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On the existence of holomorphic solutions of the Cauchy problem for nonlinear first order partial differential equations*

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Abstract. The paper deals with the Cauchy problem for general nonlinear first order partial differential equations, and gives a sufficient condition on the following assertion: if the problem has a formal power series solution, it has a holomorphic solution.

In 1968, M. Artin [1] has proved that if a system of analytic equations has a formal power series solution, then it has a convergent power series solution. In this note we are proving similar results for the Cauchy problem for nonlinear first order partial differential equations.

§ 1. Notations and Definitions.

Let $(t, x) \in C^2$, $X_i \in C$ $(i=1, 2, 3)$, let $F(t, x, X_1, X_2, X_3)$ be a holomorphic function in (t, x, X_1, X_2, X_3) defined in a neighborhood of $(0, 0, a_1, a_2, a_3) \in C^5$ satisfying $F(0, 0, a_1, a_2, a_3)=0$, and let $u_0(x)$ be a holomorphic function defined near $x=0$ satisfying $u_0(0)=a_1$ and $(\partial u_0/\partial x)(0)=a_3$. In this paper we are dealing with the following Cauchy problem:

$$\text{(E)} \qquad \begin{cases} F\left(t, x, u, \dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}\right)=0, \\ u(0, x)=u_0(x) \quad \text{and} \quad \dfrac{\partial u}{\partial t}(0, 0)=a_2, \end{cases}$$

where $u=u(t, x)$ is an unknown function.

We denote by $C[[t, x]]$ the ring of formal power series in (t, x), and by $C\{t, x\}$ the subring of convergent power series in (t, x). Note that

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any $f(t, x) \in \boldsymbol{C}[[t, x]]$ can be expressed in the form

$$f(t, x)=\sum_{i=0}^{\infty} f_i(x) t^i, \quad f_i(x) \in \boldsymbol{C}[[x]] \qquad (i \geqq 0).$$

DEFINITION 1. The valuation $v_t(f)$ of $f(t, x)$ in t is defined as follows:

(i) When $f \equiv 0$, then $v_t(f)=\infty$;
(ii) When $f \not\equiv 0$, then $v_t(f)=\min\{i; f_i(x) \not\equiv 0\}$.

DEFINITION 2. The Borel transform $B_t(f)$ of $f(t, x)$ in t is defined by

$$B_t(f)=\sum_{i=0}^{\infty} \frac{1}{i!} f_i(x) t^i.$$

For simplicity, we write $Du=\left(u, \dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}\right)$. For any $u(t, x) \in \boldsymbol{C}[[t, x]]$ we put

$$q_i(u)=v_t\left(\frac{\partial F}{\partial X_i}(t, x, Du)\right), \qquad i=1, 2;$$

$$q_i(u, 0)=v_t\left(\frac{\partial F}{\partial X_i}(t, 0, Du(t, 0))\right), \qquad i=1, 2;$$

$$\rho(u; t)=-\frac{t\left(\dfrac{\partial F}{\partial X_1}(t, 0, Du(t, 0))\right)}{\dfrac{\partial F}{\partial X_2}(t, 0, Du(t, 0))}.$$

Note that $q_i(u)$, $q_i(u, 0)$, $\rho(u; t)$ are determined depending only on $u \in \boldsymbol{C}[[t, x]]$. If $q_2(u, 0)<\infty$ and $q_2(u, 0) \leqq q_1(u, 0)+1$ hold, then $\rho(u; 0)=\rho(u; t)|_{t=0}$ is well defined.

Now let us introduce new variables $y_1, y_2, \cdots, z_1, z_2, \cdots$ and put

$$Y=\sum_{i=1}^{\infty} y_i t^i, \qquad Z=\sum_{i=1}^{\infty} z_i t^i.$$

Then, by using the formal Taylor expansion in t we define $p_i(\mathrm{E})$ $(i=0, 1, 2)$ by

$$p_0(\mathrm{E})=v_t\Big(F\Big(t,x,u_0(x)+Y,\frac{\partial Y}{\partial t},\frac{\partial u_0(x)}{\partial x}+Z\Big)$$
$$-F\Big(t,x,u_0(x)+Y,\frac{\partial Y}{\partial t},\frac{\partial u_0(x)}{\partial x}\Big)\Big)$$

and for $i=1, 2$

$$p_i(\mathrm{E})=v_t\Big(\frac{\partial F}{\partial X_i}\Big(t,x,u_0(x)+Y,\frac{\partial Y}{\partial t},\frac{\partial u_0(x)}{\partial x}+Z\Big)$$
$$-\frac{\partial F}{\partial X_i}\Big(t,x,u_0(x)+Y,\frac{\partial Y}{\partial t},\frac{\partial u_0(x)}{\partial x}\Big)\Big).$$

LEMMA 1. *Let* $p_i(\mathrm{E})$ $(i=0, 1, 2)$ *be as avove. Then we have*:

(1) $p_i(\mathrm{E})\geqq 1$ $(i=0, 1, 2)$;
(2) $p_1(\mathrm{E})\geqq p_0(\mathrm{E})-2$;
(3) $p_2(\mathrm{E})\geqq p_0(\mathrm{E})-1$;
(4) $v_t\Big(\frac{\partial F}{\partial X_3}\Big(t,x,u_0(x)+Y,\frac{\partial Y}{\partial t},\frac{\partial u_0(x)}{\partial x}+Z\Big)\Big)\geqq p_0(\mathrm{E})-1.$

The proof of this lemma will be given in § 3.

§ 2. Results.

By using the above notations, we have the following Artin's type theorem:

THEOREM 1. *If* (E) *has a formal power series solution* $\hat{u}(t,x)\in C[[t,x]]$ *and if the conditions*

i) $q_2(\hat{u})=q_2(\hat{u},0)<\infty$,

ii) $q_2(\hat{u})\leqq \min\Big\{q_1(\hat{u})+1, \frac{p_0(\mathrm{E})-1}{2}\Big\}$,

iii) $\rho(\hat{u};0)-q_2(\hat{u})\notin\{2,3,4,\cdots\}$

are satisfied, then (E) *has a convergent solution* $w(t,x)\in C\{t,x\}$.

REMARK 1. When $(\partial F/\partial X_2)(0,0,a_1,a_2,a_3)\neq 0$ holds, the Cauchy-Kowalewski theorem says that (E) has a unique holomorphic solution $u(t,x)$. In this case, we have $q_2(u)=q_2(u,0)=0$ and $\rho(u;0)=0$; therefore the conditions i), ii), iii) in Theorem 1 are trivially satisfied.

When $q_1(\hat{u})+1<q_2(\hat{u})$, (E) has not a convergent solution in general (see Example 4 given below). Though, still we have:

THEOREM 2. *If* (E) *has a formal power series solution* $\hat{u}(t,x)\in \boldsymbol{C}[[t,x]]$ *and if the conditions*

i) $q_1(\hat{u})=q_1(\hat{u},0)<\infty$,

ii) $q_1(\hat{u})+1\leqq \min\left\{q_2(\hat{u})-1, \dfrac{p_0(\mathrm{E})-1}{2}\right\}$

are satisfied, then (E) *has a formal solution* $w(t,x)=\sum_{i=0}^{\infty}w_i(x)t^i$, $w_i(x)\in \boldsymbol{C}\{x\}$ $(i\geqq 0)$, *such that the Borel transform* $B_t(w)$ *of* $w(t,x)$ *in* t *is convergent near the origin of* $\boldsymbol{C}^2$.

Example 1. Let us consider

$$(2.1)\qquad \begin{cases} t^2\left(\dfrac{\partial u}{\partial t}\right)^2-u^2+t^2\left(\dfrac{\partial u}{\partial x}\right)^3=0, \\ u(0,x)=0. \end{cases}$$

By a calculation we can see that for any $\phi(x)\in \boldsymbol{C}[[x]]$ satisfying $\phi(0)\neq 0$ (2.1) has a unique formal solution $u(\phi)$ of the form $u(\phi)=\phi(x)t+t^2U(t,x)$, $U(t,x)\in \boldsymbol{C}[[t,x]]$. In this case we have $p_0=5$, $q_1(u(\phi))=1$, $q_2(u(\phi))=q_2(u(\phi),0)=2$, $\rho(u(\phi);0)=1$ and therefore we can apply Theorem 1 to this case. Note that $u(\phi)(t,x)$ is convergent if we take $\phi(x)\in \boldsymbol{C}\{x\}$.

Example 2. Let us consider

$$(2.2)\qquad \begin{cases} t^2\left(\dfrac{\partial u}{\partial t}\right)^2-3u^2+t^2\left(\dfrac{\partial u}{\partial x}\right)^3+(1+x)^2t^2=0, \\ u(0,x)=0. \end{cases}$$

By a calculation we can see that for any $\phi(x)\in \boldsymbol{C}[[x]]$ (2.2) has two formal solution $u(\phi,\pm)$ of the form

$$u(\phi,\pm)=\pm(1+x)t+\phi(x)t^3+t^4U(t,x),\quad U(t,x)\in \boldsymbol{C}[[t,x]].$$

In this case we have $p_0=5$, $q_1(u(\phi,\pm))=1$, $q_2(u(\phi,\pm))=q_2(u(\phi,\pm),0)=2$, $\rho(u(\phi,\pm);0)=3$ and therefore we can apply Theorem 1 to this case.

Example 3. Let us consider

$$(2.3)\qquad \begin{cases} xt\dfrac{\partial u}{\partial t}-u+t+t^2\dfrac{\partial u}{\partial x}=0, \\ u(0,x)=0. \end{cases}$$

By a calculation we can see that (2.3) has a unique formal solution $\hat{u}(t,x)\in \boldsymbol{C}[[t,x]]$ and that it is divergent. In this case we have $p_0=3$, $q_1(\hat{u})=q_1(\hat{u},0)=0$, $q_2(\hat{u})=1$, $q_2(\hat{u},0)=\infty$. Note that ii) in Theorem 1 is satisfied but i) in Theorem 1 is not satisfied.

Example 4. Let us consider

$$(2.4)\qquad \begin{cases} t^2\dfrac{\partial u}{\partial t}-u+t+t^2\dfrac{\partial u}{\partial x}=0, \\ u(0,x)=0. \end{cases}$$

By a calculation we can see that (2.4) has a unique formal solution $\hat{u}(t,x)\in \boldsymbol{C}[[t,x]]$ and it is given by $\hat{u}=\sum_{k=1}^{\infty}((k-1)!)t^k$. In this case we have $p_0=3$, $q_1(\hat{u})=q_1(\hat{u},0)=0$, $q_2(\hat{u})=2$ and therefore we can apply Theorem 2 to this case.

Example 5. Let us consider

$$(2.5)\qquad \begin{cases} \left(t\dfrac{\partial u}{\partial t}-u\right)\left(t^2\dfrac{\partial u}{\partial t}-u+t^2\right)+t^3\left(\dfrac{\partial u}{\partial x}\right)^2=0 \\ u(0,x)=0. \end{cases}$$

By a calculation we can see that for any $\phi(x)\in \boldsymbol{C}[[x]]$ satisfying $\phi(0)\neq 0$ (2.5) has a unique formal solution $u(\phi)$ of the form $u(\phi)=\phi(x)t+t^2U(t,x)$, $U(t,x)\in \boldsymbol{C}[[t,x]]$. In this case we have $p_0=5$, $q_1(u(\phi))=1$, $q_2(u(\phi))=q_2(u(\phi),0)=2$, $\rho(u(\phi);0)=1$ and therefore we can apply Theorem 1 to this case. Note that the equation (2.5) is not regular singular in t, since (2.5) has a divergent formal solution $\hat{u}=\sum_{k=2}^{\infty}((k-2)!)t^k$.

In this paper, we are discussing only the case of two independent variables. Though, we can also deal with the case of $n+1$ independent variables $(t,x_1,\cdots,x_n)$ by the same argument as in this paper.

§ 3. Proof of Lemma 1.

For simplicity, we write

$$\eta(t, x, Y, Z) = \left(t, x, u_0(x) + Y, \frac{\partial Y}{\partial t}, \frac{\partial u_0(x)}{\partial x} + Z \right).$$

Since $Z|_{t=0} = 0$ holds, we have

$$(F(\eta(t, x, Y, Z)) - F(\eta(t, x, Y, 0)))|_{t=0} \equiv 0$$

which implies $p_0(\mathrm{E}) \geqq 1$. The condition $p_i(\mathrm{E}) \geqq 1$ $(i=1, 2)$ may be proved in the same way.

Let us show (2) and (3). By the definition of $p_0(\mathrm{E})$ we have

$$F(\eta(t, x, Y, Z)) \equiv F(\eta(t, x, Y, 0)) \qquad (\operatorname{mod} t^{p_0(\mathrm{E})}). \tag{3.1}$$

Applying $\partial/\partial y_i$ $(i=1, 2)$ on both sides of (3.1) we obtain

$$\begin{aligned} &\frac{\partial F}{\partial X_1}(\eta(t, x, Y, Z))t^i + \frac{\partial F}{\partial X_2}(\eta(t, x, Y, Z))it^{i-1} \\ &\equiv \frac{\partial F}{\partial X_1}(\eta(t, x, Y, 0))t^i + \frac{\partial F}{\partial X_2}(\eta(t, x, Y, 0))it^{i-1} \qquad (\operatorname{mod} t^{p_0(\mathrm{E})}), \end{aligned}$$

that is, we obtain

$$\begin{aligned} &\frac{\partial F}{\partial X_1}(\eta(t, x, Y, Z))t^2 \equiv \frac{\partial F}{\partial X_1}(\eta(t, x, Y, 0))t^2 \qquad (\operatorname{mod} t^{p_0(\mathrm{E})}), \\ &\frac{\partial F}{\partial X_2}(\eta(t, x, Y, Z))t \equiv \frac{\partial F}{\partial X_2}(\eta(t, x, Y, 0))t \qquad (\operatorname{mod} t^{p_0(\mathrm{E})}), \end{aligned}$$

which imply (2) and (3) respectively.

Similarly, by applying $\partial/\partial z_1$ on both sides of (3.1) we have

$$\frac{\partial F}{\partial X_3}(\eta(t, x, Y, Z))t \equiv 0 \qquad (\operatorname{mod} t^{p_0(\mathrm{E})}),$$

which implies (4). Thus, (4) is also proved.

§ 4. Proof of Theorem 1.

Assume that (E) has a formal solution $\hat{u}(t, x) \in \boldsymbol{C}[[t, x]]$ satisfying the conditions i), ii), iii) in Theorem 1. Set $\hat{u} = \sum_{i=0}^{\infty} \hat{u}_i(x) t^i$ with $\hat{u}_0(x) = u_0(x)$ and $\hat{u}_i(x) \in \boldsymbol{C}[[x]]$ $(i \geqq 1)$.

Case: $q_2(\hat{u}) = 0$. By i) we have $q_2(\hat{u}, 0) = 0$ and this is equivalent to the condition $(\partial F/\partial X_2)(0, 0, a_1, a_2, a_3) \neq 0$: hence by the Cauchy-Kowalewski theorem we obtain the convergent solution $w(t, x) \in C\{t, x\}$.

Case: $q_2(\hat{u}) \geqq 1$. Put $q_2 = q_2(\hat{u})$, $\rho = \rho(\hat{u}; 0)$ and

$$\hat{\varphi} = \sum_{i=0}^{q_2+1} \hat{u}_i(x) t^i.$$

Then, it is easy to see that

$$\frac{\partial F}{\partial X_i}(t, x, D\hat{u}) \equiv \frac{\partial F}{\partial X_i}(t, x, D\hat{\varphi}) \qquad (\mathrm{mod}\, t^{q_2+1}) \tag{4.1}$$

and therefore by i), ii) we have

$$\begin{cases} q_2 = q_2(\hat{\varphi}) = q_2(\hat{\varphi}, 0), \\ q_2 - 1 \leqq q_1(\hat{\varphi}), \end{cases} \tag{4.2}$$

which is equivalent to

$$\begin{cases} \dfrac{\partial F}{\partial X_1}(t, x, D\hat{\varphi}) \equiv 0 \qquad (\mathrm{mod}\, t^{q_2-1}), \\ \dfrac{\partial F}{\partial X_2}(t, x, D\hat{\varphi}) \equiv 0 \qquad (\mathrm{mod}\, t^{q_2}), \\ \left(t^{-q_2} \dfrac{\partial F}{\partial X_2}(t, 0, D\hat{\varphi}(t, 0)) \right)\Big|_{t=0} \neq 0. \end{cases} \tag{4.3}$$

Moreover, by (4.1), (4.2) and the definition of $\rho(\hat{u}; 0)$ we have

$$\rho(\hat{\varphi}; 0) = \rho. \tag{4.4}$$

Put $\hat{U} = t^{-q_2-1}(\hat{u} - \hat{\varphi})$. Then $\hat{u} = \hat{\varphi} + t^{q_2+1}\hat{U}$ and $\hat{U}(0, x) \equiv 0$. Since $\hat{u}$ is a formal solution of (E), by using Taylor expansion we get the equality

$$\begin{aligned} 0 = {} & F(t, x, D\hat{\varphi}) + \frac{\partial F}{\partial X_1}(t, x, D\hat{\varphi}) t^{q_2+1} \hat{U} \\ & + \frac{\partial F}{\partial X_2}(t, x, D\hat{\varphi}) t^{q_2} \Big(t\frac{\partial}{\partial t} + q_2 + 1 \Big) \hat{U} + \frac{\partial F}{\partial X_3}(t, x, D\hat{\varphi}) t^{q_2+1} \Big(\frac{\partial \hat{U}}{\partial x} \Big) \\ & + \sum_{|\alpha| \geqq 2} \frac{1}{\alpha!} \Big(\frac{\partial}{\partial X} \Big)^{\alpha} F(t, x, D\hat{\varphi}) t^{|\alpha|(q_2+1) - \alpha_2} \hat{U}^{\alpha_1} \Big(\Big(t\frac{\partial}{\partial t} + q_2 + 1 \Big) \hat{U} \Big)^{\alpha_2} \Big(\frac{\partial \hat{U}}{\partial x} \Big)^{\alpha_3}, \end{aligned} \tag{4.5}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\Big(\dfrac{\partial}{\partial X} \Big)^{\alpha} = \Big(\dfrac{\partial}{\partial X_1} \Big)^{\alpha_1} \Big(\dfrac{\partial}{\partial X_2} \Big)^{\alpha_2} \Big(\dfrac{\partial}{\partial X_3} \Big)^{\alpha_3}$.

Since

$$v_t\Big(\frac{\partial F}{\partial X_1}(t, x, D\hat{\varphi})t^{q_2+1}\Big)=q_1(\hat{\varphi})+q_2+1\geqq 2q_2,$$

$$v_t\Big(\frac{\partial F}{\partial X_2}(t, x, D\hat{\varphi})t^{q_2}\Big)=q_2(\hat{\varphi})+q_2=2q_2,$$

$$v_t\Big(\frac{\partial F}{\partial X_3}(t, x, D\hat{\varphi})t^{q_2+1}\Big)\geqq (p_0(\mathrm{E})-1)+q_2+1\geqq 3q_2+1,$$

$|\alpha|(q_2+1)-\alpha_2\geqq|\alpha|q_2\geqq 2q_2$ and $v_t(\hat{U})\geqq 1$ hold, by comparing the valuation in each term of (4.5) we get

$$F(t, x, D\hat{\varphi})\equiv 0 \qquad (\mathrm{mod}\, t^{2q_2+1}), \tag{4.6}$$

and after simplification by t^{2q_2} (4.5) is reduced to the form

$$\begin{aligned} &a(t, x, D\hat{\varphi})\hat{U}+b(t, x, D\hat{\varphi})\Big(t\frac{\partial}{\partial t}+q_2+1\Big)\hat{U} \\ &=th(t, x, D\hat{\varphi})+G\Big(t, x, D\hat{\varphi}; \hat{U}, \Big(t\frac{\partial}{\partial t}+q_2+1\Big)\hat{U}, \frac{\partial\hat{U}}{\partial x}\Big), \end{aligned} \tag{4.7}$$

where

$$a(t, x, D\hat{\varphi})=t^{1-q_2}\frac{\partial F}{\partial X_1}(t, x, D\hat{\varphi}),$$

$$b(t, x, D\hat{\varphi})=t^{-q_2}\frac{\partial F}{\partial X_2}(t, x, D\hat{\varphi}),$$

$$h(t, x, D\hat{\varphi})=-t^{-2q_2-1}F(t, x, D\hat{\varphi}),$$

$$\begin{aligned} G(t, x, D\hat{\varphi}; Y_1, Y_2, Y_3)&=-t^{1-q_2}\frac{\partial F}{\partial X_3}(t, x, D\hat{\varphi})Y_3 \\ &-\sum_{|\alpha|\geqq 2}\frac{1}{\alpha!}\Big(\frac{\partial}{\partial X}\Big)^{\alpha}F(t, x, D\hat{\varphi})t^{|\alpha|(q_2+1)-\alpha_2-2q_2}(Y_1)^{\alpha_1}(Y_2)^{\alpha_2}(Y_3)^{\alpha_3}. \end{aligned}$$

Note that $q_2=q_2(\hat{\varphi}, 0)$ implies $b(0, 0, D\hat{\varphi}(0, 0))\neq 0$ and that $\rho=\rho(\hat{\varphi}; 0)=-(a/b)(0, 0, D\hat{\varphi}(0, 0))$ holds.

Now, let us find a convergent solution $w(t, x)\in \boldsymbol{C}\{t, x\}$ of (E) in the following form:

$$\begin{cases} w=\varphi+t^{q_2+1}W, \\ \varphi=u_0(x)+\sum_{i=1}^{q_2+1} w_i(x)t^i, \ w_i(x)\in \boldsymbol{C}\{x\}, \\ W(t, x)\in \boldsymbol{C}\{t, x\}, \ W(0, x)\equiv 0, \end{cases}$$

where $w_1(x), \cdots, w_{q_2+1}(x)$ and $W(t, x)$ are unknown functions.

To do so, let us first consider the following system of equations with respect to $w_1(x), \cdots, w_{q_2+1}(x)$:

$$\begin{cases} F(t, x, D\varphi)\equiv 0 \quad (\mathrm{mod}\, t^{2q_2+1}), \\ \dfrac{\partial F}{\partial X_1}(t, x, D\varphi)\equiv 0 \quad (\mathrm{mod}\, t^{q_2-1}), \\ \dfrac{\partial F}{\partial X_2}(t, x, D\varphi)\equiv 0 \quad (\mathrm{mod}\, t^{q_2}). \end{cases} \tag{4.8}$$

Since $2q_2+1\leqq p_0(\mathrm{E})$ is assumed, by Lemma 1 we have $q_2-1\leqq p_0(\mathrm{E})-2\leqq p_1(\mathrm{E})$ and $q_2\leqq p_0(\mathrm{E})-1\leqq p_2(\mathrm{E})$; therefore it follows from the definition of $p_i(\mathrm{E})$ $(i=0, 1, 2)$ that the system (4.8) is nothing but a system of analytic equations with respect to $w_1(x), \cdots, w_{q_2+1}(x)$. Moreover, by (4.3) and (4.6) we know that (4.8) has a formal power series solution $\hat{u}_1(x), \cdots, \hat{u}_{q_2+1}(x)$. Hence, by Artin's theorem we obtain a convergent solution $w_i(x)\in \boldsymbol{C}\{x\}$ $(i=1, \cdots, q_2+1)$ of (4.8). In addition we may assume

$$(D\varphi)(t, 0)=(D\hat{\varphi})(t, 0) \tag{4.9}$$

(see Artin [1]).

Then, by (4.8) and (4.9) it is easy to see that

$$\begin{cases} q_2=q_2(\varphi)=q_2(\varphi, 0), \\ q_2-1\leqq q_1(\varphi), \\ \rho=\rho(\varphi; 0)=-(a/b)(t, x, D\varphi)\Big|_{\substack{t=0\\x=0}} \end{cases}$$

holds and therefore by the same argument as in (4.7) the equation (E) is reduced to the following equation with respect to $W(t, x)$

$$\begin{aligned} &a(t, x, D\varphi)W+b(t, x, D\varphi)\Big(t\frac{\partial}{\partial t}+q_2+1\Big)W \\ &\quad =th(t, x, D\varphi)+G\Big(t, x, D\varphi;\ W, \Big(t\frac{\partial}{\partial t}+q_2+1\Big)W, \frac{\partial W}{\partial x}\Big), \end{aligned} \tag{4.10}$$

which is a partial differential equation of Briot-Bouquet type in t discussed in [2, 3]. Since iii) implies $\rho-q_2-1\notin\{1, 2, 3, \cdots\}$, by the result given in [2, 3] we see that (4.10) has a unique holomorphic solution $W(t, x)$ satisfying $W(0, x)\equiv 0$.

Thus, we have proved that (E) has a convergent solution $w(t, x)\in C\{t, x\}$.

§ 5. Proof of Theorem 2.

Assume that (E) has a formal solution $\hat{u}(t, x)\in C[[t, x]]$ satisfying the conditions i), ii) in Theorem 2. Put $q_1=q_1(\hat{u})$. Then, the formal solution $w(t, x)$ desired in Theorem 2 is obtained in the following way.

First, we put

$$\begin{cases} w=\varphi+t^{q_1+2}W, \\ \varphi=u_0(x)+\sum_{i=1}^{q_1+2} w_i(x)t^i, \quad w_i(x)\in C\{x\}, \\ W(t, x)\in C[[t, x]], \quad W(0, x)\equiv 0. \end{cases}$$

Next, by applying Artin's theorem to the system of analytic equations with respect to $w_1(x), \cdots, w_{q_1+2}(x)$:

$$(5.1)\qquad \begin{cases} F(t, x, D\varphi)\equiv 0 & (\mathrm{mod}\, t^{2q_1+3}), \\ \dfrac{\partial F}{\partial X_1}(t, x, D\varphi)\equiv 0 & (\mathrm{mod}\, t^{q_1}), \\ \dfrac{\partial F}{\partial X_2}(t, x, D\varphi)\equiv 0 & (\mathrm{mod}\, t^{q_1+2}), \\ D\varphi(t, 0)\equiv D\hat{u}(t, 0) & (\mathrm{mod}\, t^{q_1+2}), \end{cases}$$

we get a solution $w_i(x)\in C\{x\}$ $(i=1, \cdots, q_1+2)$.

Then, the equation (E) is reduced to the following equation with respect to $W(t, x)$:

$$(5.2)\qquad A(t, x, D\varphi)W = tH(t, x, D\varphi)+R\Big(t, x, D\varphi; W, \Big(t\frac{\partial}{\partial t}+q_1+2\Big)W, \frac{\partial W}{\partial x}\Big),$$

where

$$A(t, x, D\varphi)=t^{-q_1}\frac{\partial F}{\partial X_1}(t, x, D\varphi),$$

$$H(t, x, D\varphi) = -t^{-2q_1-3}F(t, x, D\varphi),$$
$$R(t, x, D\varphi; Y_1, Y_2, Y_3)$$
$$= -t^{-q_1-1}\frac{\partial F}{\partial X_2}(t, x, D\varphi)Y_2 - t^{-q_1}\frac{\partial F}{\partial X_3}(t, x, D\varphi)Y_3$$
$$- \sum_{|\alpha|\geqq 2}\frac{1}{\alpha!}\left(\frac{\partial}{\partial X}\right)^{\alpha}F(t, x, D\varphi)t^{|\alpha|(q_1+2)-\alpha_2-2q_1-2}(Y_1)^{\alpha_1}(Y_2)^{\alpha_2}(Y_3)^{\alpha_3}.$$

Since $A(0, 0, D\varphi(0, 0)) \neq 0$ is assumed by i) and since

$$v_t\left(t^{-q_1-1}\frac{\partial F}{\partial X_2}(t, x, D\varphi)\right) \geqq q_2 - q_1 - 1 \geqq 1,$$
$$v_t\left(t^{-q_1}\frac{\partial F}{\partial X_3}(t, x, D\varphi)\right) \geqq (p_0(\mathrm{E}) - 1) - q_1 \geqq 2q_1 + 2 - q_1 \geqq 2,$$
$$|\alpha|(q_1+2) - \alpha_2 - 2q_1 - 2 \geqq |\alpha|(q_1+1) - 2q_1 - 2 \geqq 0$$

hold, by the Maillet's type result in [4, 5] we see that (5.2) has a unique formal solution $W(t, x) \in \boldsymbol{C}[[t, x]]$ such that $W(0, x) \equiv 0$ and $B_t(W) \in \boldsymbol{C}\{t, x\}$ hold.

Since the argument is quite parallel to that in § 4, the justification of the above construction of $w(t, x)$ may be left to readers.

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