor's kind guidance, for the purpose of extending his investigations. The following paper contains the results of the experiments on torsion, and its principal object is to show how great the *defect of Hooke's law* is and how great *hysteresis* there exists in the relation of torsion to couple. In mathematical part, a *formula for the hysteresis-function* due to the elastic yielding (Elastische Nachwirkung or Weber's Effect) is deduced. Lastly, as an appendix, the well-known wide difference between the velocities of the tremors and those of the principal schocks in an earthquake is explained, as an effect of the elastic yielding of the rocks through which the seismic waves are propagated.

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(1) Bevan. Phil. Transactions. 1829.

(2) Bevan. Phil. Magazine. Vol. LXVIII., 1826.

(3) Rondelt. Traité théorique et pratique de l'Art de Batir. 1830–32.

(4) H. Nagaoka. E. C. of Rocks and the velocity of the Seismic Waves. The Pub. of the E. I. C. in Foreign Languages. No. 4. 1900. Phil. Mag. 1900.

ARRANGEMENT AND TWISTING APPARATUS.

The arrangement for the experiment was, on the whole, one and the same as that used by Professor H. Nagaoka,[1] but with such adaptations as were necessary to make the stress cyclical. Preliminary experiments, as the professor noted in his publication above cited, showed a great defect of Hooke's law and the existence of remarkable hysteresis. Fig. 1, Pl. I., shows the general features of the relation of stress to strain in a piece of sandstone. Thus, it will be obvious that, in the ordinary sense of the term, it is nonsense to speak of the modulus of rigidity of rocks. The value suggested by the mathematical theory as the limiting one is to take the tangent of the angle which the tangent at the origin to the stress-strain curve makes with the strain axis. But, it is extremely difficult to determine this angle, because the experimental error due to the non-delicacy of the apparatus is greatest where the stress is vanishingly small. Possibly sound-experiments might be the best method of ascertaining this limiting value. The result deduced from such infinitely small strains as occur in sound vibrations is no doubt of great value as regards the elastic property of matter ; but it must be far from what we have to consider in the study of the geological phenomena and seismic waves.

The chief features of my improvements of the arrangement were ;—firstly, to twist the specimen cyclically with increasing and decreasing couple *passing through zero continuously* ; secondly, to get rid of the influence of the friction between the parts of the instrument ; and thirdly, to eliminate any external disturbance such as the yielding of the scale-support or minute displacements of the telescope.

(1) H. Nagaoka. The Pub. of the Earthquake Invest. Committee in Foreign Languages. No. 4. 1900.

Figures in Pl. II. show the twisting apparatus thus improved, together with a rough sketch of the whole arrangement. Here it is only necessary to remark that any couple whatever, positive or negative, could be produced without any increase of load to be supported by the knife-edge.

The order of observation was as follows :—

1. To begin with, a specimen, whose dimensions were nearly $1 \times 1 \times 15$ centimetres, was firmly clamped in horizontal position and at right angles to the plane of the pulleys.
2. The directions of the two mirrors attached to the specimen were so adjusted that the two images of the vertical scale, whose distance from the specimen was 2.716 metres, stood side by side within the field of one and the same telescope, mounted on a tripod.
3. Equal weights, each $\frac{1}{2}$ M_0, were put on the pans Q_1 and Q_2, which, as a matter of course, gave no couple to the specimen.
4. Zero-readings were taken firstly for the right image which was reflected from the mirror attached close to the fixed end of the specimen, then as fast as I could for the left image which corresponded to the free end.
5. A definite number of pieces, which constituted the weight $\frac{1}{2}$ M_0, say $\frac{1}{2}$ M were taken off from one of the pans and put on the other. The resulting couple was obviously equal to Mgr, if r represent the radius of the twisting pully and g the value of gravity. A time-record corresponding to this transposition of weight was taken.
6. After a certain definite time, the readings were taken for both images, as in the case of zero-readings.
7. Second transposition of weight ; the time recorded ; scale

readings taken : and so on till a definite amount of couple was reached.

8. Then, the weight was transposed in the opposite way so that the couple diminished gradually and ultimately became oppositely directed. In this way, a series of observations was made to complete the cycle many times.

9. From the difference of the deviations of the two images, the amount of twist due to the corresponding weight difference was calculated. One mm. of scale division $=1.845\times10^{-4}$ radian. One gram of mass $=6.712\times10^{3}$ c.g.s. unit of couple.

Thus, it will be easily seen that the result of observation was at no time affected by any external disturbance such as the yielding of the scale-support, or minute displacements of the telescope, or flexure of the floor on which the twisting apparatus rested. One instance is given in Pl. III. for the case of a piece of clay slate. As might be expected, the irregularity of the course of the curve A is enormous, while the course of the curve B is quite free from it.

Here it is necessary to explain why, in the third operation above mentioned, I put a superfluous weight $\frac{1}{2}$ M_0 on the pans giving no couple. When the two weights, whose respective masses are m_1 and m_2, are put on the pans Q_1 and Q_2 respectively, the resulting couple is given by

$$\mathfrak{C}=\left[\{m_1-m_2\}\mp\sigma\{m_1+m_2\}\right]\text{rg.}$$

the upper or the lower of the double sign being selected according as $\{m_1-m_2\}$ is increasing or decreasing ; where σ is a numerical coefficient depending on a dissipation of couple due to the imperfectness of the arrangement ; e.g. friction of the pulley, imperfect flexibility and extensibility of the strings. For future reference, $m_1-m_2=M$ and $m_1+m_2=M_0$ will be called the *Effective Mass* and the *Resisting Mass* respectively. Thus

$$\mathfrak{C}=\{M\mp\sigma M_0\}rg=\mathrm{Mrg}\mp\mathrm{constant},\qquad \text{provided } -M_0<M<+M_0,$$

whence $\dfrac{d\mathfrak{C}}{dM}=\mathrm{rg}=\mathrm{constant}=6.712\times10^3$ per gram.

Or graphically, the relation between M and $\mathfrak{C}$ may be expressed by a narrow parallelogram whose shorter sides are parallel to the axis of M and *the angle contained by the adjacent sides is independent of the imperfection of the apparatus.* The value of σ determined by direct experiments is:

from the first experiment,	$\sigma_1=0{\cdot}0128$;
from the second experiment,	$\sigma_2=0{\cdot}0132$;
mean of the two,	$\sigma=0{\cdot}0125$;

Thus for instance, if the resisting mass be two kilograms, the couple due to the effective mass of N grams is

either $\mathfrak{C}_i=\{N-25\}\mathrm{rg}$, for the increasing couple

or $\mathfrak{C}_d=\{N+25\}\mathrm{rg}$, for the decreasing couple.

GENERAL INVESTIGATION OF THE STRESS-STRAIN RELATION.

Two specimens, serpentine and pyroxenite, whose limits of elasticity seemed to be comparatively wide, were tested by a series of cycles of positive and negative couples which gradually increased in their absolute amount.

Half the difference between the maximum and minimum couples in a cyclical process of the twisting and untwisting will be called the *Amplitude* of the cycle. Half the sum of these two couples will be called the *Centre* of the cycle. For instance, if the couple varies cyclically between $j+\theta$ and $j-\theta$, then θ and j are called the amplitude and the centre of the cycle respectively.

In the present experiments, the centre being at origin i.e. $j=o$, the specimens were twisted to and fro cyclically up to the amplitude θ given below :—

$\theta=2{\cdot}68$; $=6{\cdot}71$; $=10{\cdot}07$; $=16{\cdot}11$; $=20{\cdot}14$ $\times 10^6$ c.g.s. units.

The result of the experiments is plotted in Pl. IV. It will be seen, at a glance, that

1. If the curve be divided into several sections at the positions where the variation of couple changes its sign, each section, except the one which starts from the origin, will be approximately straight, provided a comparatively small portion immediately after the turning point is left out of consideration.
2. None of the sections except the one which starts from the origin, pass through the origin. Each section for which the variation of couple is a positive, or on-curve, lies below the origin, i.e. it passes through the fourth quadrant : while each section for which the variation is a negative, or off-curve, passes through the second quadrant.
3. Suppose a straight line to be drawn coinciding as nearly as possible with each section. Of these representatives, that which represents the section which belongs to a cycle of greater amplitude lies further from the origin.
4. That which lies further from the origin makes a greater angle with the axis of couple.

Of the above facts, the first shows that Hooke's law is approximately fulfilled, provided the change of couple always takes place in *one direction* only ; and that there must be a certain disturbing cause, prevailing especially in the neighbourhood of the starting point and at the turning point of the course. The second shows that there is certain dissipation of energy for each cycle, due either to the imperfect elasticity of the twisted rocks or to the imperfection of the twisting apparatus. Observe that the latter is a constant. The third teaches us that the dissipation above stated increases more and more when the amplitude of the cycle becomes greater and greater. This increase of the dissipation must necessarily be due to the imperfect elasticity of the specimen. The fourth means that the specimen becomes less and less rigid when it is twisted further and further. Thus, the ordinary conception of the modulus of rigidity is necessarily vague and uncertain. In future, the *actual resistance* to the deformation in any state whatever, be it elastic or plastic at that state, will be taken as the measure of rigidity at that state. Hence: *The Rigidity-Modulus of a substance in a given state is measured by the increase of stress required to give a unit increase of deformation to the substance in that state:* i.e. the trigonometrical tangent of the angle which the tangent to the stress-strain curve at a point corresponding to that state, makes with the strain axis.

The following numerical calculations show the above facts quantitatively:—The equation of the representative straight line is either $\alpha+\beta M=\delta$ or $\beta\{M-\gamma\}=\delta$;
where M and δ being the effective mass and the corresponding deflection of the image respectively, α is proportional to the residual twist surviving the couple which is proportional to M, and β is inversely proportional to the rigidity-modulus of the specimen at that state: while γ is the effective mass which is required to bring the specimen into the state of no torsion. The couple required to detort the specimen is, as G. Wiedemann[1] noted in his experiment on metallic wire, obviously less than the couple which produced that tort. The following table giving the relation between α, β, γ and θ proves the above statements.

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(1) G. Wiedemann. Pogg. Annalen. Bd. CVI. 1859.

TABLE I.

Specimen.	No. 8_2. Serpentine from Chichibu.			No. 32_1. Pyroxenite from Gumma.		
Amplitude \ Constant	α	β	γ	α	β	γ
0.0	∓0·01105	0·255	±0·0434	∓0·0074	0·249	±0·0390
0·4	∓0·0295	0·424	±0·0691	∓0·0322	0·428	±0·0752
1·0	∓0·0529	0·485	±0·1091	∓0·0538	0·497	±0.1082
1·5	∓0·0636	0·508	±0·1254	∓0·0873	0·504	±0·1732
2·4	∓0·0987	0·513	±0·1924	∓0·1229	0·529	±0·2324
3·0	∓0·1212	0·527	±0·2300	(M_0=3300grams)		

Here, of the double sign, the upper or the lower must be selected according as the required twist corresponds to the increasing or decreasing state of the effective mass.

Now plotting the value of γ against the value of θ, as is shown in Fig. 9 Pl. V., we see that γ may be expressed as a linear function of θ, so that

$$\gamma = \sigma M_0 + h\,\theta$$

where h is a constant depending on the imperfect elasticity of the specimen. Numerical values of σ and h obtained from the above experiments are :—

No. 8_2. $\sigma_1 = 0{\cdot}0128$, $h_1 = 0{\cdot}0817$

No. 32_1. $\sigma_2 = 0{\cdot}0132$, $h_2 = 0{\cdot}0619$

The values of σ_1 and σ_2 which must be equal to each other, coincide tolerably well within the error of observation. The difference of plasticity of several rocks will be indicated by the relative values of the constant h.

THE DOUBLY INDEFINITE CHARACTER OF THE MODULUS OF RIGIDITY.

From the experiments, some of which are graphically shown in Pls. IV. and VI., we may easily conclude that

1. When the maximum amount of stress, to which the specimen has just been subjected, is given, the modulus of rigidity is a function of the stress actually acting upon it.
2. When the stress acting upon it is given, the modulus of rigidity is a function of the maximum amount of the stress to which the specimen has just been subjected.

Of these two, the first expresses simply that Hooke's law is defective, while the second shows the existence of torsional hysteresis. One numerical example is given in the following table to show how the modulus of rigidity, even in the limiting case of a vanishingly small stress, varies with the magnitude of the stress just previously applied.

TABLE II.

Modulus of rigidity, taken at the state of a vanishingly small stress.

Previous couple (Arbit. unit) / Specimen	1	30	75	115	185
Serpentine	$5{\cdot}22\times10^{11}$	3·14	2·74	2·62	2·59
Pyroxenite	$4{\cdot}90\times10^{11}$	2·84	2·45	2·41	2·30

The relation is also graphically shown in Fig. 11_a, Pl. V.

Here, it may be necessary to remark that, even when Hooke's law is approximately fulfilled, the factor of proportionality may entirely depend on its previous history. To give one instance, in the result of the experiment shown in Figs. 12 and 13, Pl. VII., there is little or no objection to assuming Hooke's law, provided each series of observations, be it either in the case of an on or an off curve, is considered independently from the other. Moreover, the factor of proportionality for the on-curve is nearly equal to that for the corresponding off-curve. It, however, never remains the same for the two different series.

The numerical values are :— {Specimen No. 4_2. Sandstone}.

$$\mu_1=0{\cdot}72\times10^{10} \quad ; \quad \mu_2=1{\cdot}25\times10^{10}$$

There is no reason to reject either of them, since they were measured in one and the same manner under equal surrounding conditions. Nothing but the difference of previous history can account for the cause of such an ambiguity in the modulus of rigidity.

YIELDING AND RECOVERY FROM THE YIELDING.

Attention was first drawn by Weber[(1)] to this subject. He called it "*Prolongatio vel contractio secundaria.*" It requires a certain duration as well as magnitude of stress ; it disappears if the stress be removed for a certain period. Ignace Giulio[(2)], in his experiments on torsion of metallic wire, recognized also the existence of both the yielding and the recovery ; but as he could not find any quantitative relation, so he concluded his memoir with the following question :—" Quelle èst la fonction de la tension et du temps suivant laquelle ces altérations se produisent et disparaissent ? "

YIELDING :—In most rocks, the yielding is considerable. Indeed, it is doubtful whether there is any limit to the elastic yielding of certain rocks, at least, when the twisting couple is sufficiently great. In the following experiments, equal weights were put on the two pans, which gave no couple, and the readings corresponding to that state were taken. At a known instant, a definite amount of weight was transported from one pan to the other. Then the readings were taken from instant to instant. The results of the experiments are shown in Fig. 14, in Pl. VIII. In these two experiments, the effective mass was the same but with different resisting masses.

(1) Weber. W. Weber's Werke: her. v. d. König. Gesel. d. Wiss. zu Göttingen. Bd. I. pp. 475–488.

(2) Ignace Giulio. Memorie della reale Accademia delle Scienze di Torino. Serie II. Tom. IV, 1842.

The influence of the different magnitudes of the constant couple (the resisting mass being the same) upon the yielding was then examined. In successive experiments, the constant couples were in the ratio of 3 : 5 : 7 9 : 11 : 13 : 15. The curves in Pl. IX. show the result. As a matter of course, all curves are of similar form, but there exists the remarkable difference that of any two curves, the one whose couple was the smaller, approaches to horizontality more quickly than the other. Here it must be observed that the absolute amount of twist is not known, since the so-called zero-reading is nothing but the reading which corresponds, as the case may be, to a certain unknown twisting couple lying between $+M_0\sigma gr$ and $-M_0\sigma gr$. In other words, all curves in Pl. IX. are not referred to one and the same origin of coordinates.

RATE OF THE YIELDING :—From the values of the twist and of the corresponding time, the rate of the yielding may be calculated by the formula

$$\frac{\triangle\tau}{\triangle t}=\frac{\tau_{(t'')}-\tau_{(t')}}{t''-t'}$$

where $\tau_{(t)}$ represents the amount of twist at the instant t. Plotting the result, obtained from eight observations, as shown in Fig. 16, Pl. X., we may, at once, perceive that the general relation between the rate of yielding and the time-element much resembles something like a rectangular hyperbola. To test whether this is true or not, instead of $\frac{\triangle\tau}{\triangle t}$ and t themselves, their logarithms are plotted in Fig. 17 on the same plate. The most probable curve and also the simplest at the same time, would be a straight line. One straight line, whose equation is

$$\log\frac{\triangle\tau}{\triangle t}+log\ t=log\frac{1}{10}$$

is drawn in the figure. Then, the equation of the corresponding hyperbola is

$$t\frac{\triangle\tau}{\triangle t}=\frac{1}{10}$$

which is also traced in the corresponding figure. {Eight of such straight lines and hyperbolas should be drawn in the figures to correspond to all the points dotted there.}

Assuming that the relation between the rate of yielding and the time-element is given by the above equation, we may write.

$$d\tau=k\frac{dt}{t}$$

which when integrated becomes

$$\tau=k\ log\ t+\text{constant}.$$

Let the value of τ at the time $t=1$ be represented by τ_0, and we have

$$\eta=\tau-\tau_0=k\ log\ t$$

as the value of the twist due to the yielding, provided the yielding is counted after the lapse of a unit time. In Fig. 18, Pl. IX., the curves are traced for two different values of k. Here it must be observed that the constant k must depend on the amount of the constant couple as well as on the nature of the rocks. As C. F. Dietzel[1] found in his experiments on vulcanized caoutchouc the yielding is, most probably, proportional to the stress, so that we may put

$$k=\nu.\ \mathfrak{C}$$

or $$\eta=\nu.\ \mathfrak{C}.\ log\ t.$$

where ν is a constant depending on the nature of the rock.

Now it may be doubted whether the yielding can actually proceed without limit in time. To ascertain this point, a specimen was subjected under a constant couple for a long time, and then the time variation of its twist was observed. The result is shown in

(1) C. F. Dietzel. Polytechnisches Centralblatt: Jahrgang 1857. Leipzig.

Figs. 19–22, Pl. XI., which shows steady increase of the twist even after the lapse of many thousands of minutes.

RECOVERY FROM THE YIELDING:—To examine whether the yielding above investigated is elastic or permanent, a specimen was subjected under a constant couple during one hundred and sixty-seven hours, and then it was suddenly released from the couple. The amount of residual was observed from instant to instant with the corresponding time-record. The result is shown in Figs. 23 and 24 in Pl. VIII. This curve of recovery resembles, as a whole, that of yielding.

Again, to investigate the influence of the recovery on the cycles, a specimen, which had been subjected under a constant couple during three hundred and six and a half hours, was twisted to and fro cyclically. Gradual recovery, as is shown in Fig. 25, Pl. XII., may be traced along the whole cycle. Comparing this with that drawn in Pl. VI., both of which were treated in one and the same manner as regards the cyclic process, the effect of the recovery will be more clearly understood. Fig. 26 in Pl. XIII., shows the result of the next experiment conducted on the same specimen in a similar manner. Here, the form of curve, as a whole, has a centre of symmetry coinciding with that of the cyclic process. No possible explanation for this non-coincidence of the last two curves can be found but that, in the first case, in which the specimen had been kept under the couple during very long time, the total yielding having been very great, the specimen had a greater tendency than in the other case to recover from the yielding when the couple was withdrawn.

THE HYSTERESIS FUNCTION.

From what has been explained in the foregoing pages, we see that the strain produced by the shearing stress is of a very complicated nature. Though the greater part of the strain occurs almost instantaneously, the total amount of it gradually increases with time: i.e. say $S=S_0+S_t$. When the couple is removed, the greater part disappears instantaneously and the remaining part diminishes gradually, and the limiting value for infinite time is generally supposed not to be zero. Thus we have $S=S'_0+S'_t+S_r$. The suffixes 0 and t mean that the strain requires no time and a certain time respectively to appear or disappear, while r means that it remains for an infinite time. These facts have long been noticed by many experimentalists, as cited in the introduction above. Several distinguished elasticians have endeavoured to establish a relation between these different kinds of strains.

Since there is no reason for assuming $S_0=S'_0$ and also as S'_t can not be equal to S_t unless S_r is zero, the general expression for the strain must be of a form

$$S=S_{0\cdot 0}+S_{0\cdot t}+S_{0\cdot r}+S_{t\cdot 0}+S_{t\cdot t}+S_{t\cdot r}$$

so that $$S_0=S_{0\cdot 0}+S_{0\cdot t}+S_{0\cdot r}\ ;\ S'_0=S_{0\cdot 0}+S_{t\cdot 0}\ ;\ S_t=S_{t\cdot 0}+S_{t\cdot t}+S_{t\cdot r}\ ;$$

$$S'_t=S_{0\cdot t}+S_{t\cdot t}\ ;\ \text{and}\ S_r=S_{0\cdot r}+S_{t\cdot r}.$$

To establish a complete relation between stress, strain and time, we must find a relation between the stress and each term of the strain above mentioned.

The simplest is the case where all the terms except the first are negligibly small. Such a body is generally said to be perfectly elastic. Within proper limits, which are called the limits of elasticity, this is the case for most hard solids. As regards the relation between stress and strain under this condition, Hooke's investigation was most satisfactory and the result expressed in the law, well known by his name—Hooke's law—is so closely associated with perfect elasticity that some writers have taken "*defect of Hooke's Law*" to mean "*defect of perfect elasticity*," which is obviously absurd.

When other terms are not negligible, the stress-strain relation is so complicated that we have not yet any established law, notwithstanding the utmost endeavours of several distinguished elasticians and engineers. A. F. W. Brix[1] recognised only two terms $S_{0.0}$ which followed Hooke's law and $S_{0 \cdot r}$ for which he could discover no law. G. Wertheim[2] divided the angle of torsion into two parts which correspond to S_0 and S_r respectively, but he disregarded the terms corresponding to S_t and S'_t. W. Wundt[3] concluded that S, and not $S_{0 \cdot 0}$, was proportional to the stress that produced it; but reachad no definite conclusion as to the term S_t or S'_t.

A. W. Volkmann[4] found for silk and nerve, the stress-strain relation to be *hyperbolic*, but for muscle to be *elliptic*. He thought this relation something peculiar to organic bodies. In F. E. Neumann's paper[5], we find a consideration of *set* which literally corresponds to the term S_r. His conclusion is that the principal sets can be taken as linear functions of the principal elastic strain. More recently, Voigt[6] discussed the relation between $S_{0 \cdot 0}$ and S_r for the case of bending; while James Muir[7] experimented on the *recovery from overstain* which corresponds to S'_t, but he arrived at no quantitative relation.

After all, it is no easy matter to state any exact physical relation between stress, strain and time-element. In the following pages, the

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(1) A. F. W. Brix. Abhandlungen über die Cohäsions- und Elasticitäts-Verhältnisse einiger......Eisendrähte......; Berlin, 1837.

(2) G. Wertheim. Annale de chemie et de physique. Tom. 50. 1857,

(3) W. Wundt. Archiv für Anatomie, Physiologie und Wiss. Medicin. Jahrgang 1857.

(4) A. W. Volkmann. Archiv für A, P, u. s. w. herausgegeben v. C. B. Reichert und E. de Bois-Reymond. Bd. I. 1859.

(5) F. E. Neumann. Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers. 1885.

(6) Voigt. Untersuchung der Elasticitätsverhältnisse des Steinsalzes. Leipzig. 1874.

(7) J. Muir. Phil. Transactions. 1900.

yielding appears necessarily to be *elastic*; or recovery is complete if the stress be removed for an *infinite time.* Thus, *absolute set* is disregarded here. It may have, no doubt, more or less magnitude; —in magnetic hysteresis, indeed, we recognize its existence as very common. I doubt, however, whether it is really as great as it is repeatedly reported to be in the writings of experimentalists. What many experimentalists have called *set* appears to have been in greater part *not absolute set but elastic yielding.* For instance, when E. Chevandier and G. Wertheim(1) considered the strain to consist of two parts, i.e. an elastic part and permanent part, or when G. Wiedeman(2) speaks of the temporary torsion and the torsional sets, the elastic yielding obviously comes within the latter category. To cite the best example, Ignace Giulio(3) whose experimental discussion of set is very interesting, says himself: "On voit encore que ce que j'ai nommé jusqu'ici *Allongement Permanent.*......... *disparait* en grande partie après un temps suffisamment long...... "

The following pages contain some mathematical investigations concerning the *Hysteresis Function* due to the elastic yielding. Assume that the strain consists of two parts of which the first follows Hooke's law, being independent of time, and the second, though it is also proportional to the stress, depends on a time-element in a manner given by the relation established in the former experiments. Then an interesting formula for the hysteresis function may be deduced, from which the expressions for the amount of yielding as well as for the amount of residual after any number of reversals of twisting and untwisting, and all other properties of the torsional hysteresis follow at once.

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(1) E. Chevandier and G. Wertheim. Mémoire sur les propriétés mécaniques du bois. 1848.

(2) G. Wiedemann. Pogg. Annalen. Bd. CVI. 1859.

(3) I. Giulio. Memorie della reale Accademia delle Scienze di Torino. Serie II, Tom. IV. 1842.

Let the principle of superposition be assumed to hold good for the case of yielding, and take for granted the following relation, no matter how or from what hypothesis it may have been obtained,

$$\eta = k \log t. * \qquad (1).$$

To simplify the matter, let us suppose that a definite amount of couple begins to act at the origin of time and remains acting for an infinite time ; and also that after each unit of time—in my experiments it was one minute—the couple is increased by the same amount. Then, the total amount of the yielding at the instant $T=p$ is given by

$$\eta = \eta_p + \eta_{p-1} + \ldots\ldots + \eta_2 + \eta_1 = k \log \Gamma\{p+1\} \qquad (2).$$

where Γ is a well-known symbol for Gamma-function.

If a negative couple were to act at the instant $T=p+1$ and afterwards, increasing in its absolute amount step by step like the positive couple, the yielding due to this negative couple at the instant $T=p+r$ would evidently be

$$\eta = -k \log \Gamma\{r+1\}$$

Hence, if the couple increases for the first p minutes and then remains constant for the following r minutes, the yielding at the instant $T=p+r$ is

$$\eta = k \log \Gamma\{p+r+1\} - k \log \Gamma\{r+1\} = k \log \frac{\Gamma\{p+r+1\}}{\Gamma\{r+1\}}. \qquad (3).$$

Again, if the couple after an increase for the first p minutes, remains constant for the next r minutes, and then decreases step by step during last n minutes, the yielding at the instant $T=p+r+n$, as it may easily be seen, is given by

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* In Page 15, we had the formula $\eta = \tau - \tau_0 = k \log t$, where τ_0 is the value of τ at the time $t=1$, so that the expression for the yielding i.e. $\eta = k \log t$ holds good only for $t > 1$. When the present paper is under the press, it is kindly noted by Mr. S. Sano, that it is better to take a form $\eta = k \log \frac{t+\theta}{\theta}$, where θ is a constant, than to take the form given in the text. It is very good of him to have given me so much of other valuable remarks. The author would like to thank him for all his kindness.

$$\eta = k\log\frac{\Gamma\{p+r+n+1\}}{\Gamma\{r+n+1\}\Gamma\{n+1\}}. \tag{4}$$

Lastly, if the couple remains constant after this moment, then the yielding at the instant $T=p+r+n+t$ is given by

$$\eta = k\log\frac{\Gamma\{p+r+n+t+1\}\Gamma\{t+1\}}{\Gamma\{r+n+t+1\}\Gamma\{n+t+1\}}. \tag{5}$$

Proceeding in this way, we may find the value of η after any number of torsional cycles. I give here its general form. Suppose we start from the origin, at the instant $T=o$, which corresponds to the neutral state of the specimen, and for the sake of simplicity also suppose that the change of couple takes place by unit amount per unit of time.

1. Increasing the couple step by step we reach a couple$=I$ at the instant $T=i_1$, so that $I=i_1$;

2. for the next r_1 minutes, the couple remains constant to the instant $T=i_1+r_1$;

3. then it decreases step by step and ultimately, going even to the negative direction, becomes equal to II at the instant $T=i_1+r_1+i_2$, so that $II=i_1-i_2$;

4. here the couple remains constant and equal to II. during the next r_2 minutes;

5. again, the couple increases once more till it becomes equal to III at the instant $T=i_1+r_1+i_2+r_2+i_3$, so that $III.=i_3-i_2+i_1$;

etc. etc. etc.

where I, II, III,.........N represent certain definite amounts of couple, positive or negative as the case may be. Let p and n denote any given stage in the cycle on the increasing or decreasing procedures respectively, and t the time during which the couple remains constant at the last stage p or n; also let ν be the number of reversals of the couple from increasing to decreasing or from decreasing to increasing, so that N is the amount of couple in which the ν^{th} reversal occurs.

These premised, we may write the general expression for the hysteresis function as follows :—

If ν is even, the amount of yielding, η_p, is given by the expression

$$k\log\frac{\Gamma\Big\{p-N+t+1\Big\}\Gamma\Big\{p-N+r_\nu+t+1\Big\}\overset{\frac{\nu}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho-1}^{\nu}\sigma(i_\sigma+r_\sigma)+p-N+t+1\Big\}\Big]\overset{\frac{\nu}{2}-1}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho+1}^{\nu}\sigma(i_\sigma+r_\sigma)+r_{2\rho}+p-N+t+1\Big\}\Big]}{\Gamma\Big\{t+1\Big\}\overset{\frac{\nu}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho}^{\nu}\sigma(i_\sigma+r_\sigma)+p-N+t+1\Big\}\Big]\overset{\frac{\nu}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho}^{\nu}\sigma(i_\sigma+r_\sigma)+r_{2\rho-1}+p-N+t+1\Big\}\Big]} \tag{6}$$

If ν is odd, the amount of yielding, η_n, is given by the expression

$$k\log\frac{\Gamma\Big\{t+1\Big\}\overset{\frac{\nu+1}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho-1}^{\nu}\sigma(i_\sigma+r_\sigma)+N-n+t+1\Big\}\Big]\overset{\frac{\nu-1}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho+1}^{\nu}\sigma(i_\sigma+r_\sigma)+r_{2\rho}+N-n+t+1\Big\}\Big]}{\Gamma\Big\{N-n+t+1\Big\}\Gamma\Big\{N-n+r_{\nu-2}+t+1\Big\}\overset{\frac{\nu-1}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho}^{\nu}\sigma(i_\sigma+r_\sigma)+N-n+t+1\Big\}\Big]\overset{\frac{\nu-1}{2}}{\underset{1}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{\sum_{2\rho}^{\nu}\sigma(i_\sigma+r_\sigma)+r_{2\rho-1}+N-n+t+1} \tag{7}$$

As a particular case, when the twist and untwist occurs λ times cyclically within a constant amplitude $\mathfrak{A}$, with origin as centre and the couple not remaining constant at the moment of reversal—i.e. $r_\rho=o$, the amount of yielding at the increasing stage p is given by

$$\eta_p=k\log\frac{\Gamma\Big\{4(\lambda+1)\mathfrak{A}+p+1\Big\}\overset{\lambda}{\underset{0}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{4(\lambda+1)\mathfrak{A}-(4\rho+3)\mathfrak{A}+p+1\Big\}\Big]^2}{\overset{\lambda}{\underset{0}{\overline{\left|\rho\right|}}}\Big[\Gamma\Big\{4(\lambda+1)\mathfrak{A}-(4\rho+1)\mathfrak{A}+p+1\Big\}\Big]^2} \tag{8}$$

Let us now investigate some general properties of this hysteresis function in the following pages ;—To begin with, it is necessary to remark that the strain corresponding to any stress is here supposed to consist of two terms, of which the principal term is simply proportional to the actual stress while the other is given by the hysteresis function, η ; thus

$$\tau=\tau_0+\eta. \qquad (9).$$

Let the specimen be freed from any external couple after it has been once acted upon by a couple whose amount is given by p, so that in the equation (4) $n=p$, and then let it be again acted upon by a couple. The yielding *apparently due to* unit couple, no matter whether it be positive or negative, is given by

$$\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=n}=k\log\frac{\{2p+r+1\}}{\{p+r+1\}\{p+1\}}=-\tau_2 \quad \text{say} \qquad (10).$$

which is always negative. Now, if the newly applied couple be negative, the principal term due to this negative couple is necessarily negative ; let it be $-\tau_1$ so that the *increase of twist* apparently due to this negative unit couple is

$$\tau_n=-\{\tau_1+\tau_2\} \qquad (11).$$

On the other hand, if it be positive, the principal term is also positive and, by Hooke's law, equal to $+\tau_1$; so that the *increase of twist* due to this positive couple is

$$\tau_p=\tau_1-\tau_2 \qquad (12).$$

Thus, the absolute value of τ_n being greater than that of τ_p, we have following interesting result :

Proposition I. *If after withdrawing the whole couple applied to a piece of rock, we begin to reapply it, the specimen must apparently be more rigid in one direction than in the other.*

In the equation (5) put $p=n$, and we have

$$\sigma=k\log\frac{\Gamma\{2p+r+t+1\}\Gamma\{t+1\}}{\Gamma\{p+r+t+1\}\Gamma\{p+t+1\}} \qquad (13).$$

which is the *expression for the residual* surviving the couple, since in this case n being equal to p the specimen is free from any external couple.

Taking the difference of σ for two consecutive values of t, we have

$$\frac{\triangle\sigma}{\triangle t}=k\, log\frac{\{2p+r+t+1\}\{t+1\}}{\{p+r+t+1\}\{p+t+1\}} \quad <0 \qquad (14).$$

which is evidently negative, because any rectangle is smaller than a square when their periphery is given. Also, the limiting value of σ as well as of $\frac{\triangle\sigma}{\triangle t}$ is zero for $t=\infty$. Thus we have.

PROPOSITION II. *The residual diminishes with the lapse of time and ultimately disappears wholly after an infinite time. Consequently, the yielding is elastic, and recovers wholly if the couple is removed for an infinite duration.*

The differential for the unit increase of r is

$$\frac{\triangle\sigma}{\triangle r}=k\, log\frac{\{2p+r+t+1\}}{\{p+r+t+1\}} \quad >0 \qquad (15).$$

which is positive since the fraction is greater than unity; but the second differential being negative. We have.

PROPOSITION III. *The residual—so-called set—increases with the increase of the time during which the couple acted on the specimen.*

Here it must be remarked that the *expression for the recovery* may be deduced from that for the residual. It is given by

$$\rho=k\, log\frac{\Gamma\{2p+r+1\}\Gamma\{p+r+t+1\}\Gamma\{p+t+1\}}{\Gamma\{p+r+1\}\Gamma\{p+1\}\Gamma\{2p+r+t+1\}\Gamma\{t+1\}} \qquad (16).$$

where r expresses the time-element during which the specimen remained acted by a constant couple p. Thus it is evident that the curve of recovery is a little different from that of yielding whose equation is (1). For a particular case, $p=10$ and $r=0$, the curve of recovery is traced in Fig. 27, Pl. VIII.

Again, from the equation (8), the difference between the yieldings at the two consecutive stages p and $p+1$ is

$$\frac{\Delta\eta}{\Delta p}=k\log\frac{\left\{4(\lambda+1)\mathfrak{A}+p+1\right\}\prod_{\rho=0}^{\lambda}\left[4(\lambda+1)\mathfrak{A}-(4\rho+3)\mathfrak{A}+p+1\right]^2}{\prod_{\rho=0}^{\lambda}\left[4(\lambda+1)\mathfrak{A}-(4\rho+1)\mathfrak{A}+p+1\right]^2}=k\log\varepsilon \quad \text{say.} \tag{17}$$

Putting $p=-\mathfrak{A}$, the expression for ε may be written in a form, if we write $\mathfrak{B}$ for $4(\lambda+1)\mathfrak{A}$,

$$\varepsilon=\frac{\{\mathfrak{B}+1-\mathfrak{A}\}\{\mathfrak{B}+1-4\mathfrak{A}\}}{\{\mathfrak{B}+1-2\mathfrak{A}\}^2}\;\frac{\{\mathfrak{B}+1-4\mathfrak{A}\}\{\mathfrak{B}+1-8\mathfrak{A}\}}{\{\mathfrak{B}+1-6\mathfrak{A}\}^2}\cdots\cdots\frac{\{1+4\mathfrak{A}\}\;1}{\{1+2\mathfrak{A}\}^2}<1$$

which is less than unity since each group of the component fractions is less than unity; whence for this value of p,

$$\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=-\mathfrak{A}}<0. \tag{18}$$

Putting $p=\mathfrak{A}$, it may also be written in a form

$$\varepsilon=\frac{\{\mathfrak{B}+1+\mathfrak{A}\}}{\{\mathfrak{B}+1\}}\;\frac{\{\mathfrak{B}+1-2\mathfrak{A}\}^2}{\{\mathfrak{B}+1\}\{\mathfrak{B}+1-4\mathfrak{A}\}}\cdots\cdots\frac{\{1+6\mathfrak{A}\}^2}{\{1+8\mathfrak{A}\}\{1+4\mathfrak{A}\}}\;\frac{\{1+2\mathfrak{A}\}^2}{\{1+4\mathfrak{A}\}\;1}>1$$

which is greater than unity, so that for this value of p we have

$$\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=\mathfrak{A}}>0 \tag{18'}$$

From what has been proved just above, (18) and (18′), we may infer

PROPOSITION IV. *When a specimen is twisted cyclically, the twist may increase notwithstanding the decrease of the applied couple, and vice versa.*

The equation (17) may be written in a form, if we put $\mathfrak{R}=4(\lambda+1)\mathfrak{A}+p+1$ for the sake of brevity,

$$\frac{\Delta\eta}{\Delta p}=k\log\frac{\mathfrak{R}\{\mathfrak{R}-3\mathfrak{A}\}^2\cdots\cdots\{p+1+\mathfrak{A}\}^2}{\{\mathfrak{R}-\mathfrak{A}\}^2\{\mathfrak{R}-5\mathfrak{A}\}^2\cdots\cdots\{p+1+3\mathfrak{A}\}^2}$$

whose differential with respect to the amplitude $\mathfrak{A}$ is

$$\frac{\Delta^2\eta}{\Delta\mathfrak{A}\Delta p}=k\ log\Big[\frac{\{\mathfrak{R}+4(\lambda+1)\}\{\mathfrak{R}-\mathfrak{A}\}^2\{\mathfrak{R}+4(\lambda+1)-3(\mathfrak{A}+1)\}^2\ldots\ldots\ldots}{\mathfrak{R}\{\mathfrak{R}+4(\lambda+1)-(\mathfrak{A}+1)\}^2\{\mathfrak{R}-3\mathfrak{A}\}^2\ldots\ldots\ldots}$$

$$\frac{\ldots\{p+1+3\mathfrak{A}\}^2\{p+1+\mathfrak{A}+1\}^2}{\ldots\{p+1+3(\mathfrak{A}+1)\}^2\{p+1+\mathfrak{A}\}^2}\Big]>0 \qquad (19).$$

which is always positive since we have

$$\frac{\mathfrak{R}+4(\lambda+1)}{\mathfrak{R}}>1\,;$$

$$\frac{\{\mathfrak{R}-\mathfrak{A}\}\{\mathfrak{R}+4(\lambda+1)-3(\mathfrak{A}+1)\}}{\{\mathfrak{R}+4(\lambda+1)-(\mathfrak{A}+1)\}\{\mathfrak{R}-3\mathfrak{A}\}}=1+\frac{6\mathfrak{A}-(p+1)}{\{\mathfrak{R}-\mathfrak{A}+4\lambda+3\}\{\mathfrak{R}-3\mathfrak{A}\}}>1\,;$$

etc. etc.

PROPOSITION V. *When the centre of cycle is fixed, the hysteresis curve at any stage p after λ cycles becomes more and more steep when the amplitude of the cycle becomes greater and greater.*

Let the number of cycles be increased from λ to $\lambda+1$, then we have

$$\frac{\Delta^2\eta}{\Delta\lambda\Delta p}=k\ log\frac{\{(4\lambda+6)\mathfrak{A}+p+1\}\{[(4\lambda+6)\mathfrak{A}+p+1]^2-3\mathfrak{A}^2\}+2\mathfrak{A}^3}{\{(4\lambda+6)\mathfrak{A}+p+1\}\{[(4\lambda+6)\mathfrak{A}+p+1]^2-3\mathfrak{A}^2\}-2\mathfrak{A}^3}>0 \qquad (20).$$

which is necessarily positive. Again take a further differential :—

$$\frac{\Delta^3\eta}{\{\Delta\lambda\}^2\Delta p}=k\ log\frac{\{\mathfrak{R}^2+8\mathfrak{A}\mathfrak{R}\}\{\mathfrak{R}^2+8\mathfrak{A}\mathfrak{R}+15\mathfrak{A}^2\}^2}{\{\mathfrak{R}^2+8\mathfrak{A}\mathfrak{R}+7\mathfrak{A}^2\}^2\{\mathfrak{R}^2+8\mathfrak{A}\mathfrak{R}+16\mathfrak{A}^2\}^2}<0 \qquad (21).$$

which is evidently negative.

From what has been just proved, (20) and (21), follows

PROPOSITION VI. *Provided the centre of cycle is fixed at origin, the hysteresis curve becomes more and more steep when the cycle is repeated over and over again and the curve asymptotically approaches a closed one.*

Suppose that after $\varLambda$ cycles of the amplitude $\mathfrak{A}$ about the neutral state, another smaller cycle of the amplitude $\mathfrak{a}$, whose centre is situated at j, is completed λ times, and then it is just in the stage p. Then, the yielding at this instant is, by the general equation (6), writing $\mathfrak{D}$ for $4\varLambda\mathfrak{A}+4(\lambda+1)\mathfrak{a}+1$,

$$\eta=k\log\frac{\Gamma\{\mathfrak{D}-p\}\left[\Gamma\{\mathfrak{D}-p-3\mathfrak{A}\}\right]^2\cdots\cdots\left[\Gamma\{(4\lambda+1)\mathfrak{a}+j-p+1\}\right]^2\cdots\left[\Gamma\{\mathfrak{a}+j-p+1\}\right]^2}{\left[\Gamma\{\mathfrak{D}-p-\mathfrak{A}\}\right]^2\left[\Gamma\{\mathfrak{D}-p-5\mathfrak{A}\}\right]^2\cdots\left[\Gamma\{(4\lambda+3)\mathfrak{a}+j-p+1\}\right]^2\cdots\left[\Gamma\{3\mathfrak{a}+j-p+1\}\right]^2} \qquad (22).$$

Hence the difference between the amounts of yielding at two consecutive stages is

$$\frac{\Delta\eta}{\Delta p}=k\log\frac{\{\mathfrak{D}-p\}\left[\{\mathfrak{D}-p-3\mathfrak{A}\}\right]^2\cdots\cdots\left[4(\lambda+1)\mathfrak{a}+j-p+1\right]^2\cdots\cdots\left[\mathfrak{a}+j-p+1\right]^2}{\left[\mathfrak{D}-p-\mathfrak{A}\right]^2\left[\mathfrak{D}-p-5\mathfrak{A}\right]^2\cdots\cdots\cdots\cdots\cdots\left[3\mathfrak{a}+j-p+1\right]^2} \qquad (23).$$

Putting herein $p=j$, we have the increase of yielding at the stage corresponding to the centre of the smaller cycle in its increasing stage. The result is

$$\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=j}=k\log\frac{\{\mathfrak{D}-j\}\left[\mathfrak{D}-j-3\mathfrak{A}\right]^2\cdots\cdots\left[4(\lambda+1)\mathfrak{a}+1\right]^2\cdots\cdots\left[\mathfrak{a}+1\right]^2}{\left[\mathfrak{D}-j-\mathfrak{A}\right]^2\left[\mathfrak{D}-j-5\mathfrak{A}\right]^2\cdots\cdots\cdots\cdots\left[3\mathfrak{a}+1\right]^2}\lesseqqgtr 0 \qquad (24).$$

whose sign wholly depends on the relative values of $\mathfrak{A}$, $\mathfrak{a}$ and j. This is a more general case of the fourth proposition.

Taking its differential with respect to j, we have

$$\frac{\Delta}{\Delta j}\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=j}=k\log\frac{\{\mathfrak{D}-j-1\}\left[\mathfrak{D}-j-\mathfrak{A}\right]^2\left[\mathfrak{D}-j-1-3\mathfrak{A}\right]^2\cdots\cdots\left[\mathfrak{A}+4(\lambda+1)\mathfrak{a}-j-1\right]^2}{\{\mathfrak{D}-j\}\left[\mathfrak{D}-j-1-\mathfrak{A}\right]^2\left[\mathfrak{D}-j-3\mathfrak{A}\right]^2\cdots\cdots\left[\mathfrak{A}+4(\lambda+1)\mathfrak{a}-j\right]^2}<0 \qquad (25).$$

which is always negative since we have

$$\frac{\mathfrak{D}-j-1}{\mathfrak{D}-j}<1\,;\ \frac{\{\mathfrak{D}-j-\mathfrak{A}\}\{\mathfrak{D}-j-1-3\mathfrak{A}\}}{\{\mathfrak{D}-j-1-\mathfrak{A}\}\{\mathfrak{D}-j-3\mathfrak{A}\}}=\frac{H}{H+2\mathfrak{A}}<1\,;\ \text{etc. etc.}$$

PROPOSITION VII. *The hysteresis curve, whose amplitude is given, tends to become more and more horizontal when its centre becomes more and more remote from the neutral state of the specimen.*

In the equation (8) put $\lambda=0$ and $p=0$, then the value of η, which corresponds to the residual after the first cycle, is

$$\eta_1=k\log\frac{\Gamma(4\mathfrak{A}+1)[\Gamma(\mathfrak{A}+1)]^2}{[\Gamma(3\mathfrak{A}+1)]^2}$$

while, putting $\lambda=1$ and $p=0$, the residual after the second cycle is

$$\eta_2 = k log \frac{\Gamma(8\mathfrak{A}+1)[\Gamma(5\mathfrak{A}+1)]^2[\Gamma(\mathfrak{A}+1)]^2}{[\Gamma(7\mathfrak{A}+1)]^2[\Gamma(3\mathfrak{A}+1)]^2};$$

so that the difference of the two is

$$\eta'_2 = k\ log \frac{\Gamma(8\mathfrak{A}+1)[\Gamma(5\mathfrak{A}+1)]^2}{[\Gamma(7\mathfrak{A}+1)]^2\Gamma(4\mathfrak{A}+1)}.$$

Now, the value given by η_1 is what is called *set* by engineers and the value of η'_2 is its increase caused by the second cycle. Comparing the absolute values of these two, we easily see that the latter is very small. In a particular case, indeed, where $\mathfrak{A}=10$, the value of η_1 is $\{3{\cdot}825\ klog10\}$ while that of η'_2 is only $\{0{\cdot}238\ klog10\}$, which falls almost within the limits of error of observation, or which may be neglected without any serious error. Thus we have

PROPOSITION VIII. *Suppose that we give a set to a specimen by twisting it through a definite angle. A second twisting through the same angle causes little or no further set.*

Lastly, referring to the equations (6) and (7) we see that, since η is a function of several variables, it may have any value whatever, within certain limits, when the amount of couple actually acting upon the specimen i.e. p or n is given. Also, from (17) we see that, when p or n and η are given, $\frac{\Delta\eta}{\Delta p}$ or $\frac{\Delta\eta}{\Delta n}$ may have several different values. Thus we have :

PROPOSITION IX. *Not only may the specimen be brought to any twisted state, within wide limits, by a given definite couple, but it may have more than one gradient in passing through that state.*

As a particular case of the above, if both p or n and η are zero, the specimen is actually free from any external couple and also it is free from any residual. In every respect, there is no external difference between such a piece and a virgin one. Tested with couple, however, it retains latent traces of the twist from which it was lately released, and it is much more easily twisted in one direction than in the other. This is shown by the fact that $\frac{\Delta\eta}{\Delta p}$ and $\frac{\Delta\eta}{\Delta n}$ have different values at this stage. Moreover, in this case, though both p or n and η are zero, $\frac{\Delta\eta}{\Delta t}$ is not generally zero ; so that we have

PROPOSITION X. *A lately twisted specimen, which is actually in the non-twisted state, becomes gradually twisted with the lapse of time without any application of external couple.*

After all, a specimen which has been, twisted at least once can not be really neutral one like a virgin piece. The neutral state is the state of *no internal stress* which is the same as that which Saint-Venant used as a means of deducing the uniqueness of the solution of the elastic equations under the name " L'état dit naturel ou primitif." An interesting explanation for the internal state of molecular equiliblium by Sir W. Thomson(1) may be cited here ; "........the outer particles will be strained in the direction opposite to that in which it was twisted, and the inner ones in the same direction as that of the twisting, the two sets of opposite couples thus produced among the particles of the bar balancing one another."

The further nature of the Hysteresis function is to be most clearly comprehended by tracing the curve representing it. The result of a laborious calculation is graphically shown in the figure in Pl. XIV. It represents the Hysteresis function for a particular case ;—$\mathfrak{A}=10$; $\lambda=0, 1, 2$; and $-10 \leqq (p \text{ or } n) \leqq +10$, as well as for two other cases where the amplitude for the one and the centre for the other was changed for different cycles.

As to the Hysteresis curve, it may be necessary to remark that,

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(1) Sir W. Thomson. Mathematical and Physical papers. Vol. III.

since the twist consists of two terms, the form of any actual stress-strain curve must be greatly affected in its features by the relative value of the two terms. When the first term, which fulfils Hooke's law, is negligible relatively to the yielding, the curve takes the form given in Pl. XIV. On the other hand, if the yielding is negligible as compared with the principal term, then, as a matter of course, the curve shrinks into a straight line. For any other relative values, the curve takes an intermediate form. Figures in Pl. XV., which have been drawn from the results of actual calculation, show several intermediate forms. The inclined straight line in each figure represents the curve to which it shrinks if the yielding be disregarded. In the following pages, all these mathematical deductions will be experimentally proved.

VERIFICATION BY EXPERIMENTS.

On gradually releasing a twisted piece of rock from the couple, the specimen shows a tendency to persist in its twisted state, so that there remains some residual twist when all the couple is withdrawn. Moreover, if after withdrawing the whole couple, we begin to reapply it, we find that the specimen is more rigid in one direction than in the other. These facts are graphically shown in Pl. XVI. for the case of sandstone.

When a piece of rock is twisted cyclically within any definite limits of couple, its stress-strain curve approaches a closed one of simple and regular form. The on-curve, however, differs entirely from the off-curve, though they are so related to each other that one, when it is turned through two right angles, nearly coincides with the other. These facts will be better understood from the curves traced in Figs 35-46, Pls. XVII. and XVIII.

The figure in Pl. XVI. gives further illustrations of the effects of torsional hysteresis in causing a loop to be formed on the curves of twist when the couple experiences several cyclic changes with different amplitudes and centres. Here it is to be seen that the greater the amplitude of the cycle, the more the loop tends to become vertical.

Starting from a strained state, a specimen was twisted cyclically, the centre of each cycle being different from that of the others. The result is shown in Pl. XIX., all of the loops produced by the several cycles being nearly similar to one another, though they are widely different in their dimensions. We see, however, that when the amplitude of the cycle is given with regard to the amount of the applied couple, the loop whose centre is further from zero-stress is more horizontal than the other.

Other particulars with respect to the natures of the torsional hysteresis will be learned from the several figures in the annexed plates which have been carefully drawn from experimental results. Comparing the figures in Pls. XVII. and XVIII. with those in Pl. XV., we can not but be struck with the very close coincidence between theory and experiment. Here it is necessary to remark that, though a piece which has been twisted at least once can not be really neutral, yet the process of demagnetization in the case of magnetic hysteresis suggests a means by which any twisted piece can again be brought to a practically neutral state. One example of this process is shown in Fig. 48, Pl. XX. To begin with, in any of the above experiments, this process was applied to get rid of the latent traces of previous history which the specimen may have acquired.

To give a general idea of the comparative magnitudes of the rigidity of several rocks, some of the stress-strain curves for different rocks are drawn, Fig. 49, Pl. XXI., on one and the same scale as that for soft iron. Corrections due to some differences between their respective dimensions are not taken into account. The horizontal line corresponds to a rigid substance, while the vertical, to an ideal fluid. Numerical values of the modulus of rigidity for different rocks are also added in the annexed table. They serve, however, only to indicate their order of magnitude.

TABLE III.

No.	Rock	Locality	Kind	Mod. of Rigidity	Density	Velocity of Trans. W.
			ARCHÆAN ROCKS.	$\times 10^{10}$		Kilom.÷Sec.
18	Chlorite-schist	Chichibu	Metamorphic	20·0—24·1	2·82	2·70—2·89
8	Serpentine	Chichibu	Eruptive (Altered)	24·8—52·2	2·71	3·03—4·40
31	Quartz-schist	Gumma	Metamorphic	24·5—28·9	2·64	3·06—3·29
42	Mica-schist	Ibaraki	Metamorphic	1·77	2·64	0.82
7	Peridotite (Serpentinized)	Kuji	Eruptive (Altered)	22·3	2·61	2·93
			PALÆOZOIC ROCKS.			
32	Pyroxenite	Gumma	Sedimentary (Metamorphized)	23·0—49·0	2·90	2·82—4·33
12	Clay-slate	Shiga	Sedimentary	2·93	2·74	1·03
21	Limestone	Chichibu	Sedimentary ?	7·73	2·64	1.71
6	Marble	Kuji	Sedimentary (Metamorphized)	8·63—9·15	2·64	1·82—1·85
13	Clay-slate	Shiga	Sedimentary	10·82	2·58	2·05
10	Granite	Kagawa	Eruptive	5·71	2·57	1·49
9	Granite	Mikage	Eruptive	16·9	2·54	2·58
14	Red Schalstein	Shiga	Sedimentary	7·79	2·43	1·79
			TERTIARY ROCKS.			
5	Rhyolite	Yechizen	Eruptive	2·74—3·12	2·36	1·07—1·16
4	Sandstone	Ki-i	Sedimentary	0·72—1·25	2·21	0·57—0·94
3	Sandstone	Chiba	Sedimentary	0·41—2·64	2·20	0·43—1·09
2	Tuff	Izu	Sedimentary	5·73—6·18	1·91	1·74—1·80
			DILUVIUM ROCKS.			
17	Andesite	Gumma	Eruptive	8·09	2·63	1·75
1	Andesite	Sagami	Eruptive	8·02	2·59	1·76
16	Andesite (Porous)	Gumma	Eruptive	2·17	2·32	0·97

THE EFFECT OF TEMPERATURE.

The present arrangement being unsuitable for heating and cooling the specimen, my intention was rather to ascertain only the order of the experimental error due to the temperature-variation in the laboratory. To get rid of the effect of the yielding, the specimen to be tested was subjected to a constant couple during three days and nights. Then, the creeping due to the yielding having become comparatively small, the temperature-variation of the twist was observed. The time-variations of both the temperature and the twist were, as they are shown in Figs. 50–52 in Pl. XXII., similar to each other. In Fig. 53 in the same plate, we see the increase of twist plotted against the corresponding rise of temperature. The curve, as a whole, expresses the *simple proportionality between the two elements.* We find, however, one remarkable fact that the temperature-variation of the twist has a *minimum value* in the neighbourhood of 9°C. It may be questionable whether this is a general property of all rocks or merely special to that specimen alone. By the way, it may be cited here that, from the experiment of G. Wertheim(1), the stretch modulus of iron and steel seems to have a maximum value at or a little below 100°C., since at 200° it is sensibly the same as at —15°.

Lastly, we must not neglect to remark here that though the temperature-variation of the twist is unexpectedly great, yet it is almost negligible as compared with the total amount of the twist and also with that of the yielding. Indeed, the increase of twist per degree of temperature-rise is less than 3×10^{-3} of its total amount in sandstone.

(1) G. Wertheim. Annales de chémie. T. XV. 1845.

APPENDIX.

Relation between the Velocity of Propagation and the Amplitude of Seismic Waves.

It is a well known fact that, in any earthquake, the principal shocks are always preceded by tremors of small amplitude. On the supposition that the waves of the tremors and of the principal shocks are all generated simultaneously at one and the same origin, the above facts show that the velocities of the tremors are much greater than that of the principal shocks. Indeed, according to Professor Omori[1], the velocities of the first and second tremors are equal to 12·8 and 7·2 kilometres per second, while that of the principal shocks is only about 3·3 kilometres per second. An explanation of this fact is given by Professor H. Nagaoka in his essay above cited, from the consideration that there exists a stratum of maximum velocity of propagation in the earth's crust. The causes of this simple fact must possibly be of a very complicated nature. In so far, however, as the wave-velocity is a function of the elastic constants, the elastic yielding can never be disregarded in the determination of the velocities of seismic waves. In the case of sound waves, the *Newtonian Velocity* deduced from Boyle's law is much smaller than that found by observation. Taking, however, the heating by condensation and cooling by rarefaction into account, Laplace obtained a result agreeing with experiment. So, in the case of seismic waves, the greater part, at least, of the discrepancy must be due to the assumption of Hooke's law. Even in the weakest earthquake, the strain associated with the principal shocks must be, I think, far beyond the limits of elasticity.

If what I have stated above be the case, the effect of the yielding

(1) F. Ōmori. Publi. of the E. I. Committee in Foreign Languages. No. 5. 1901.

would be, at least, *one of the principal causes of the diminution of velocity*. In the equation (17) put $p=0$ and $\lambda=1$, then

$$\left\{\frac{\Delta\eta}{\Delta p}\right\}_{p=0,\ \lambda=1}=k\ log\frac{\{8\mathfrak{A}+1\}[5\mathfrak{A}+1]^2[\mathfrak{A}+1]^2}{[7\mathfrak{A}+1]^2[3\mathfrak{A}+1]^2}=\tau_2 \quad \text{say.}$$

This value of $\frac{\Delta\eta}{\Delta p}$, expressed as a function of the amplitude $\mathfrak{A}$, represents the amount of the increase of twist due to the yielding, per unit increase of couple in the stage of zero couple after one cycle of twisting through an amplitude $\mathfrak{A}$. Now, the actual amount of twist is the sum of two terms ; i.e.

$$\tau=\tau_1+\tau_2$$

where τ_1 is the term which fulfils Hooke's law. Let v and c be the velocity and factor of proportionality respectively, then we have

$$v=c\{\tau_1+\tau_2\}^{-\frac{1}{2}}=v_0\left\{1+\frac{\tau_2}{\tau_1}\right\}^{-\frac{1}{2}}$$

If the yielding be disregarded, then neglecting $\frac{\tau_2}{\tau_1}$ we have

$$v=v_0$$

so that *all waves with different amplitudes propagate with a common velocity if, and only if, the stratum of rocks through which they propagate does not yield.*

When the yielding is enormous, which is the case for a loose stratum such as the earth's crust, the velocity decreases very rapidly when the amplitude of the wave increases. The quantitative relation between these two elements is given in the following table.

TABLE IV.

$\mathfrak{A}$ Particular unit	1	2	3	4	5	6	7	8	9	10
$\frac{v}{v_0}$ (Ideal)	1·00	0·68	0·55	0·50	0·46	0·44	0·42	0·40	0.39	0·38
$\frac{v}{v_0}$ (Marble)	1·00	0·96	0·94	0·92	0·91	0·90	0·89	0·88	0·87	0·86

The values of $\frac{v}{v_0}$ given in the second row correspond to the case where one may be neglected as compared with $\frac{\tau_2}{\tau_1}$; while those in the last row correspond to the case where the waves propagate through an infinitely extended uniform stratum of marble. The last is calculated using a value of k determined experimentally, the result being traced in Fig. 38, Pl. XVIII.

Quite independently of the above hysteresis function, the ratio of velocities of several waves through a stratum of serpentine and pyroxenite are obtained from the experiments shown in Pls. IV_a. and IV_b.

TABLE V:

Ratio of Amplitudes.		1	30	75	115	185
Ratio of Velocities.	Serpentine	1·00	0·78	0·73	0·71	0·71
	Pyroxenite.	1·00	0·76	0·71	0·70	0·69

Here we can not omit to remark that, though the values given in TABLE IV. may be absurd as it stands on the foundation of the hysteresis function, those in TABLE V. can not be disputed by any one even with the result of the most subtle analysis, provided that the foundamental formula, which expresses that the velocity varies as the square root of the rigidity-modulus of the medium through which the wave propagates, be granted.

By the equation (25), it can not be disregarded that the velocity of propagation increases when the stratum of rock, through which the wave propagates, is in a strained state. Thus, for example, the seismic wave would probably propagate more quickly along a mountain chain than in uniform plain land. This variation, however, is much smaller than the former.

From what has been above discussed, it may safely be said that of several waves, the one whose amplitude is smaller has necessarily a greater velocity than the other. When the yielding predominates, the velocity may become two or three times smaller as the amplitude becomes some ten times greater. The disturbance of smallest amplitude will first make its appearance as the beginning of the preliminary tremor, followed by waves of greater amplitudes in succession. Other disturbances, propagating through different strata or of other origin, would probably appear intermixed with the former, giving a somewhat irregular record on the seismograph.

In TABLE III., the velocities of propagation of transversal waves through several rocks are also added. They serve only to give a general idea of the order of magnitude, since the velocity is never constant for any specimen, as there exists more or less yielding.

In conclusion, I wish to express my great indebtedness to Mr. Fukuchi for valuable information concerning the geological and petrological character of the specimens examined in the present experiment. My best thanks, however, are due to Professor H. Nagaoka and also to Professor A. Tanakadate, without whose valuable advice and most kind guidance I could scarcely have succeeded in carrying out this experiment.

EXPLANATION OF PLATES.

Remarks. Unit of couple $= 6{\cdot}712 \times 10^6$ c.g.s. unit; unit of twist $= 1{\cdot}845 \times 10^{-3}$ radian; unit of time = one minute.

Pl. I.

Fig. 1. Preliminary experiment on Specimen No. 3_3 shows great deviation from Hooke's law and the existence of torsional hysteresis.

Pl. II.

Figs. 2 and 3. Front and side views respectively of the twisting apparatus.

„ 4. Plan of the whole arrangement.

„ 5. Framework carrying the mirror.

Pl. III.

Fig. 6. Curve A. Yielding of the Support. Curve B. Twist of Specimen No. 12_1. This shows that the result of observation is not disturbed by the yielding of the support.

Pl. IV_a. and IV_b.

Figs. 7 and 8. Effect of cyclical application of positive and negative couples with their several amplitudes. They show the existence of torsional hysteresis, even in the case where Hooke's law is tolerably fulfilled. Specimens, No. 32_1 and 8_2.

Pl. V.

Fig. 9. Relation between the negative couple required to annihilate the residual twist and the original couple. Deduced from the above observations.

„ 11a. Relation between the modulus of rigidity and the amplitude: Specimens No. 32_1 and 8_2.

„ 11b. Relation between the residual twist and the amplitude.

Pl. VI.

Fig. 10. Torsional hysteresis in Specimen No. 3_3. Showing the indefiniteness of the modulus of rigidity.

Pl. VII.

Figs. 12 and 13. On-curve and Off-curve respectively. Each series of observations admits of being connected by a straight line, so that Hooke's law is fulfilled. The difference of inclinations, however, shows the ambiguity of the modulus of rigidity. Specimen No. 4_2.

Pl. VIII.

Fig. 14. Yielding of Specimen No. 3_2 under constant couple. The resisting masses are different for the two curves.

„ 23 and 24. Recovery from the yielding. Specimen No. 4_1 was subjected to constant couple during a week.

„ 27. Theoretical curve for recovery, deduced from hysteresis function.

Pl. IX.

Fig. 15. Yielding of Specimen No. 4_1 under constant couples. Constant couple has different values for different curves.

„ 18. Theoretical curves for yielding.

Pl. X.

Figs. 16 and 17. Relation between the rate of yielding and the time-element. Specimen No. 4_1. Full lines are theoretical curves. Eight straight lines and hyperbolas should be drawn to correspond to all points.

Pl. XI.

Figs. 19-22. Yielding after a long time. Specimen No. 4_1.

Pl. XII.

Fig. 25. Effect of the elastic recovery on the cycles. The specimen, No. 3_3, was subjected under a constant couple during 306.5 hours, and then that strained state was taken as the centre of cycle.

Pl. XIII.

Fig. 26. Twisting and untwisting—no negative couple applied—of the specimen No. 3_3.

Pl. XIV.

Fig. 28. Hysteresis function (due to the elastic yielding).

Pl. XV.

Figs. 29-33. Examples of the hysteresis due to the elastic yielding, calculated from the hysteresis function by giving different values to the constant involved in the function. The inclined straight line in each curve shows how the curve shrinks if the yielding be disregarded.

Pl. XVI.

Fig. 34. Torsional hysteresis of a piece of sandstone. No. 3_3.

Pls. XVII. and XVIII.

Figs. 35-46. Examples of torsional hysteresis in several rocks of different kinds. Specimens: No. 3_2; 3_1; 5_2; 6_1; 6_2; 10_1; 9_1; 8_1; 7_2; 31_2; 18_1; 17_2. These are to be compared with the theoretical curves in Pl. XV.

Pl. XIX.

Fig. 47. This shows that the modulus of rigidity increases when the specimen is in a strained state. No. 3_3

Pl. XX.

Fig. 48. Neutralization of a non-virgin piece. No. 16_1.

Pl. XXI.

Fig. 49. Gradual diminution of rigidity, from rigid substance to perfect fluid. Axes of abscissa and ordinate correspond to rigid substance and perfect fluid respectively. Specimens: soft iron; No. 7_1; No. 17_2; No. 14_1; No. 2_1; No. 16_1; No. 3_1; No. 42_1.

Pl. XXII.

Figs. 50-52. In each figure, the upper curve shows the variation of temperature with time, while the lower that of twist, the couple being c-on stant. Specimen No. 4_1.

Fig. 53. Relaticn between the temperature and the amount of twist.

ERRATA.

Page	9	Line	4.	*For*	Pl. IV.	*read*	Pls. IV_a. and IV_b.
,,	11	,,	last.	,,	IV.	,,	IV_a., IV_b.
,,	14	,,	4.	,,	7 9	,,	7 : 9.
,,	18	,,	12	,,	reachad	,,	reached.
,,	40	,,	16.	,,	c-on	,,	con-

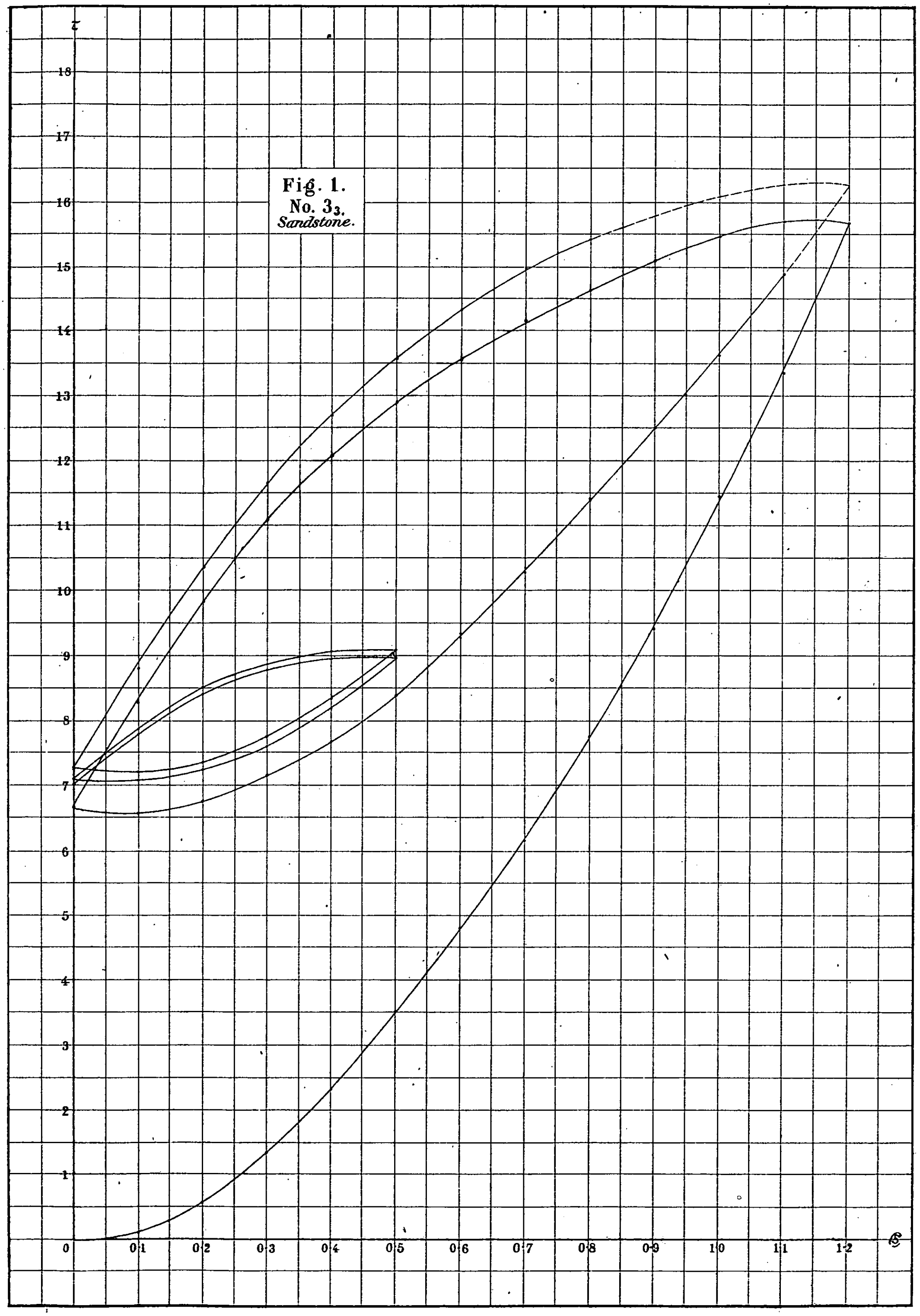


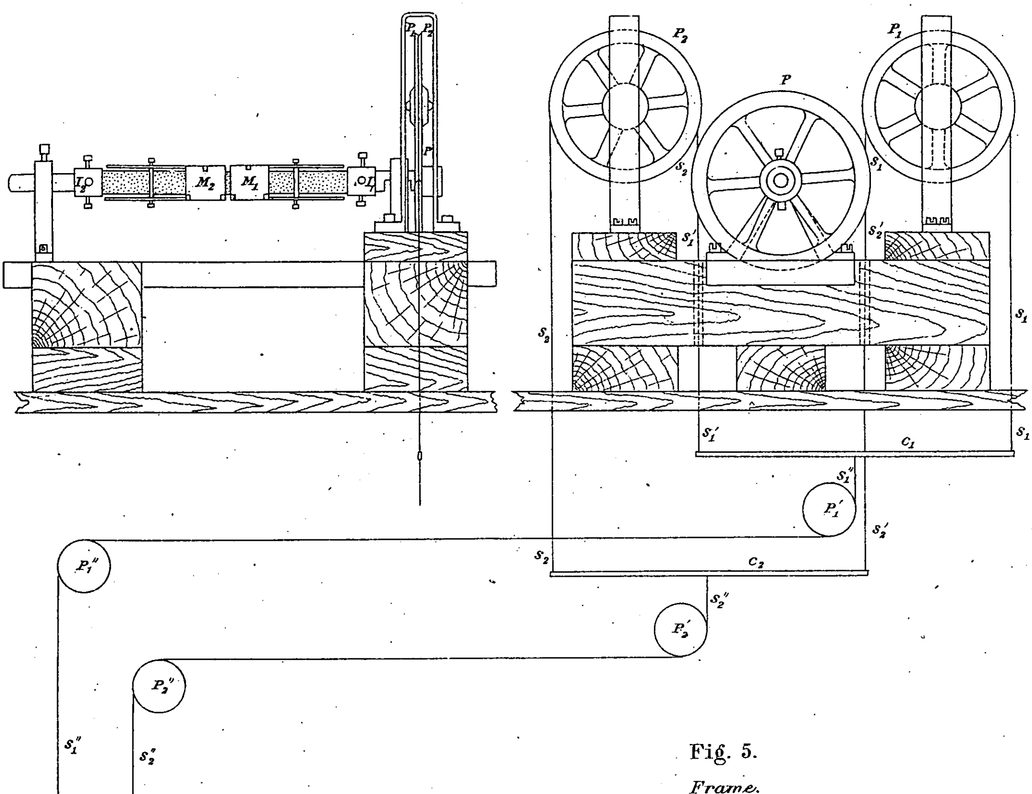
Fig. 1.
No. 3₃.
Sandstone.
τ
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
1·1
1·2
θ

Fig. 2.

Front-view

Fig. 3.

Side-view.



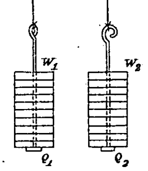


Fig. 5.

Frame.

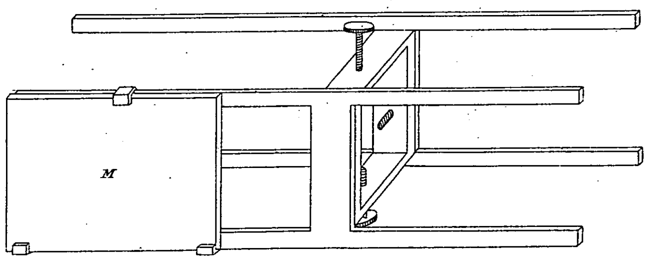


Fig. 4.

Plan of the arrangement.

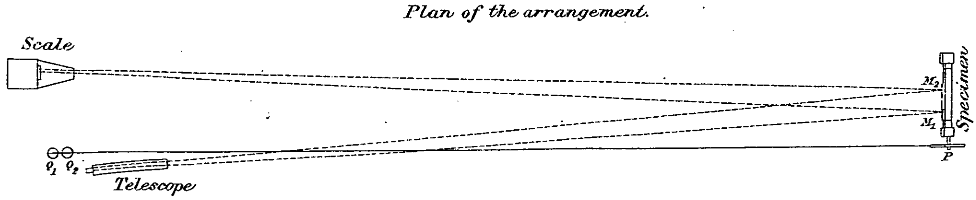
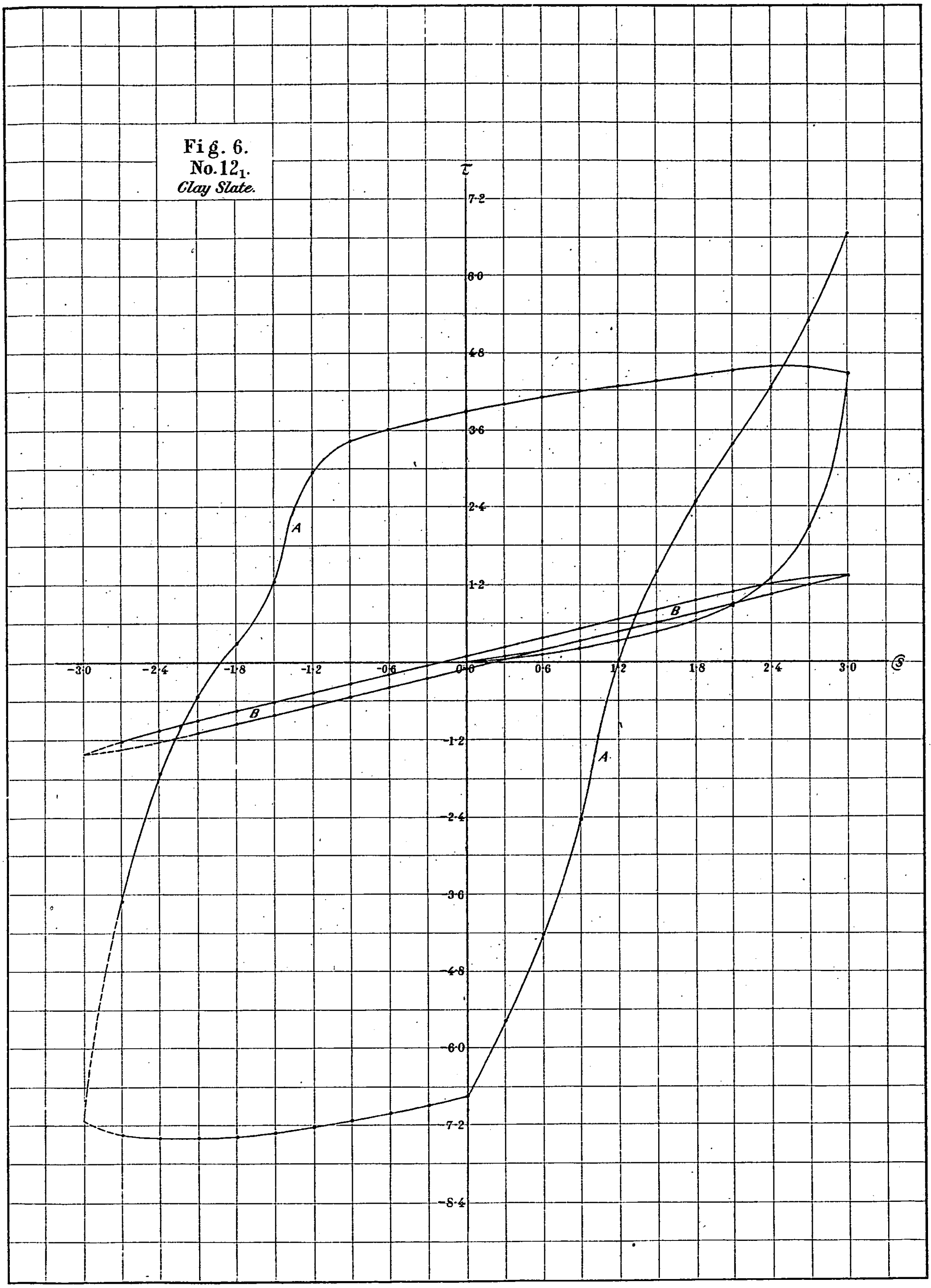
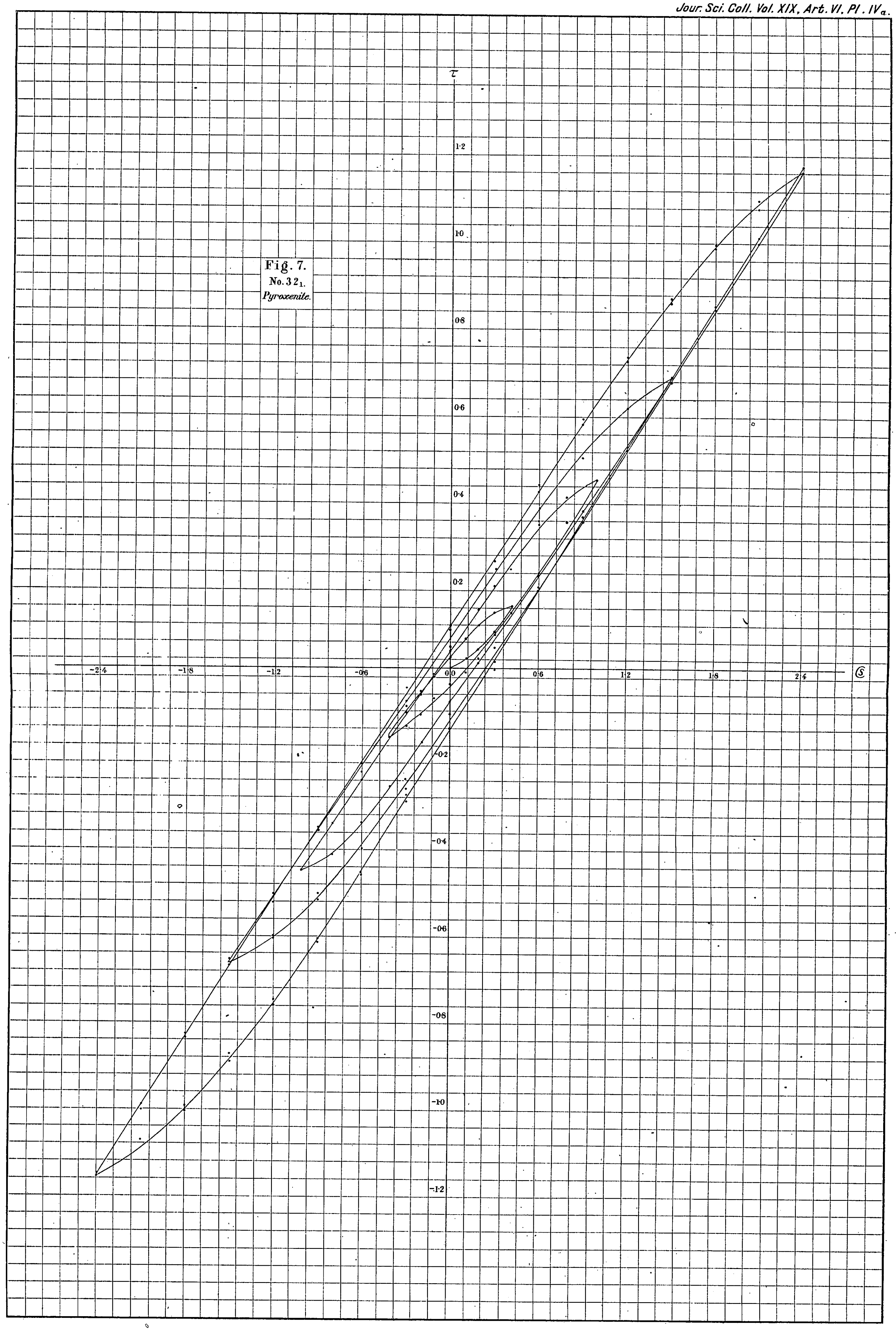


Fig. 6.
No. 12_1.
Clay Slate.




τ
1·2
1·0
0·8
0·6
0·4
0·2
0·0
-0·2
-0·4
-0·6
-0·8
-1·0
-1·2
-2·4
-1·8
-1·2
-0·6
0·6
1·2
1·8
2·4
𝔖
Fig. 7.
No. 3 2_1.
Pyroxenite.

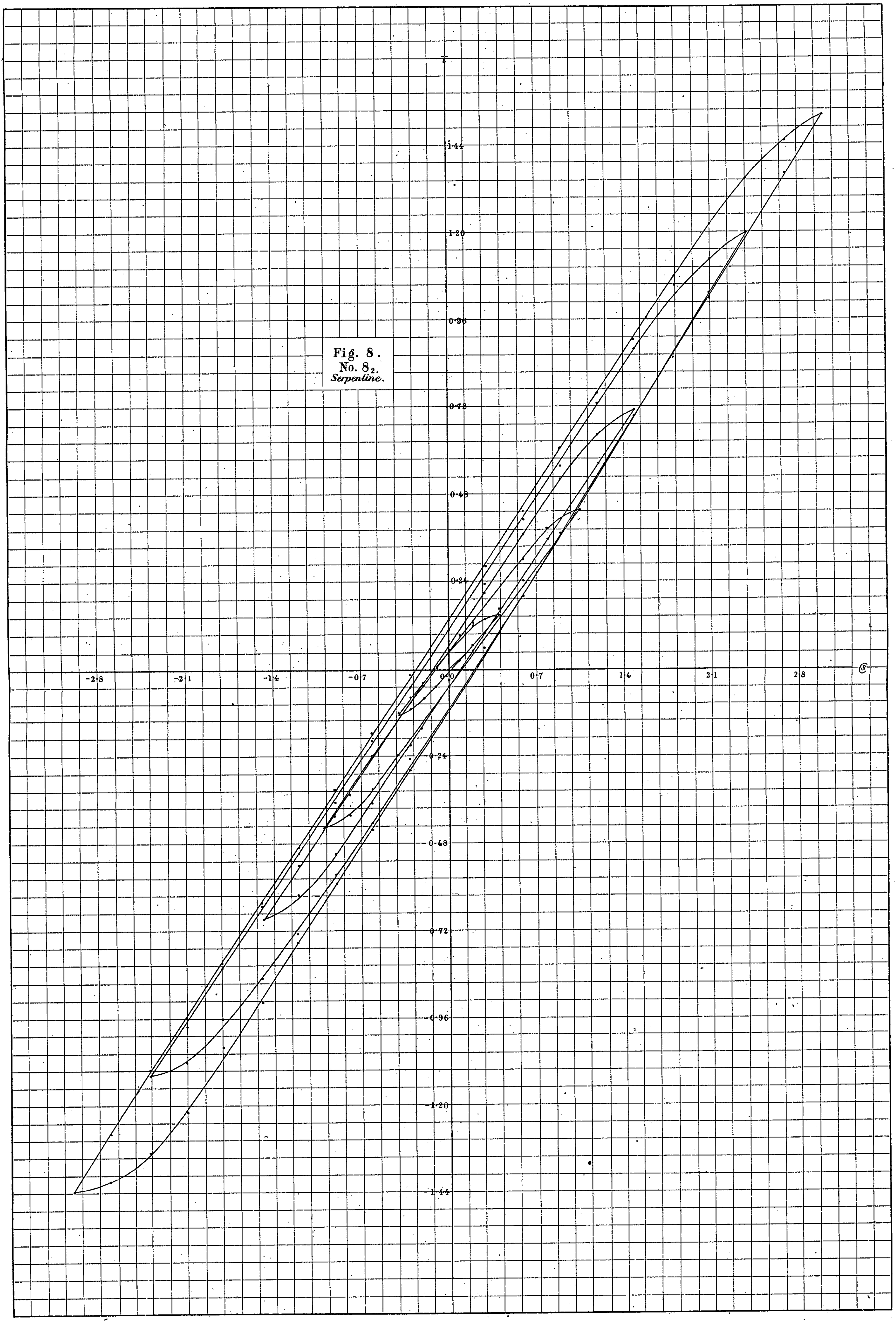
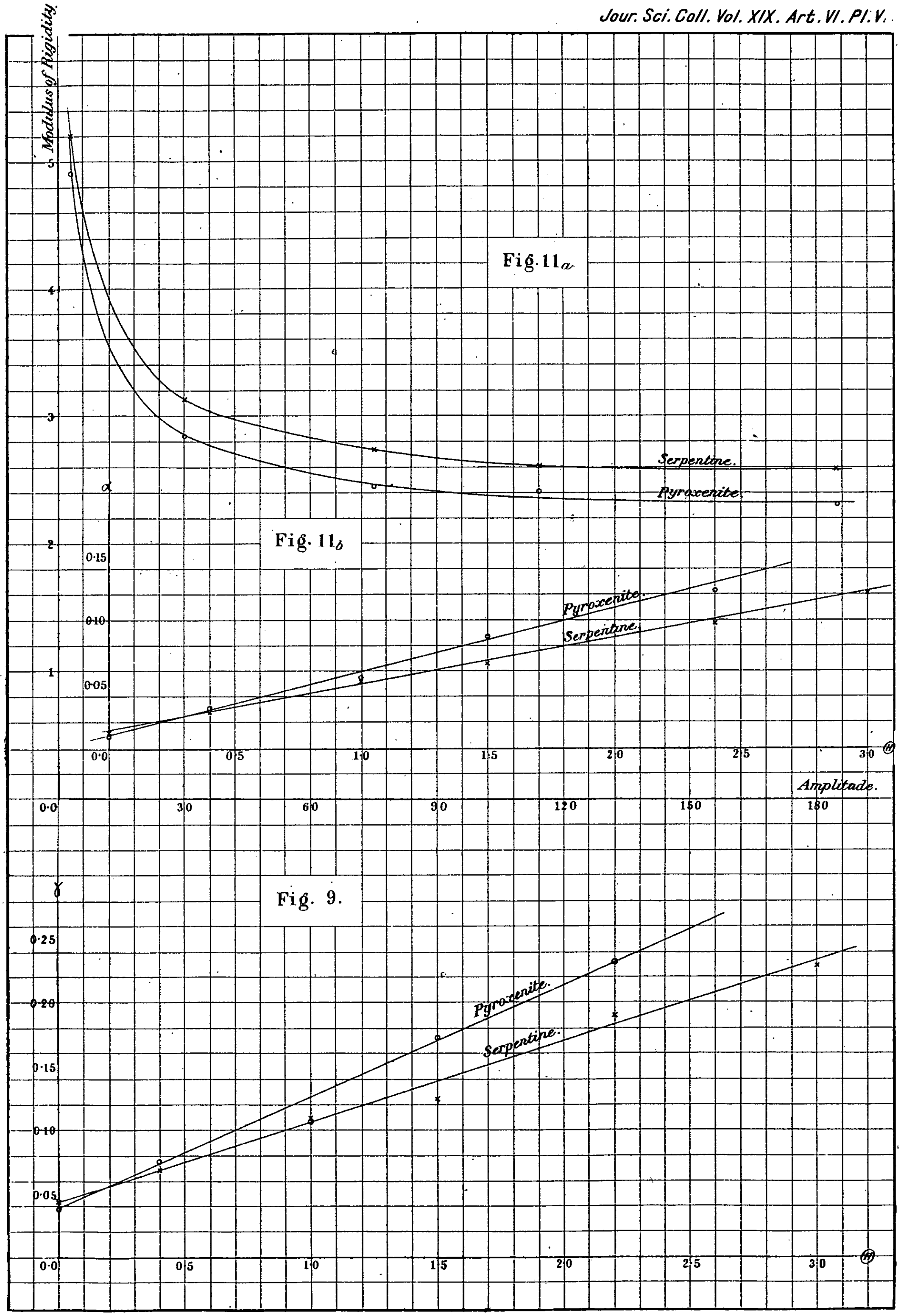


Fig. 8.
No. 8_2.
Serpentine.
τ
1·44
1·20
0·96
0·72
0·48
0·24
0·0
-0·24
-0·48
-0·72
-0·96
-1·20
-1·44
-2·8
-2·1
-1·4
-0·7
0·7
1·4
2·1
2·8
S


Modulus of Rigidity.
Fig. 11a
Serpentine.
Pyroxenite.
α
Fig. 11b
Pyroxenite.
Serpentine.
Amplitude.
Fig. 9.
γ
Pyroxenite.
Serpentine.
5
4
3
2
1
0·0
30
60
90
120
150
180
0·15
0·10
0·05
0·0
0·5
1·0
1·5
2·0
2·5
3·0
0·25
0·20
0·15
0·10
0·05
0·0
0·5
1·0
1·5
2·0
2·5
3·0

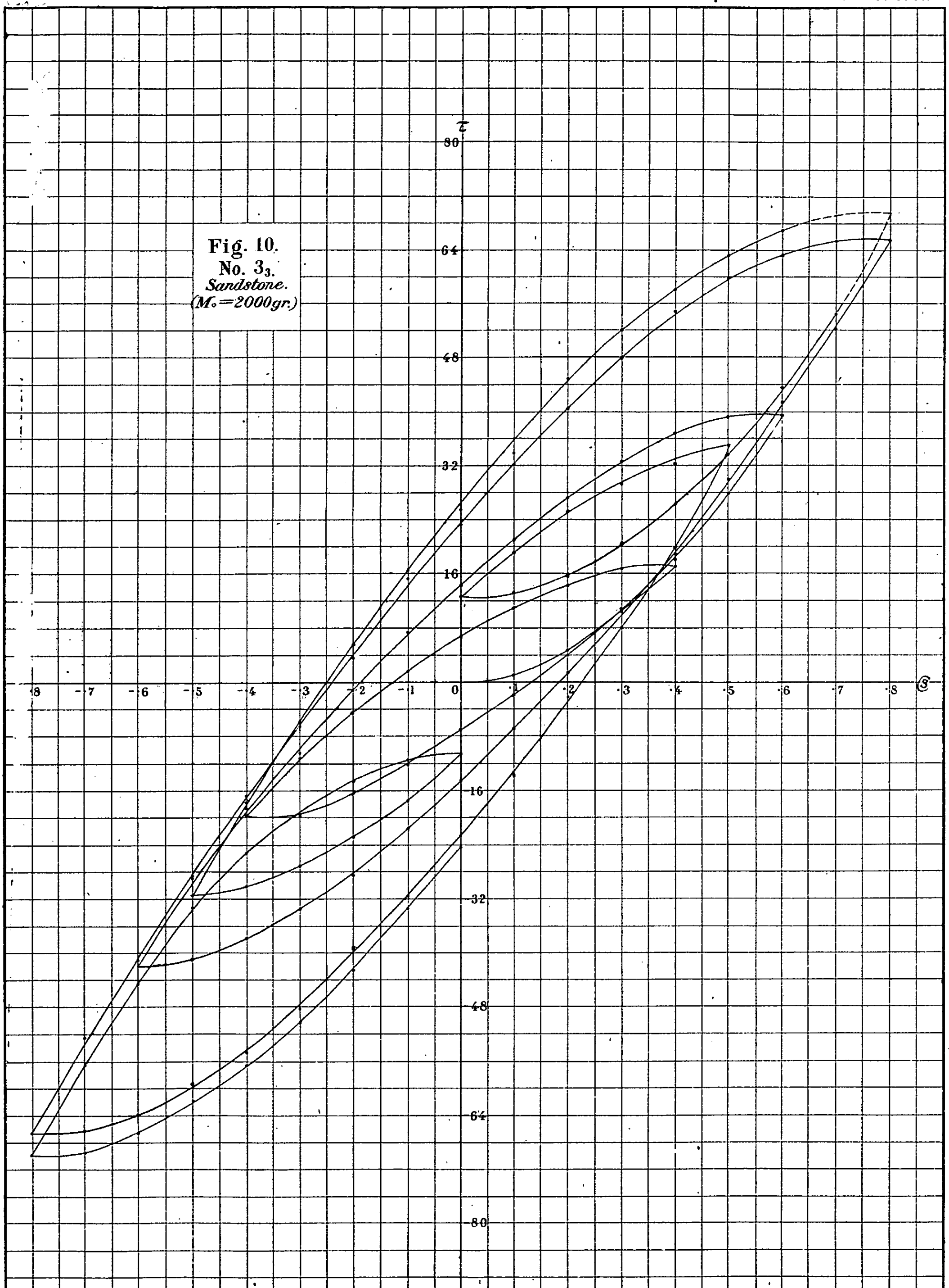
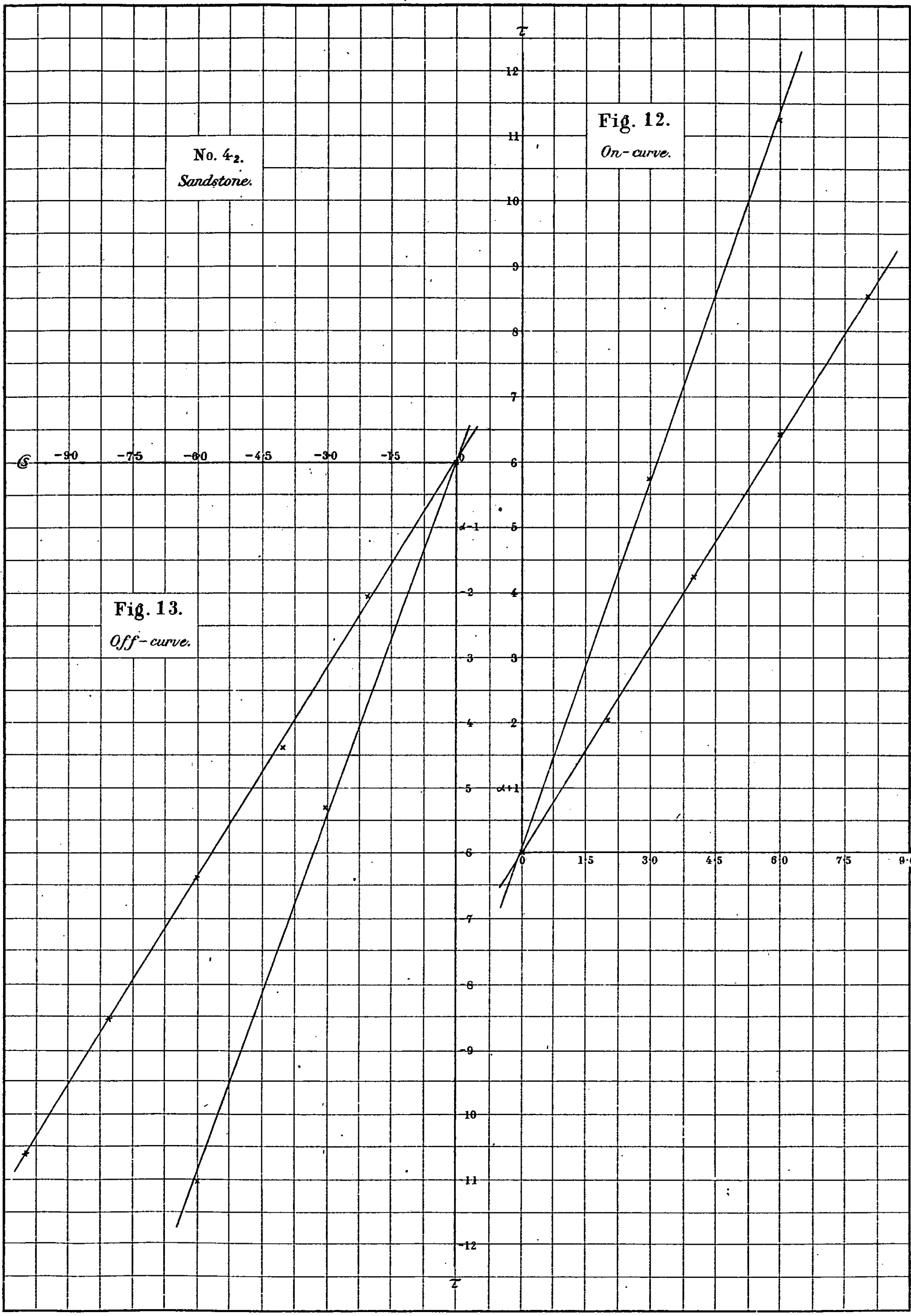


Fig. 10.
No. 3_3.
Sandstone.
(M_0=2000gr.)
τ
80
64
48
32
16
16
32
48
64
80
·8
-·7
-·6
-·5
-·4
-·3
-·2
-·1
0
·1
·2
·3
·4
·5
·6
·7
·8
S


No. 4₂.
Sandstone.
Fig. 12.
On-curve.
Fig. 13.
Off-curve.

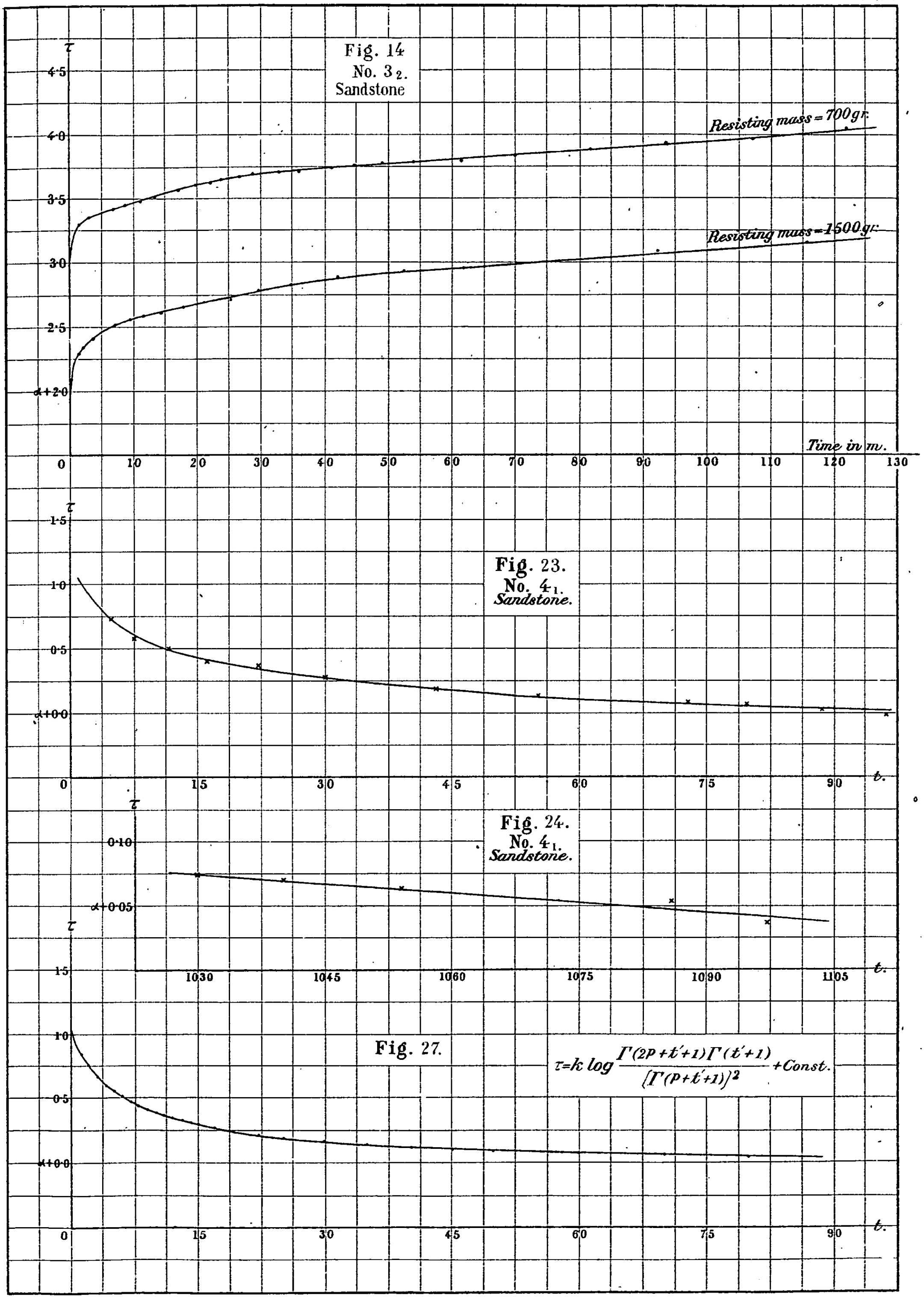


Fig. 14. No. 3_2. Sandstone

Fig. 23. No. 4_1. *Sandstone.*

Fig. 24. No. 4_1. *Sandstone.*

Fig. 27.

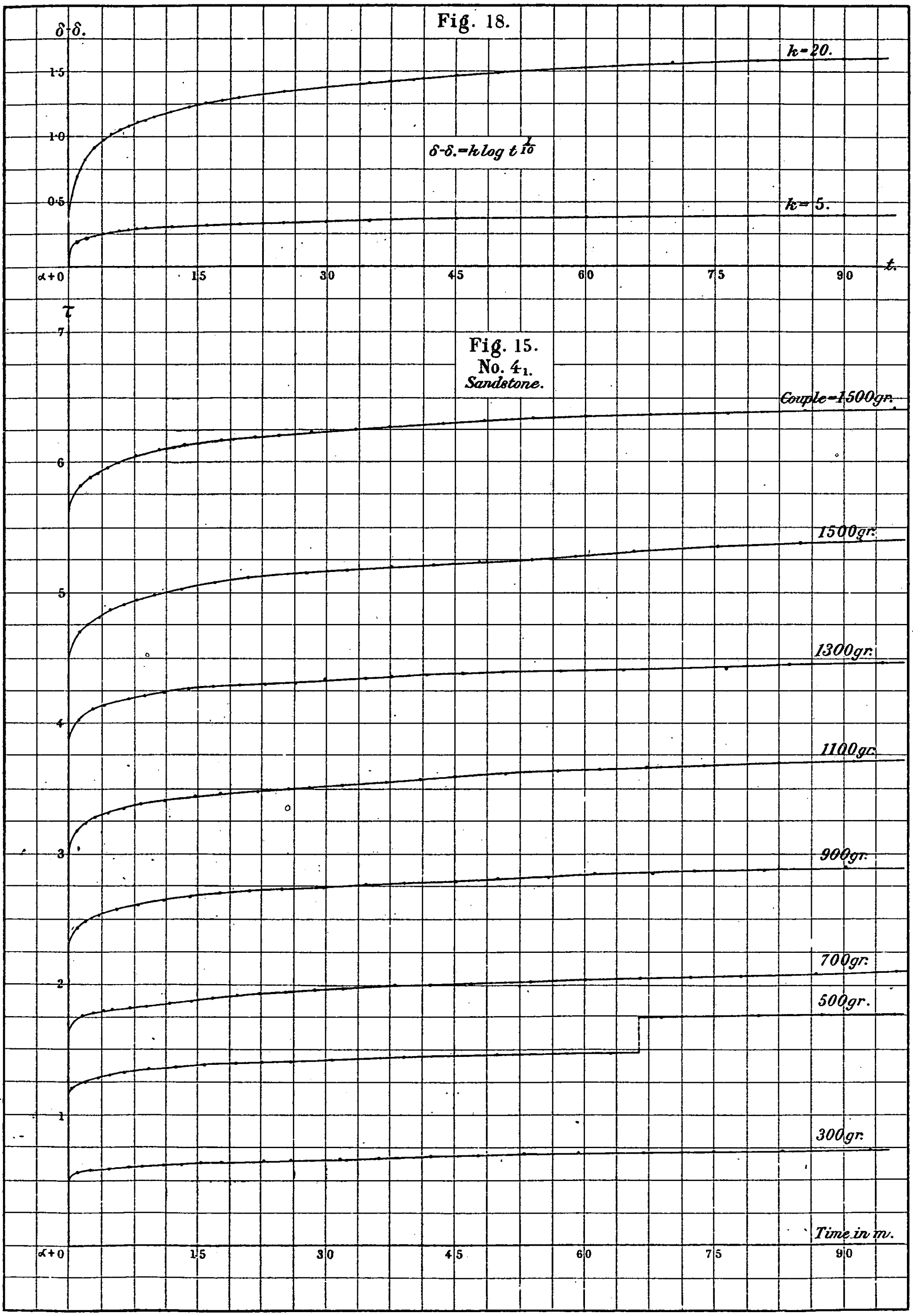


Fig. 18.
δ-δ.
k=20.
δ-δ.=k log t^1/10
k=5.
1·5
1·0
0·5
α+0
15
30
45
60
75
90
t.
τ
Fig. 15.
No. 4₁.
Sandstone.
Couple=1500gr.
1500gr.
1300gr.
1100gr.
900gr.
700gr.
500gr.
300gr.
Time in m.
7
6
5
4
3
2
1

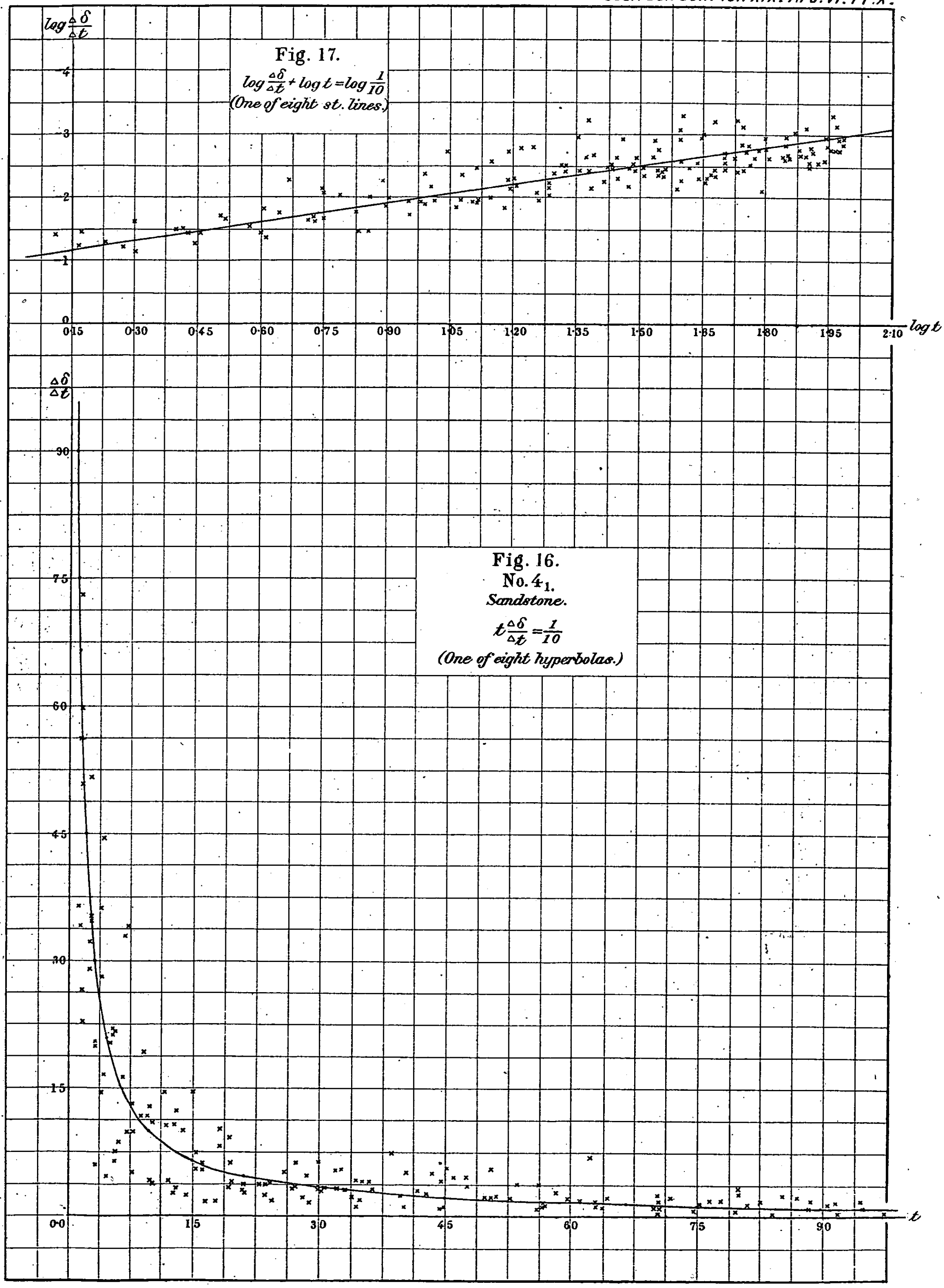


Fig. 17.
$\log\frac{\Delta\delta}{\Delta t} + \log t = \log\frac{1}{10}$
(One of eight st. lines.)
$\log\frac{\Delta\delta}{\Delta t}$
$\log t$
Fig. 16.
No. 4_1.
Sandstone.
$t\frac{\Delta\delta}{\Delta t} = \frac{1}{10}$
(One of eight hyperbolas.)
$\frac{\Delta\delta}{\Delta t}$
t

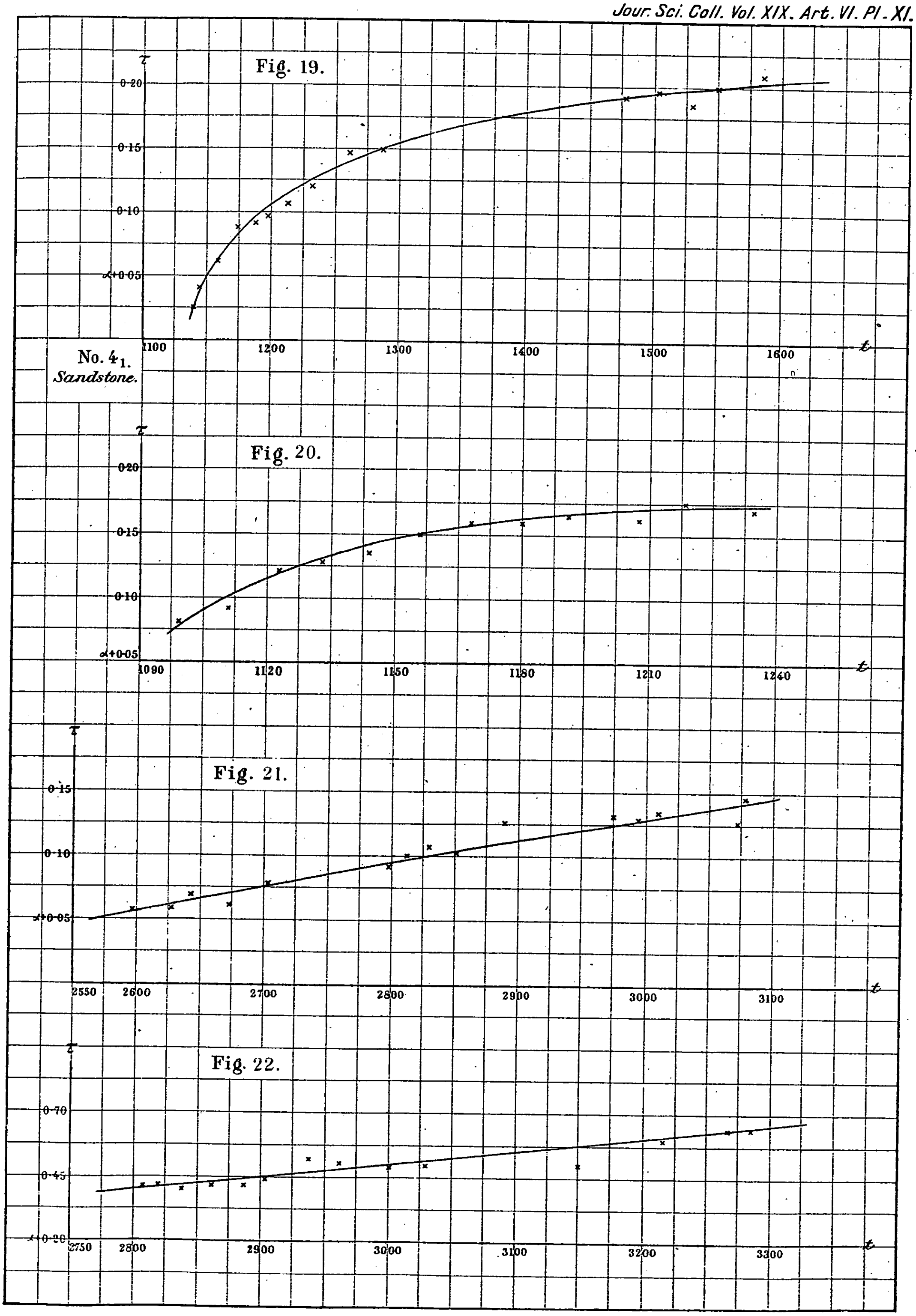


Fig. 19.
τ
0·20
0·15
0·10
α+0·05
1100 1200 1300 1400 1500 1600
t
No. 4_1.
Sandstone.
Fig. 20.
τ
0·20
0·15
0·10
α+0·05
1090 1120 1150 1180 1210 1240
t
Fig. 21.
τ
0·15
0·10
α+0·05
2550 2600 2700 2800 2900 3000 3100
t
Fig. 22.
τ
0·70
0·45
α+0·20
2750 2800 2900 3000 3100 3200 3300
t

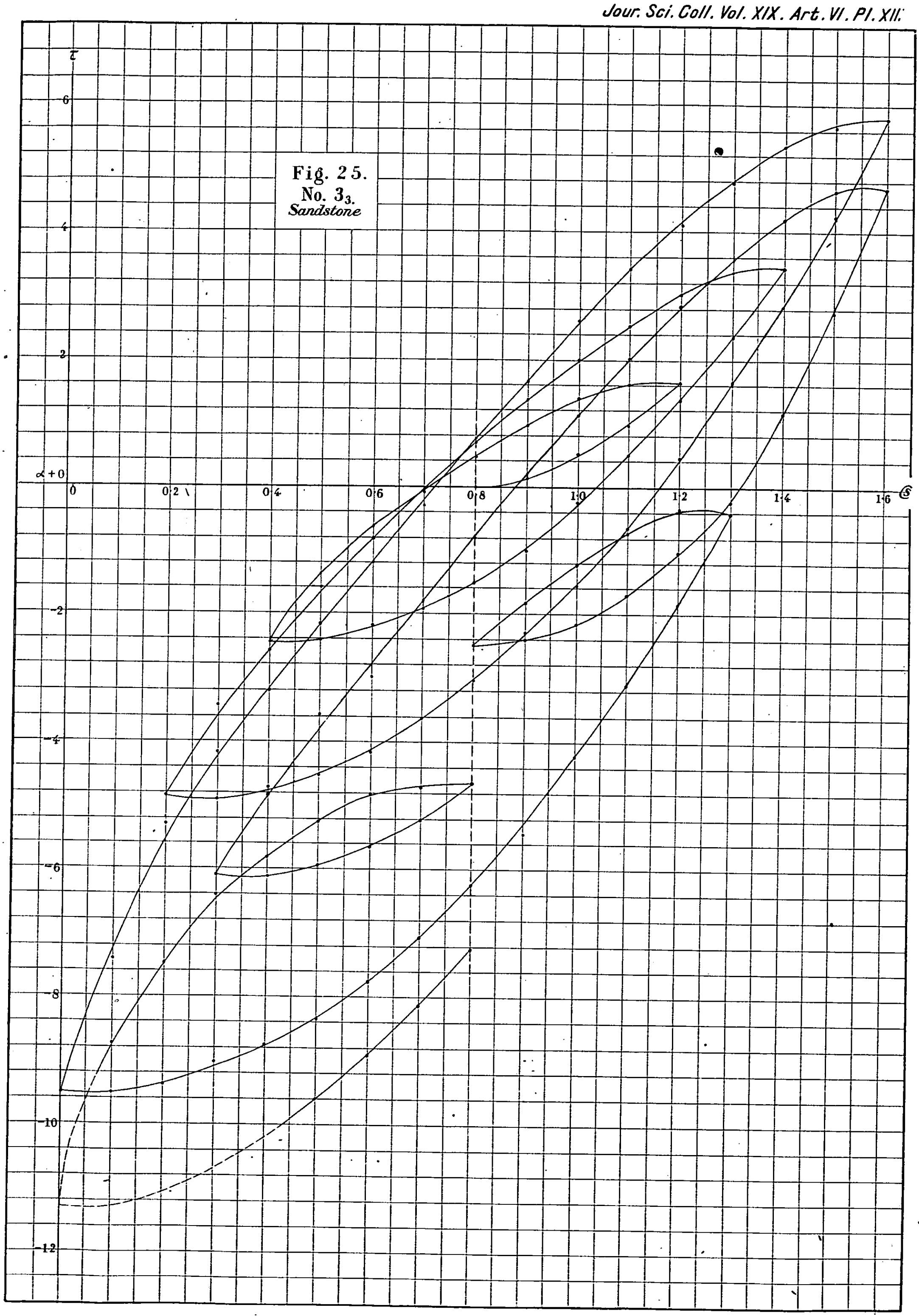


Fig. 25.
No. 3_3.
Sandstone
τ
6
4
2
α+0
-2
-4
-6
-8
-10
-12
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
S

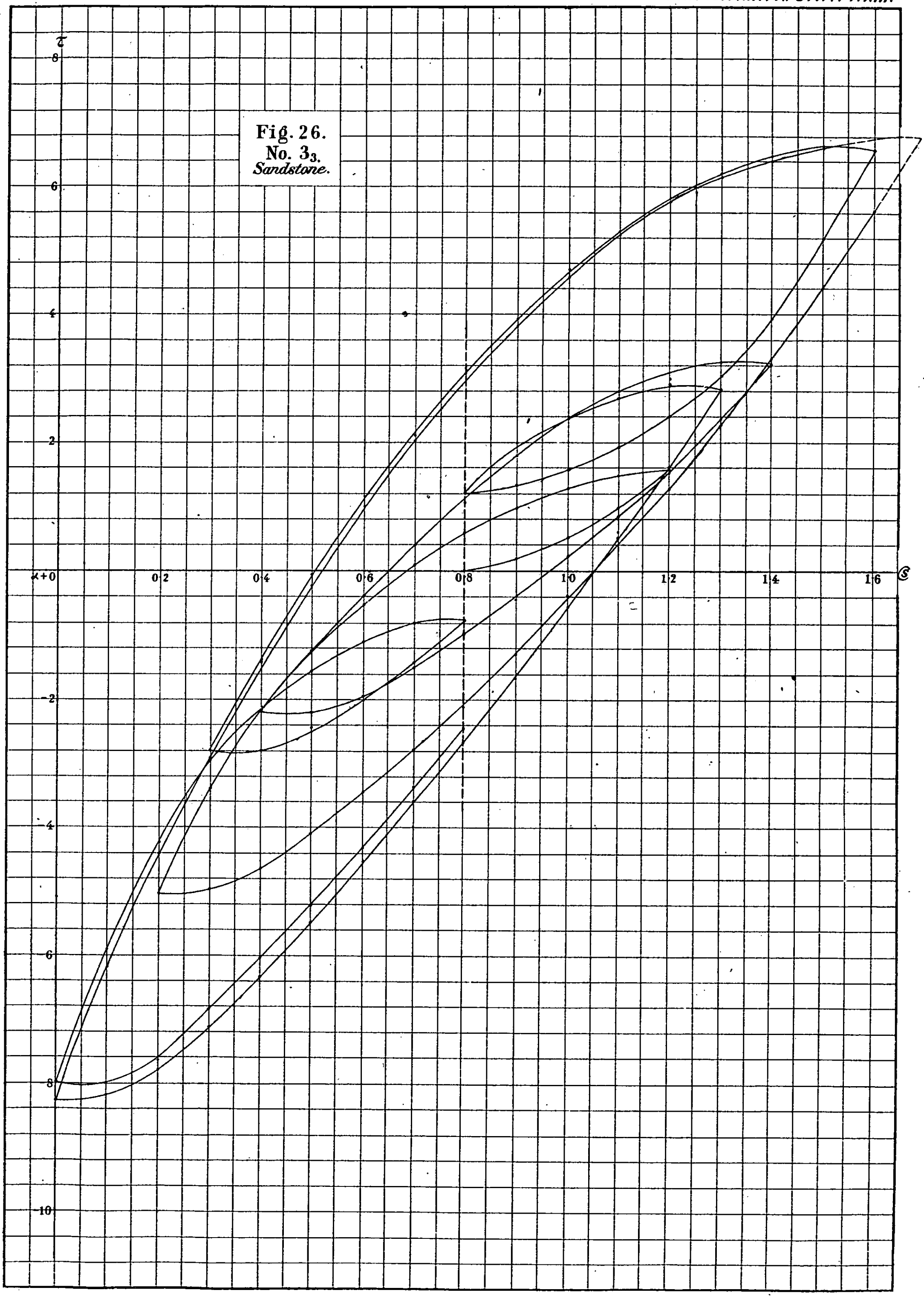


Fig. 26.
No. 3_3.
Sandstone.
τ
8
6
4
2
-2
-4
-6
-8
-10
x+0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
𝔖

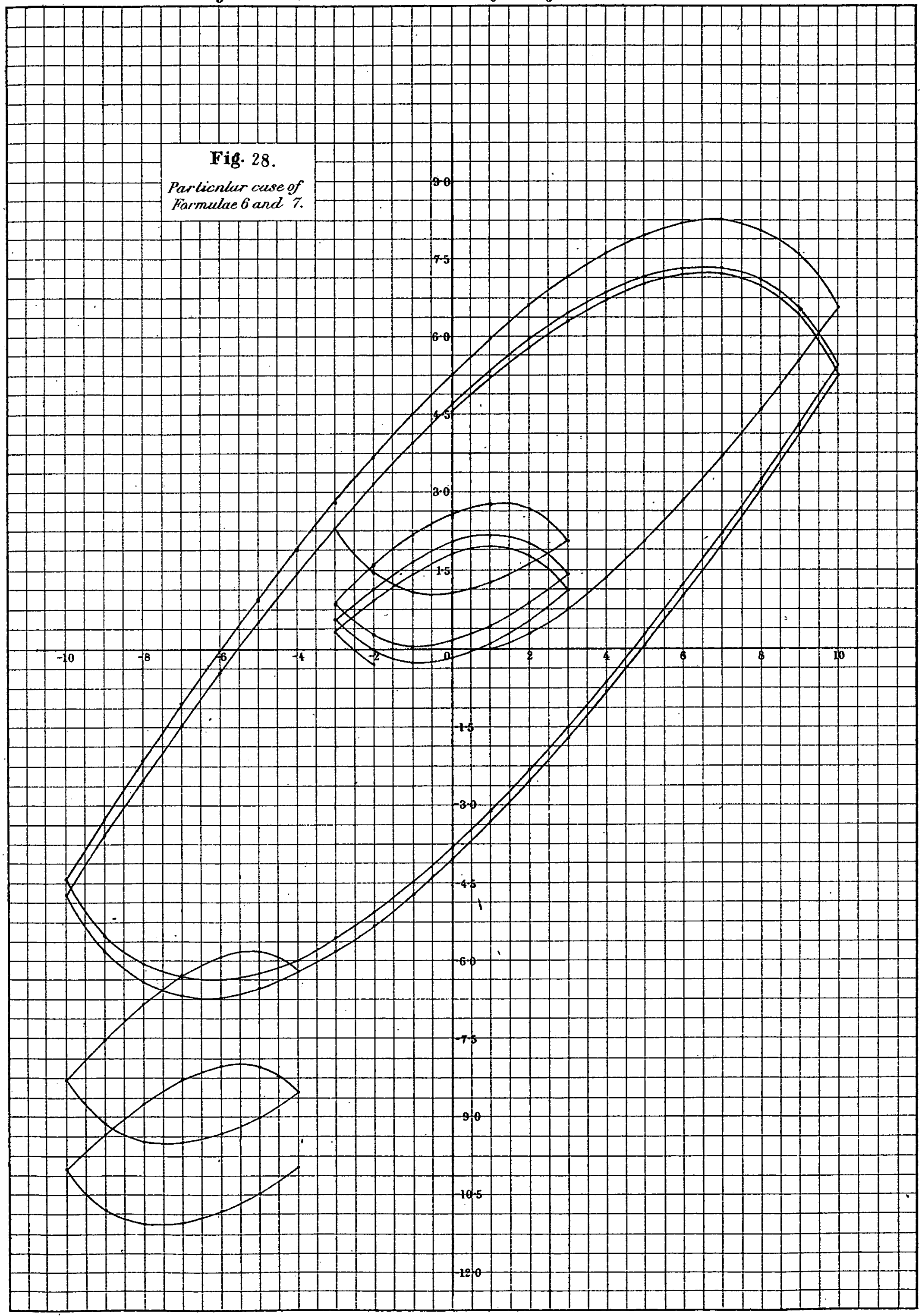


Fig. 28.
Particular case of Formulae 6 and 7.
9·0
7·5
6·0
4·5
3·0
1·5
-10
-8
-6
-4
-2
0
2
4
6
8
10
1·5
3·0
4·5
6·0
7·5
9·0
10·5
12·0

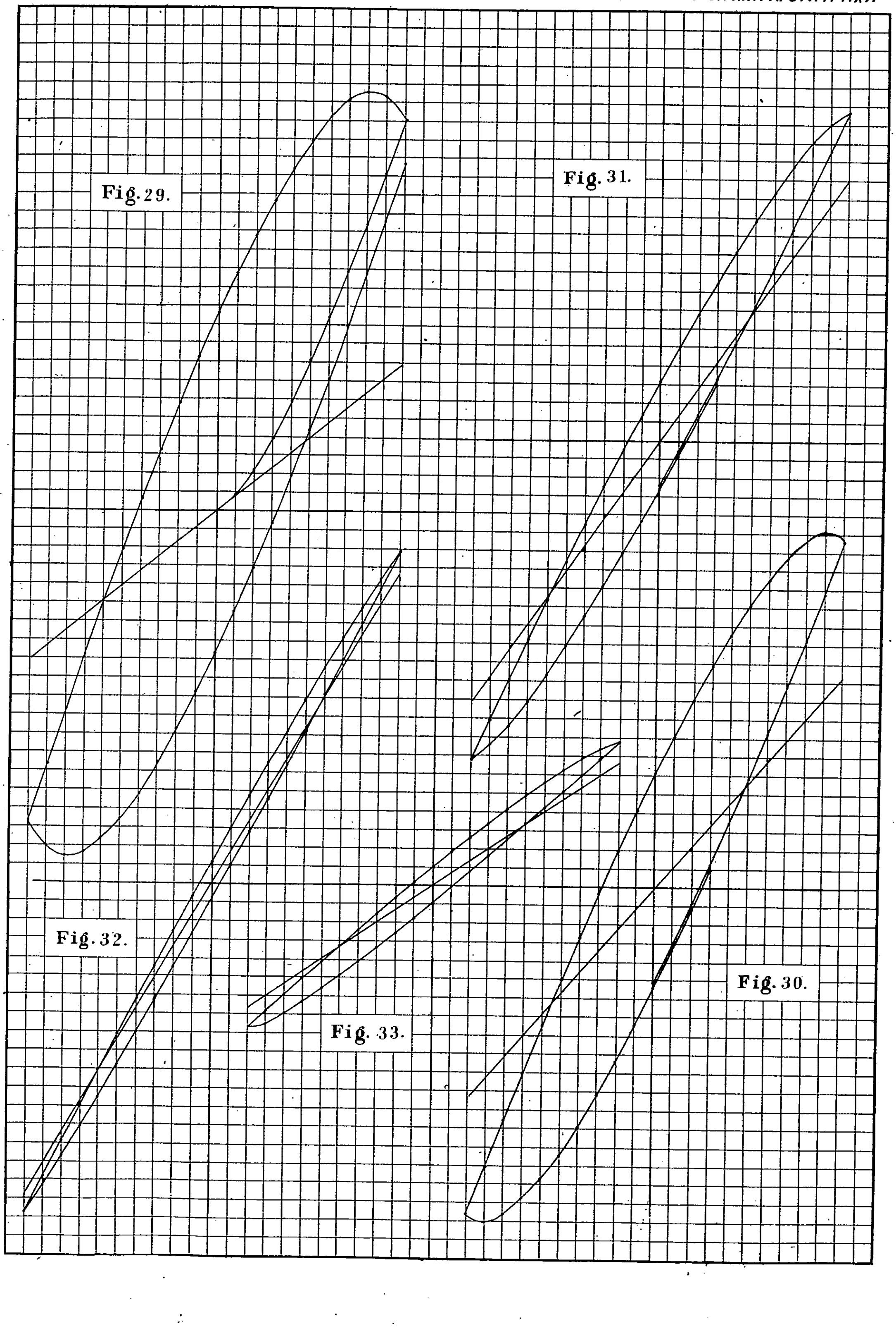


Fig. 29.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 30.

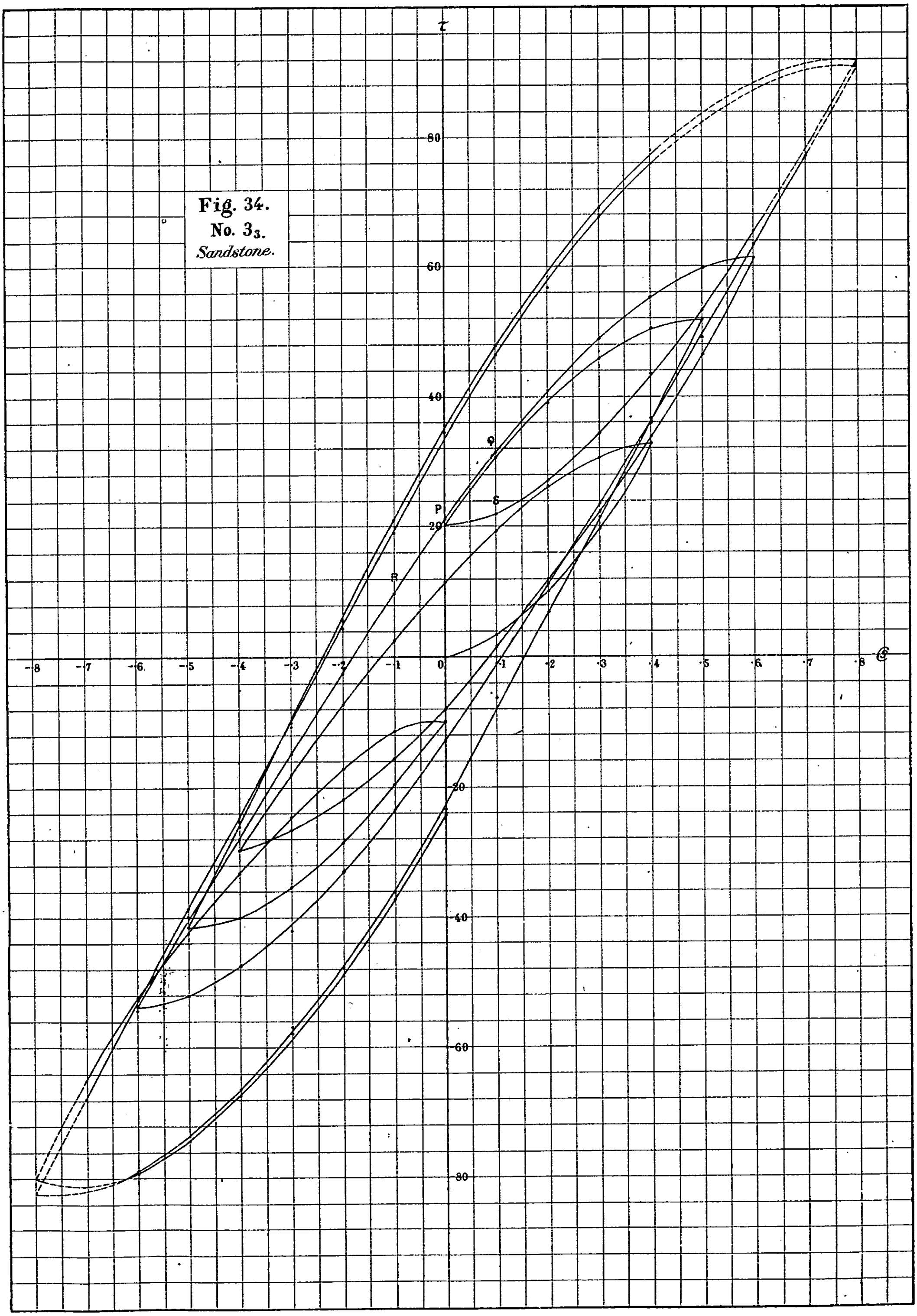


Fig. 34.
No. 3_3.
Sandstone.
τ
80
60
40
20
P
Q
S
R
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
𝔖
20
40
60
80

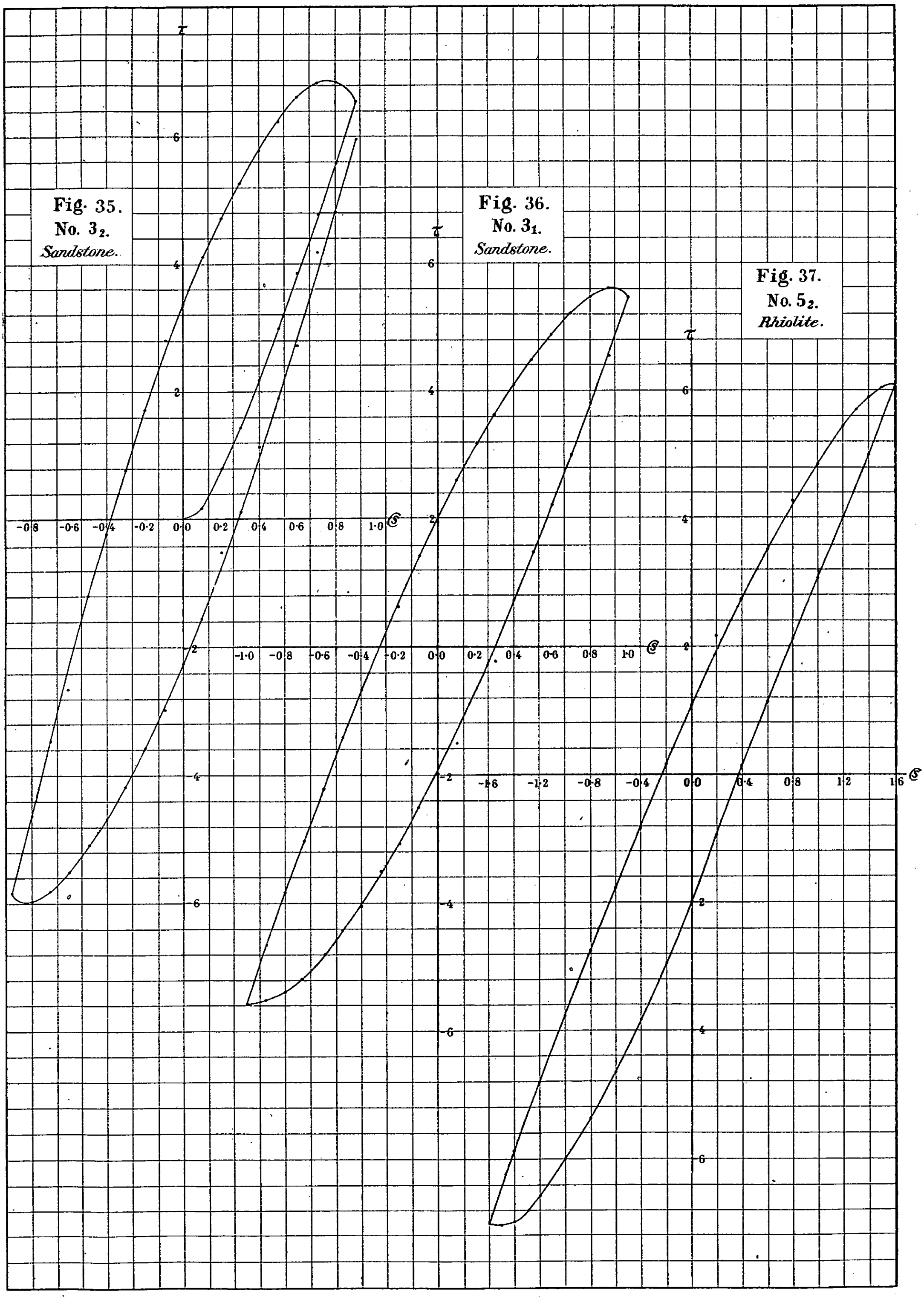


Fig. 35.
No. 3_2.
Sandstone.
Fig. 36.
No. 3_1.
Sandstone.
Fig. 37.
No. 5_2.
Rhiolite.
τ
𝔊

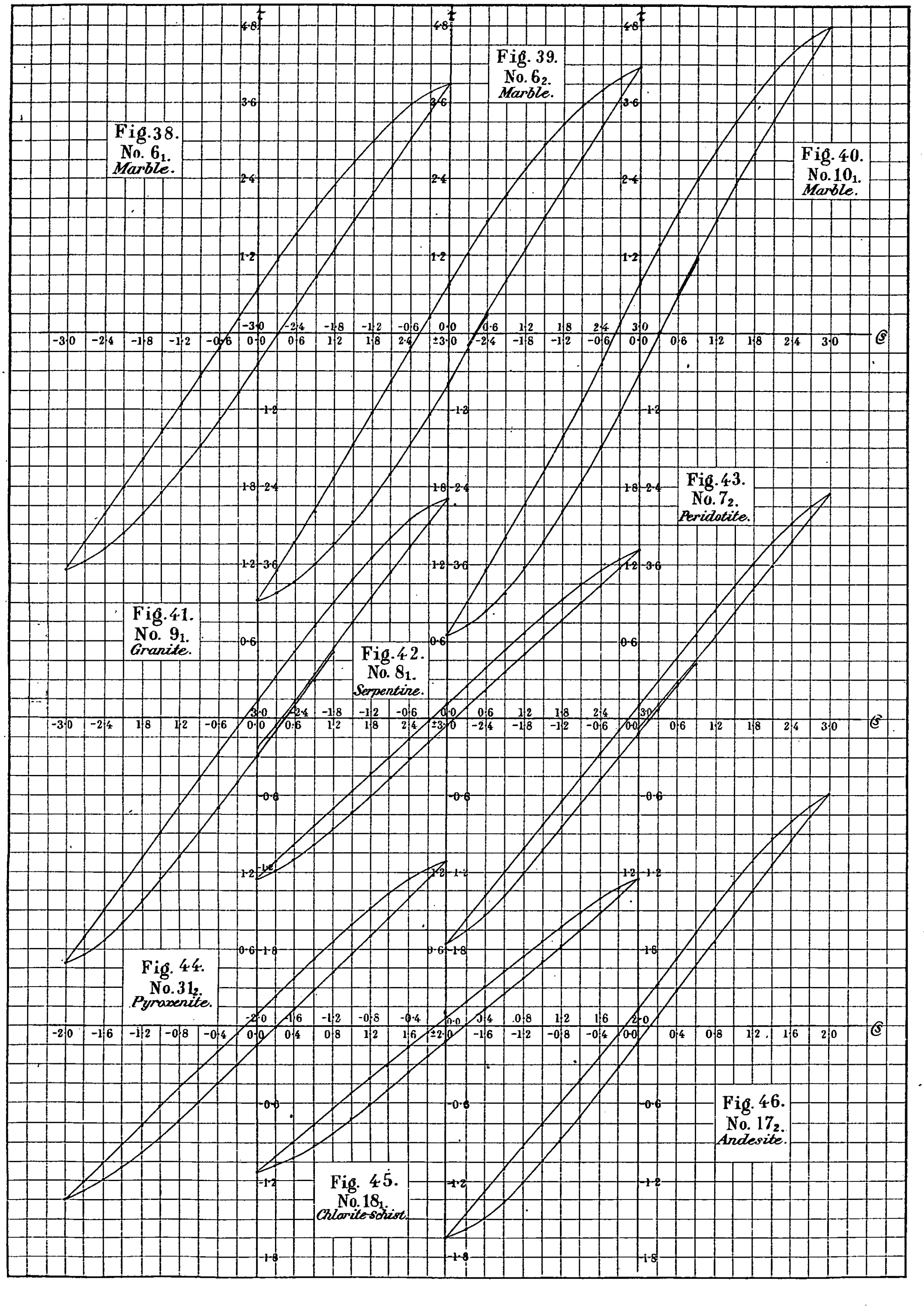


Fig. 38. No. 6_1. Marble.
Fig. 39. No. 6_2. Marble.
Fig. 40. No. 10_1. Marble.
Fig. 41. No. 9_1. Granite.
Fig. 42. No. 8_1. Serpentine.
Fig. 43. No. 7_2. Peridotite.
Fig. 44. No. 31_2. Pyroxenite.
Fig. 45. No. 18_1. Chlorite-schist.
Fig. 46. No. 17_2. Andesite.

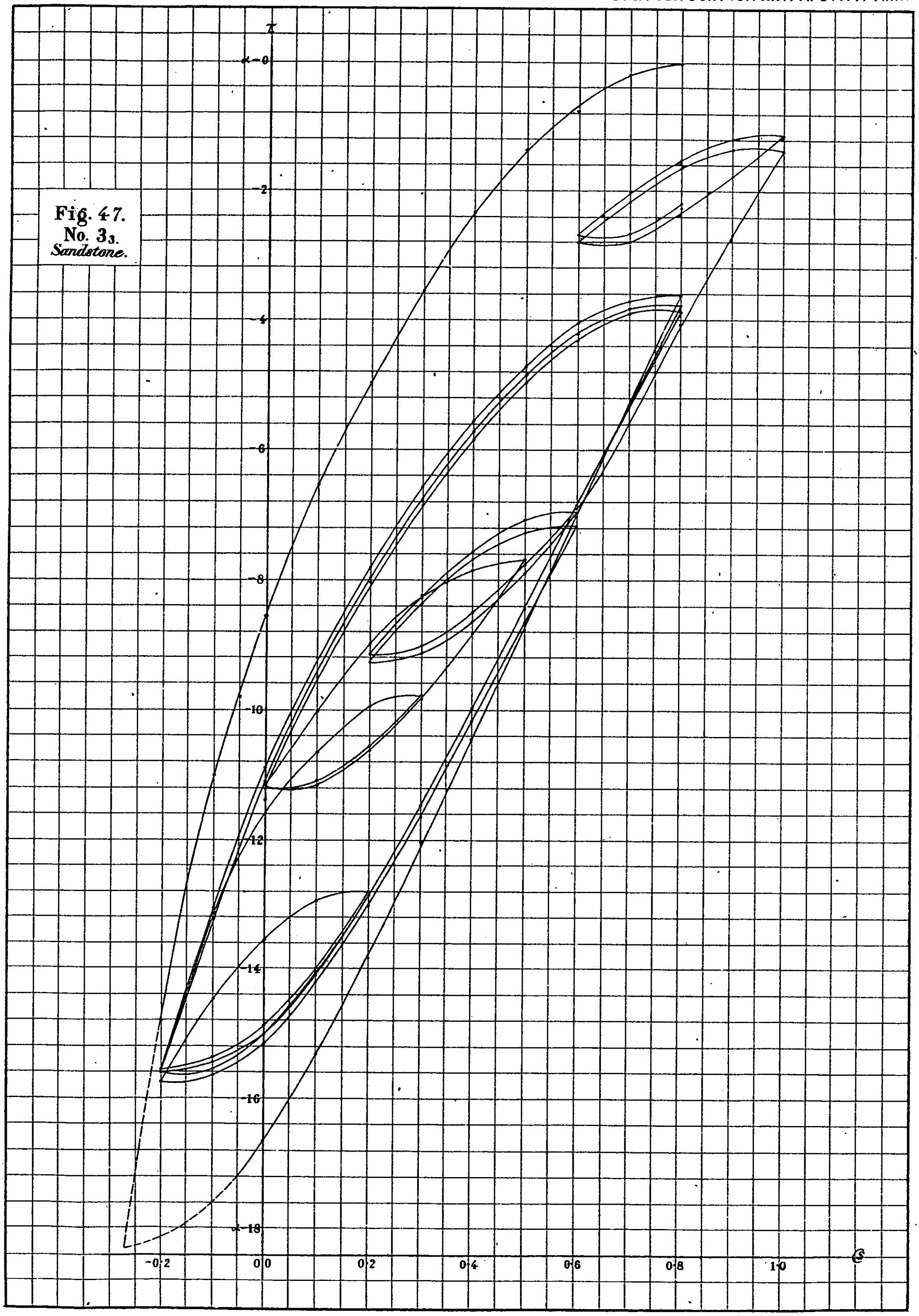


Fig. 47.
No. 33.
Sandstone.
0
2
4
6
8
10
12
14
16
18
-0·2
0·0
0·2
0·4
0·6
0·8
1·0

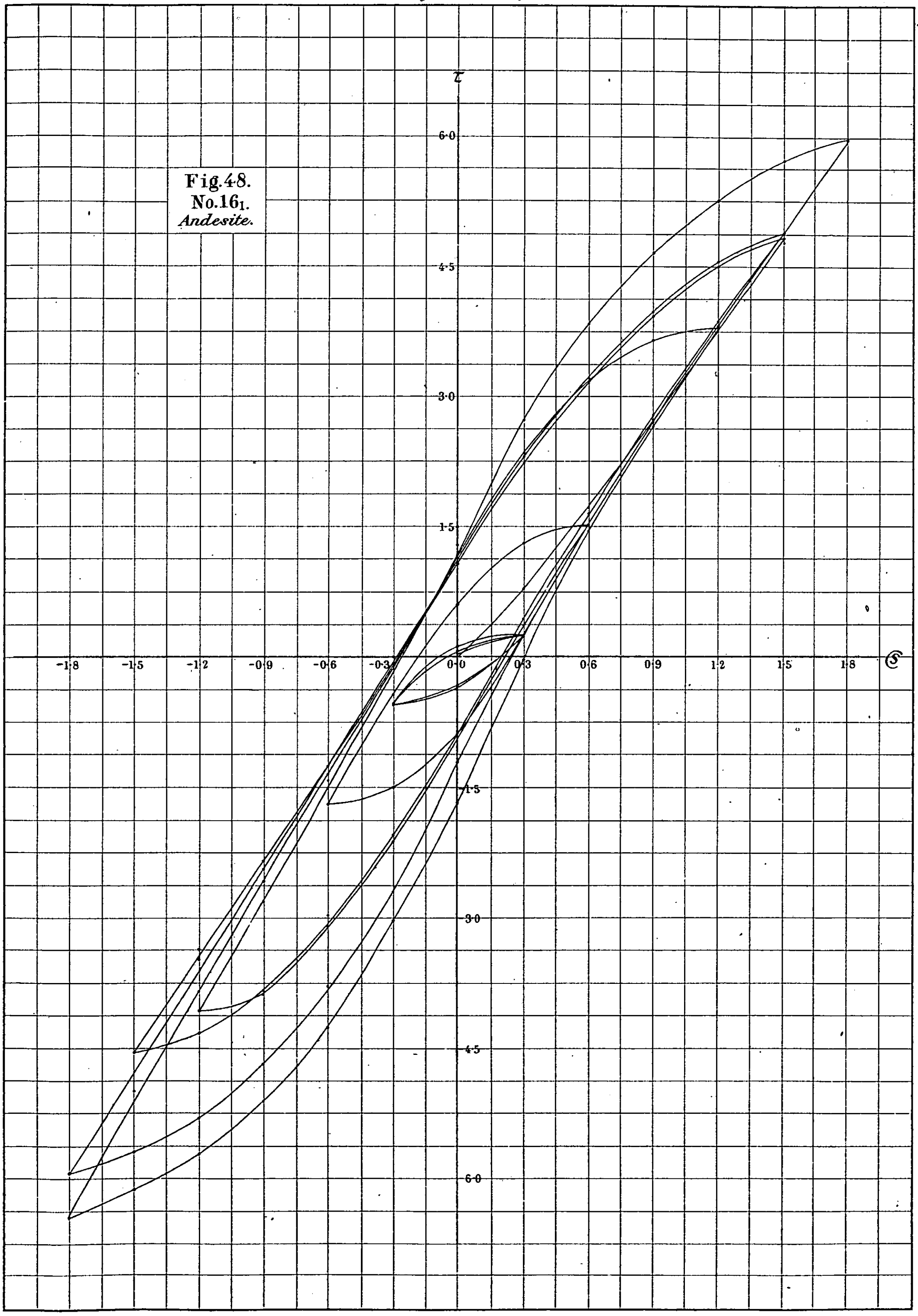


Fig. 48.
No. 16₁.
Andesite.
τ
6·0
4·5
3·0
1·5
-1·5
-3·0
-4·5
-6·0
-1·8
-1·5
-1·2
-0·9
-0·6
-0·3
0·0
0·3
0·6
0·9
1·2
1·5
1·8
𝔖

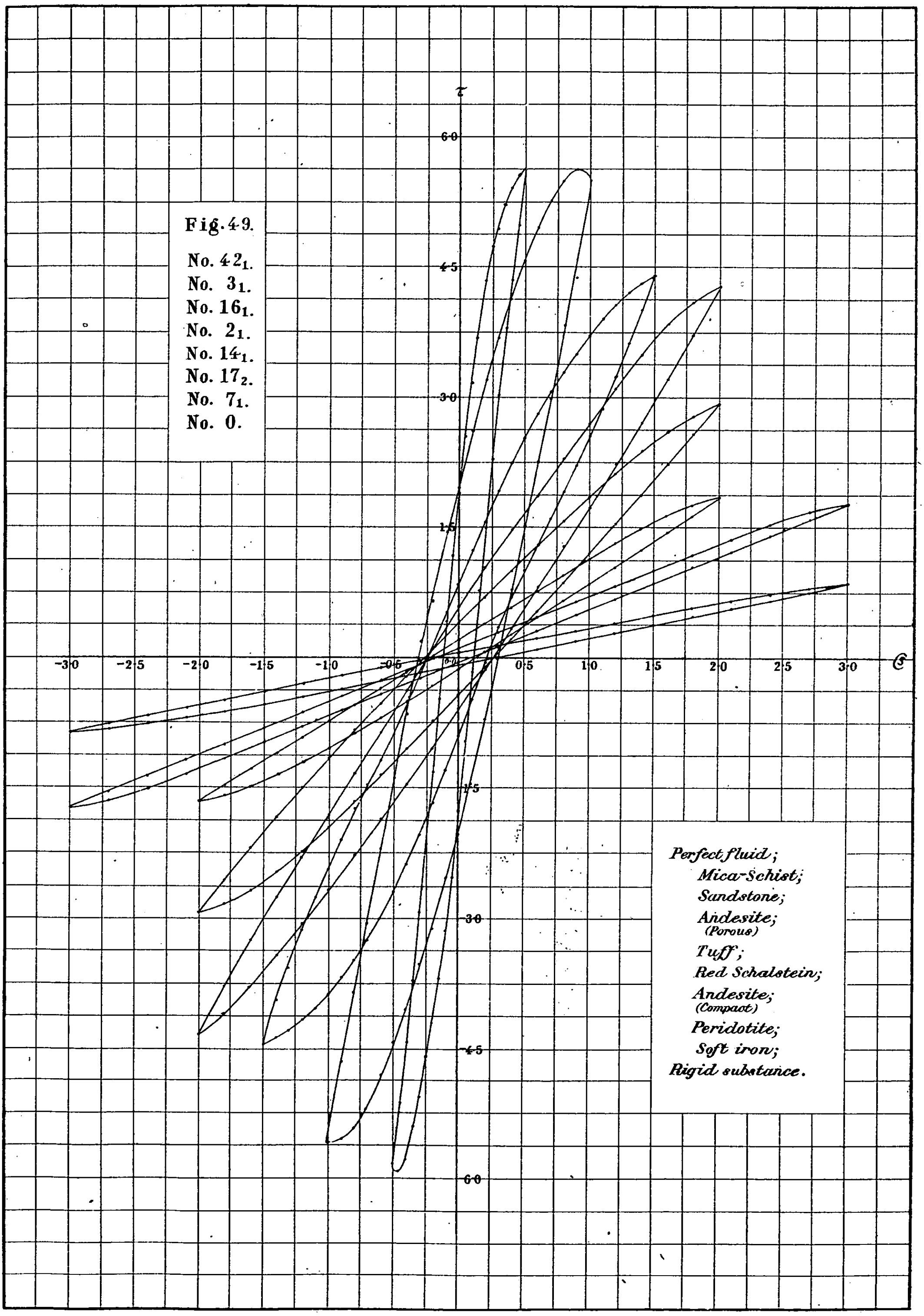


Fig. 49.
No. 42₁.
No. 3₁.
No. 16₁.
No. 2₁.
No. 14₁.
No. 17₂.
No. 7₁.
No. 0.
Perfect fluid;
Mica-Schist;
Sandstone;
Andesite; (Porous)
Tuff;
Red Schalstein;
Andesite; (Compact)
Peridotite;
Soft iron;
Rigid substance.
τ
ε
6·0
4·5
3·0
1·5
0·0
-3·0
-2·5
-2·0
-1·5
-1·0
-0·5
0·5
1·0
1·5
2·0
2·5
3·0

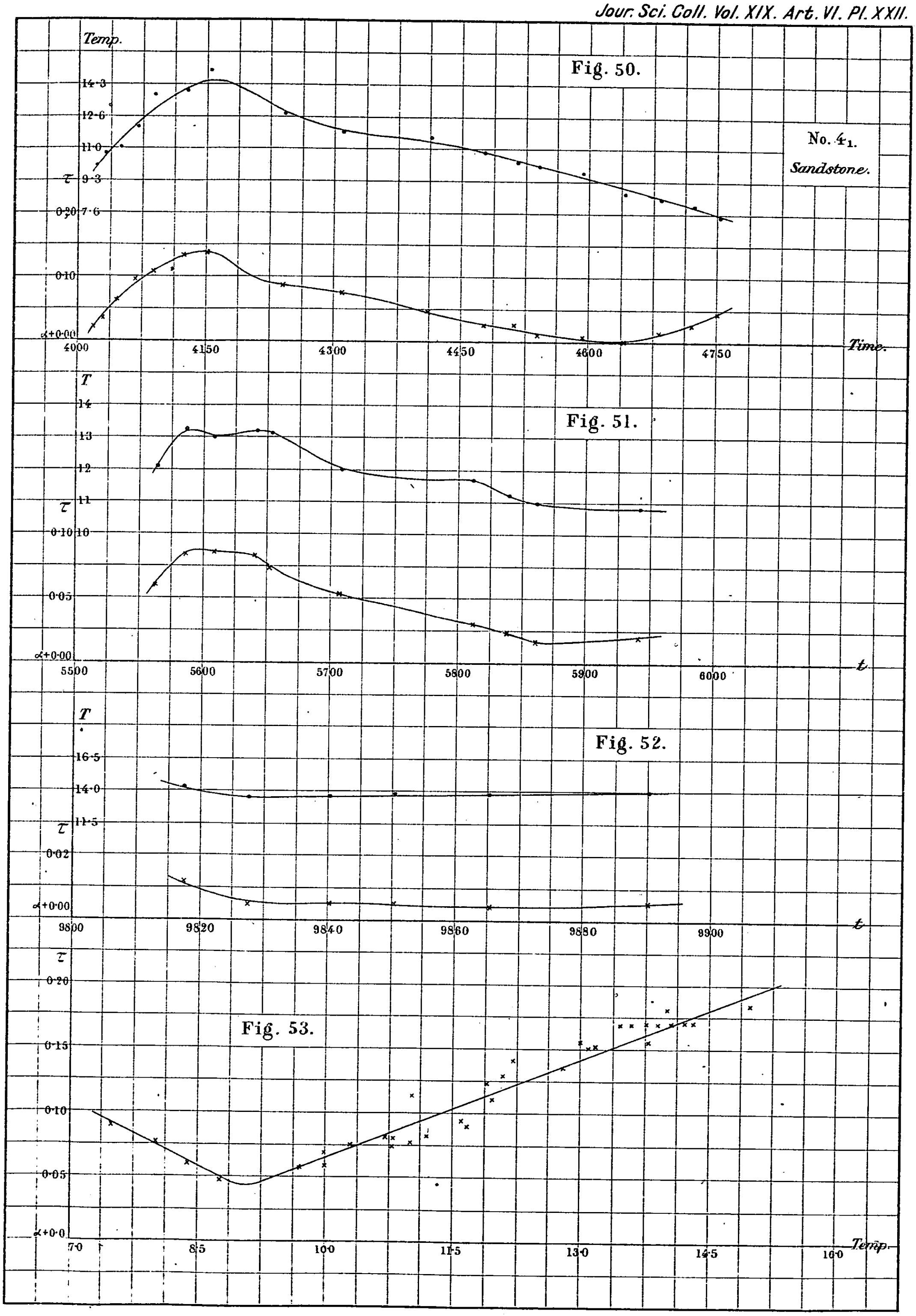


Fig. 50.
No. 4₁.
Sandstone.
Temp.
14·3
12·6
11·0
9·3
7·6
τ
0·20
0·10
α+0·00
4000
4150
4300
4450
4600
4750
Time.
Fig. 51.
T
14
13
12
11
10
τ
0·10
0·05
α+0·00
5500
5600
5700
5800
5900
6000
t
Fig. 52.
T
16·5
14·0
11·5
τ
0·02
α+0·00
9800
9820
9840
9860
9880
9900
t
Fig. 53.
τ
0·20
0·15
0·10
0·05
α+0·0
7·0
8·5
10·0
11·5
13·0
14·5
16·0
Temp.