Theories.

Time-lag due to Inertia.

In problems on torsion the couple which acts to twist the specimen is generally considered abstractly, and the problem is solved by the statical method, so that the solution corresponds to its limiting result after infinite time. When considered from the kinetic point of view, the very word "Motion" can not be understood without the notion of Time-lag. Any phenomenon which relates to motion of matter, though it may occur at the very instant when the causal agent acted, can never be completed without certain lapse of time. Again according to Newton, it requires other opposing causal agent to stop further progress of the phenomenon, provided it is due to motion of matter. If the opposing agent is one which takes place as an effect of further progress of the phenomenon, the opposing agent can never become sufficiently great to stop its further progress during finite time. It is necessarily prolonged to infinite time, though it may become infinitely small at the same time. In problems on torsion, the same is the case, as it will be seen from the following lines.

Let τ=twist at any time t,

τ_0=limiting value of τ at $t=\infty$,

$\mathfrak{C}_0$=applied twisting couple,

μ=a constant, proportional to rigidity$=\mathfrak{C}_0\,\tau^{-1}$,

M_0=mass of the weights by means of which the couple is produced,

r=radius of the twisting pulley,

$\mathfrak{C}$=actual couple acting within the twisting specimen at any instant t,

$\mathfrak{J}$=moment of inertia of the twisted bar and its subsidiaries.

Then we have for the equation of motion

$$\frac{d\tau}{dt}=\frac{\mathfrak{C}}{\mathfrak{J}+rM_0}t.$$

Now, when the bar becomes actually twisted through any angle τ, there is an opposing couple $\mu\tau$ due to the elastic force of the bar. Hence we have

$$\mathfrak{C}=\mathfrak{C}_0-\mu\tau$$

i.e.
$$\frac{d\tau}{dt}=\frac{\mathfrak{C}_0-\mu\tau}{\mathfrak{J}+rM_0}t.$$

Which becomes after integration

$$\tau=\tau_0\left\{1-e^{-\frac{\mu t^2}{2(\mathfrak{J}+rM_0)}}\right\}$$

where
$$\tau_0=\frac{\mathfrak{C}_0}{\mu},$$

or
$$t=\sqrt{\left\{\frac{2(\mathfrak{J}+rM_0)}{\mu}\right\}}\sqrt{\left\{\log\frac{1}{1-\frac{\tau}{\tau_0}}\right\}}$$

i.e.
$$t=k\sqrt{\left\{\log\frac{1}{1-\frac{\tau}{\tau_0}}\right\}}$$

where
$$k=\sqrt{\{2(\mathfrak{J}+rM_0)\mu^{-1}\}}.$$

As a matter of course, τ is zero for $t=0$ and τ becomes τ_0 for $t=$infinity. Here, in putting $\mathfrak{C}=\mathfrak{C}_0-\mu\tau$ and $\tau_0=\mathfrak{C}_0\ \mu^{-1}$, Hooke's law is evidently assumed so that the limiting value of twist after infinite time is simply proportional to the applied couple.

Table XII in the end of this article gives the corresponding values of t and $\frac{\tau}{\tau_0}$; value of the constant k depends upon the rigidity of the specimen and the total mass to be moved. To illustrate this point more clearly, I have plotted this result of calculation in Figs. 51 and 52, Pl. XXV. Fig. 51 corresponds to $k=\frac{1}{\log 10}$ while Fig. 52 to $k=\frac{3}{\log 10}$: *i.e.* the value of $\mathfrak{J}+rM_0$ is three times greater in the latter than in the former.

Elastic Yielding.

In the above, I have assumed the modulus of rigidity to remain constant for all twisted state. No matter, however, what the constitution of solid is, the modulus of rigidity must be affected somehow with increase of twist, at least, in so far as the elastic force is due to mutual action between two consecutive parts, as the mutual relation may probably be affected by the torsion. To ascertain how this diminution of elastic force is connected with the amount of twist, is no easy matter. For the present case, let us assume that the modulus of rigidity remains constant and equal to μ_0 till the amount of twist reaches a certain definite value $\tau=\tau_1$, and that when the twist goes further beyond that limit the modulus of rigidity μ at any twisted state is given by

$$\mu=\mu_0\{1-e^{-\nu\tau}\}.$$

Under this supposition, we have

$$\frac{d\tau}{dt}=\frac{1}{k}\left\{\mathfrak{C}_0-\mu_0(1-e^{-\nu\tau})\tau\right\}t$$

$$=\frac{1}{k}\left\{\mathfrak{C}_0-\mu_0\tau\right\}t+\frac{\mu_0}{k}\tau e^{-\nu\tau}t.$$

Now, the first term of the second member is due to the moment of inertia, and it has already been discussed in the last section. The second term is due to the elastic-yielding.

Let η be the amount of twist due to the elastic-yielding so that

$$\eta=\tau-\tau_1$$

then $$\frac{d\eta}{dt}=\frac{\mu_0\tau_1}{k}e^{-\nu\tau_1}e^{-\nu\eta}t=Re^{-\nu\eta}t$$

Integrating this differential equation, we have

$$e^{\nu\eta}=\frac{\nu R}{2}t^2+\text{const.}$$

Now, let the value of ν be so chosen that it satisfies the relation

$$2k\nu e^{\nu\tau_1}=\mu_0\tau_1$$

then $$\eta=\frac{2}{\nu}\log t$$

where ν is a function of the modulus of rigidity in virgin piece and the initial twist.

This relation between the elastic yielding and the time is graphically shown in Pl. VIII., Fig. 14, taking 1 and 4 for the value of ν which correspond to the Specimen No. 4_1, a piece of sandstone. The table giving its numerical values is omitted, since it is nothing but a logarithmic table of natural numbers.

Hysteresis due to the Elastic Yielding.

The law according to which the elastic yielding of rocks proceeds is not yet strictly known, so that the hysteresis due to it can never be strictly calculated. For the present case, however, as a first approximation, assuming the relation above established as an actual one, we may arrive at a result which seems not at all to be unplausible.

Taking the relation, no matter how and from what hypothesis it was obtained,

$$\eta = k \log t \qquad (1).$$

for granted, let us further assume that as to the elastic yielding, the principle of superposition holds good, so that if η_1 and η_2 are the amounts of elastic yielding due to two couples $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively, then the elastic yielding due to the resultant couple $\mathfrak{C} = \mathfrak{C}_1 \pm \mathfrak{C}_2$ is given by $\eta = \eta_1 \pm \eta_2$.

To simplify the matter, let us suppose that a definite couple begins to act at the instant $t=0$, and remains constantly acting for ever: and also that after each unit time, say per one minute, the couple is increased by the amount equal to the first one. Then, the total amount of the elastic-yielding at the instant $t=p$ is given by

$$\eta_p = \eta_{(p)} + \eta_{(p-1)} + \eta_{(p-2)} + \ldots + \eta_{(2)} + \eta_{(1)}$$
$$= k\{\log p + \log(p-1) + \ldots + \log 2 + \log 1\} = k \log \Gamma\{p+1\} \qquad (2).$$

where Γ is the well-known symbol for Gamma-function.

If then the increasing of the couple is stopped at the instant $t=p$, and the couple remains constant for the following q minutes, the total yielding at the instant $t=p+q$ may be found as follows:— Instead of stopping the increase of the couple, if it were continued to increase as before, the total yielding at the instant $t=p+q$ would have been given by

$$\eta' = k \log \Gamma\{p+q+1\}.$$

Again, if a negative couple were to act at the instant $t=p+1$ and afterwards, increasing in its absolute amount step by step as the positive couple, the total yielding due to this at the same instant $t=p+q$ would be evidently

$$\eta'' = -k \log \Gamma\{q+1\}.$$

Let these two couples act at the same time, then it is obvious that the resultant couple increases to the instant $t=p$, but after that instant it remains constant, since both positive and negative couples increase by equal amount. The resultant yielding due to this resultant couple is obviously given by

$$\eta = \eta' + \eta'' = k \log \Gamma\{p+q+1\} - k \log \Gamma\{q+1\} = k \log \frac{\Gamma\{p+q+1\}}{\Gamma\{q+1\}} \quad (3).$$

If the couple increases for the first p minutes, and then, instead of remaining constant, it decreases step by step during n minutes, the total yielding, as it may easily be seen, at the instant $t=p+n$ is given by

$$\eta = \eta_p - \eta_n = k \log \frac{\Gamma\{p+n+1\}}{\Gamma\{n+1\}} - k \log \Gamma\{n+1\} = k \log \frac{\Gamma\{p+n+1\}}{\Gamma\{n+1\}^2}. \quad (4).$$

If the couple remains constant since this moment, then the total yielding at the instant $t=p+n+r$ is

$$\eta = k \log \frac{\Gamma\{p+n+r+1\}\Gamma\{r+1\}}{[\Gamma\{n+r+1\}]^2}. \quad (5).$$

As a particular case, put $p=n$ then

$$\sigma = k \log \frac{\Gamma\{2p+r+1\}\Gamma\{r+1\}}{[\Gamma\{p+r+1\}]^2}; \quad (6).$$

this value of σ represents the value of residual twist surviving the couple in a piece which was twisted during first p minutes by an increasing couple, then released gradually from it during next p minutes, and then remained free from any couple for the last r minutes.

Here it must be observed that the equation (6), which expresses the amount of residual twist, is also *the equation of elastic-recovery.* That is to say, if t' represents the time which has elapsed since the specimen has been made free from the twisting couple, then the twist due to the elastic-recovery is given by

$$\rho = k\left\{\log\frac{\Gamma\{2p+1\}}{[\Gamma\{p+1\}]^2} - \log\frac{\Gamma\{2p+t'+1\}\Gamma\{t'+1\}}{[\Gamma\{p+t'+1\}]^2}\right\};$$

or

$$\rho = k\log\frac{\Gamma\{2p+1\}\,[\Gamma\{p+t'+1\}]^2}{[\Gamma\{p+1\}]^2\,\Gamma\{2p+t'+1\}\,\Gamma\{t'+1\}}. \qquad (7).$$

For a particular value of p, I have calculated this value of ρ which is given in Table XIII, at the end of this article, and is also graphically shown in Fig. 53, Pl. VII.

Proceeding in this way, we may find the value of η after any cycle of twist. I give here its general expression. Suppose we start from the origin, $t=0$, which corresponds to the neutral state of the specimen, and for the sake of simplicity also suppose that the change of couple takes place by unit amount per unit of time.

1. Increasing the couple step by step we reach couple = I at the instant $t=i_1$;
2. Then decreasing the couple and ultimately going even to negative, we reach couple = II at the instant $t=i_1+i_2$, so that i_2 = I—II.
3. Again, increasing the couple once more we reach couple = III at the instant $t=i_1+i_2+i_3$; etc. etc.;

where I, II, ...N represent certain definite number, positive or negative in sign, as the case may be. Let p and n denote any given stage in the cycle on the increasing and decreasing procedures respectively. Also let $\eta_{\text{I}\cdot\text{II}\cdot p}$ express the total yielding at the increasing stage p, after the specimen was twisted up to I in the positive direction and then untwisted, or twisted in opposite direction, up to II; similar meaning for $\eta_{\text{I}\cdot\text{II}\cdot\text{III}\cdot n}$. Here both p and n may be either positive or negative. For instance, $\eta_{5\cdot -7\cdot +6}$ expresses the yielding, at the instant $t=2\times5+2\times7+6=30$, of a specimen which was treated in the following way:—Starting from the neutral state, the couple was increased up to 5 then it was decreased up to -7 and once more it is increased up to $+6$ just now.

These premised, we may write the general expression for the Hysteresis-function as follows:—

$$\eta_{\mathrm{I.\,II.\ldots.N.}\,p} = k \log \frac{\Gamma\left\{\sum_{\sigma=1}^{N'} i_\sigma + p+1-\mathrm{N}\right\}\left[\Gamma\{p+1-\mathrm{N}\}\right]^2 \prod_{\rho=1}^{\frac{N'}{2}-1}\left[\Gamma\left\{\sum_{\sigma=2\rho+1}^{N'} i_\sigma + p+1-\mathrm{N}\right\}\right]^2}{\prod_{\rho=1}^{\frac{N'}{2}}\left[\Gamma\left\{\sum_{\sigma=2\rho}^{N'} i_\sigma + p+1-\mathrm{N}\right\}\right]^2};$$

$N'=$ even; (8).

$$\eta_{\mathrm{I.\,II.\ldots.N.}\,n} = k \log \frac{\Gamma\left\{\sum_{\sigma=1}^{N'} i_\sigma + \mathrm{N}+1-n\right\} \prod_{\rho=1}^{\frac{N'-1}{2}}\left[\Gamma\left\{\sum_{\sigma=2\rho+1}^{N'} i_\sigma + \mathrm{N}+1-n\right\}\right]^2}{\left[\Gamma\{\mathrm{N}+1-n\}\right]^2 \prod_{\rho=1}^{\frac{N'-1}{2}}\left[\Gamma\left\{\sum_{\sigma=2\rho}^{N'} i_\sigma + \mathrm{N}+1-n\right\}\right]^2};$$

$N'=$ odd. (9).

where the symbol $\sum_{\sigma=2\rho}^{N'}$ represents, as usual, the sum of all terms obtained by giving a series of natural numbers from 2ρ to N' to the value of σ; while the other $\prod_{\rho=1}^{\frac{N'}{2}}$ expresses that the product of all the terms is to be taken which are obtained from the general term preceded by the symbol, by giving a series of natural numbers from 1 to $\frac{N'}{2}$ to the value of ρ; and N' is the number of alterations of the couple from an increasing or decreasing state to a decreasing or increasing state respectively.

As a particular case, when the twisting and untwisting occurs λ times cyclically with a constant amplitude, $\pm A$, the total yielding at the increasing stage p is given by

$${}_{\lambda A}\eta_p = k \log \frac{\Gamma\{4(\lambda+1)A+p+1\} \prod_{\rho=0}^{\lambda}\left[\Gamma\{4(\lambda+1)A-(4\rho+3)A+p+1\}\right]^2}{\prod_{\rho=0}^{\lambda}\left[\Gamma\{4(\lambda+1)A-(4\rho+1)A+p+1\}\right]^2}.$$

................................(10).

To give general idea of the nature of the expression, I undertook a laborious calculation to evaluate this for the case $A=10$, $\lambda=0,1,2$;

and p having any value between -10 and $+10$. The result is given in Tables XIV.—XVIII. at the end of the article.

Now the difference between the yieldings at two consecutive stages p and $p+1$ is given by

$$\triangle_p \{{}_{\lambda.A}\eta_p\} = {}_{\lambda.A}\eta_{p+1} - {}_{\lambda.A}\eta_p$$

$$= k \log \frac{\{4(\lambda+1)A+p+1\} \prod_{\rho=0}^{\lambda} \left[4\{\lambda+1\}A - \{4\rho+3\}A+p+1 \right]^2}{\prod_{\rho=0}^{\lambda} \left[4\{\lambda+1\}A - \{4\rho+1\}A+p+1 \right]^2} = k \log \varepsilon \;;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11).$$

Here it may be remarked that, since there is no negative factor in the expression ε, the difference of yielding has always real value: but it is neither necessarily positive nor necessarily negative. For, writing $(4\lambda+1)A=B$ for the sake of brevity, if we put $p=-A$, the expression ε may be written in the form

$$\varepsilon = \frac{\{B+1-A\}\{B+1-4A\}}{\{B+1-2A\}^2} \frac{\{B+1-4A\}\{B+1-8A\}}{\{B+1-6A\}^2} \cdots \frac{\{1+4A\}.1}{\{1+2A\}^2} < 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12).$$

which is less than unity, since each group of factors is less than unity. Observe that if α and β are two real quantities, the product of the two is always less than the square of their arithmetical mean. Again, if we put $p=A$, the expression ε may be written in the form

$$\varepsilon = \frac{\{B+1+A\}}{\{B+1\}} \frac{\{B+1-2A\}^2}{\{B+1\}\{B+1-4A\}} \cdots \frac{\{1+6A\}^2}{\{1+8A\}\{1+4A\}} \frac{\{1+2A\}^2}{\{1+4A\}.1} > 1 \quad (13).$$

which is obviously greater than unity, since as was already remarked, geometrical mean is less than the arithmetical. Thus, the value of $\triangle_p . {}_{\lambda.A}\eta_p$ is negative for $p=-A$ while it is positive for $p=A$, so that when a specimen is twisted to and fro cyclically, the twist may increase, although the couple is decreasing or vice versa.

In the next place, the influence of amplitude on the amount of yielding may be investigated as follows. Let the constant amplitude of the cycle, whose centre is at the origin, be $\pm A$, then writing the equation (11) in the form

$$\triangle_{p\cdot}\{_{\lambda\cdot A}\eta_p\}=k\log\frac{[R][R-3A]^2\ldots[p+1+A]^2}{[R-A][R-5A]^2\ldots[p+1+3A]^2}$$

we have, for the variation of amplitude from A to $A+1$

$$\triangle_A(\triangle_{p\cdot}\,{}_{\lambda\cdot A}\eta_p)=\triangle_{p\cdot}\,{}_{\lambda\cdot A+1}\eta_p-\triangle_{p\cdot}\,{}_{\lambda\cdot A}\eta_p$$

$$=k\log\frac{[R+4\overline{\lambda+1}][R-A]^2[R+4\overline{\lambda+1}-3(A+1)]\ldots[p+1+3A]^2[p+1+A+1]^2}{[R][R+4\overline{\lambda+1}-(A+1)]^2[R-3A]^2\ldots[p+1+3(A+1)]^2[p+1+A]^2}>0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (14).$$

Now,

$$\frac{R+4\overline{\lambda+1}}{R}>1;\ \frac{[R-A][R+4\overline{\lambda+1}-3(A+1)]}{[R+4(\lambda+1)-(A+1)][R-3A]}=1+\frac{6A-(p+1)}{[R-A+4\lambda+3][R-3A]}>1;$$

etc. etc.

so that the product of all factors is greater than unity. Thus, when the centre of cycle is at the origin, the curve of elastic yielding at any stage p after λ cycles of twisting and untwisting becomes steeper as the amplitude of the cycle is increased.

In the next place, let the number of cycles be increased from λ to $\lambda+1$; then we have

$$\triangle_\lambda\{\triangle_p({}_{\lambda\cdot A}\eta_p)\}=k\log\frac{\{4(\lambda+2)A+p+1\}\prod_{\rho=0}^{\lambda+1}\left[4\{\lambda+2\}A-\{4\rho+3\}A+p+1\right]^2}{\prod_{\rho=0}^{\lambda+1}\left[4\{\lambda+2\}A-\{4\rho+1\}A+p+1\right]^2}$$

$$-k\log\frac{\{4(\lambda+1)A+p+1\}\prod_{\rho=0}^{\lambda}\left[4\{\lambda+1\}A-\{4\rho+3\}A+p+1\right]^2}{\prod_{\rho=0}^{\lambda}\left[4\{\lambda+1\}A-\{4\rho+1\}A+p+1\right]^2}$$

$$=k\log\frac{\{R+4A\}\prod_{\rho=0}^{\lambda+1}\left[R-\{4\rho-1\}A\right]^2\prod_{\rho=0}^{\lambda}\left[R-\{4\rho+1\}A\right]^2}{\{R\}\prod_{\rho=0}^{\lambda+1}\left[R-\{4\rho-3\}A\right]^2\prod_{\rho=0}^{\lambda}\left[R-\{4\rho+3\}A\right]^2}$$

$$=k\log\frac{\{R+4A\}[R+A]^2}{\{R\}[R+3A]^2}$$

$$=k\log\frac{\{(R+2A)+2A\}[\{R+2A\}-A]^2}{\{(R+2A)-2A\}[\{R+2A\}+A]^2}$$

$$=k\log\frac{\{(R+2A)+2A\}\{[R+2A]^2-3A^2\}+2A^3}{\{(R+2A)+2A\}\{[R+2A]^2-3A^2\}-2A^3}>0..(15)$$

Thus, the value of $\triangle_p({}_{\lambda.A}\eta_p)$ increases with λ: *i.e.* the curve of hysteresis function at any stage p becomes steeper as the number of cycles of twisting and untwisting is increased.

Again, taking the second differential of $(\triangle_p\eta)$ with respect to λ, we have

$$\triangle^2_\lambda\{\triangle_p({}_{\lambda.A}\eta_p)\}=k\log\frac{\{R+8A\}[R+5A]^2[R+3A]^2\{R\}}{[R+4A]^2[R+7A]^2[R+A]^2}$$

$$=k\log\frac{R\{R+8A\}[R\{R+8A\}+15A^2]^2}{[R\{R+8A\}+16A^2][R\{R+8A\}+7A^2]^2}.<0..(16)$$

Now, for the sake of brevity, put

$$R\{R+8A\}=D.$$

then

$$R\{R+8A\}[R\{R+8A\}+15A^2]^2=D\{D+15A^2\}^2;$$

$$[R\{R+8A\}+16A^2][R\{R+8A\}+7A^2]^2$$
$$=\{D+16A^2\}\{D+7A^2\}^2$$
$$=\{D+15A^2+A^2\}\{D^2+15A^2D+49A^4-A^2D\}$$
$$=\{D+15A^2\}\{D^2+15A^2D\}+A^2(D+7A^2)^2+\{D+15A^2\}\{49A^4-A^2D\}$$
$$=D\{D+15A^2\}^2+16A^4\{49A^2+3D\}.$$

Thus, of the fraction under the sign of logarithm, the numerator is less than the denominator: *i.e.* the logarithm is negative. Hence, though the value of $\triangle_p\{{}_{\lambda.A}\eta_p\}$ increases with λ, its rate of increase diminishes with the increase of λ. Its physical interpretation is:—

When a piece of rock is twisted to and fro cyclically about its neutral state, the curve traced by the cycle approaches asymptotically to a closed one.

Lastly, if the amplitude is given and remains constant, but the centre of symmetry of the cyclic process is changed from one stage to the other, the influence on the inclination of the curve of yielding is as follows:—Suppose that after N cycles of amplitude $\mathfrak{A}$ round the neutral state, other smaller cycles of amplitude $\mathfrak{a}$, whose centre of symmetry is at the stage j, is completed λ times, and then it is in the stage p just now. The total yielding is given by

$$
{}_j\eta_p = k\log\frac{\Gamma\{a-p\}[\Gamma\{a-p-3\mathfrak{A}\}]^2\ldots[\Gamma\{(4\lambda+1)\mathfrak{a}+j-p+1\}]^2\ldots[\Gamma\{\mathfrak{a}+j-p+1\}]^2}{[\Gamma\{a-p-\mathfrak{A}\}][\Gamma\{a-p-5\mathfrak{A}\}]^2\ldots[\Gamma\{(4\lambda+3)\mathfrak{a}+j-p+1\}]^2\ldots[\Gamma\{3\mathfrak{a}+j-p+1\}]^2} \quad \ldots\ldots (17).
$$

where $a=\{4\mathrm{N}-1\}\mathfrak{A}+\mathfrak{A}+\mathfrak{a}-j+2\{2\lambda+1\}\mathfrak{a}+j+\mathfrak{a}+1$
$=4\mathrm{N}\mathfrak{A}+4\{\lambda+1\}\mathfrak{a}+1.$

Hence, difference between the amounts of yielding at two consecutive stages p and $p-1$ is given by

$$
\triangle_p\ {}_j\eta_p = {}_j\eta_{p-1} - {}_j\eta_p
$$

$$
= k\log\frac{\{a-p\}[a-p-3\mathfrak{A}]^2\ldots[(4\lambda+1)\mathfrak{a}+j-p+1]^2\ldots[\mathfrak{a}+j-p+1]^2}{[a-p-\mathfrak{A}][a-p-5\mathfrak{A}]^2\ldots[(4\lambda+3)\mathfrak{a}+j-p+1]^2\ldots[3\mathfrak{a}+j-p+1]^2} \quad \ldots\ldots (18).
$$

Put $p=j$, then we have the increase of yielding at the stage corresponding to the centre of smaller cycle in the increasing state; the result is

$$
\triangle_p\ {}_j\eta_j = k\log\frac{\{a-j\}[a-j-3\mathfrak{A}]^2\ldots[(4\lambda+1)\mathfrak{a}+1]^2\ldots[\mathfrak{a}+1]^2}{[a-j-\mathfrak{A}][a-j-5\mathfrak{A}]^2\ldots\ldots\ldots\ldots[3\mathfrak{a}+1]^2} \lesseqgtr 0 \quad \ldots\ldots (19).
$$

It may easily be seen that the value of $\triangle_p\ {}_j\eta_j$ is not fixed in its sign, but it depends wholly upon the relative magnitudes of $\mathfrak{a}$ to $\mathfrak{A}$ and j.

To know the influence of the change of the centre of cyclic process on the amount of yielding, we only require to know how the value of $\triangle_p.\ {}_j\eta_j$ changes when the centre of cycle is changed from j to $j+1$ Thus,

$$
\triangle_j\{\triangle_p.\ {}_j\eta_j\} = \triangle_p.\ {}_{j+1}\eta_{j+1} - \triangle_p\ {}_j\eta_j
$$

$$
= k\log\frac{\{a-j-1\}[a-j-\mathfrak{A}]^2[a-j-1-3\mathfrak{A}]^2\ldots[\mathfrak{A}+4(\lambda+1)\mathfrak{a}-j-1]^2}{\{a-j\}[a-j-1-\mathfrak{A}]^2[a-j-3\mathfrak{A}]^2\ldots\ldots[\mathfrak{A}+4(\lambda+1)\mathfrak{a}-j]^2} < 0 \quad (20).
$$

Now looking at the factors under the sign of logarithm, we see that

$$
\frac{a-j-1}{a-j} < 1\,;\ \frac{\{a-j-\mathfrak{A}\}\{a-j-1-3\mathfrak{A}\}}{\{a-j-\mathfrak{A}-1\}\{a-j-3\mathfrak{A}\}} = \frac{[a-j-\mathfrak{A}]^2-\{a-j-\mathfrak{A}\}\{1+2\mathfrak{A}\}}{[a-j-\mathfrak{A}]^2-\{a-j-\mathfrak{A}\}\{1+2\mathfrak{A}\}+2\mathfrak{A}}
$$

$$
= \frac{H^2}{H^2+2\mathfrak{A}} < 1\,; \text{ etc. etc.}
$$

Thus, the product of all the factors is less than unity so that its logarithm is necessarily negative: *i.e.*

$$\triangle_{p\ j+1}\eta_{j+1} < \triangle_{p\ j}\eta_j.$$

This shows that the curve traced by the cycle of given amplitude tends to become more and more horizontal when its centre of symmetry of the cyclic process becomes more and more remote from the neutral state of the specimen.

Here it must be remarked that if $\mathfrak{a}$ is very small compared with $\mathfrak{A}$ and j so that in (19) the relation $\triangle_{p\ j}\eta_j < 0$ exists, then since both $\triangle_{p\ j}\eta_j$ and $\triangle_{p\ j+1}\eta_{j+1}$ are negative, the inequality (20) expresses that

the absolute value of $\triangle_{p\ j+1}\eta_{j+1}$ > the absolute value of $\triangle_{p\ j}\eta_j$.

In the above, all discussions were made with respect to the yielding in the increasing stage, η_p; but similar process may be applied to discuss η_n which corresponds to the decreasing stage. One important difference is that the value of $\triangle_{p\ j}\eta_j$ differs more and more from that of $\triangle_{n\ j}\eta_j$ as the value of j becomes greater and greater: indeed they may be even of different signs. The result of this is that the curve of twist, when the centre of cycle is remote from the neutral state, is far from being a closed one. It rather resembles the projection of a helix on a plane inclined to its axis.

To illustrate these results more clearly, it is best to show the results of actual calculation by the above formulæ. The tables in the end of this article contain them. They are also plotted in Fig. 54 of Pl. XXVI. Here it may be necessary to remark that the total amount of twist in any stage p or n consists of two terms, of which one may be calculated from Hooke's law by the formula $\tau_0 = \frac{p}{\mu_0}$ or $\tau_0 = \frac{n}{\mu_0}$, and the other is given by η_p or η_n; so that for the actual amount of twist we have

$$\tau = \tau_0 + \eta.$$

Since η is proportional to k which is a function of μ_0, the curve expressing the relation between the twist τ and the twisting couple $\mathfrak{C}$ must be greatly affected in its feature by the relative value of τ_0

and η. If η is negligible compared to τ_0, then the curve shrinks into a straight line. On the other hand, when τ_0 is very small compared with η, the curve takes a form given in Fig. 54, Pl. XXVI. For any other relative values, the curve takes an intermediate form. Figures in Pl. XXVII, which I have drawn from actual calculation show several intermediate ones. When we compare these figures with those in Pls. XIV and XV., we are not but struck with closest coincidence between theory and experiment.

Other term of hysteresis which is independent of time remains still out of solution.