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Note on the Potential and the Lines of Force of a Circular Current.

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§ 1. The potential of a circular electric current or of a vortex ring of small section, has been investigated by Lord Kelvin,[1] Maxwell[2] Hicks,[3] and Minchin.[4] They all express the solid angle subtended by the circle by means of elliptic integrals or in terms of zonal spherical harmonics. The lines of force of a circular current are usually obtained from the expression for the mutual potential energy (denoted by M) of two coaxial circular coils. By using F. Neumann's formula, M may be expressed by means of elliptic integrals, or developed in terms of zonal harmonics, which is sometimes advantageous for calculating the action between thick coils. Maxwell has also given a table of the coefficients of mutual induction, when the coils are near each other. In these calculations we are always in need of Legendre's tables. It is very curious that so

1) Lord Kelvin, Trans. R. S. E., 1869.
2) Maxwell, Treatise on Electricity and Magnetism II. Chap. 14.
3) Hicks, Phil. Trans. for 1881, p. 628.
4) Minchin, Phil. Mag. vol. 35, 1893.

little use has been made of Jacobi's q-series. Mathy[1] uses Weierstrass's $\wp$-function in evaluating M, but he seems to be inclined to the use of a hypergeometric series, rather than to the reduction of these integrals to a rapidly converging q-series, to which the expression can be easily transformed.

The problem can, however, be attacked from another point of view. In the following, I proceed by finding the Newtonian potential of an uniform circular disc, and derive the expression for the potential and the lines of force for a circular current by simple differentiation. Finally M is expressed by means of a simple q-series, of which a single term will generally suffice to secure a practically accurate value; the force between two coaxial coils can also be expressed in a similar manner.

§ 2. The whole investigation rests on the following lemma.

The potential U of an homogeneous body of rotation (about z-axis) satisfies Laplace's equation outside the body, which in this case is given by

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial x^2}+\frac{1}{x}\frac{\partial U}{\partial x}=0.$$

x being the radial coordinate. Thus

$$\frac{\partial^2 U}{\partial z^2}=-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial U}{\partial x}\right)$$

and

$$\frac{\partial^2 U}{\partial x\,\partial z}=\frac{1}{x}\frac{\partial}{\partial z}\left(x\frac{\partial U}{\partial x}\right)$$

If the potential φ of a certain distribution symmetrical about z-axis be derivable from U by differentiation with respect to z, so that

$$\varphi=\frac{\partial U}{\partial z} \qquad \text{(I.)}$$

1) Mathy, Journal de Physique, tom. 10, p. 33, 1901.

and if $$\psi = x\frac{\partial U}{\partial x}, \qquad \text{(II.)}$$

then since

$$\frac{\partial \varphi}{\partial x}\,\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial z}\,\frac{\partial \psi}{\partial z} = 0.$$

$$\psi = \text{const.}$$

must represent lines of force.

The potential of a circular current is equivalent to that of a magnetic shell, which is derivable from the Newtonian potential of a uniform disc, by differentiation with respect to the normal.

§ 3. *Potential of a uniform circular disc.*—Let the surface density $=1$; then taking x, y axes in the plane of the disc, and z axis perpendicular to it through the centre, the potential at point x, o, z is evidently given by

$$U = \int_0^a \int_0^{2\pi} \frac{\rho\, d\rho\, d\theta}{\sqrt{x^2 - 2\,x\,\rho\cos\theta + \rho^2 + z^2}} \qquad (1)$$

ρ and θ being polar coordinates, and a the radius of the disc.

Writing $R^2 = x^2 - 2x\,\rho\cos\theta + \rho^2$, and making use of Lipschitz's integral concerning the Bessel's function, we obtain

$$\frac{1}{\sqrt{x^2 - 2x\,\rho\cos\theta + \rho^2 + z^2}} = \frac{1}{\sqrt{R^2 + z^2}} = \int_0^\infty e^{-\lambda z} J_0(R\lambda)\, d\lambda. \qquad (2)$$

The addition theorem of Bessel's functions gives

$$J_0(\lambda R) = J_0(\lambda x)\, J_0(\lambda\rho) + 2\sum_1^\infty J_n(\lambda x)\, J_n(\lambda\rho)\cos n\theta. \qquad (3)$$

Substituting in (2), we get

$$U = \int_0^\infty \int_0^a \int_0^{2\pi} \rho\, e^{-\lambda z}\, \{J_0(\lambda x)\, J_0(\lambda\rho) + 2\sum_1^\infty J_n(\lambda x)\, J_n(\lambda\rho)\cos n\theta\}\, d\lambda\, d\rho\, d\theta.$$

$$= 2\pi \int_0^\infty \int_0^a e^{-\lambda z}\, \rho\, J_0(\lambda x)\, J_0(\lambda\rho)\, d\lambda\, d\rho.$$

But
$$\rho\, J_0(\lambda\rho)=\frac{d\,(\rho\, J_0(\lambda\rho))}{\lambda\, d\rho}$$

Thus the potential of a circular disc is given by

$$U=2\,\pi\, a\int_0^\infty \frac{e^{-\lambda z}}{\lambda}\, J_0(\lambda x)\, J_1(\lambda a)\, d\lambda. \tag{4}$$

This expression was first obtained by H. Weber[1] in a somewhat different manner.

§ 4. *Potential and lines of force of a uniform circular magnetic shell.*—The potential of a circular magnetic shell of unit strength is evidently given by

$$\varphi=-\frac{\partial U}{\partial z}=2\,\pi\, a\int_0^\infty e^{-\lambda z}\, J_0(\lambda\, x)\, J_1(\lambda\, a)\, d\,\lambda, \tag{A}$$

and for the function giving lines of force

$$\psi=-x\frac{\partial U}{\partial x}=2\pi a x\int_0^\infty e^{-\lambda z}\, J_1(\lambda\, x)\, J_1(\lambda\, a)\, d\,\lambda. \tag{B}$$

The two expressions (A) and (B) can be greatly simplified by using the addition theorem for $J_0(R\,\lambda)$ and $J_1(R\,\lambda)$.

Differentiating (3) with respect to ρ and integrating between the limits 0 and π of θ,
we obtain

$$J_0(\lambda x)\, J_1(\lambda a)=\frac{1}{\pi}\int_0^\pi \frac{J_1(\lambda R)}{R}\,(a-x\cos\theta)\, d\,\theta.$$

Similarly

$$J_1(\lambda x)\, J_1(\lambda a)=\frac{2}{\pi}\int_0 J_0(\lambda R)\cos\theta\, d\theta.$$

1) H. Weber, Crelle's Journal, Bd. 75, p. 88.

Remembering that

$$\int_0^\infty e^{-\lambda z} J_1(\lambda R)\, d\lambda = \frac{1}{R} - \frac{z}{R\sqrt{R^2+z^2}} \quad ,$$

$$\int_0^\infty e^{-\lambda z} J_0(\lambda R)\, d\lambda = \frac{1}{\sqrt{R^2+z^2}} \quad ,$$

we find by simple substitution

$$\varphi = 2a\int_0^\pi \frac{a-x\cos\theta}{R^2} d\theta - 2az\int_0^\pi \frac{(a-x\cos\theta)}{R^2\sqrt{R^2+z^2}} d\theta \qquad (5)$$

and

$$\psi = 4ax\int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{R^2+z^2}} \qquad (6)$$

But we easily find that

$$\int_0^\pi \frac{(a-x\cos\theta)\, d\theta}{R^2} = \frac{\pi}{a} \quad ,$$

whence by (5)

$$\varphi = 2\pi - 2az\int_0^\pi \frac{(a-x\cos\theta)\, d\theta}{(a^2+x^2-2ax\cos\theta)\sqrt{a^2+x^2+z^2-2ax\cos\theta}} \qquad (A')$$

The above expression gives the solid angle subtended by the disc at point (x, o, z).

Evidently the coëfficient of mutual induction M of two parallel coaxial coils is connected with ψ by the relation

$$\pi\psi = M. \qquad (7)$$

Consequently, (6) gives

$$M = 4\pi ax\int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{a^2+x^2+z^2-2ax\cos\theta}} \qquad (B')$$

This expression coincides with that obtained in the usual manner from F. Neumann's formula.

§ 5. *Evaluation of φ or solid angle subtended by a circle.*—Denoting the integral entering in (A') by Ω, we have

$$\Omega = 2\,az\int_0^\pi \frac{(a - x\cos\theta)\,d\theta}{(a^2+x^2-2ax\cos\theta)\sqrt{a^2+x^2+z^2-2ax\cos\theta}} \tag{8}$$

Putting $\cos\theta = As + B$, where

$$A = \left(\frac{2}{ax}\right)^{\frac{1}{3}}, \tag{9}$$

$$B = \frac{a^2+x^2+z^2}{ax},$$

we easily find

$$\sin\theta\sqrt{a^2+x^2+z^2-2\,ax\cos\theta} = \sqrt{4\,(s-e_1)(s-e_2)(s-e_3)}$$

$$= \sqrt{4\,s^3 - g_2\,s - g_3} = \sqrt{S}$$

where

$$e_1 = \frac{2B}{A}\ , \quad e_2 = \frac{1-B}{A}\ , \quad e_3 = -\frac{(1+B)}{A}. \tag{10}$$

$$\frac{1}{4}\,g_2 = \frac{3B^2-1}{A^2}\ , \quad \frac{1}{4}\,g_3 = \frac{2\,B\,(B^2-1)}{A^3}\ ,$$

$$G = \left\{\frac{(a^2+x^2+z^2)^2 - 4\,a^2\,r^2}{4\,ar}\right\}^2,$$

whence

$$k^2 = \frac{2}{1+3B} = \frac{4\,ax}{(a+x)^2+z^2},$$

$$k'^2 = \frac{3\,B-1}{3\,B+1} = \frac{(a-x)^2+z^2}{(a+x)^2+z^2}. \tag{11}$$

Thus, putting

$$u=\int^{\infty}\frac{ds}{\sqrt{S}} \qquad \text{or} \qquad s=\wp(u)$$

$$\Omega=A\, z\int_{\omega_2}^{\omega_3}du+z\,(a^2-x^2)\int_{\omega_2}^{\omega_3}\frac{du}{a^2+x^2-2a\,x\,(A\,\wp(u)+B)}. \tag{12}$$

Let

$$\wp(a)=\frac{2\,(a^2+x^2)-z^2}{6\,ax\,A},$$

then

$$\wp'(a)=-\sqrt{4\,\wp^3(a)-g_2\,\wp(a)-g_3}=-\,i\,\frac{z\,(a^2-x^2)}{2\,ax}$$

and

$$\Omega=\omega_1\,A\,z+i\int_{\omega_2}^{3}\frac{\wp'(a)}{\wp(a)-\wp(u)}\,du. \tag{13}$$

Evaluating the last integral, we arrive at the result

$$\Omega=\frac{2\omega_1}{e_2-e_3}\,z+2\,i\left\{\eta_1 a-\omega_1\frac{\sigma'}{\sigma}(a)\right\}. \tag{14}$$

Thus the potential of a circular current or of a vortex ring is given by

$$\varphi=2\,\pi-2\left[\frac{\omega_1\,z}{e_2-e_3}+i\left\{\eta_1\,a-\omega_1\frac{\sigma'}{\sigma}(a)\right\}\right] \tag{A''}$$

$$=2\,\pi-\frac{\pi\,\vartheta_2^2(o)}{\sqrt{e_2-e_3}}\,z-i\,\frac{\vartheta_1'\,(v_a i)}{\vartheta_1\,(v_a i)} \qquad \text{where} \quad v_a=\frac{a}{2\omega_1}i$$

The form of integral (A′) is somewhat different from that given by Hicks and Minchin, but it leads to the same result. The process of reduction from the expressions given by the above mentioned authors is more laborious.

In practical calculation, the part requiring the most painstaking is the evaluation of a, for which the expressions given in Prof. Schwarz's "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen" p. 71. will be found very useful. For convenience in calculation, the following values of $\wp(a)-e_\lambda$ have been tabulated

$$\wp(a)-e_1=-\frac{1}{A}\quad\frac{z^2}{2\,ax}=-\frac{z^2}{4\,ax}(e_2-e_3)$$

$$\wp(a)-e_2=\frac{1}{A}\quad\frac{(a-x)^2}{2\,ax}=\frac{(a-x)^2}{4\,ax}(e_2-e_3) \qquad (15)$$

$$\wp(a)-e_3=\frac{1}{A}\quad\frac{(a+x)^2}{2\,ax}=\frac{(a+x)^2}{4\,ax}(e_2-e_3)$$

Thus $e_2<\wp(a)<e_1$.

$$e_2-e_3=\frac{2}{A}=2\left(\frac{ax}{2}\right)^{\frac{1}{3}}$$

$$e_1-e_3=\frac{1}{A}\quad\frac{(a+x)^2+z^2}{2\,ax}=\frac{(a+x)^2+z^2}{4\,ax}(e_2-e_3)$$

$$e_1-e_2=\frac{1}{A}\quad\frac{(a-x)^2+z^2}{2\,ax}=\frac{(a+x)^2+z^2}{4\,ax}(e_2-e_3)$$

Putting

$$\frac{\sqrt[4]{e_1-e_3}\sqrt{\wp(a)-e_2}-\sqrt[4]{e_1-e_2}\sqrt{\wp(a)-e_3}}{\sqrt[4]{e_1-e_3}\sqrt{\wp(a)-e_2}+\sqrt[4]{e_1-e_2}\sqrt{\wp(a)-e_3}}=lt \quad \text{where} \quad l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}}$$

and

$$\mathfrak{L}_0=1+\left(\frac{1}{2}\right)^2 l^4+\left(\frac{1.3}{2.4}\right)^2 l^8+\left(\frac{1.3.5}{2.4.6}\right)^2 l^{12}+\cdots\cdots$$

$$\mathfrak{L}_{0,1}=\left(\frac{1}{2}\right)^2 l^4+\left(\frac{1.3}{2.4}\right)^2 l^8+\left(\frac{1.3.5}{2.4.6}\right)^2 l^{12}+\cdots\cdots$$

$$\mathfrak{L}_{0,2}=\left(\frac{1.3}{2.4}\right)^2 l^8+\left(\frac{1.3.5}{2.4.6}\right)^2 l^{12}+\cdots\cdots$$

....

we find after a simple calculation, that

$$\frac{a}{2\omega_1}\pi i = v_a \pi$$

$$= -\frac{1}{2} \log (t + \sqrt{t^2-1}) - \frac{1}{2}\sqrt{t^2-1} \left(\frac{\mathfrak{L}_{0,1}}{\mathfrak{L}_0} t + \frac{2}{3} \frac{\mathfrak{L}_{0,2}}{\mathfrak{L}_0} t^3 + \ldots\ldots \right)$$

Since $\mathfrak{L}_{0,1}$. $\mathfrak{L}_{0,2}, \ldots\ldots$ are generally small quantities, the second term will be very small as compared with the first. Further we notice that

$$i \left\{ \eta a - \omega_1 \frac{\sigma'}{\sigma}(a) \right\} = \frac{\pi}{2} \left\{ \frac{e^{v_a \pi} + e^{-v_a \pi}}{e^{v_a \pi} - e^{-v_a \pi}} \right.$$

$$\left. + \sum{}' \frac{2q^{2n} e^{-2v_a \pi}}{1-q^{2n} e^{-2v_a \pi}} - \sum{}' \frac{2q^{2n} e^{2v_a \pi}}{1-q^{2n} e^{2v_a \pi}} \right\}$$

Thus the calculation of the solid angle subtended by a circle can be easily undertaken without the use of special tables.

§ 6. *Evaluation of M or the coefficient of mutual induction of two coaxial coils.*—As has already been noticed, M=const. gives magnetic lines of force about a circular current, or stream lines about a circular vortex ring of infinitely small section. Reverting to (B′), (9), (10), and (11), we find that

$$M = 4\pi ax \int_0^\pi \frac{\cos\theta \, d\theta}{\sqrt{a^2+x^2+z^2-2ax\cos\theta}}$$

$$= 4\pi ax A \int_{\omega_3}^{\omega_2} (A \wp(u) + B)\, du \qquad (16)$$

Thus the expression for M can be written in either of the following forms :—

$$M = 4\pi ax \frac{A^2}{\omega_1}\left(\frac{e_1 \omega_1^2}{2} - \eta_1 \omega_1 \right) \qquad (B_1'')$$

or

$$M = 4\pi ax \frac{A^2}{\omega_3}\left(\frac{e_1 \omega_1 \omega_3}{2} - \eta_1 \omega_3 \right) \qquad (B_2'')$$

§ 7. *M expressed in terms of ϑ-functions.*—Since

$$e_1\,\omega_1^2 = -\eta_1\,\omega_1 - \frac{1}{4}\,\frac{\vartheta_2''(o)}{\vartheta_2(o)}\quad,$$

$$\eta_1\,\omega_1 = -\frac{1}{12}\,\frac{\vartheta_1'''(o)}{\vartheta_1'(o)}\quad,$$

and
$$\frac{A^2}{\omega_1} = \frac{4}{\pi\sqrt{ax}\,\vartheta_2^2(o)}$$

we find from (B_1'')

$$M = \frac{2\sqrt{ax}}{\vartheta_2^2\,(o)}\left\{\frac{\vartheta_1'''(o)}{\vartheta_1'(o)} - \frac{\vartheta_2''(o)}{\vartheta_2(o)}\right\} \tag{17}$$

Using the relations

$$\frac{\vartheta_1'''(o)}{\vartheta_1'(o)} = -4\,\pi^2\,\frac{\partial\,log\,\vartheta_1'(o)}{\partial\,log\,q}\quad,$$

$$\frac{\vartheta_2''(o)}{\vartheta_2(o)} = -4\,\pi^2\,\frac{\partial\,log\,\vartheta_2(o)}{\partial\,log\,q}$$

$$\vartheta_1'(o) = \pi\vartheta_0(o)\;\vartheta_2(o)\;\vartheta_3(o)\quad,$$

we can write

$$M = -\frac{2\sqrt{ax}}{\vartheta_2^2(o)}\left(\frac{\vartheta_0''(o)}{\vartheta_0\,(o)} + \frac{\vartheta_3''(o)}{\vartheta_3(o)}\right) \tag{18}$$

Utilizing the relation

$$\eta_1\,\omega_3 = \eta_3\,\omega_1 - \frac{1}{2}\pi\,i$$

we may put (B_2'') in the form

$$M = 4\pi\,ax\frac{A^2}{\omega_3}\left\{\frac{\omega_1}{\omega_3}(e_1\,\omega_3^2 - \eta_3\,\omega_3) + \frac{1}{2}\,\pi\,i\right\}$$

Writing $\tau_1 = -\frac{\omega_1}{\omega_3}$, we easily find that

$$M=4\pi ax\frac{A^2}{\omega_3}\left\{\frac{1}{8\tau}\left(\frac{\vartheta_2''(o)}{\vartheta_2(o)}+\frac{\vartheta_3''(o)}{\vartheta_3(o)}\right)+\frac{1}{2}\pi i\right\}$$

$$=\frac{8\sqrt{ax}}{\pi\vartheta_0^2(o\mid\tau_1)}\left\{\pi^2-\frac{\log\left(\frac{1}{q_1}\right)}{4}\left(\frac{\vartheta_2''(o\mid\tau_1)}{\vartheta_2(o\mid\tau_1)}+\frac{\vartheta_3''(o\mid\tau_1)}{\vartheta_3(o\mid\tau_1)}\right)\right\} \tag{19}$$

The expressions (17), (18), and (19) are of great practical importance, as will be shown in another section.

§ 8. *Expression for* $\frac{\partial M}{\partial z}$. In addition to M, we shall have to find $\frac{\partial M}{\partial z}$, which represents the force acting between two coaxial coils Since

$$\frac{\partial M}{\partial z}=4\pi axz\int_0^\pi\frac{\cos\theta\, d\theta}{(a^2+x^2+z^2-2ax\cos\theta)^{\frac{3}{2}}}, \tag{20}$$

we easily find that

$$\frac{1}{4\pi ax}\frac{\partial M}{\partial z}=-\frac{A^4 z}{4}\int_{\omega_3}^{\omega_2}\frac{A\wp(u)+B}{A\wp(u)-2B}du$$

$$=-\frac{A^4 z}{4}\left\{\omega_1-\frac{3e_1}{2(e_1-e_2)(e_1-e_3)}(\eta_1+e_1\omega_1)\right\}. \tag{21}$$

Expressing e_1, e_1-e_2, e_1-e_3, by means of ϑ-functions,

$$\frac{\partial M}{\partial z}=-\frac{\pi z}{\sqrt{ax}}\left\{\vartheta_2^2(o)+\frac{1}{2\pi^2}\left(\frac{1}{\vartheta_0^4(o)}+\frac{1}{\vartheta_3^4(o)}\right)\vartheta_2''(o)\vartheta_2(o)\right\} \tag{22}$$

§ 9. *M expressed in q-series.*—For reducing the ϑ-functions in (17) and (18), we can conveniently make use of the expansions given by Jacobi (Fundamenta Nova p. 104–105, Gesammelte Werke Bd. 1. p. 161). As the result of expansion, we easily find that

$$\frac{M}{4\pi\sqrt{ax}}=4\pi q^{\frac{3}{2}}(1+*+3q^4-4q^6+9q^8-12q^{10}\ldots\ldots) \tag{23}$$

Putting

$$3q^4-4q^6+9q^8-12q^{10}+\cdots\cdots=\varepsilon,$$

we can conveniently write

$$\frac{M}{4\pi\sqrt{ax}} = 4\,\pi\, q^{\frac{3}{2}}(1+\varepsilon) \tag{24}$$

Since ε is a very small quantity, we can, with tolerable accuracy, put

$$M \doteq 16\,\pi^2\sqrt{ax}\,q^{\frac{3}{2}} \tag{25}$$

Expressing (19) by means of q_1,

$$\frac{M}{4\pi\sqrt{ax}}=\frac{1}{2(1-2\,q_1+2\,q_1^4-2\,q_1^9)^2}\left[log\left(\frac{1}{q_1}\right)\Big\{1+8\,q_1(1-q_1+4\,q_1^2-5\,q_1^3 +6\,q_1^4-4\,q_1^5+8\,q_1^6-13\,q_1^7+\ldots\ldots)\Big\}-4\right] \tag{26}$$

The above expression is useful when the coils are very near each other. In such cases, q_1 is a very small quantity, so that by putting

$$32\,q_1^3-40\,q_1^4+48\,q_1^5-32\,q_1^6=\varepsilon_1$$

$$\frac{M}{4\pi\sqrt{ax}}=\frac{1}{2(1-2\,q_1+2q_1^4)^2}\left[log\left(\frac{1}{q_1}\right)\Big\{1+8\,q_1(1-q_1)+\varepsilon_1\Big\}-4\right] \tag{27}$$

For $\gamma>70^0$, where $sin\,\gamma=k$, q_1^3 is negligibly small, and

$$\frac{M}{4\pi\sqrt{ax}}\doteq\frac{1}{2\,(1-2\,q_1)^2}\left[log\left(\frac{1}{q_1}\right)\Big\{1+8\,q_1+\varepsilon_1'\Big\}-4\right] \tag{28}$$

$$\text{where } \varepsilon_1'=-8\,q_1^2+\varepsilon_1$$

Finally (22) gives

$$\frac{\partial M}{\partial z}=\frac{192\,\pi^2 z}{\sqrt{ax}}\,q^{\frac{5}{2}}(1+20\,q^2+225\,q^4+1840\,q^6+12120\,q^8+\cdots\cdots) \tag{29}$$

It will not be altogether out of place to digress on the practical utility of the several formulae above obtained.

In the first place, there is no need of finding γ from k or k', which is given at once from the dimensions and configuration of the coils. Instead of finding γ, we shall have to calculate

$$q=\frac{1}{2}l+2\left(\frac{1}{2}l\right)^5+15\left(\frac{1}{2}l\right)^9+\cdots\cdots$$

where $$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}},$$

of which the first term is generally sufficient,[1] as the following short table for $q-\frac{l}{2}$ will show. For practical purposes, a single calculation according to formula (25) or (27) gives at once the value of M. In order to shew the rapid convergence of the q-series, it will be sufficient to indicate the smallness of the corrections ε and ε_1, ε_1' entering in the above two formulae. For this purpose, the following short tables of corrections have been calculated.

Table of $\varepsilon=3q^4-4q^6+9q^8-12q^{10}$ and $q-\frac{l}{2}$

q.	Approximate value of γ=arc sin k.	ε	$q-\frac{l}{2}$
0.01	22°.6	0.000 0000	0.000 00000
0.02	31°.6	0.000 0005	0.000 00001
0.03	38°.1	0.000 0024	0.000 00005
0.04	43°.5	0.000 0077	0.000 00020
0.05	48°.0	0.000 0187	0.000 00063
0.06	51°.9	0.000 0387	0.000 00156
0.07	55°.3	0.000 0726	0.000 00336
0.08	58°.4	0.000 1218	0.000 00655
0.09	61°.1	0.000 1947	0.000 01181
0.10	63°.6	0.000 2961	0.000 02000
0.11	65°.9	0.000 4323	0.000 03220
0.12	67°.9	0.000 6105	0.000 04974
0.13	69°.8	0.000 8382	0.000 07421
0.14	71°.5	0.001 1234	0.000 10746

1) For the table of q, see Jacobi, Crelle's Journal Bd. 26, p. 93; Gesammelte Werke, Bd. 1, p. 363.

γ=arc sin k.	ε
10°	0.000 00000
20°	0.000 00000
30°	0.000 00031
40°	0.000 00367
45°	0.000 01044
50°	0.000 02738
55°	0.000 06773
60°	0.000 16098
65°	0.000 37397
70°	0.000 86566

Table of $\varepsilon_1 = 32\, q_1^3 - 40\, q_1^4 + 48\, q_1^5 - 32\, q_1^6$

$$\varepsilon_1' = -8 q_1^2 + \varepsilon_1.$$

q_1	Approximate value of γ=arc sin k.	ε_1	ε_1'
0.010	67°.4	0.000 0316	−0.000 7684
0.009	68°.5	0.000 0231	−0.000 6249
0.008	69°.7	0.000 0162	−0.000 4958
0.007	71°.0	0.000 0109	−0.000 3811
0.006	72°.4	0.000 0069	−0.000 2811
0.005	7 3°9	0.000 0040	−0.000 1960
0.004	75°.6	0.000 0020	−0.000 1260
0.003	77°.5	0.000 0010	−0.000 0710
0.002	79°.8	0.000 0003	−0.000 0317
0.001	82°.8	0.000 0000	−0.000 0080

The table shows that the error in M calculated by the formula

$$\frac{M}{4\pi\sqrt{ax}} \doteqdot 4\pi\, q^{\frac{3}{2}}$$

is only 0.001 per cent for $\gamma=45^0$, and 0.09 per cent for $\gamma=70^0$. When the coils are near each other, the approximation can be carried still further by using (27). In all these calculations, Legendre's table of elliptic integrals may be dispensed with; a somewhat tedious operation lies in finding $log\, nat\left(\frac{1}{q_1}\right)$.

It will not be out of place to give a numerical value for a single instance, in order to shew the rapid convergence of (23). For $k=\sin 70^0$,

$$q=0.1309845\left(=\frac{1}{2}l\right)+0.0000771\left(=2\left(\frac{1}{2}l\right)^5\right)+0.0000002\left(=15\left(\frac{1}{2}l\right)^9\right)$$

$$=0.1310618$$

$$log\frac{M}{4\pi\sqrt{ax}}=\bar{1}.7754242\left(=log\, 4\pi q^{\frac{3}{2}}\right)+0.0003758\left(=log\,(1+\varepsilon)\right)$$

$$=\bar{1}.7758000.$$

which coincides with the value given by Maxwell. It is to be noticed, that the above is the most unfavorable case in which (24) may be applied.

§ 10. It will be worth while to mention that the expression for the solid angle subtended by a circle, in terms of zonal harmonics, can be deduced from the formulae already obtained.

The potential of a magnetie shell φ is given by (A).

$$\varphi=2\pi a\int_0^\infty e^{-\lambda z}\, J_0(\lambda x)\, J_1(\lambda a)\, d\lambda$$

By expanding $J_1(\lambda a)$ according to ascending powers of λa, and remembering that

$$\int_0^\infty e^{-\lambda z} J_0(\lambda z)\, \lambda^{2m+1} d\lambda = 1.2.3.\ldots.(2m+1)(x^2+z^2)^{-m+1} P_{2m+1}\left(\frac{z}{\sqrt{x^2+z^2}}\right),$$

where P_{2m+1} denotes zonal harmonics of $2m+1$th order, we arrive at the ordinary expression for the solid angle in terms of spherical harmonics. M can be similarly expressed by using (B′).

It is needless to remark that such expansions converge very slowly. What I wish to show in the present paper is that we may sometimes arrive at a convenient and practical result by using a q-series, instead of falling into the grooves of spherical harmonics.