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Stationary Surface Tremors

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With Plates I.

§ 1. The problem of elastic surface waves on an isotropic solid was first treated by Lord Rayleigh* in a paper communicated to the London Mathematical Society in 1885. The important bearing of this class of waves on earthquakes was recognised by him, but the result of analysis in its practical aspect has scarcely been discussed, inasmuch as the hypothesis of isotropy of the medium is hardly compatible with the structure of the earth's crust. When the great complexity which will be introduced into the boundary conditions if the stratigraphical structure such as is actually met with is taken into account, the advantage gained does not easily compensate for the mathematical difficulty which necessarily accrues in the solution of the problem. But if the result of calculation based on a simple abstraction as to the nature of the medium be interpreted in the light of a simple comment as to the character of the motion which is capable of being excited on the surface of the elastic solid, the analysis would not be a useless piece of mathematical play, as the conception of the phenomena is thereby greatly facilitated. A mere quantitative discrepancy will not prove imperfection of the theory, but one notices at a glance that the hypothesis made at the outset of calculation necessarily deviates from what is found in nature, where the intricacy and heterogeneity far surpass the power of modern analysis. The greatest objec-

* Lord Rayleigh, Proc. London Mathem. Soc., Vol 17, 4, 1887.

tion which can perhaps be raised against the present treatment of the problem is the neglect of the influence of gravity in modifying the surface wave.

§ 2. Take the xy-plane at the horizontal boundary, and z-axis vertically downwards, and denote the component displacements by u, v, w; then the equations of motion becomes.

$$(1) \qquad \rho\frac{\partial^2 u}{\partial t^2}=(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\Delta\theta$$

with similar equations for v and w. Here ρ denotes the density, λ, μ Lamé's constants, and

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

$$\Delta\theta=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}$$

For the free vibration of period $\frac{2\pi}{p}$, the equations can be easily reduced to the form

$$(2) \qquad \begin{cases}(\Delta+k^2)\,u=\left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial x}\\ (\Delta+k^2)\,v=\left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial y}\\ (\Delta+k^2)\,w=\left(1-\frac{k^2}{h^2}\right)\frac{\partial\theta}{\partial y}\end{cases}$$

where

$$(3) \qquad \begin{cases}h^2=\frac{\rho}{\lambda+\mu}p^2\\ k^2=\frac{\rho}{\mu}p^2\end{cases}$$

The equations can therefore be satisfied by

$$(4) \qquad \begin{cases}u=-\frac{1}{h^2}\frac{\partial\theta}{\partial x}\\ v=-\frac{1}{h^2}\frac{\partial\theta}{\partial y}\\ w=-\frac{1}{h^2}\frac{\partial\theta}{\partial z}\end{cases}$$

to which we shall have to add the complementary solutions satisfying

(5) $$(\varDelta+k^2)u=0,\quad (\varDelta+k^2)v=0,\quad (\varDelta+k^2)w=0.$$

with the condition

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

For the surface wave on the plane $z=0$, the displacements are proportional to

$$e^{i(fx+gy+pt)}$$

where f, g are constants to be determined by the boundary conditions.

By differentiating (4) with respect to x, y, z, we find

(6) $$(\varDelta+h^2)\theta=0.$$

or

(6′) $$\frac{\partial^2\theta}{\partial z^2}-(f^2+g^2-h^2\theta)=0.$$

Thus $\theta=Pe^{-rz}$ where $r^2=f^2+g^2-h^2$

The equation (5) can be written

(7) $$\frac{\partial^2 u}{\partial z^2}-(f^2+g^2-k^2)u=0.$$

or $u=Ae^{-sz}$ where $s^2=f^2+g^2-k^2$

Thus (4) and (5) give for u, v, w

(8) $$\begin{cases} u=-\dfrac{if}{h^2}Pe^{-rz}+Ae^{-sz} \\ v=-\dfrac{if}{h^2}Pe^{-rz}+Be^{-sz} \\ w=-\dfrac{r}{h^2}Pe^{-rz}+Ce^{-sz} \end{cases}$$

where P, A, B, C are all proportional to $e^{i(fx+gy+pt)}$ and A, B, C are connected by the relation

(8′) $$i(fA+gB)-sC=0$$

which is equivalent to $\theta=0$. in the complementary term.

The absence of shear at the horizontal boundary $z=0$ leads to the relation

(9) $$(s^2+f^2+g^2)h^2C+2r(f^2+g^2)\,P=0\,,$$

while the evanescence of the normal surface traction is expressed by

(10) $$(k^2-2h^2)P-2(r^2+sh^2C)=0.$$

Eliminating C and P between (9) and (10), we obtain

(11) $$\{2(f^2+g^2)-k^2\}^4=16(f^2+g^2)^2(f^2+g^2-h^2)(f^2+g^2-k^2)$$

Remembering that

$$\frac{h^2}{k^2}=\frac{\mu}{\lambda+2\mu},$$

and putting $$\gamma^2=\frac{k^2}{f^2+g^2}=\frac{\rho}{\mu(f^2+g^2)}p^2$$

we arrive at the equation

(12) $$\gamma^6-8\gamma^4+24\gamma^2-16-16\left(\frac{\mu}{\lambda+2\mu}\right)(\gamma^2-1)=0.$$

for finding the frequency p of the free vibration, as given by Rayleigh.

The following real roots of γ have been calculated for different values of $\frac{\mu}{\lambda+2\mu}$ (with corresponding ratio σ of lateral contraction to longitudinal elongation).

$\frac{h^2}{k^4}=\frac{\mu}{\lambda+2\mu}$	σ	$\gamma=\frac{k}{\sqrt{f^2+g^2}}$
.0	$\frac{1}{2}$	0.9554
.1	$\frac{4}{9}$	0.9482
.2	$\frac{3}{8}$	0.9387
.3	$\frac{2}{7}$	0.9251

.4	$\frac{1}{6}$	0.9052
.5	0	0 8741
.6	$-\frac{1}{4}$	0.8237

γ being determined for a given boundary, the displacements u, v, w are given by

$$(13)\quad \begin{cases} u=a\frac{if}{h^2}\left\{e^{-rz}+\frac{2rs}{s^2+f^2+g^2}e^{-sz}\right\}e^{i(pt+fx+gy)} \\ v=a\frac{ig}{h^2}\left\{e^{-rz}+\frac{2rs}{s^2+f^2+g^2}e^{-sz}\right\}e^{i(pt+fx+gy)} \\ w=a\frac{r}{h^2}\left\{e^{-rz}-\frac{2(f^2+g^2)}{s^2+f^2+g^2}e^{-sz}\right\}e^{i(pt+fx+gy)} \end{cases}$$

a being a constant to be found from the initial condition.

When there are several values of u_n, v_n, w_n, for u, v, w corresponding to different values of f and g, and consequently p, the displacements will be given by

$$(14)\qquad u=\Sigma u_u,\ v=\Sigma v_n,\ w=\Sigma w_n.$$

§ 3. When the elastic solid is bounded by two parallel planes, the discussion of waves can be generally reduced to that in two dimensions. If the bounding planes be given by

$$x=0,\ x=a,\ \text{and}\ z=0,$$

the surface wave would be generally given by the components u and w only.

Three different cases are to be distinguished.

(I) When both planes $x=0$, $x=a$ are clamped; then u varies as $\sin fx$ and

$$f=\frac{n\pi}{a},$$

consequently

(15) $$p_n=\gamma\frac{n\pi}{a}\sqrt{\frac{\mu}{\rho}} \qquad (n=1,\ 2,\ 3,\ldots.)$$

and the complete period is given by

(15′) $$T_n=\frac{2\pi}{p_n}=\frac{2a}{n\gamma\sqrt{\frac{\mu}{\rho}}}$$

Writing $$a\frac{f}{h^2}\left(-e^{-rz}+\frac{2rs}{s^2+f^2}e^{-sz}\right)=A_n$$

$$a\frac{r}{h^2}\left(e^{-rz}-\frac{2f^2}{s^2+f^2}e^{-sz}\right)=C_n$$

where r, s are expressed by the above value of f,

(16) $$\begin{cases} u_n=A_n \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} p_n t \sin n\pi\dfrac{x}{a} \\ w_n=C_n \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} p_n t \sin n\pi\dfrac{x}{a} \end{cases}$$

satisfy the specified condition, so that the sum of these expressions will also represent the possible forms of surface waves.

II. When one boundary is clamped and the other is free,

$$u=0 \quad \text{for} \quad x=0$$

and $$\frac{\partial u}{\partial x}=0 \quad ,, \quad x=a.$$

Thus

$$u \propto \sin fx$$

where

(17) $$f=\frac{2n+1}{2}\frac{\pi}{a} \qquad (n=0,\ 1,\ 2,\ 3,\ldots\ldots)$$

and $$p_n=\frac{2n+1}{2}\frac{\pi}{a}\gamma\sqrt{\frac{\mu}{\rho}}$$

or $$T_n=\frac{4}{2n+1}\frac{a}{\gamma\sqrt{\frac{\mu}{\rho}}}$$

The displacements are given by

$$(18)\qquad \begin{cases} n_n = A_n' \substack{\cos \\ \sin} p_n t \sin\left(\dfrac{2n+1}{2}\right)\dfrac{\pi x}{a} \\ \\ w_n = C_n' \substack{\sin \\ \cos} p_n t \sin\left(\dfrac{2n+1}{2}\right)\dfrac{\pi x}{a} \end{cases}$$

III. When both planes $x=0$, $x=a$ are free,

$$\frac{\partial u}{\partial x}=0 \quad \text{for} \quad \begin{matrix} x=0 \\ x=a \end{matrix}$$

and

$$u \propto \cos fx.$$

Evidently

$$(19)\qquad f=\frac{n\pi}{a} \qquad\qquad (n=1,\ 2,\ 3,\ldots\ldots)$$

and

$$p_n=\frac{n\pi}{a}\gamma\sqrt{\frac{\mu}{\rho}}$$

or

$$T_n=\frac{2a}{n\gamma\sqrt{\dfrac{\mu}{\rho}}}$$

which is the same as for clamped boundaries.

Consequently

$$(20)\qquad \begin{cases} u_n = A_n \substack{\sin \\ \cos}\, p_n t \cos\dfrac{n\pi}{a}x. \\ \\ w_n = C_n \substack{\cos \\ \sin}\, p_n t \cos\dfrac{n\pi}{a}x. \end{cases}$$

§ 4. The case which is really worth consideration is when the boundary is yielding to a certain amount. As the surface soil is characterised by its plasticity, and the unevenness of the bounding surface makes the approach to the ideal case somewhat difficult, the general discussion of the yielding boundary in its application to geophysics may with propriety be postponed till the main cause of the surface tremors is well settled by observation. If we allow for the deviations in the actual case from what is assumed in the above discussion, the first case would correspond to the vibrations of a plain between parallel mountain ranges, the second case to those of a plain isolated by a ridge on one side and a mountain range on the other, and the third case to those of a high plateau or of a plain between two parallel rivers.

To give a numerical instance, let us suppose that $a=10$ km. $\sqrt{\frac{\mu}{\rho}}=3\frac{\text{km}}{\text{sec}}$, $\chi=0.9$, then

$$\begin{cases} T_1 = 7^s.4 \\ T_2 = 3.7 \end{cases} \quad \text{for (I) and (III)}$$

and

$$\begin{cases} T_0 = 14.8 \\ T_1 = 4.9 \end{cases} \quad \text{,, (II).}$$

The above result shows that the vibration is quite analogous to the longitudinal vibration of rods; the nodes and loops are to be found in places which are aliquot parts of the breadth. The velocity of propagation is not so simply defined as in the case of rods.

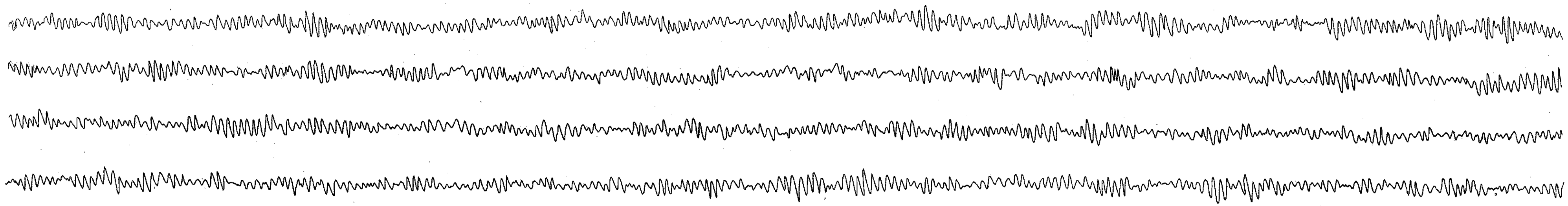
The velocity of transversal wave $\sqrt{\frac{\mu}{\rho}}$ is perhaps very small for the uppermost strata of the earth's crust and probably much less than $3\frac{\text{km}}{\text{sec}}$. When the plain on which the surface wave is excited is of considerable extent, the period of stationary wave may attain a tolerably large value. If such motion be stimulated by progressive seismic waves, there is every possibility of the motion being traced on seismographs. One great drawback in seismometers is the impossibility of the discrimination of the progressive from the stationary waves. I am not aware if such waves have ever been observed, but it would not be out of place here to notice the probable appearance of vibrations of long periods.

The tremors accompanying the barometric change generally indicate persistence of surface waves for some intervals of time. As the surface soil is strongly damped, the evanescence will be quick; but the surface pressure is generally accompanied with strains, as already shown in the preceding paper, in regions quite remote from the place of actual application. The extent to which the horizontal component of the surface strain is felt is linearly related to the radius of the circular region subjected to stress. As is often met with on the earth's surface, the region of low pressure or of strong rainfall is not limited to a narrow space, but extends over hundreds or sometimes thousands of kilometres; the consequence is that the region in which the surface wave is liable to be excited is of vast extent. In countries which abound with mountains, interspersed with plains between, the surface waves may be either propagating or stationary, so long as the disturbing force is acting, whether the point under consideration be within the pressure domain or outside it.

The inspection of the diagram of tremors at once indicate the existence of beats in the vibrations. These are of exactly the same nature as is observed in the sea waves on calm days, where the bottom is irregular. These beats are probably due to the superposition of two vibrations of nearly equal periods and amplitudes. In the case of earth tremors, the existence of two such vibrations is to be traced to the nonuniformity of the surface strata, when either the elastic constants or the thickness of the horizontal strata are slightly different, there will be superposition of two waves giving rise to beats analogous to those well-known in light and sound.

§ 6. The fulfilment of the boundary condition in the present problem offers insurmountable difficulty to exact calculation, except for those boundaries which are conformable to rectangular coordinates or circles. The result of calculation in its practical application is only a rough approximation, and probably represents a part of the phenomena of earth tremors in its broadest aspect. The discussion of the present problem with special analogies to sea waves will not fail to be of special interest to those pursuing the study of geophysics, as there are so many points of resemblance between the two. I have therefore thought it advisable to give a diagram of sea waves observed at Odawara on a calm day, to show the close kinematical analogies which may exist between these two phenomena.

Tremor.



Sea wave.