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Fundamental Theorems in Global Knot Theory

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Contents

Introduction

A knot K in an m-dimensional manifold M^m is a submanifold of M^m diffeomorphic (resp. homeomorphic or PL homeomorphic) to the $(m-2)$-sphere S^{m-2} in smooth (resp. topological or PL) category. The knots in manifolds are the object of global knot theory which is a generalization of the knot theory in spheres. Global knot theory in each manifold is a kind of the inner world of the manifold and varies depending on the topological property of the manifold.

A knot K in M^m is local if K is covered by an m-disk imbedded in M^m. Criterion theorems of unknottedness and localness for knots in manifolds are fundamental in global knot theory as the unknotting theorem of Papakyriakopoulos and that of Levine are fundamental in the classical knot theory and the higher dimensional knot theory respectively ([14], [11]).

Professor Itiro Tamura (1926-1991) passed away on February 21, 1991. After his death the manuscript of this article was found in his office well prepared and ready to be published. Because of its importance, the editorial board decided, with Mrs. Tamura's approval, to publish it in J. Fac. Sci. Univ. Tokyo. Sect. IA. Mathematics.

Professor Koichi Yano carefully read the manuscript prior to publication, and gave the board some useful comments. The Editorial board thanks Professor Yano for his benevolent efforts. Two footnotes in §2 are due to him.

In this paper we shall prove the unknotting and the localness theorems for knots in highly connected smooth manifolds by means of knot modules (Theorems 3 and 4 in § 6), generalizing the unknotting theorem for higher dimensional knots in the spheres by Levine. In § 8, genus 1 knots in $S^n \times S^{n+1}$ will be studied, and the localness and the unknottedness of them will be determined by computing their knot modules (Theorem 5), making use of fundamental theorems. The results (i) and (ii) in Theorem 5 reveal the intrinsic difference between knot theory in spheres and global knot theory, and the latter seems to have an unexpected and ample nature. In § 9, knot cobordisms will be considered and the nullity of the knot cobordism $C_{2n}(S^{n+1} \times S^{n+1})$ will be proved for $n \geqq 2$.

The study of global knot theory arose from the two previous results by the author. One is the solution of the Schoenflies theorem in smooth manifolds in which the localness theorem for knots of codimension one was considered (Tamura [18]). The fundamental theorems answer problems in [18]. The other is the existence of fibred knots in $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifolds which was used to construct foliations of codimension one of them (Tamura [16]).

Recently several works on global knot theory in lower dimensions appeared (see, for example, Lee [9], Suzuki [15], Yano [21]).

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§ 1. Difinitions and notations.

Let M^m be a connected m-dimensional smooth mainifold ($m \geqq 3$). A submanifold K of M^m diffeomorphic to the q-sphere S^q is called a *knot of codimension* $m-q$ in M^m for $1 \leqq q \leqq m-1$. Two knots K and K' of codimension $m-q$ in M^m are *equivalent* if there exists a diffeomorphism $h\colon M^m \longrightarrow M^m$ such that $h(K)=K'$.

A knot K of codimension $m-q$ in M^m is said to be *unknotted* or *trivial* if K bounds a $(q+1)$-disk smoothly imbedded in M^m. A knot K of codimension $m-q$ in M^m is said to be *local* or *engulfable* if there exists an m-disk D^m smoothly imbedd in M^m such that

$$D^m \supset K.$$

It is obvious that a local knot K of codimension $m-q$ in M^m is inessential in M^m, that is, the inclusion map $K \longrightarrow M^m$ is homotopic to a constant map.

A knot of codimension 2 in M^m is simply called a *knot* in M^m. Let K be a knot in M^m and let $N(K)$ be a tubular neighborhood of K in M^m. The complement $M^m - K$ is denoted by X_0, and $M^m - \operatorname{Int} N(K)$ is called the *exterior* of K and is denoted by X. Obviously the natural inclusion map $X \longrightarrow X_0$ is a homotopy equivalence.

In this paper, homology groups $H_*(\ )$ and cohomology groups $H^*(\ )$ denote always the singular homology and cohomology groups with integer coefficient unless otherwise stated.

The homomorphism $H_q(X) \longrightarrow H_q(M^m)$ induced by the inclusion map of the exterior X of K into M^m is injective if $q \neq 1$ and is bijective if $q \neq 1, 2, m$.

A knot K in M^m is said to be *fibred* if the normal bundle of K is trivial and there exists a smooth fibration $\bar{p}: X \longrightarrow S^1$ such that the restriction $\bar{p}|(X \cap N(K)): X \cap N(K) \longrightarrow S^1$ is the projection onto a fibre of the fibration $\partial N(K) \longrightarrow K$.

A knot K in M^m is said to be *r-simple* if the homotopy groups $\pi_i(X)$ are as follows:

$$\pi_1(X) \cong Z,$$
$$\pi_i(X) = 0 \qquad \text{for } 2 \leqq i \leqq r,$$

that is, $\pi_i(X) \cong \pi_i(S^1)$ for $1 \leqq i \leqq r$, where $r \geqq 1$.

In case M^m is an $(n-1)$-connected closed $(2n+1)$-dimensional (resp. an $(n-1)$-connected closed $2n$-dimensional) smooth manifold, an $(n-1)$-simple (resp. $(n-2)$-simple) knot K in M^m is said to be *simple*, where $n \geqq 2$ (resp. $n \geqq 3$). This terminology is a generalization of one used in the higher dimensional knot theory in the spheres (see, for example, [8]).

Let $\mathcal{K}(M^m)$ denote the equivalence classes of knots in M^m. $\mathcal{K}(S^m)$ is simply denoted by $\mathcal{K}^m$. For $K \in \mathcal{K}(M^m)$ and $K' \in \mathcal{K}^m$, we denote by $K' \cdot K$ the knot $K' \# K$ in $S^m \# M^m = M^m$, where $\#$ denotes the connected sum. Thus $\mathcal{K}^m$ operates on $\mathcal{K}(M^m)$. For a trivial knot K_0 in M^m, $\mathcal{K}^m \cdot K_0 = \{K' \cdot K_0;\ K' \in \mathcal{K}^m\}$ is the set of local knots. We remark that the operation $K' \cdot K$ depends on the choice of orientations.

§ 2. Seifert surfaces.

Let K be a knot in a connected m-dimensional smooth manifold M^m ($m\geqq 3$). A compact connected $(m-1)$-dimensional submanifold V of M^m such that $\partial V=K$ is called a *Seifert surface* for K if V is transversally orientable (that is, the normal bundle of V in M^m is trivial).

Now we prove the existence of a Seifert surface.

PROPOSITION 1. *Let M^m be a connected closed m-dimensional smooth manifold such that $m\geqq 4$ and $H^2(M^m)=0$, and let K be a knot in M^m. Then there exists a Seifert surface for K.*

PROOF. Let $N(K)$ be a tubular neighborhood of K in M^m. If $m\geqq 5$, the normal bundle of K in M^m is trivial, since $\pi_{m-3}(O(2))=0$. In case $m=4$, if the normal bundle of K in M^4 is not trivial, we have

$$H_2(N(K),\ \partial N(K);\boldsymbol{R})\cong \boldsymbol{R},$$
$$H_q(\partial N(K);\boldsymbol{R})=0 \quad q=1,2,$$

which imply that $H^2(M^4)\neq 0$. Thus the normal bundle of K must be trivial for $m=4$. Therefore we have

$$\partial N(K)\cong S^{m-2}\times S^1 \qquad (m\geqq 4).$$

Let $f\colon \partial N(K)\longrightarrow S^1$ be the projection onto a fibre of the fibration $\partial N(K)\longrightarrow K$. The only obstruction to extending f over the exterior X lies in $H^2(X,\partial X;\pi_1(S^1))\cong H^2(M^m, N(K))$. By the cohomology exact sequence of $(M^m, N(K))$, it follows from the hypothesis $H^2(M^m)=0$ and $m\geqq 4$ that

$$H^2(M^m, N(K))=0.$$

Thus there exists an extension $\bar{f}\colon X\longrightarrow S^1$ of f. We can choose $\bar{f}$ so that $\bar{f}$ is smooth and transversally regular at a point $p\in S^1$. Then the connected component of $\bar{f}^{-1}(p)$ containing K determines a Seifert surface for K. Thus this proposition is proved.

Let K be a knot in a connected m-dimensional smooth manifold and let V be a Seifert surface for K. The complement M^m-V of V is denoted by Y_0. Let $h\colon V\longrightarrow \boldsymbol{R}$ be a non-negative smooth function on V such that

$$h^{-1}(0)=\partial V=K.$$

Define $N=\{(x,t)\in V\times \boldsymbol{R};\ |t|\leqq h(x)\}$, $N_-=\{(x,t)\in V\times \boldsymbol{R};\ t=-h(x)\}$, $N_+=\{(x,t)\in V\times \boldsymbol{R}; t=h(x)\}$. Then N is a submanifold of $V\times \boldsymbol{R}$ with a corner[1] at $K\times\{0\}$ such that

$$\partial N=N_-\cup N_+,\ N_-\cap N_+=\partial N_-=\partial N_+=K\times\{0\}.$$

There exists an imbedding $\iota\colon N\longrightarrow M^m$ satisfying $\iota(x,0)=x$ for $x\in V$. The images $\iota(N)$, $\iota(N_-)$ and $\iota(N_+)$ will be denoted by $N(V)$, V_- and V_+ respectively. Obviously we have

$$K=\partial V=V_-\cap V_+=\partial V_-=\partial V_+.$$

There exists a homotopy equivalence $r\colon N(V)\longrightarrow V$ such that $r|V$ is the identity map. We may consider $N(V)$ as a tubular neighborhood of V in M^m. The inclusion maps $V_-\longrightarrow N(V)$ and $V_+\longrightarrow N(V)$ are homotopy equivalences. Let $\iota'_-\colon V\longrightarrow V_-$ and $\iota'_+\colon V\longrightarrow V_+$ denote natural diffeomorphisms.

Obviously it holds that

$$H_q(\partial N(V))\cong H_q(V_-)\oplus H_q(V_+)\qquad (1\leqq q\leqq m-3).^{2)}$$

Now consider the homology exact sequence of $(N(V),\ \partial N(V))$:

$$\cdots\longrightarrow H_{q+1}(N(V),\ \partial N(V))\stackrel{\partial}{\longrightarrow}H_q(\partial N(V))\stackrel{\iota_*}{\longrightarrow}H_q(N(V))\longrightarrow\cdots,$$

where $\iota\colon \partial N(V)\longrightarrow N(V)$ is the inclusion map. Since, for the homomorphism $(\iota'_+\circ r)_*: H_q(N(V))\longrightarrow H_q(\partial N(V))$, the composition $\iota_*\circ(\iota'_+\circ r)_*$ is the identity if $1\leqq q\leqq m-3$,[2] the above exact sequence splits into a short exact sequence

$$0\longrightarrow H_{q+1}(N(V),\ \partial N(V))\stackrel{\partial}{\longrightarrow}H_q(\partial N(V))\stackrel{\iota_*}{\longrightarrow}H_q(N(V))\longrightarrow 0$$

for $1\leqq q\leqq m-3$. The kernel of ι_* above is the subgroup of $H_q(\partial N(V))$ consisting of the elements

$$(\iota'_+)_*\alpha-(\iota'_-)_*\alpha\qquad (\alpha\in H_q(V)).$$

1) For N to be a submanifold with corner in the usual sense, it seems to need the condition that the derivative of h to the normal direction at boundary point of V are not zero.

2) Since ∂N is a closed $(m-1)$-dimensional manifold, the homomorphism $H_{m-1}(\partial N)\to H_{m-2}(S^{m-2})$ is surjective. So the Mayer-Vietries exact sequence shows that this isomorphism holds also for $q=m-2$ as well as for $1\leqq q\leqq m-3$. This will be needed in the proof that the short exact sequence on this page holds for $q=m-3$.

Let τ': $\ker \iota^* \longrightarrow H_q(V)$ be the isomorphism defined by

$$\tau'((\iota'_+)_*\alpha - (\iota'_-)_*\alpha) = \alpha$$

and let $\tau = \tau' \circ \partial$. Then we have isomorphisms

$$(*) \qquad \tau\colon H_{q+1}(N(V),\ \partial N(V)) \longrightarrow H_q(V)$$

for $1 \leqq q \leqq m-3$.

Let $Y = M^m - \operatorname{Int} N(V)$, then $\partial Y = \partial N(V) = V_- \cup V_+$ and Y is homotopy equivalent to $Y_0 = M^m - V$. We may consider that Y is the manifold obtained from M^m by cutting at V. Y is called the *exterior* of V in M^m.

Let

$$(**) \qquad \varepsilon\colon H_q(M^m,\ Y) \longrightarrow H_q(N(V),\ \partial N(V))$$

be the excision isomorphisms. Consider the homology exact sequence of (M^m, Y):

$$\cdots \longrightarrow H_{q+1}(M^m) \xrightarrow{\iota'_*} H_{q+1}(M^m,\ Y) \xrightarrow{\partial} H_q(Y) \xrightarrow{\iota_*} H_q(M^m) \longrightarrow \cdots,$$

where ι: $Y \longrightarrow M^m$ and ι': $(M^m, \varnothing) \longrightarrow (M^m, Y)$ are inclusion maps. Then, by the isomorphisms (*) and (**), this exact sequence becomes as follows for $1 \leqq q \leqq m-3$:

$$\cdots \longrightarrow H_{q+1}(M^m) \xrightarrow{\theta} H_q(V) \xrightarrow{\bar\partial} H_q(Y) \xrightarrow{\iota_*} H_q(M^m) \longrightarrow \cdots,$$

where $\theta = \tau \circ \varepsilon \circ \iota'_*$ and $\bar\partial = \partial \circ \varepsilon^{-1} \circ \tau^{-1}$. This exact sequence is called the *fundamental exact sequence* for the Seifert surface V.

Let α be an element of $H_q(V)$ $(1 \leqq q \leqq m-3)$, then, by the definition of τ, it holds that

$$\bar\partial(\alpha) = (\iota_+)_*\alpha - (\iota_-)_*\alpha,$$

where $\iota_-: V \longrightarrow Y$, $\iota_+: V \longrightarrow Y$ are inclusion maps induced from ι'_-: $V \longrightarrow V_-$, $\iota'_+: V \longrightarrow V_+$ respectively.

Let M^m be a simply connected m-dimensional smooth manifold $(m \geqq 4)$ and let K be a 1-simple knot in M^m. We denote by p: $\tilde X \longrightarrow X$ the universal covering of X. Now suppose that there exists a Seifert surface V for K such that V is simply connected. Then the exterior Y of V is also simply connected by the Van Kampen theorem. Let $Y^{(i)}$, $V_-^{(i)}$ and $V_+^{(i)}$ $(i \in Z)$ be copies of Y, V_- and V_+ indexed by the integers respectively. In this situation, the universal covering $\tilde{X}$ is the quotient space of the disjoint union $\coprod_{i\in Z} Y^{(i)}$ by the natural identification of $V_-^{(i-1)}$ with $V_+^{(i)}$ for all $i\in Z$. The subset of $\tilde{X}$ obtained from $V_-^{(i-1)}$ and $V_+^{(i)}$ by the identification is denoted by $V^{(i)}$, thus we have $Y^{(i-1)}\cap Y^{(i)}=V^{(i)}$, and we denote by

$$\iota_+\colon V^{(i)}\longrightarrow Y^{(i)},\ \iota_-\colon V^{(i)}\longrightarrow Y^{(i-1)}$$

the inclusion maps induced from $\iota_+\colon V\longrightarrow Y$ and $\iota_-\colon V\longrightarrow Y$.

We can compute $H_*(\tilde{X})$ by the Mayer-Vietoris exact sequence of $(\tilde{X}; \coprod_{i:\text{odd}} Y^{(i)}, \coprod_{i:\text{even}} Y^{(i)})$:

$$\cdots\longrightarrow\Sigma H_q(V^{(i)})\xrightarrow{\phi}\Sigma H_q(Y^{(i)})\xrightarrow{\Sigma\iota_*}H_q(\tilde{X})\xrightarrow{\partial}\cdots,$$

where

$$\phi(\bar{\alpha})=(\iota_+)_*(\bar{\alpha})-(\iota_-)_*(\bar{\alpha}) \qquad (\bar{\alpha}\in H_q(V^{(i)})).$$

§ 3. Seifert surfaces for knots in highly connected manifolds.

The following proposition shows that, in case M^m is highly connected, the Seifert surfaces for simple knots in M^m can be chosen highly connected. The proof is similar to that of the case of higher dimensional knots in spheres (Levine [11, Theorem 2]).

PROPOSITION 2. *Let M^m be a q-connected closed m-dimensional smooth manifold and let K be a knot in M^m, where $m\geqq 6$, $2\leqq q\leqq [m/2]-1$.*

(a) *If K is k-simple for $1\leqq k\leqq q$ (resp. $1\leqq k\leqq q-1$) in case m is odd (resp. m is even), then there exists a Seifert surface for K which is k-connected.*

(b) *Conversely if there exists a k'-connected Seifert surface for K such that $1\leqq k'\leqq q$, then the knot K is k'-simple.*

PROOF. By Proposition 1, there exists a Seifert surface for K. In order to prove (a), we perform surgeries on V to make it k-connected. Firstly we assume that K is 1-simple. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ be a set of generators of $\pi_1(V)$ and let

$$g^i\colon S^1\longrightarrow V \qquad (i=1, 2, \cdots, r)$$

be imbeddings such that g_i represents α_i and $g_i(S^1)\cap g_j(S^1)=0$ $(i\neq j)$.

Since $\pi_1(M^m-K)\cong H_1(M^m-K)\cong Z$ and the algebraic intersection number of $g_i(S^1)$ and V is zero, g^i: $S^1\longrightarrow V$ is null homotopic in M^m-K. Thus there exist imbeddings

$$g_i\colon D^2\longrightarrow M^m-K \qquad (i=1, 2, \cdots, r)$$

such that $g_i|\partial D^2=g_i$ and $g_i(D^2)\cap g_j(D^2)=0$ $(i\neq j)$, and that $g_i(D^2)$ intersects Int V transversally.

For an innermost connected component of $g_r^{-1}(g_r(D^2)\cap V)$, say Σ_r, we denote by $\hat{D}_r$ the compact connected subset of D^2 diffeomorphic to the 2-disk such that $\partial\hat{D}_r=\Sigma_r$. We take an imbedding

$$\bar{\varphi}\colon D^2\times D^{m-2}\longrightarrow M^m-K$$

such that $\bar{\varphi}(D^2\times\{0\})=g_r(\hat{D}_r)$ and $\bar{\varphi}(D^2\times D^{m-2})\cap V=\bar{\varphi}(\partial D^2\times D^{m-2})$. Then

$$V'=(V-\bar{\varphi}(\partial D^2\times D^{m-2}))\cap\bar{\varphi}(D^2\times\partial D^{m-2})$$

is the submanifold of M^m obtained from V by the surgery $\chi(\bar{\varphi}|S^1\times D^{m-2}))$ and is a Seifert surface for K. By applying the surgery as above for connected components of $g_i(D^2)\cap V$ $(i=1, 2, \cdots, r)$ successively, a 1-connected Seifert surface V_1 for K is obtained.

Now we assume that K is k-simple and there exists a $(k-1)$-connected Seifert surface V_{k-1} for K, where $2\leqq k\leqq q$.

Let Y_{k-1} denote the exterior of V_{k-1}. Then, by the fundamental exact sequence (§ 2), Y_{k-1} is $(k-1)$-connected. Suppose that $\beta\in\pi_k(V_{k-1})$ is contained in the kernel of $(\iota_+)_*\colon \pi_k(V_{k-1})\longrightarrow\pi_k(Y_{k-1})$. Then there exists an imbedding

$$\varphi'\colon (D^{k+1}, S^k)\longrightarrow(M^m, \mathrm{Int}\ V_{k-1})$$

such that $\varphi'|S^k$ represents β and $\varphi'(D^{k+1})\cap V_{k-1}=\varphi'(S^k)$. Let

$$\bar{\varphi}'\colon (D^{k+1}\times D^{m-k-1}, S^k\times D^{m-k-1})\longrightarrow(M^m, \mathrm{Int}\ V_{k-1})$$

be an imbedding such that

$$\bar{\varphi}'|(D^{k+1}\times\{0\})=\varphi',$$
$$\bar{\varphi}'(D^{k+1}\times D^{m-k-1})\cap V_{k-1}=\bar{\varphi}'(\partial D^{k+1}\times D^{m-k-1}),$$

and let

$$V'_{k-1}=(V_{k-1}-\bar{\varphi}'(\partial D^{k+1}\times D^{m-k-1}))\cup\bar{\varphi}'(D^{k+1}\times\partial D^{m-k-1}).$$

Then V'_{k-1} is a $(k-1)$-connected Seifert surface for K and we have

$$\pi_k(V'_{k-1})=\pi_k(V_{k-1})/(\beta).$$

Thus, by applying the surgeries as above for the kernel of $(\iota_+)_*$: $\pi_k(V_{k-1})\longrightarrow\pi_k(Y_{k-1})$ and $(\iota_-)^*$: $\pi_k(V_{k-1})\longrightarrow\pi_k(Y_{k-1})$ successively, we obtain a $(k-1)$-connected Seifert surface $\overline{V}_{k-1}$ for K such that the exterior $\overline{Y}_{k-1}$ of $\overline{V}_{k-1}$ is $(k-1)$-connected and that

$$(\iota_+)_*\colon \pi_k(\overline{V}_{k-1})\longrightarrow\pi_k(\overline{Y}_{k-1}),$$
$$(\iota_-)_*\colon \pi_k(\overline{V}_{k-1})\longrightarrow\pi_k(\overline{Y}_{k-1})$$

are injective.

Let β' be an element of $\pi_k(\overline{V}_{k-1})$. Then, since $\pi_k(M^m-K)=0$, there exists an imbedding

$$g'_k\colon (D^{k+1}, S^k)\longrightarrow(M^m-K, \operatorname{Int}\overline{V}_{k-1})$$

such that $g'_k|S^k$ represents β' and $g'_k(S^k)$ intersects $\overline{V}_{k-1}$ transversally. Suppose that $g_k'^{-1}(g'_k(\operatorname{Int}D^{k+1})\cap\overline{V}_{k-1})\neq\emptyset$. We choose an innermost conuected component W of it. Then there exists a compact connected submanifold $\hat{W}$ of $\operatorname{Int}D^{k+1}$ such that $\partial\hat{W}=W$. Since $\overline{V}_{k-1}$ is $(k-1)$-connected, the only obstruction to extending $g'_k|W\colon W\longrightarrow\overline{V}_{k-1}$ to a map $\hat{W}\longrightarrow\overline{V}_{k-1}$ is an element of

$$H^{k+1}(\hat{W}, W; \pi_k(\overline{V}_{k-1}))\cong\pi_k(\overline{V}_{k-1}),$$

say σ. Moreover, since $\overline{Y}_{k-1}$ is $(k-1)$-connected, the obstruction to extending the map $g'_k|W\colon W\longrightarrow\overline{V}_{k-1}$ to a map $\hat{W}\longrightarrow\overline{Y}_{k-1}$ is $(\iota_+)_*\sigma$ or $(\iota_-)_*\sigma$. The map $g'_k|W\colon W\longrightarrow\overline{Y}_{k-1}$ is such an extension. Thus $(\iota_+)_*\sigma$ or $(\iota_-)_*\sigma$ is zero. Since $(\iota_\pm)_*\colon \pi_k(\overline{V}_{k-1})\longrightarrow\pi_k(\overline{Y}_{k-1})$ are injective, it holds that $\sigma=0$. This implies that $g'_k|W\colon W\longrightarrow\overline{V}_{k-1}$ has an extension $W\longrightarrow\overline{V}_{k-1}$. Therefore we can cancel W from $g_k'^{-1}(g'_k(\operatorname{Int}D^{k+1})\cap\overline{V}_{k-1})$. By applying the above method successively, we may suppose that β' is contained in the kernel of $(\iota_+)_*$ or $(\iota_-)_*$. Since $(\iota_\pm)_*$ are injective, β' should be zero. Thus $\overline{V}_{k-1}$ is k-connected. This proves (a).

Now let V be a Seifert surface for K satisfying the assumption of (b) and let Y be the exterior of V. Then it follows from the Van Kampen theorem that Y is simply connected and $\pi_1(X)\cong\pi_1(M^m-K)\cong Z$. Furthermore, by the fundamental exact sequence (§ 2), it holds that

$$H_i(Y)=0 \qquad 1\leqq i\leqq k',$$

which implies, by the Mayer-Vietoris exact sequence of $(\tilde{X}; \coprod_{i:\text{odd}} Y^{(i)}, \coprod_{i:\text{even}} Y^{(i)})$ (see §2), that

$$H^i(\tilde{X})=0 \qquad 1\leqq i\leqq k'.$$

Thus (b) is proved.

§4. Knot modules.

Let K be a knot in a 2-connected compact m-dimensional smooth manifold M^m. The complement X_0 or the exterior X of K is the most important invariant of K. In order to get useful and calculable invariants from X, we define knot modules $A_q(K; M^m)$ $q=1, 2, 3, \cdots$. In the following we assume that $m\geqq 5$. Remark that $H_1(X)\cong Z$.

Let p: $\tilde{X}\longrightarrow X$ be the maximal abelian covering corresponding to the kernel of the surjection $\pi_1(X)\longrightarrow H_1(X)$ and let t denote a multiplicative generator of the covering transformation group $H_1(X)$. The homomorphism $H_q(\tilde{X})\longrightarrow H_q(\tilde{X})$ induced by t: $\tilde{X}\longrightarrow\tilde{X}$ is also denoted by the same notation t, and we denote $t(\alpha)$ by $t\alpha$ for $\alpha\in H_q(\tilde{X})$. The integral group ring Λ of $H_1(X)$ is the ring $Z[t, t^{-1}]$ of Laurent polynomials in t.

The homology modules $A_q(K; M^m)=H_q(\tilde{X})$ $(q\geqq 1)$ are called the *knot modules* of K. $A_q(K; M^m)$ is a finitely generated module over Λ. In case $M^m=S^m$, $A_q(K; S^m)$ is simply denoted by $A_q(K)$.

An intersection pairing I of $H_q(\tilde{X})$ is defined by fixing an orientation of $\tilde{X}$, making use of the triangulation and the dual one of $\tilde{X}$ (Blanchfield [1]). Thus we have

$$I\colon A_q(K; M^m)\otimes A_{m-q}(K; M^m)\longrightarrow Z,$$
$$I(\alpha, \beta)=\alpha\cdot\beta \quad (\alpha\in A_q(K; M^m),\ \beta\in A_{m-q}(K; M^m)).$$

Let K be a 1-simple knot in a 2-connected closed m-dimensional smooth manifold M^m $(m\geqq 5)$. Then p: $\tilde{X}\longrightarrow X$ above is the universal covering. Suppose that there exists a simply connected Seifert surface V for K. Let $Y^{(i)}$, $V^{(i)}$ $(i\in Z)$ and $\tilde{X}=\bigcup_{i\in Z} Y^{(i)}$ be as in §2. Then t operates on $Y^{(i)}$ and $V^{(i)}$ so that

$$t(Y^{(i)})=Y^{(i+1)},\quad t(V^{(i)})=V^{(i+1)}.$$

The homomorphisms $C_q(Y^{(i)}) \longrightarrow C_q(Y^{(i+1)})$, $C_q(V^{(i)}) \longrightarrow C_q(V^{(i+1)})$, $H_q(Y^{(i)}) \longrightarrow H_q(Y^{(i+1)})$ and $H_q(V^{(i)}) \longrightarrow H_q(V^{(i+1)})$ induced by t are also denoted by the same notation t.

Now let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n \geqq 3)$, and let K be a simple (i.e. $(n-1)$-simple) knot in M^{2n+1}. Then, by Proposition 2, there exists a Seifert surface V for K which is $(n-1)$-connected. By the Poincaré-Lefschetz duality theorem of $(V, \partial V)$, it holds that

$$H_q(V)=0 \quad \text{if} \quad q \neq 0, n,$$

and that $H_n(V)$ is a finitely generated free abelian group. It follows from the fundamental exact sequence that

$$H_q(Y)=0 \qquad q \neq 0, n, n+1,$$

thus Y is $(n-1)$-connected. Therefore the fundamental exact sequence becomes as follows:

$$0 \longrightarrow H_{n+1}(Y) \xrightarrow{\iota_*} H_{n+1}(M^{2n+1}) \xrightarrow{\theta} H_n(V) \xrightarrow{\partial} H_n(Y) \xrightarrow{\iota_*} H_n(M^{2n+1}) \longrightarrow 0.$$

Since $H_{n+1}(M^{2n+1})$ and $H_n(V)$ are free abelian, $H_{n+1}(Y)$ is also free abelian and $\iota_*(H_{n+1}(Y))$ is a direct summand of $H_{n+1}(M^{2n+1})$.

By the observation above, it follows from the Mayer-Vietoris exact sequence in § 2 that

$$A_q(K; M^{2n+1})=0 \qquad q \neq 0, n, n+1,$$

and that $A_n(K; M^{2n+1})$ and $A_{n+1}(K; M^{2n+1})$ satisfy the exact sequence

$$\begin{aligned} 0 \longrightarrow \sum_{i \in Z} H_{n+1}(Y^{(i)}) &\xrightarrow{\Sigma \iota_*} A_{n+1}(K; M^{2n+1}) \xrightarrow{\partial} \sum_{i \in Z} H_n(V^{(i)}) \\ &\xrightarrow{\phi} \sum_{i \in Z} H_n(Y^{(i)}) \xrightarrow{\Sigma \iota_*} A_n(K; M^{2n+1}) \longrightarrow 0. \end{aligned}$$

Let M^{2n} be an $(n-1)$-connected closed $2n$-dimensional smooth manifold $(n \geqq 3)$ and let K be a simple (i.e. $(n-1)$-simple) knot in M^{2n}. Then, by Proposition 2, there exists a Seifert surface V for K which is $(n-2)$-connected. By the Poincaré-Lefschetz duality theorem of $(V, \partial V)$, it holds that

$$H_q(V)=0 \quad \text{if} \quad q \neq 0, n-1, n.$$

It follows from the fundamental exact sequence that

$$H_q(Y)=0 \qquad q\neq 0, n-1, n,$$

thus Y is $(n-1)$-connected. Therefore the fundamental exact sequence becomes as follows:

$$0\longrightarrow H_n(V)\xrightarrow{\partial} H_n(Y)\xrightarrow{\iota_*} H_n(M^{2n})\xrightarrow{\partial} H_{n-1}(V)\xrightarrow{\partial} H_{n-1}(Y)\longrightarrow 0.$$

By the observation above, it follows from the Mayer-Vietoris exact sequence in § 2 that

$$A_q(K; M^{2n})=0 \qquad q\neq 0, n-1, n, n+1,$$

and that $A_{n-1}(K; M^{2n})$, $A_n(K; M^{2n})$ and $A_{n+1}(K; M^{2n})$ satisfy the exact sequence

$$0\longrightarrow A_{n+1}(K; M^{2n})\xrightarrow{\partial} \sum_{i\in \boldsymbol{Z}} H_n(V^{(i)})\xrightarrow{\phi} \sum_{i\in \boldsymbol{Z}} H_n(Y^{(i)})\xrightarrow{\Sigma\iota_*} A_n(K; M^{2n})$$
$$\xrightarrow{\partial} \sum_{i\in \boldsymbol{Z}} H_{n-1}(V^{(i)})\xrightarrow{\phi} \sum_{i\in \boldsymbol{Z}} H_{n-1}(Y^{(i)})\xrightarrow{\Sigma\iota_*} A_{n-1}(K; M^{2n})\longrightarrow 0.$$

In the following let us consider knot modules of local knots. Let K be a local knot in a 2-connected closed m-dimensional smooth manifold M^m, and let $\hat{D}^m$ be an m-disk imbedded in M^m such that $K\subset\hat{D}^m$. K can be considered to be a knot in the double $\hat{D}^m\cup D^m=S^m$. The knot in S^m thus obtained will be denoted by $\hat{K}$. Then it is obvious that the knot modules $A_q(K; M^m)$ $(q=1, 2, \cdots, m-1)$ are given by

$$A_q(K; M_m)\cong A_q(\hat{K})\oplus(H_q(M^m)\otimes\Lambda).$$

§ 5. Fibred knots in highly connected manifolds.

The following theorem was proved in order to show the existence of foliations of codimension one ([16, Theorem 8], [17]). The terminology "specially spinnable structures" was used there instead of fibred knots.

THEOREM 1. *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n\geqq 3)$. Then there exists a knot K in M^{2n+1} which is simple and fibred such that the exterior $X=M^{2n+1}-N(K)$ of K is a fibre bundle over S^1 with $(n-1)$-connected $2n$-dimensional smooth manifold F as fibre.*

For the proof, refer to [16]. The construction of a fibred knot in the special case of $M^{2n+1}=S^n\times S^{n+1}$ (n: odd) in Theorem 2 below will reveal the proof of the above theorem.

Let K be a simple, fibred knot in an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold M^{2n+1} ($n\geqq 3$) such as in Theorem 1. Then the knot modules $A_q(K; M^{2n+1})$ $(q=1, 2, \cdots)$ are as follows:

$$A_q(K; M^{2n+1})=0 \qquad q\neq 0, n$$
$$A_n(K; M^{2n+1})\cong H_n(F).$$

By a result in § 4, this shows that the knot K is not local in case M^{2n+1} is not a homotopy $(2n+1)$-sphere.

Now we shall construct a simple, fibred knot in $S^{2q-1}\times S^{2q}$ ($q\geqq 2$), making use of the method in section 3 of [16]. Some miswriting contained there will be corrected in the following arguments.

Let $q\geqq 2$ and let x_0 be a point of S^{2q-1}. We denote the subsets $S^{2q-1}\times\{x_0\}$, $\{x_0\}\times S^{2q-1}$ and the diagonal $\{(x, x); x\in S^{2q-1}\}$ of $S^{2q-1}\times S^{2q-1}$ by $\bar{a}$, $\bar{b}$ and $\bar{d}$ respectively. Let us specify orientations of $S^{2q-1}\times S^{2q-1}$ and submanifolds $\bar{a}$, $\bar{b}$ and $\bar{d}$ so that

$$[\bar{d}]=[\bar{a}]+[\bar{b}], \ I([\bar{a}], [\bar{b}])=1.$$

where $[\bar{a}]$, $[\bar{b}]$ and $[\bar{d}]$ are homology classes of $H_{2q-1}(S^{2q-1}\times S^{2q-1})$ represented by $\bar{a}$, $\bar{b}$ and $\bar{d}$ respectively.

Let $S_i^{2q-1}\times D_i^{2q}$ $(i=0, 1, 2, 3, 4, 5)$ be 6 copies of $S^{2q-1}\times D^{2q}$ and let

$$W=(S_0^{2q-1}\times D_0^{2q})\natural(S_1^{2q-1}\times D_1^{2q})\natural\cdots\natural(S_5^{2q-1}\times D_5^{2q})$$

be the boundary connected sum of them, where we give $S^{2q-1}\times D^{2q}$ the orientation so that the boundary orientation coincides with that of $S^{2q-1}\times S^{2q}$. Let $\bar{a}_i$, $\bar{b}_i$ and $\bar{d}_i$ denote oriented submanifolds of $S_i^{2q-1}\times S_i^{2q-1}$ corresponding to $\bar{a}$, $\bar{b}$ and $\bar{d}$ of $S^{2q-1}\times S^{2q-1}$ respectively. We my suppose that

$$\bar{a}_i, \bar{b}_i, \bar{d}_i\subset\partial W \qquad (i=0, 1, 2, \cdots, 5).$$

Let W' be a copy of W and let $S^{2q-1}\times S^{2q}=W\cup W$ be the Heegaard decomposition of $S^{2q-1}\times S^{2q}$ such that $[a_0]$ represents a generator of $H_{2q-1}(S^{2q-1}\times S^{2q})\cong \boldsymbol{Z}$ and that $[a_i]$, $[b_i]$ $(i=1, 2, 3, 4, 5)$ and $[b_0]$ are zero in $H_{2q-1}(S^{2q-1}\times S^{2q})$.

Let us consider $(2q-1)$-dimensional submanifolds $\bar{a}_0, \bar{d}_1, \bar{b}_1\#\bar{d}_2, \bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3$ in ∂W (Fig. 1). The intersection numbers of $[\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3]$ with $[\bar{d}_1]$ and

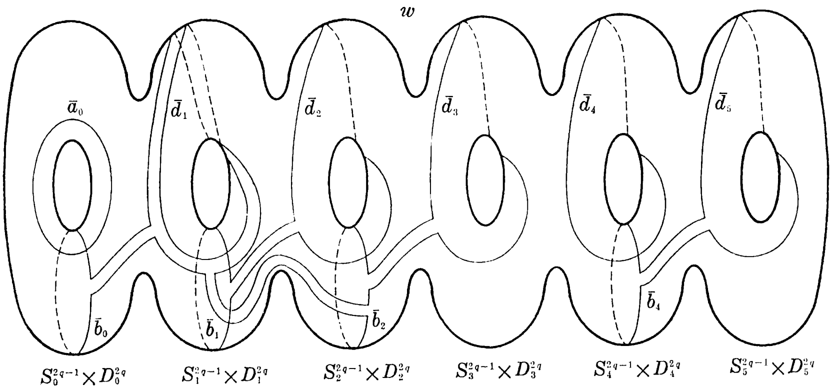

Fig. 1.

$[\bar{b}_1\#\bar{d}_2]$ are

$$I([\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3],\ [\bar{d}_1])=0,$$
$$I([\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3],\ [\bar{b}_1\#\bar{d}_2])=0.$$

Thus we can cancel geometrical intersections $(\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3)\cap\bar{d}_1$ and $(\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3)\cap(\bar{b}_1\#\bar{d}_2)$ by an isotopy movement of $\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3$. We denote by $\bar{e}$ the submanifold isotopic to $\bar{b}_0\#\bar{d}_1\#\bar{b}_2\#\bar{d}_3$ thus obtained:

$$\bar{e}\cap\bar{d}_1=\emptyset,\ \bar{e}\cap(\bar{b}_1\#\bar{d}_2)=\emptyset.$$

We may suppose that $\bar{e}$ and $\bar{a}_0$ intersect transversally at one point, since $I([\bar{e}],\ [\bar{a}_0])=1$.

Let V_1, V_2 and V_3 be smooth regular neighborhoods of $\bar{e}\cup\bar{a}_0$, $\bar{d}_1\cup(\bar{b}_1\#\bar{d}_2)$ and $\bar{d}_4\cup(\bar{b}_4\#\bar{d}_5)$ in ∂W respectively such that they are disjoint each other, and let $V=V_1\natural V_2\natural V_3$ be the boundary connected sum of V_1, V_2 and V_3 in ∂W. V is homotopy equivalent to the bouquet of 6 copies of S^{2q-1}. Let $K=\partial V$, then the following lemma holds.

LEMMA 1. *K is a simple knot in $S^{2q-1}\times S^{2q}$ and V is a Seifert surface for K.*

PROOF. V_1 is the plumbing of tubular neighborhoods of $\bar{e}$ and $\bar{a}_0$ in ∂W, and the tubular neighborhood of $\bar{a}_0$ is diffeomorphic to $S^{2q-1}\times D^{2q-1}$. Thus ∂V_1 is diffeomorphic to S^{4q-3}. On the other hand, since V_2 and V_3 are diffeomorphic, ∂V_2 and ∂V_3 are both diffeomorphic to the natural $(4q-3)$-shere or the Kervaire sphere. Thus $\partial V_2\#\partial V_3$ is diffeomorphic to

S^{4q-3}. Therefore $\partial V=\partial V_1\#\partial V_2\#\partial V_3$ is diffeomorphic to S^{4q-3}. It follows from Proposition 2, (b) that K is simple. This proves Lemma 1.

As is easily verified, the inclusion maps

$$V\longrightarrow W,\quad V\longrightarrow W'$$

are homotopy equivalences. Thus, according to the relative h-cobordism theorem, W and W' are both diffeomorphic to $V\times I$. This implies that $S^{2q-1}\times S^{2q}-\operatorname{Int}N(K)$ is a fibre bundle over S^1 with V as fibre, and, thus, K is a fibred knot in $S^{2q-1}\times S^{2q}$.

Now the following lemma holds:

LEMMA 2. *K is inessential in $S^{2q-1}\times S^{2q}$.*

PROOF. Since $\bar{d}_1, \bar{b}_1\#\bar{d}_2, \bar{d}_4$ and $\bar{b}_4\#\bar{d}_5$ are inessential in $S^{2q-1}\times S^{2q}$, ∂V_2 and ∂V_3 are inessential. The homotopy class $\{\partial V_1\}$ represented by ∂V_1 in $S^{2q-1}\times S^{2q}$ is the Whitehead product of the homotopy classes $\{\bar{e}\}$ and $\{\bar{a}_0\}$ (see [20]), and $\{\bar{e}\}$ is inessential in $S^{2q-1}\times S^{2q}$. Thus ∂V_1 is inessential in $S^{2q-1}\times S^{2q}$. This proves Lemma 2.

By Lemmas 1 and 2 and the remark on knot modules of fibred knots at the begining of this section, we have the following theorem.

THEOREM 2. *There exists a simple knot K in $S^{2q-1}\times S^{2q}$ $(q\geqq 2)$ such that K is fibred, thus not local, and inessential.*

The inessentiality is necessary for a knot to be local. But this theorem shows that it is not sufficient. This answers Problem 1 in [18]. The similar result will be again obtained in Corollary of Theorem 5 (§ 8).

Similarly, examples of such knots can be constructed in $S^{2q-1}\times S^{2q+1}$ $(q\geqq 2)$, making use of the arguments in the section 2 of [16], and also, in some $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifolds $(n\geqq 3)$.

As is easily verified, by considering Euler numbers, an $(n-1)$-connected closed $2n$-dimensional smooth manifold does not admit any simple fibred knot if it is not a homotopy $2n$-sphere.

§ 6. Localness Theorem and Unknotting Theorem.

The following theorem gives a localness criterion by means of knot modules for simple knots in highly connected closed smooth manifolds.

THEOREM 3 (Localness Theorem). (I) *Let M^{2n} be an $(n-1)$-connected closed $2n$-dimensional smooth manifold $(n\geqq 3)$ and let K be a simple knot in M^{2n}. Then K is local if and only if the knot module $A_n(K; M^{2n})$ contains a direct summand $\bar{A}$ satisfying the following conditions*:

(a) *$\bar{A}$ is a Λ-free module of rank b, where b is the n-th Betti number of M^{2n}.*

(b) *There exists a Λ-basis $\{\omega_1, \omega_2, \cdots, \omega_b\}$ for $\bar{A}$ such that the $b\times b$ matrix $(I(\omega_i, \omega_j))$ is unimodular and that*

$$I(\omega_i, t^k\omega_j)=0 \qquad k\neq 0; i, j=1, 2, \cdots, b.$$

(II) *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n\geqq 3)$ such that $H_n(M^{2n+1})$ is torsion free, and let K be a simple knot in M^{2n+1}. Then K is local if and only if the knot modules $A_n(K; M^{2n+1})$ and $A_{n+1}(K; M^{2n+1})$ contain direct summands $\bar{B}$ and $\bar{C}$ respectively which satisfy the following conditions*:

(a) *$\bar{B}$ and $\bar{C}$ are Λ-free modules of rank b, where b is the n-th and the $(n+1)$-th Betti number of M^{2n+1}.*

(b) *There exists a Λ-basis $\{\xi_1, \xi_2, \cdots, \xi_b\}$ for $\bar{B}$ and a Λ-basis $\{\zeta_1, \zeta_2, \cdots, \zeta_b\}$ for $\bar{C}$ such that*

$$I(\xi_i, \zeta_j)=\delta_{ij}, \quad I(\xi_i, t^k\zeta_j)=0 \qquad i, j=1, 2, \cdots, b; k\neq 0.$$

where δ_{ij} is Kronecker's delta.

PROOF OF THE "ONLY IF" PART. Suppose that a simple knot K in M^{2n} is local: $\hat{D}^{2n}\supset K$. Then $A_n(K; M^{2n})\cong A_n(\hat{K})\oplus(H_n(M^{2n})\otimes\Lambda)$ (see § 4). We put $\bar{A}=H_n(M^{2n})\otimes\Lambda$.

By Proposition 2, there exists a simply connected Seifert surface V for K such that $\hat{D}^{2n}\supset V$.

Let $\omega_1', \omega_2', \cdots, \omega_b'$ be generators of $H_n(M^{2n})$ represented by maps $\varphi_i'\colon S^n\longrightarrow M^{2n}-\hat{D}^{2n}$ $(i=1, 2, \cdots, b)$, and let $\tilde{\varphi}_i'\colon S^n\longrightarrow\tilde{X}$ $(i=1, 2, \cdots, b)$ be the lifts of φ_i' such that $\tilde{\varphi}_i'(S^n)\subset Y^{(0)}$, where $\tilde{X}=\bigcup_{j\in Z} Y^{(j)}$ as in § 2. Then the homology classes $\omega_1, \omega_2, \cdots, \omega_b$ represented by $\tilde{\varphi}_1', \tilde{\varphi}_2', \cdots, \tilde{\varphi}_b'$ respectively, satisfy the condition (I), (b). Therefore the conditions (I)(a), (b) are satisfied. Thus the "only if" part of (I) is proved. Similarly the "only if part of (II) is proved.

Proof of the "if" part of (I) and (II) will be given in the next section.

We remark that the 'only if" part of Theorem 3 is true even in the case of $n=2$.

The following theorem is a direct consequence of the Localness Theorem above and the Unknotting Theorem for knots in the spheres by Levine [11].

THEOREM 4 (Unknotting Theorem). (I) *Let M^{2n} be an $(n-1)$-connected closed $2n$-dimensional smooth manifold $(n\geqq 3)$ and let K be a 1-simple knot in M^{2n}. Then K is unknotted if and only the knot modules $A_q(K; M^{2n})$ $(2\leqq q\leqq n)$ satisfy the following conditions*:

(a) $A_q(K; M^{2n})=0 \qquad 2\leqq q\leqq n-1.$

(b) *$A_n(K; M^{2n})$ is a free Λ-module of rank b with a Λ-basis $\{\omega_1, \omega_2, \cdots, \omega_b\}$ such that the $b\times b$ matrix $(I(\omega_i, \omega_j))$ is unimodular and that*

$$I(\omega_i, t^k\omega_j)=0 \qquad k\neq 0;\ i, j=1, 2, \cdots, b,$$

where b is the n-th Betti number of M^{2n}

(II) *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n\geqq 3)$ such that $H_n(M^{2n+1})$ is torsion free, and let K be a 1-simple knot in M^{2n+1}. Then K is unknotted if and only if the knot modules $A_q(K; M^{2n+1})$ $(2\leqq q\leqq n+1)$ satisfy the following conditions*:

(a) $A_q(K; M^{2n+1})=0 \qquad 2\leqq q\leqq n-1.$

(b) *$A_n(K; M^{2n+1})$ and $A_{n+1}(K; M^{2n+1})$ are free Λ-modules of rank b with Λ-basis $\{\xi_1, \xi_2, \cdots, \xi_b\}$ and $\{\zeta_1, \zeta_2, \cdots, \zeta_b\}$ respeotively such that*

$$I(\xi_i, \zeta_j)=\delta_{ij},\ \ I(\xi_i, t^k\zeta_j)=0 \qquad i, j=1, 2, \cdots, b;\ k\neq 0,$$

where b is the n-th and the $(n+1)$-th Betti number of M^{2n+1}.

PROOF. First suppose that a 1-simple knot K in M^{2n} is unknotted. Then it is obvious that

$$A_q(K; M^{2n})\cong H_q(M^{2n})\otimes\Lambda \qquad 1\leqq q\leqq 2n-1.$$

The homology classes $\omega_1, \omega_2, \cdots, \omega_b\in H_n(\tilde{X})$ as in the proof of the "only if" part of Theorem 3, (I) satisfy the condition I, (b). Thus the condition (I) is satisfied.

Conversely, if the assumption of (I) is satisfied, then K is simple, since $A_q(K; M^{3n})=0$ for $1\leqq q\leqq n-1$. Therefore, by the Localness Theorem, the knot K is local: $K\subset\hat{D}^{2n}$, and, thus we have

$$A_q(K; M^{2n})\cong A_q(\hat{K})\oplus(H_q(M^{2n})\otimes\Lambda) \qquad q=1, 2, \cdots, n,$$

where $\hat{K}$ is a knot in S^{2n} determined by K in $\hat{D}^{2n}$ (§ 4). It follows from the assumption on $A_q(K; M^{2n})$ that

$$A_q(\hat{K})=0 \qquad q=1, 2, \cdots, n.$$

Since $\hat{K}$ is 1-simple, by the unknotting theorem of Levine, this implies that $\hat{K}$ is unknotted in S^{2n}. that is, K bounds an imbedded $(2n-1)$-disk in $\hat{D}^{2n}$. Thus (I) is proved.

The proof of (II) is similar.

REMARK. An alternative proof for Theorem 4 may be possibly given using the idea of Matsumoto [12].

§ 7. Proof of Localness Theorem.

In this section, we shall give the proof of the "if" part of Theorem 3. What we are going to do is the modification of situations of Seifert surfaces by surgeries.

First we prove Theorem 3, (I). By Proposition 2, there exists a Seifert surface V for K which is $(n-2)$-connected. Let Y denote the exterior of V. Then Y is also $(n-2)$-connected (§ 4). Let us consider the universal covering $\tilde{X}$ of the exterior X of K and the decomposition $\tilde{X}=\bigcup_{j\in Z} Y^{(j)}$ such that $Y^{(j)}\cap Y^{(j+1)}=V^{(j+1)}$, where $Y^{(j)}$ and $V^{(j)}$ are copies of Y and V respectively (see § 2).

Let z be a singular n-cycle of $\tilde{X}$: $z\in C_n(\tilde{X})$. The minimal non-negative integer l such that

$$z\in C_n(Y^{(j)}\cup Y^{(j+1)}\cup\cdots\cup Y^{(j+l)})$$

for some j is said to be the *length* of z with respect to V and is denoted by $l(z)$. Let ω be an n-th homology class of $\tilde{X}$: $\omega\in H_n(\tilde{X})$. The minimal non-negative integer in $\{l(z); \omega=[z]\}$ is said to be the *length* of ω with respect to V and is denoted by $l(\omega)$.

(*Step* 1). Suppose that the length $l(\omega_k)$ of ω_k with respect to V is $l\geqq 1$, where ω_k is a homology class in the assumption of Theorem 3, (I). Then there exists an n-cycle $\bar{z}_k$ of $\tilde{X}$ such that $\omega_k=[\bar{z}_k]$ and that

$$\bar{z}_k \in C_n(Y^{(j)} \cup Y^{(j+1)} \cup \cdots \cup Y^{(l+j)}).$$

We may assume that $\bar{z}_k$ is decomposed as follows (Fig. 2):

$$\bar{z}_k = \bar{c}_k^{(j)} + \bar{c}_k^{(j+1)} + \cdots + \bar{c}_k^{(j+l)},$$

where $\bar{c}_k^{(m)} \in C_n(Y^{(m)})$ $(m = j, j+1, \cdots, j+l)$. Then we have

$$[\bar{c}_k^{(j+l)}] \in H_n(Y^{(j+l)}, V^{(j+l)}),$$
$$[\bar{c}_k^{(j)}] \in H_n(Y^{(j)}, V^{(j+1)}).$$

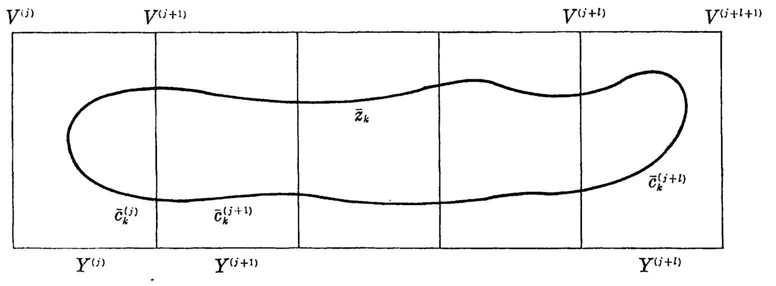

Fig. 2.

LEMMA 3. *There exists a continuous map*

$$g\colon (D^n, S^{n-1}) \longrightarrow (Y^{(j+l)}, V^{(j+l)})$$

such that $g_*([D^n, S^{n-1}]) = [\bar{c}_k^{(j+l)}]$.

PROOF. Since $V^{(j+l)}$ is $(n-2)$-connected $(n \geqq 3)$, the homology class $\partial[\bar{c}_k^{(j+l)}] \in H_{n-1}(V^{(j+l)})$ is spherical, where $\partial\colon H_n(Y^{(j+l)}, V^{(j+l)}) \longrightarrow H_{n-1}(V^{(j+l)})$ is the boundary homomorphism. Furthermore, since $V^{(j+l)}$ and $Y^{(j+l)}$ are $(n-2)$-connected $(n \geqq 3)$, the Hurewicz homomorphisms

$$h_{n-1}\colon \pi_{n-1}(V^{(j+l)}) \longrightarrow H_{n-1}(V^{(j+l)}),$$
$$h_{n-1}\colon \pi_{n-1}(Y^{(j+l)}) \longrightarrow H_{n-1}(Y^{(j+l)})$$

are bijective. Thus we have

$$(\iota_+)_*((h_{n-1})^{-1}(\partial[c_k^{(j+l)}])) = 0,$$

where $\iota_+\colon V^{(j+l)} \longrightarrow Y^{(j+l)}$ is the inclusion map. It follows from this that $(h_{n-1})^{-1}(\partial[c^{(j+l)}])$ is contained in the image of the boundary homomorphism $\partial\colon \pi_n(Y^{(j+l)}, V^{(j+l)}) \longrightarrow \pi_{n-1}(V^{(j+l)})$, that is,

$$\partial\mu_l=(h_{n-1})^{-1}(\partial[\bar{c}^{(j+l)}]) \quad \text{for} \quad \mu_l\in\pi_n(Y^{(j+l)}, V^{(j+l)}).$$

Then, $[\bar{c}^{(j+l)}]-h_n(\mu_l)\in H_n(Y^{(j+l)}, V^{(j+l)})$ is contained in the image of $H_n(Y^{(j+l)})\longrightarrow H_n(Y^{(j+l)}, V^{(j+l)})$, where h_n is the Hurwicz homomorphism. Since $Y^{(j+l)}$ is $(n-2)$-connected, every element of $H_n(Y^{(j+l)})$ is spherical. Thus this lemma is proved.

We may suppose that the map g in Lemma 3 is a proper imbedding transversal to $V^{(j+l)}$.

Similarly, there exists a proper imbedding

$$g'\colon (D^n, S^{n-1})\longrightarrow(Y^{(j)}, V^{(j+1)})$$

transversal to $V^{(j+l)}$ such that

$$g'_*([D^n, S^{n-1}])=[\bar{c}_k^{(j)}].$$

Since $I(t^{-l}\omega_k, \omega_k)=0$ by the assumption, the algebraic intersection number of $t^{-l}(g(D^n))$ and $g'(D^n)$ in $Y^{(j)}$ is zero (Fig. 3). Thus, by the Whitney trick, we can cancel the geometrical intersection of $t^{-l}(g(D^n))$ and $g'(D^n)$ by an isotopy movement of g'.

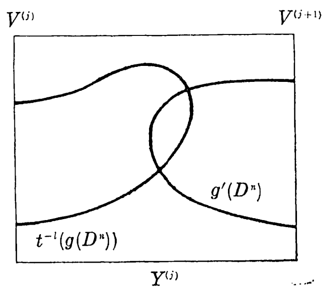

Fig. 3.

We perform the surgery on V by $p\circ q(D^n)$ or $p\circ g'(D^n)$, where $p\colon \tilde{X}\longrightarrow X$ is the projection. That, is, let

$$g\colon (D^n\times D^n, S^{n-1}\times D^n)\longrightarrow(M^{2n}, V)$$

be an imbedding such that $g|(D^n\times\{0\}, S^{n-1}\times\{0\})=p\circ g$ and $g(D^n\times D^n)\cap V=g(S^{n-1}\times D^n)$, and let

$$V_1=(V-g(\partial D^n\times D^n))\cup g(D^n\times\partial D^n)$$

be the manifold obtained from V by the surgery $\chi(g|(S^{n-1}\times D^n))$. Then,

V_1 is an $(n-2)$-connected Seifert surface for K.

Let Y_1 be the exterior of V_1 and let $\tilde{X}=\cup Y_1^{(i)}$ be the decomposition of $\tilde{X}$ by copies of Y_1. Then it is easy to see that

$$\bar{z}_k \in C_n(Y_1^{(j)} \cup Y_1^{(j+1)} \cup \cdots \cup Y_1^{(j+l-1)}),$$

where $Y_1^{(m)}$ is the the copy of Y_1 obtained from $Y^{(m)}$ by the lift of the above surgery. This implies that the length of ω_k with respect to V_1 is less than l. We remark that the surgery on V by $p\circ g'(D^n)$ can be equally used in order to make the length of ω_k less than l. By applying the above method to V successively, we obtain an $(n-2)$-connected Seifert surface V_l for K with respect to which the length of ω_k is zero.

(*Step* 2). Suppose that the length $l(\omega_i)$ of ω_i $(i=1, 2, \cdots, k-1)$ with respect to V are all zero and that there exist n-cycles $\bar{z}_i$ $(i=1, 2, \cdots, k-1)$ of $\tilde{X}$ such that

$$\omega_i=[\bar{z}_i],\ \bar{z}_i \in C_n(Y^{(0)}) \qquad i=1, 2, \cdots, k-1.$$

Since $Y^{(0)}$ is $(n-2)$-connected, the cycles $\bar{z}_i$ $(i=1, 2, \cdots, k-1)$ are spherical. Let

$$\bar{\varphi}_i\colon S^n \longrightarrow Y^{(0)} \qquad (i=1, 2, \cdots, k-1)$$

be imbeddings representing ω_i.

Let the length $l(\omega_k)$ of ω_k with respect to V is $l \geqq 1$ and let

$$\omega_k=[\bar{z}_k],\ \bar{z}_k \in C_n(Y^{(j)} \cup Y^{(j+1)} \cup \cdots \cup Y^{(j+l)})$$

as in Step 1. In case $Y^{(j+l)} \neq Y^{(0)}$, we perform the surgery on V by $p\circ g(D^n)$ as in Step 1. Let V_1 be an $(n-2)$-connected Seifert surface for K obtained by this surgery. Then the length $l(\omega_k)$ of ω_k with respect to V_1 is less than l. Since, by the assumption, it holds that

$$I(t^{j+l}\omega_i, \omega_k)=0 \qquad i=1, 2, \cdots, k-1,$$

we may suppose that, for g used in the surgery in (Step 1), the following holds:

$$p\circ\bar{\varphi}_i(S^n) \subset Y-g(D^n \times D^n) \qquad i=1, 2, \cdots, k-1.$$

This implies that

$$\bar{z}_i \in C_n(Y_1^{(0)}) \qquad i=1, 2, \cdots, k-1,$$

where $\omega_i=[\bar{z}_i]$ and $Y_1^{(0)}$ is the copy of the exterior Y_1 of V_1 obtained from $Y^{(0)}$ by the lift of the surgery. In case $Y^{(j)}\neq Y^{(0)}$, by the surgery on V by $p'\circ g(D^n)$ as in Step 1, we obtain a Seifert surface for K with the same property.

Therefore, by applying the above method successively, we obtain an $(n-2)$-connected Seifert surface V_l for K such that

$$\begin{aligned}&\bar{z}_k\in C_n(Y_l^{(r)}),\\ &\bar{z}_i\in C_n(Y_l^{(0)}) \qquad i=1,2,\cdots,k-1,\end{aligned}$$

where $Y_l^{(j)}$ is a copy of the exterior Y_l of V_l.

In case $Y_l^{(r)}\neq Y_l^{(0)}$, we modify the Seifert surface V_l as follows. Suppose that $r\neq 0$, say $0<r$. Let $\bar{\varphi}_k$: $S^n\longrightarrow Y^{(r)}$ be an imbedding representing ω_k. By the assumption, it holds that

$$I(t^{-r}\omega_k,\omega_i)=0 \qquad i=1,2,\cdots,k-1.$$

Thus we may assume that

$$t^{-r}(\bar{\varphi}_k(S^n))\cap\bar{\varphi}_i(S^n)=\emptyset \qquad i=1,2,\cdots,k-1.$$

Let $N(\bar{\varphi}_k)$ be a tubular neighborhood of $\bar{\varphi}_k(S^n)$ in $Y_l^{(r)}$ and let $V_{l+1}=p(V_l^{(r)}\#\partial N(\bar{\varphi}_k))$ be the image of the connected sum of $V^{(r)}$ and $\partial N(\bar{\varphi}_k)$ in $Y_l^{(r)}$ by the projection p: $\tilde{X}\longrightarrow X$. Then V_{l+1} is an $(n-2)$-connected Seifert surface for K. Obviously, after an isotopy movement of $\bar{\varphi}_i$ if necessary, it holds that

$$\begin{aligned}&\bar{\varphi}_k(S^n)\subset Y_{l+1}^{(r-1)},\\ &\bar{\varphi}_i(S^n)\subset Y_{l+1}^{(0)} \qquad i=1,2,\cdots,k-1,\end{aligned}$$

where $Y_{l+1}^{(j)}$ is the copy of the exterior Y_{l+1} of V_{l+1} obtained from $Y_l^{(j)}$ by the lift of the surgery.

By applying the above method successively, we obtain an $(n-2)$-connected Seifert surface V_{l+r} such that

$$\bar{\varphi}_i(S^n)\subset Y_{l+r}^{(0)} \qquad i=1,2,\cdots,k,$$

where $Y_{l+r}^{(0)}$ is the copy of the exterior Y_{l+r} of V_{l+r} obtained from $Y^{(0)}$ by surgeries.

(*Step* 3). As the consequence of Step 2, there exists an $(n-2)$-connected Seifert surface $\hat{V}$ for K such that

$$\bar{\varphi}_i(S^n) \subset \hat{Y}^{(0)} \qquad i=1, 2, \cdots, b,$$

where $\bar{\varphi}_i$: $S^n \longrightarrow \hat{Y}^{(0)}$ is an imbedding representing ω_i and $\hat{Y}^{(0)}$ is a copy of the exterior $\hat{Y}$ of $\hat{V}$.

Let us consider homology classes $p_*(\omega_1), p_*(\omega_2), \cdots, p_*(\omega_b) \in H_n(X)$. Since $\omega_i \in H_n(\hat{Y}^{(0)})$ $(i=1, 2, \cdots, b)$, we have

$$(I(p_*(\omega_i), p_*(\omega_j))) = (I(\omega_i, \omega_j)).$$

For the inclusion map ι: $X \longrightarrow M^{2n}$, this implies by the assumption I, (b) that the homology classes $\iota_* \circ p_*(\omega_1), \iota_* \circ p_*(\omega_2), \cdots, \iota_* \circ p_*(\omega_b)$ form a basis of $H_n(M^{2n})$.

Thus, as is easily verified, there exists a handle-body with b n-handles

$$HB = D^{2n} \cup (D^n \times D^n)_1 \cup (D^n \times D^n)_2 \cup \cdots \cup (D^n \times D^n)_b$$

in $\hat{Y}$ such that the inclusion map $HB \longrightarrow M^{2n}$ induces the isomorphism $H_n(HB) \longrightarrow H_n(M^{2n})$.

Since the boundary of HB is simply connected, this implies that $M^{2n} - \operatorname{Int} HB$ is diffeomorphic to D^{2n}, making use of the h-cobordism theorem. Therefore there exists a $2n$-disk D^{2n} imbedded in M^{2n} such that

$$D^{2n} \supset \hat{V} \supset K.$$

This completes the proof of Theorem 3, (I).

In the following we prove the "if" part of Theorem 3, (II). Let M^{2n+1} and K be a $(2n+1)$-dimensional smooth manifold and a knot in M^{2n+1} satisfying the assumptions of Theorem 3, (II). By Proposition 2, there exists a Seifert surface V for K which is $(n-1)$-connected. Let Y be the exterior of V, then Y is also $(n-1)$-connected (§ 4).

Let us consider the decomposition $\tilde{X} = \bigcup_{j \in Z} Y^{(j)}$ of the universal covering $\tilde{X}$ of the exterior X of K in M^{2n+1} such that $Y^{(j)} \cap Y^{(j+1)} = V^{(j+1)}$, where $Y^{(j)}$ and $V^{(j)}$ are copies of Y and V respectively.

Let $z \in C_n(\tilde{X})$ (resp. $z' \in C_{n+1}(\tilde{X})$) be a singular n-chain (resp. $(n+1)$-chain) of $\tilde{X}$ and let $\xi \in H_n(\tilde{X})$ (resp. $\zeta \in H_{n+1}(\tilde{X})$). Then the length of z and ξ (resp. z' and ζ) with respect to V can be defined in the same way as the length of $\bar{z}$ and ω in the proof of Theorem 3, (I). The length of z, ξ, z' and ζ are denoted by $l(z), l(\xi), l(z')$ and $l(\zeta)$ respectively.

(*Step* 1'). Suppose that the length $l(\xi_k)$ of ξ_k with respect to V is $l\geqq 1$, where ξ_k is a homology class in the assumption of Theorem 3, (II). Then there exists a singular n-cycle z_k such that

$$\xi_k=[z_k],\ z_k\in C_n(Y^{(j)}\cup Y^{(j+1)}\cup\cdots\cup Y^{(j+l)}).$$

We may suppose that z_k is decomposed as follows:

$$z_k=c_k^{(j)}+c_k^{(j+1)}+\cdots+c_k^{(j+l)},$$

where $c_k^{(m)}\in C_n(Y^{(m)})$ $(m=j, j+1, \cdots, j+l)$. Then we have

$$[c_k^{(j+l)}]\in H_n(Y^{(j+l)},\ V^{(j+l)})$$
$$[c_k^{(j)}]\in H_n(Y^{(j)},\ V^{(j+1)}).$$

Since $V^{(j+l)}$, $Y^{(j+l)}$, $V^{(j+1)}$ and $Y^{(j)}$ are $(n-1)$-connected, the homology classes $[c_k^{(j+l)}]$ and $[c_k^{(j)}]$ are represented by imbeddings

$$h\colon (D^n, S^{n-1})\longrightarrow(Y^{(j+l)},\ V^{(j+l)}),$$
$$h'\colon (D^n, S^{n-1})\longrightarrow(Y^{(j)},\ V^{(j+1)})$$

respectively. We may assume that

$$t^l(h'(D^n))\cap h(D^n)=0.$$

Let $\bar{h}$: $(D^n\times D^{n+1}, S^{n-1}\times D^{n+1})\longrightarrow(M^{2n+1}, V)$ be an imbedding such that $\bar{h}|(D^n\times\{0\}, S^{n-1}\times\{0\})=p\circ h$ and $\bar{h}(D^n\times D^{n+1})\cap V=\bar{h}(S^{n-1}\times D^{n+1})$, and let

$$V_1=(V-\bar{h}(\partial D^n\times D^{n+1}))\cup\bar{h}(D^n\times\partial D^{n+1})$$

be the manifold obtained from V by the surgery $\chi(\bar{h}|(S^{n-1}\times D^{n+1}))$. Then V_1 is an $(n-1)$-connected Seifert surface for K. As is easily verified, we have

$$z_k\in C_n(Y_1^{(j)}\cup Y_1^{(j+1)}\cup\cdots\cup Y_1^{(j+l-1)}),$$

where $Y_1^{(m)}$ denotes the copy of the exterior Y_1 of V_1 obtained from $Y^{(m)}$ by the lift of the surgery. Thus the length of ξ_k with respect to V_1 is less than l.

Furthermore, we can choose the imbedding h above so that

$$p\circ h(D^n)\cap p(|z_i|)=0 \qquad i=1, 2, \cdots, k-1, k+1, \cdots, b,$$

where z_i is a singular n-cycle such that $[z_i]=\xi_i$ and $|z_i|$ denotes the support of z_i. Then the length of ξ_i with respect to V_1 is not greater than that of ξ_i with respect to V for $i=1,2,\cdots,k-1,k+1,\cdots,b$.

Therefore, by applying the above method successively, we obtain an $(n-1)$-connected Seifert surface $\bar{V}_0$ for K such that the length $l(\xi_i)$ of ξ_i $(i=1,2,\cdots,b)$ with respect to $\bar{V}_0$ are all zero and ξ_i is respresented by an imbedding

$$\varphi_i\colon S^n \longrightarrow \bar{Y}_0^{(i')},$$

where $\bar{Y}_0$ denote the exterior of $\bar{V}_0$ and $\bar{Y}_0^{(j)}$ $(j\in \boldsymbol{Z})$ are copies of $\bar{Y}_0$ with $\tilde{X}=\bigcup_{j\in \boldsymbol{Z}} \bar{Y}_0^{(j)}$.

We remark that the result of this step can be also proved by separating $p(|z_i|)$ from the spine of the Seifert surface V.

(*Step* 2′). Let $\bar{V}_0$ be a Seifert surface for K as in Step 1′. Suppose that the length $l(\xi_i)$ of ξ_i with respect to $\bar{V}_0$ is zero for $i=1,2,\cdots,k-1$ and $l(\zeta_k)=l'\geqq 1$. Thus there exist singular $(n+1)$-cycles z'_i $(i=1,2,\cdots,k)$ such that

$$\begin{aligned} &\zeta_i=[z'_i] && i=1,2,\cdots,k,\\ &z'_i\in C_{n+1}(\bar{Y}_0^{(i'')}) && i=1,2,\cdots,k-1,\\ &z'_k\in C_{n+1}(\bar{Y}_0^{(j)}\cup \bar{Y}_0^{(j+1)}\cup\cdots\cup \bar{Y}_0^{(j+l')}) && \end{aligned}$$

(Fig. 4). Since $\bar{Y}_0$ is $(n-1)$-connected, z'_i is spherical. Thus, making use of Irwin's imbedding theorem ([5]), there exist PL imbeddings

$$\begin{aligned} &\phi_i\colon S^{n+1}\longrightarrow \bar{Y}_0^{(i'')} && i=1,2,\cdots,k-1,\\ &\phi_k\colon S^{n+1}\longrightarrow \bar{Y}_0^{(j)}\cup \bar{Y}_0^{(i+1)}\cup\cdots\cup \bar{Y}_0^{(j+l')} && \end{aligned}$$

such that $\phi_i(S^{n+1})$ is homologous to z'_i for $i=1,2,\cdots,k$.

Since $I(\xi_k,\zeta_k)=1$, it holds that

$$\varphi_k(S^n)\cap\phi_k(S^{n+1})\neq\emptyset,$$

thus, we have $j\leqq k'\leqq j+l'$.

In the case of $k'<j+l'$, we modify the Seifert surface $\bar{V}_0$ as follows. Since $I(\xi_k,\zeta_i)=0$ for $i=1,2,\cdots,k-1$ and $I(t^{j+l'-k'}\xi_k,\zeta_k)=0$ by the assumption of Theorem 3, (II), making use of the PL Whitney trick, we may assume that

$$p\circ\phi_i(S^{n+1})\subset \bar{Y}_0-p\circ\varphi_k(S^n) \qquad i=1,2,\cdots,k-1,$$

$$\phi_k(S^{n+1}) \cap t^{j+l'-k'}(\varphi_k(S^n)) = \emptyset.$$

Let $N(\varphi_k)$ denote a tubular neighborhood of $\varphi_k(S^n)$ in $\bar{Y}_0^{(k')}$ and let $\bar{V}_0^{(k'+1)} \# \partial N(\varphi_n)$ be the connected sum of $\bar{V}_0^{(k'+1)}$ and the boundary of $N(\varphi_k)$ in $\bar{Y}_0^{(k')}$ (Fig. 4).

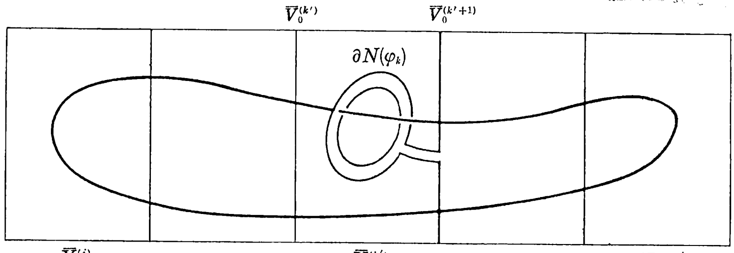

Fig. 4.

Let $\bar{V}_1 = p(\bar{V}_0^{(k'+1)} \# \partial N(\varphi_k))$. Then $\bar{V}_1$ is an $(n-1)$-connected Seifert surface for K, and as is easily verified, it holds that length $l(\xi_i)$ of ξ_i $(i=1, 2, \cdots, b)$ and $l(\zeta_i)$ of ζ_i $(i=1, 2, \cdots, k-1)$ with respect to $\bar{V}_1$ are zero and that

$$\begin{aligned} &\varphi_i(S^n) \subset \bar{Y}_1^{(i')} \qquad (i=1, 2, \cdots, k-1, k+1, \cdots, b), \\ &\varphi_k(S^n) \subset \bar{Y}_1^{(k'+1)}, \\ &\phi_k(S^{n+1}) \subset \bar{Y}_1^{(j)} \cup \bar{Y}_1^{(j+1)} \cup \cdots \cup \bar{Y}_1^{(j+l')}, \end{aligned}$$

where $\bar{Y}_1^{(m)}$ denotes the copy of the exterior $\bar{Y}_1$ of $\bar{V}_1$ obtained from $Y_0^{(m)}$ by the lift of the surgery. Therefore, by applying the above method to ξ_k successively, we obtain an $(n-1)$-connected Seifert surface $\bar{V}$ for K such that

$$\begin{aligned} &\varphi_k(S^n) \subset \bar{Y}^{(j+l')}, \\ &\phi_k(S^{n+1}) \subset \bar{Y}^{(j)} \cup \bar{Y}^{(j+1)} \cup \cdots \cup \bar{Y}^{(j+l')} \end{aligned}$$

and that the length $l(\xi_i)$ of ξ_i $(i=1, 2, \cdots, b)$ and $l(\zeta_i)$ of ζ_i $(i=1, 2, \cdots, k-1)$ with respect to $\bar{V}$ are zero, where $\bar{Y}^{(m)}$ is a copy of the exterior $\bar{Y}$ of $\bar{V}$ and $\tilde{X} = \bigcup_{m \in Z} \bar{Y}^{(m)}$.

We may suppose that z'_k is the spherical cycle $\phi_k(S^{n+1})$ and z'_k is decomposed as follows:

$$z'_k = \hat{c}_k^{(j)} + \hat{c}_k^{(j+1)} + \cdots + \hat{c}_k^{(j+l')},$$

where $\hat{c}_k^{(m)} \in C_{n+1}(\bar{Y}^{(m)})$. Then we have

$$[\hat{c}_k^{(j+l')}] \in H_{n+1}(\bar{Y}^{(j+l')}, \bar{V}^{(j+l')}),$$
$$[\hat{c}_k^{(j)}] \in H_{n+1}(\bar{Y}^{(j)}, \bar{V}^{(j+1)}).$$

LEMMA 4. *There exists a PL imbedding*

$$f\colon (D^{n+1}, S^n) \longrightarrow (\bar{Y}^{(j+l')}, \bar{V}^{(j+l')})$$

with respect to a C^1 triangulation of $(\bar{Y}^{(j+l')}, \bar{V}^{(j+l')})$ which satisfies the following conditions:

(i) $f_*([D^{n+1}, S^n]) = [\hat{c}_k^{(j+l')}]$.

(ii) *$f(D^{n+1})$ and $\varphi_k(S^n)$ intersect transversally at one point.*

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \to \pi_{n+1}(\bar{Y}^{(j+l')}) & \to & \pi_{n+1}(\bar{Y}^{(j+l')}, \bar{V}^{(j+l')}) & \to & \pi_n(\bar{V}^{(j+l')}) & \to & \pi_n(\bar{Y}^{(j+l')}) \to \cdots \\
\downarrow h_{n+1}^{(Y)} & & \downarrow h_{n+1}^{(Y,V)} & & \downarrow h_n^{(V)} & & \downarrow h_n^{(Y)} \\
\cdots \to H_{n+1}(\bar{Y}^{(j+l')}) & \to & H_{n+1}(\bar{Y}^{(j+l')}, \bar{V}^{(j+l')}) & \to & H_n(\bar{V}^{(j+l')}) & \to & H_n(\bar{Y}^{(j+l')}) \to \cdots
\end{array}$$

where $h_{n+1}^{(Y)}$ etc. are the Hurewicz homomorphisms. Since $V^{(j+l')}$ and $Y^{(j+l')}$ are $(n-1)$-connected, the homomorphisms $h_n^{(V)}$ and $h_n^{(Y)}$ are bijective, and $h_{n+1}^{(Y)}$ is surjective. This implies that $h_{n+1}^{(Y,V)}$ is surjective.

Thus a continuous map

$$f''\colon (D^{n+1}, S^n) \longrightarrow (\bar{Y}^{(j+l')}, \bar{V}^{(j+l')})$$

representing $[\hat{c}_k^{(j+l')}]$ exists. Since $Y^{(j+l')}$ and $V^{(j+l')}$ are $(n-1)$-connected and $(2n+1)-(n+1)=n\geqq 3$, according to the imbedding theorem of Irwin [5], there exists a *PL* imbedding $f'\colon (D^{n+1}, S^n) \longrightarrow (\bar{Y}^{(j+l')}, \bar{V}^{(j+l')})$ homotopic to f''.

Since $I(\xi_k, \zeta_k)=1$, making use of the *PL* Whitney trick, we obtain a *PL* imbedding f satisfying the conditions of this lemma by a *PL* isotopy movement of f'. Thus this lemma is proved.

We remark that f in Lemma 4 can be taken as a smooth imbedding if $n\geqq 4$, according to the imbedding theorem of Heafliger [2]. See also Levine [10].

Let N' be a regular neighborhood of $p(f(D^{n+1})\cup\varphi_k(S^n))$ in $\bar{Y}$ and let

$$\hat{V}_0=\bar{V}_+\cup\partial N'-\mathrm{Int}(\bar{V}_+\cap\partial N')$$

and let

$$Y'=(\bar{Y}-N')\cup\hat{V}_0,$$

where $\partial\bar{Y}=\bar{V}_+\cup\bar{V}_-$ as in §2 (Fig. 5).

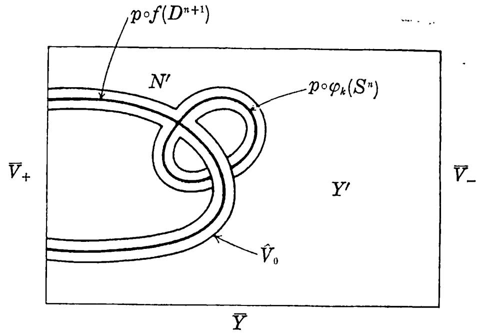

Fig. 5.

Then the following lemma holds.

LEMMA 5. (i) *$\hat{V}_0$ is a locally flat $(n-1)$-connected $2n$-dimensional PL submanifold of $\bar{Y}$ such that $\partial\hat{V}_0=K$ and $\hat{V}_0\cap\bar{V}_+$ is smooth*

(ii) *There exists a singular $(n+1)$-cycle $\hat{z}'\in C_{n+1}(Y',\bar{V}_-)$ such that*

$$\iota'_*([\hat{z}'])=p_*([\hat{c}_k^{(j)}]),$$

where ι': $(Y',\bar{V}_-)\longrightarrow(Y,\bar{V}_-)$ is the inclusion map.

(iii) *There exist singular $(n+1)$-cycles $\hat{z}'_i\in C_{n+1}(Y')$ $(i=1,2,\cdots,k-1)$ such that*

$$\iota_*([\hat{z}'_i])=p_*(\zeta_i)\qquad i=1,2,\cdots\text{‘}k-1,$$

where ι: $Y'\longrightarrow Y$ is the inclusion map.

(iv) *There exist singular n-cycles $\hat{z}_i\in C_n(Y')$ $(i=1,2,\cdots,k-1,k+1,\cdots,b)$ such that*

$$\iota_*([\hat{z}_i])=p_*(\xi_i)\qquad i=1,2,\cdots,k-1,k+1,\cdots,b.$$

(v) *$\hat{V}_0$ is smoothable. That, is, there exists a smooth submanifold $\hat{V}$ which is PL isotopic to $\hat{V}_0$ relative $\hat{V}_0 \cap \bar{V}_+$.*

PROOF. It is obvious that $\hat{V}_0$ is locally flat, since N' is the regular neighborhood. Let us consider

$$W=(\bar{V}_+\times I)\cup N',$$

where we identify $\bar{V}_+\times\{1\}$ with $\bar{V}_+$. It is easy to see that W is $(n-1)$-connected and

$$H_q(W, \bar{V}_+\times\{0\})\cong\begin{cases} Z & q=n, n+1, \\ 0 & q\neq n, n+1 \end{cases}$$

and that $H_{n+1}(W, \bar{V}_+\times\{0\})\cong H_{n+1}(N', \bar{V}_+)\cong Z$ is generated by $p_*[\hat{c}_k^{(j+l')}]=(p\circ f)_*[D^{n+1}, S^n]$. By the Poincaré-Lefschetz duality theorem, we have

$$H^q(W, \bar{V}_+\times\{0\})=H_{2n+1-q}(W, \hat{V}_0).$$

Thus the homology groups of $(W, \hat{V}_0)$ are given by

$$H_q(W, \hat{V}_0)=\begin{cases} Z & q=n, n+1, \\ 0 & q\neq n, n+1, \end{cases}$$

and, since $(p\circ f)(D^{n+1})\cap(p\circ\varphi_k)(S^n)$ consists of one point, $H_n(W, \hat{V}_0)\cong Z$ is generated by $[p\circ\varphi_k(S^n)]$.

Consider the homology exact sequence of $(W, \hat{V}_0)$:

$$\cdots\longrightarrow H_n(W)\longrightarrow H_n(W, \hat{V}_0)\longrightarrow H_{n-1}(\hat{V}_0)\longrightarrow H_{n-1}(W)\longrightarrow\cdots.$$

Since the generator $[p\circ\varphi_k(S^n)]$ of $H_n(W, \hat{V}_0)\cong Z$ is contained in the image of $H_n(W)$ in this exact sequence, it holds that $H_{n-1}(\hat{V}_0)=0$. Thus $\hat{V}_0$ is $(n-1)$-connected, since $\hat{V}_0$ is obviously $(n-2)$-connected. This proves (i).

Consider the homology exact sequence of a triple $(\bar{Y}, Y', \bar{V}_-)$:

$$\cdots\longrightarrow H_{n+1}(Y', \bar{V}_-)\xrightarrow{\iota'_*} H_{n+1}(\bar{Y}, \bar{V}_-)\xrightarrow{\iota''_*} H_{n+1}(\bar{Y}, Y')\xrightarrow{\partial}\cdots,$$

where ι': $(Y', \bar{V}_-)\longrightarrow(\bar{Y}, \bar{V}_-)$ and ι'': $(\bar{Y}, \bar{V}_-)\longrightarrow(\bar{Y}, Y')$ are inclusion maps. As is easily verified, the homology group $H_{n+1}(\bar{Y}, Y')\cong H_{n+1}(N', N'\cap\hat{V}_0)$ is isomorphic to Z and is generated by a fibre of the fibre bundle $p(N(\varphi_k))\longrightarrow p\circ\varphi_k(S^n)$ defined as a tubular neighborhood of $p\circ\varphi_k(S^n)$.

On the other hand, by the assumption of Theorem 3, (II), we have

$$I(\xi_k, t^{l'}\zeta_k)=0.$$

Thus, for $p_*[\hat{c}_k^{(j)}]\in H_{n+1}(\bar{Y}, \bar{V}_-)$, it holds that

$$\iota''_*(p_*[\hat{c}_k^{(j)}])=0.$$

This implies that $p_*([\hat{c}_k^{(j)}])$ is contained in the image of ι'_*: $H_{n+1}(Y', \bar{V}_-) \longrightarrow H_{n+1}(\bar{Y}, \bar{V}_-)$. Therefore (ii) is proved.

Similarly, by the homology exact sequence of $(\bar{Y}, Y')$ and the fact that

$$I(\xi_k, \zeta_i)=0 \qquad i=1, 2, \cdots, k-1,$$

it follows that $p_*(\zeta_i)$ is contained in the image of ι_*: $H_{n+1}(Y') \longrightarrow H_{n+1}(\bar{Y})$. This proves (iii).

Since $I(\xi_i, \zeta_k)=0$ $(i=1, 2, \cdots, k-1, k+1, \cdots, b)$, we can choose the imbeddings φ_i: $S^n \longrightarrow Y_1^{(i')}$ representing ξ_i $(i=1, 2, \cdots, k-1, k+1, \cdots, b)$ so that

$$p\circ\varphi_i(S^n)\subset Y'.$$

This proves (iv).

By the smoothing of a locally flat PL submanifold $\hat{V}_0$ in $\tilde{X}$ (Hirsch-Mazur [3, Theorem 7.4]), we obtain $\hat{V}$ as in (v). Thus this lemma is proved.

The submanifold $\hat{V}$ in Lemma 5 is an $(n-1)$-connected Seifert surface for K. By Lemma 5, as is easily verified, it holds that the length $l(\xi_i)$ of ξ_i $(i=1, 2, \cdots, b)$ and $l(\zeta_i)$ of ζ_i $(i=1, 2, \cdots, k-1)$ with respect to $\hat{V}$ are zero and that

$$z'_k\in C_{n+1}(\hat{Y}^{(j)}\cup \hat{Y}^{(j+1)}\cup \cdots \cup \hat{Y}^{(j+l'-1)}),$$

that is, the length of ζ_k with respect to $\hat{V}$ is less than l'.

(*Step* 3′). By applying Step 1′ and Step 2′ successively, we obtain an $(n-1)$-connected Seifert surface V for K such that the length $l(\xi_i)$ of ξ_i and $l(\zeta_i)$ of ζ_i with respect to V are zero $(i=1, 2, \cdots, b)$.

Let us consider homology classes $p_*(\xi_1), p_*(\xi_2), \cdots, p_*(\xi_b)\in H_n(X)$ and $p_*(\zeta_1), p_*(\zeta_2), \cdots, p_*(\zeta_b)\in H_{n+1}(X)$. Since $l(\xi_i)=l(\zeta_i)=0$ $(i=1, 2, \cdots, b)$, as is easily verified, it holds by the assumption II, (b) that

$$I(p_*(\xi_i), p_*(\zeta_j))=\delta_{ij}.$$

Thus, for the inclusion map $\bar{\iota}$: $X \longrightarrow M^{2n+1}$, the homology classes

$\bar{\iota}_* \circ p_*(\zeta_1), \bar{\iota}_* \circ p_*(\zeta_2), \cdots, \bar{\iota}_* \circ p_*(\zeta_b)$ form a basis of $H_{n+1}(M^{2n+1})$.

V has the spine S which is a bouquet of n-spheres. The inclusion map $S \longrightarrow V$ is a homotopy equivalence. The intersection numbers of $p_*(\zeta_i)$ and elements of $H_n(S)$ are always zero. Therefore S is homologous to zero and, thus, inessential in M^{2n+1}. Since the codimension of S in M^{2n+1} is $n+1 \geqq 3$ and M^{2n+1} is 2-connected, by the PL engulfing theorem [4] and the smoothing of a PL $(2n+1)$-disk in M^{2n+1} ([3]), there exists a $(2n+1)$-disk D^{2n+1} imbedded in M^{2n} such that

$$D^{2n+1} \supset S,$$

and, thus, we may suppose that

$$D^{2n+1} \supset V \supset K.$$

Therefore Theorem 3, (II) is completely proved.

§ 8. Genus 1 knots in $S^n \times S^{n+1}$.

In this section we deal with knots in $S^n \times S^{n+1}$ and prove the localness and the unknottedness by applying fundamental theorems (Theorems 3, 4). In the following we assume that $n \geqq 3$.

Let D^{2n} be an imbedded $2n$-disk in $S^n \times S^n$ and let $V_0 = S^n \times S^n - \mathrm{Int}\, D^{2n}$. Obviously V_0 is $(n-1)$-connected and $H_n(V_0) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$.

Let f_0: $V_0 \longrightarrow S^n \times S^{n+1}$ be an imbedding. Then the image $V = f_0(V_0)$ is called a *genus* 1 *Seifert surface* in $S^n \times S^{n+1}$ and $K = f_0(\partial V_0)$ is called a *genus* 1 *knot* in $S^n \times S^{n+1}$. The map f_0 is called a *defining map* of V.

Two genus 1 Seifert surfaces V and V' are equivalent if there exists a diffeomorphism h: $S^n \times S^{n+1} \longrightarrow S^n \times S^{n+1}$ such that $h(V) = V'$.

Let $\{\mu_1, \mu_2\}$ denote the set of generators of $H_n(V_0)$ represented by $S^n \times \{*\}$, $\{*'\} \times S^n$ $(*, *' \in S^n)$. $(f_0)_*(\mu_1)$ and $(f_0)_*(\mu_2)$ generate a subgroup $G(V)$ of $H_n(S^n \times S^{n+1}) \cong \boldsymbol{Z}$. Let m denote the non-negative integer such that $H_n(S^n \times S^{n+1})/G(V) \cong \boldsymbol{Z}/m\boldsymbol{Z}$. m is called the *degree* of V.

LEMMA 6. *If* $m=0$, *there exists a* $(2n+1)$-*disk* D^{2n+1} *imbedded in* $S^n \times S^{n+1}$ *such that* $D^{2n+1} \supset V$.

PROOF. If $m=0$, then $G(V)=0$. This implies that V is inessential in $S^n \times S^{n+1}$. Thus, by the PL engulfing theorem [4] and the smoothing of a PL $(2n+1)$-disk in $S^n \times S^{n+1}$ ([3]), the existence of D^{2n+1} such that $D^{2n+1} \supset V$ follows.

Let V be a genus 1 Seifert surface of degree m. Then there exist integers c_1, c_2, d_1, d_2 such that $(c_1, c_2)=1$, $(f_0)_*(c_1\mu_1+c_2\mu_2)=m[S^n]$ and that $c_1d_2-c_2d_1=1$, $(f_0)_*(d_1\mu_1+d_2\mu_2)=0$, where $[S^n]$ is a generator of $H_n(S^n\times S^{n+1})$. In general, a set of generators $\bar{\alpha}$, $\bar{\beta}$ of $H_n(V)$ is called *canonical generators* if $\iota_*(\bar{\alpha})=m[S^n]$, $\iota_*(\bar{\beta})=0$, where ι: $V\longrightarrow S^n\times S^{n+1}$ is the inclusion map. The above argument shows that a genus 1 Seifert surface admits always canonical generators.

Consider the fundamental exact sequence for V (§ 4):

$$0\longrightarrow H_{n+1}(Y)\xrightarrow{\iota_*}H_{n+1}(S^n\times S^{n+1})\xrightarrow{\theta}H_n(V)\xrightarrow{\bar{\partial}}H_n(Y)$$
$$\xrightarrow{\iota_*}H_n(S^n\times S^{n+1})\longrightarrow 0,$$

where $H_n(V)\cong Z\oplus Z$. Then the following lemma holds.

LEMMA 7. *Let V be a genus 1 Seifert surface of degree m and let $\bar{\alpha}$, $\bar{\beta}$ be canonical generators.*

If $m=0$, then it holds that

$$H_{n+1}(Y)\cong Z,\ H_n(Y)\cong Z\oplus Z\oplus Z.$$

If $m\neq 0$, then it holds that

$$H_{n+1}(Y)=0,\quad H_n(Y)\cong Z\oplus Z\oplus Z_m,$$

and that $\theta(H_{n+1}(S^n\times S^{n+1}))$ is the subgroup of $H_n(V)$ generated by $m\bar{\beta}$.

PROOF. The homomorphism θ: $H_{n+1}(S^n\times S^{n+1})\longrightarrow H_n(V)$ is the composition of ι'_*: $H_{n+1}(S^n\times S^{n+1})\longrightarrow H_{n+1}(S^n\times S^{n+1}, Y)$, ε: $H_{n+1}(S^n\times S^{n+1}, Y)\longrightarrow H_{n+1}(N(V), \partial N(V)$, ∂: $H_{n+1}(N(V), \partial N(V))\longrightarrow H_n(\partial N(V))$ and τ'_*: $\partial(H_{n+1}(N(V), \partial N(V)))\longrightarrow H_n(N(V))$ (see § 2). For a generator $[S^{n+1}]$ of $H_{n+1}(S^n\times S^{n+1})\cong Z$, the intersection numbers in $(N(V), \partial N(V))$ should be

$$I(\varepsilon\circ\iota'_*[S^{n+1}], \bar{\alpha})=\pm m, I(\varepsilon\circ\iota'_*[S^{n+1}], \bar{\beta})=0.$$

This shows that $\theta([S^{n+1}])=\tau'_*\circ\partial\circ\varepsilon\circ\iota'_*([S^{n+1}])=\pm m\bar{\beta}$. Thus this lemma follows from the fundamental exact sequence above.

Let us denote $\alpha=\bar{\partial}(\bar{\alpha})$, $\beta=\bar{\partial}(\bar{\beta})$ and let γ be an element of $H_n(Y)$ such that $\iota_*(\gamma)$ generates $H_n(S^n\times S^{n+1})$. Then, by Lemma 7, we have the following lemma.

LEMMA 8. $\{\alpha, \beta, \gamma\}$ *is a set of generators of $H_n(Y)$. In case $m\neq 0$, β is a generator of the torsion part Z_m of $H_n(Y)$.*

A genus 1 Seifert surface V is said to be *flat* if it has cannonical generators $\bar{\alpha}$, $\bar{\beta}$ such that $\bar{\alpha}$, $\bar{\beta}$ can be represented by n-spheres imbedded in V with trivial normal bundle.

Now we construct a flat genus 1 Seifert surface with degree m in $S^n\times S^{n+1}$. We specify orientations of S^n and S^{n+1}, and give $S^n\times S^{n+1}$ the product orientation.

Let $S_0^n=\{(x_1, x_2, \cdots, x_{n+1})\in S^{n+1}; x_{n+1}=0\}$ be the equator of S^{n+1} and let $u_0, u_1, \cdots, u_s, v_1, v_2, \cdots, v_t$ be $s+t+1$ points of S_0^n, where $s-t+1=m$. We give $S^n\times\{u_i\}$ $(i=0, 1, \cdots, s)$ (resp. $S^n\times\{v_i\}$ $(i=1, 2, \cdots, t)$) orientations consisting with (resp. reversing) the orientation of S^n. Let S_1^n be the connected sum of $S^n\times\{u_i\}$ $(i=0, 1, \cdots, s)$ and $S^n\times\{v_i\}$ $(i=1, 2, \cdots, t)$ in $S^n\times S_0^n$:

$$S_1^n=(S^n\times\{u_0\})\#(S^n\times\{u_1\})\#\cdots\#(S^n\times\{u_s\})$$
$$\#(S^n\times\{v_1\})\#(S^n\times\{v_2\})\#\cdots\#(S^n\times\{v_t\}).$$

We may suppose that, for a point $z_0\in S^n$, the following holds:

$$S_1^n\cap(\{z_0\}\times S_0^n)=(\bigcup_{i=0}^{s}(z_0, u_i))\cup(\bigcup_{i=0}^{t}(z_0, v_i)).$$

Let S_2^n be a submanifold of $\{z_0\}\times S^{n+1}$ deffeomorphic to the n-sphere having the following properties:

(i) $(z_0, u_0)\in S_2^n$.

(ii) Let $\hat{D}$ and $\hat{D}'$ be subsets of $\{z_0\}\times S^{n+1}$ such that $\hat{D}\cup\hat{D}'=\{z_0\}\times S^{n+1}$ and $\hat{D}\cap\hat{D}'=S_2^n$. Then we have

$$(z_0, z_1), (z_0, z_2), \cdots, (z_0, z_{s'}), (z_0, v_1), (z_0, v_2), \cdots, (z_0, v_t)\in\operatorname{Int}\hat{D},$$
$$(z_0, u_{s'+1}), (z_0, u_{s'+2}), \cdots, (z_0, u_s), (z_0, v_{t'+1}), (z_0, v_{t'+2}), \cdots, (z_0, v_t)\in\operatorname{Int}\hat{D}',$$

where $1\leqq s'\leqq s$, $1\leqq t'\leqq t$ and $s'-t'=q$.

We give $\hat{D}$ the orientation induced from S^{n+1}, and give S_2^n the boundary orientation of $\hat{D}$.

Let $N(S_1^n)$ and $N(S_2^n)$ be tubular neighborhoods of S_1^n and S_2^n in $S^n\times S^{n+1}$ respectively. Then we have

$$N(S_1^n)=S_1^n\times D_1^{n+1},\ N(S_2^n)=S_2^n\times D_2^{n+1}.$$

We give D_1^{n+1} and D_2^{n+1} the orientations so that the product orientations of $S_1^n\times D_1^{n+1}$ and $S_2^n\times D_2^{n+1}$ consistent with the orientation of $S^n\times S^{n+1}$.

Let

$$p_1'\colon N(S_1^n)\longrightarrow D_1^{n+1},\ p_2'\colon N(S_2^n)\longrightarrow D_2^{n+1}$$

be projections onto fibres such that $p_1'(N(S_1^n)\cap(S^n\times S_0^n))$ is an n-disk imbedded in D^{n+1} and $p_2'(N(S_2^n)\cap(\{z_0\}\times S^{n+1}))$ is a 1-disk imbedded in D^{n+1}.

We define imbeddings

$$g_{m,l}\colon (S^n\times D^n, \partial(S^n\times D^n))\longrightarrow(N(S_1^n), \partial N(S_1^n)),$$
$$h_{q,r}\colon (S^n\times D^n, \partial(S^n\times D^n))\longrightarrow(N(S_2^n), \partial N(S_2^n)),$$

so that they satisfy the following conditions, where $m,\ q\in \boldsymbol{Z}$ and we understand that

$$\begin{aligned}&l,\ r\in \boldsymbol{Z} &&\text{if } n=3,\ 7,\\ &l,\ r\in 2\boldsymbol{Z} &&\text{if } n \text{ is odd and } \neq 3,\ 7,\\ &l,\ r=0 &&\text{if } n \text{ is even.}\end{aligned}$$

(i) $g_{m,l}(S^n\times\{0\})=S_1^n$, $h_{q,r}(S^n\times\{0\})=S_2^n$. And $g_{m,l}|(S^n\times\{0\})$, $h_{q,r}|(S^n\times\{0\})$ are orientation-preserving.

(ii) The degree of the map $p_1'\circ g_{m,l}|(S^n\times\{y'\})\colon S^n\times\{y'\}\longrightarrow\partial D_1^{n+1}$ is l for a point $y'\in\partial D^n$ and the degree of the map $p_2'\circ h_{q,r}|(S^n\times\{y''\})\colon S^n\times\{y''\}\longrightarrow\partial D_2^{n+1}$ is r for a point $y''\in\partial D^n$ with respect to the orientations of S^n and the boundary orientations of ∂D_1^{n+1} and ∂D_2^{n+1}.

(iii) $g_{m,l}(S^n\times D^n)\cap h_{q,r}(S^n\times D^n)$ is diffeomorphic to the $2n$-disk.

The existence of $g_{m,0}$ and $h_{q,0}$ is obvious. $g_{m,l}$ and $h_{q,r}$ can be obtained by twisting the trivialization of $N(S_1^n)$ and $N(S_2^n)$. The twistings are expressed by elements of $\pi_n(SO(n+1))$. As is well known, the image of the homomorphism

$$\pi_n(SO(n+1))\longrightarrow\pi_n(SO(n+1)/SO(n))=\pi_n(S^n)\cong\boldsymbol{Z}$$

induced by the projection $SO(n+1)\longrightarrow SO(n+1)/SO(n)=S^n$ is $\boldsymbol{Z}$ if $n=3,\ 7$; $2\boldsymbol{Z}$ if n is odd and $\neq 3,\ 7$; zero if n is even (Milnor [13]). Thus $g_{m,l}$ and $h_{q,r}$ as above exist.

Let

$$V_{m,q,l,r}=g_{m,l}(S^n\times D^n)\cup h_{q,r}(S^n\times D^n)$$

be the plumbing of $g_{m,l}(S^n\times D^n)$ and $h_{q,r}(S^n\times D^n)$ (Fig. 6). Then $V_{m,q,l,r}$ is a flat Seifert surface of degree m, where the defining map $f_0\colon V_0\longrightarrow V_{m,q,l,r}$ is taken so as $f_0(S^n\times\{*\})=S_1^n$, $f_0(\{*'\}\times S^n)=S_2^n$. $V_{m,q,l,r}$ is called a

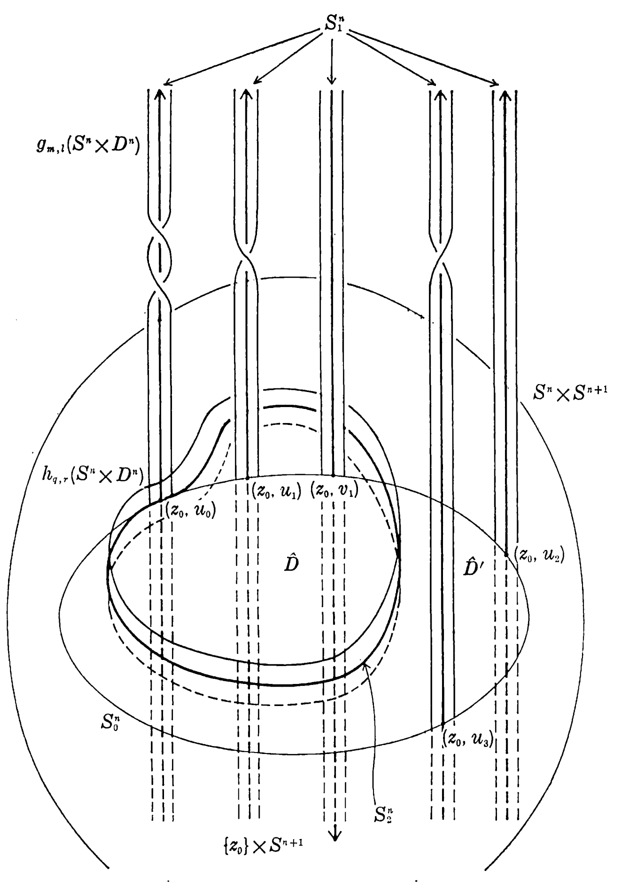

Fig. 6.

genus 1 *Seifert surface of type* (m, q, l, r). The integers l and r are said to be *twisting numbers* around S_1^n and S_2^n respectively, where l and r are integers if $n=3, 7$; even integers if n is odd and $\neq 3, 7$; and zero if n is even as above.

The boundary $\partial V_{m,q,l,r}$ of $V_{m,q,l,r}$ is called a *genus* 1 *knot of type* (m, q, l, r) in $S^n \times S^{n+1}$ and is denoted by $K_{m,q,l,r}$. By Proposition 2, (b), $K_{m,q,l,r}$ is simple. By the same argument as in the proof of Lemma 2 (§ 5), the following proposition holds:

PROPOSITION 3. *Genus* 1 *knots of type* (m, q, l, r) *are inessential in* $S^n \times S^{n+1}$.

Let Y be the exterior of $V_{m,q,l,r}$ in $S^n\times S^{n+1}$ and let $\iota_\pm$: $V_{m,q,l,r}\longrightarrow Y$ be imbeddings as in § 2, where we choose $\iota_\pm$ so that $\iota_+(z_0, u_0)\in \hat{D}$.

Let $\bar{\alpha}$, $\bar{\beta}$ be homology classes of $H_n(V_{m,q,l,r})$ represented by S_1^n and S_2^n respectively. Then $\bar{\alpha}$, $\bar{\beta}$ are canonical generators. Since $\bar{\partial}(\xi)=(\iota_+)_*\xi-(\iota_-)_*\xi$ for $\xi\in H_n(V_{m,q,l,r})$, as is easily verified, the homology classes $\alpha=\bar{\partial}(\bar{\alpha})$ and $\beta=\bar{\partial}(\bar{\beta})$ of $H_n(Y)$ are represented by $(-1)^{n+1}(\{z''\}\times\partial D_2^{n+1})$ and $-(\{z'\}\times\partial D_1^{n+1})$, where $z'\in S_1^n$, $z''\in S_2^n$ with $z', z''\notin N(S_1^n)\cap N(S_2^n)$. Furthermore, as the homology class γ of $H_n(Y)$ such that $\iota_*(\gamma)$ generates $H_n(S^n\times S^{n+1})$, we can choose the homology class represented by $S^n\times\{x_0\}$ having the orientation consistent with that of S^n for $x_0\in \hat{D}'\cap Y$. Then the following lemma holds:

LEMMA 9. (i) $(\iota_+)_*(\bar{\alpha})=(q+1)\alpha-l\beta+m\gamma$,
$(\iota_+)_*(\bar{\beta})=(-1)^{n+1}r\alpha-q\beta$

(ii) $(\iota_-)_*(\bar{\alpha})=q\alpha-l\beta+m\gamma$,
$(\iota_-)_*(\bar{\beta})=(-1)^{n+1}r\alpha-(q+1)\beta$.

PROOF. For the inclusion map ι: $Y\longrightarrow S^n\times S^{n+1}$, it is obvious that $\iota_*(\alpha)=\iota_*(\beta)=0$. Thus the coefficients of γ in $(\iota_+)_*\bar{\alpha}$ and $(\iota_-)_*\bar{\alpha}$ should be m, and that of γ in $(\iota_+)_*(\beta)$ and $(\iota_-)_*(\bar{\beta})$ should be zero. The homology class of $H_n(Y)$ represented by $S^n\times\{x_0'\}$ having the orientation consistent with that of S^n for $x_0'\in \dot{D}\cap Y$ is $\alpha+\gamma$. Thus, in the case of $V_{m,q,0,0}$, we have

$$(\iota_+)_*(\bar{\alpha})=(q+1)\alpha+m\gamma.$$

Furthermore, since $[\{z'\}\times\partial D_1^{n+1}]=-\beta$ the twisting around S_1^n changes the coefficient of β in $(\iota_+)_*(\bar{\alpha})$, and thus, $(\iota_+)_*\bar{\alpha}$ for $V_{m,q,l,r}$ is as in (i) above. Since $\alpha=\bar{\partial}(\bar{\alpha})=(\iota_+)_*(\bar{\alpha})-(\iota_-)_*(\bar{\alpha})$, $(\iota_-)_*(\bar{\alpha})$ is as in (ii) above.

In the case of $V_{m,q,0,0}$, we may consider that $\partial(\hat{D}\cap Y)$ consists of $\iota_+(S_2^n)$, $\partial D^{n+1}(z_0, u_i)$ $(i=1, 2, \cdots, s')$ and $\partial D^{n+1}(z_0, v_i)$ $(i=1, 2, \cdots, t')$, where $D^{n+1}(z_0, u_i)$ and $D^{n+1}(z_0, v_i)$ are $(n+1)$-disks in $\hat{D}$ with centers (z_0, u_i) and (z_0, v_i) having the orientations induced from $\hat{D}$ respectively. $\partial D^{n+1}(z_0, u_i)$ and $\partial D^{n+1}(z_0, v_i)$ with the boundary orientations represent homology classes $-\beta$ and β respectively. Thus we have

$$(\iota_+)_*(\bar{\beta})=-q\beta.$$

Since $[\{z''\}\times\partial D_2^{n+1}]=(-1)^{n+1}\alpha$, the twisting around S_2^n changes the coefficient of α in $(\iota_+)_*(\bar{\beta})$, and thus, $(\iota_+)_*(\bar{\beta})$ for $V_{m,q,l,r}$ is as in (i) above.

Since $\beta=\bar{\partial}(\bar{\beta})=(\iota_+)_*(\bar{\beta})-(\iota_-)_*(\bar{\beta})$, $(\iota_-)_*(\bar{\beta})$ is as in (ii) above. This completes the proof.

Now fundamental theorems (Theorems 3 and 4) enable us to determine the localness and the unknottedness of a genus 1 knot of type $(m, q, l, 0)$ as follows:

THEOROM 5. *Let $K_{m,q,l,0}$ be a genus 1 knot of type $(m, q, l, 0)$ in $S^n\times S^{n+1}$ $(n\geqq 3)$. Then the following hold:*

(i) *$K_{0,q,l,0}$ is local. And $K_{0,q,l,0}$ is unknotted if and only if $q=0$ or -1.*

(ii) *In case $m\neq 0$, $K_{m,q,l,0}$ is local if and only if m and q satisfy the following condition (*):*

(*) *Each prime factor of m divides q or $q+1$.*

Furthermore, $K_{m,q,l,0}$ $(m\neq 0)$ is unknotted if it is local.

(iii) *$K_{m,q,l,0}$ are not fibred knots.*

The knot $K_{0,q,l,0}$ is essentially a knot in S^{2n+1}. The results of (i) and (ii) reveal the contrasting property of knot theory in S^{2n+1} and knot theory in $S^n\times S^{n+1}$.

In order to prove Theorem 5, (ii), we need the following number theoretical lemma due to Y. Ihara.

LEMMA 10. *Let a, b, c and d be integers such that $ad-bc\neq 0$ and* g.c.d. $(a, b, c, d)=1$. *Then integral polynomials $F(x)$, $G(x)$ and a non-negative integer k satisfying the equation*

$$(**)\quad (ax+b)F(x)+(cx+d)G(x)=x^k$$

exist, if and only if each prime factor p of $ad-bc$ satisfies

$$(***)\quad a\equiv c\equiv 0 \bmod p \quad \textit{or} \quad b\equiv d\equiv 0 \bmod p.$$

PROOF. Let $\mathfrak{a}=(ax+b, cx+d)$ be the ideal of $\boldsymbol{Z}[x]$ generated by integral polynomials $ax+b$ and $cx+d$. By the assumption that $ad-bc\neq 0$ and g.c.d. $(a, b, c, d)=1$, each prime ideal $\mathfrak{p}$ of $\boldsymbol{Z}[x]$ containing $\mathfrak{a}$ is maximal and hence it can be written in the form $\mathfrak{p}=(p, \varphi(x))$, where p is a prime factor of $ad-bc$ and $\varphi(x)$ is an integral polynomial whose reduction mod p is irreducible in $\boldsymbol{Z}_p[x]$.

Suppose that each prime factor of $ad-bc$ satisfies (***). Then each prime ideal $\mathfrak{p}=(p, \varphi(x))$ containing $\mathfrak{a}$ can be assumed to satisfy the condition $\varphi(x)=x$. Now let

$$\mathfrak{a}=\mathfrak{q}_1\cap\mathfrak{q}_2\cap\cdots\cap\mathfrak{q}_j$$

be a primary decomposition of a and p_i be the prime ideal associated with $\mathfrak{q}_i$ $(i=1,2,\cdots,j)$. Then, since $\mathfrak{p}_i=(p_i,x)$ and each element of $\mathfrak{p}_i$ is nilpotent $\bmod \mathfrak{q}_i$, there exists a non-negative integer k_i such that

$$x^{k_i}\in\mathfrak{q}_i.$$

Thus, for $k=\max_i (k_i)$, we have

$$x^k\in\mathfrak{a}.$$

Conversely suppose that there exist integral polynomials $F(x)$ and $G(x)$ satisfying (**). Then, for each prime factor p of $ad-bc$, either the reductions of $ax+b$ and $cx+d$ $\bmod p$ have a common factor x or otherwise they are integers $\bmod p$ one of which is non-zero. Thus this lemma is proved.

PROOF OF THEOREM 5. In this proof, we write $K_{m,q,l,0}$ and $V_{m,q,l,0}$ simply by K and V respectively. Let Y be the exterior of V and let $\tilde{X}=\bigcup_{i\in Z} Y^{(i)}$ be the decomposition of the universal covering $\tilde{X}$ of $X=S^n\times S^{n+1}-K$ by copies of Y. By the argument in § 4 and Lemmas 8, 9, the knot module $A_n(K; S^n\times S^{n+1})$ is isomorphic to a Λ-module with generators $t^i\alpha$, $t^i\beta$, $t^i\gamma$ $(i\in Z)$ and relations

$$[R]\qquad \begin{aligned} q\alpha-l\beta+m\gamma&=(q+1)t\alpha-lt\beta+mt\gamma,\\ (q+1)\beta&=qt\beta, \end{aligned}$$

where we understand that α, β, γ are lifts of α, β, γ in Lemma 9 such that $\alpha,\beta,\gamma\in H_n(Y^{(0)})$.

First we consider the case of $m=0$. By Lemma 6, there exists a $(2n+1)$-disk $\hat{D}^{2n+1}$ imbedded in $S^n\times S^{n+1}$ such that $\hat{D}^{2n+1}\supset V$, and, thus, K is local.

By considering the double $\hat{D}^{2n+1}\cup D^{2n+1}=S^{2n+1}$, K is a knot in S^{2n+1} and V is a Seifert surface for K. It is obvious by the relation $[R]$ that $H_n(V)\cong Z\oplus Z$ with generators $\bar{\alpha}$ and $\bar{\beta}$ and that the knot module $A_n(K; S^{2n+1})$ is isomorphic to a Λ-module with generators $t^i\alpha$, $t^i\beta$ $(i\in Z)$ and relations

$$q\alpha-l\beta=(q+1)t\alpha-lt\beta,$$

$$(q+1)\beta = qt\beta.$$

As is easily verified, it follows from the argument in the proof of Lemma 9 that the linking numbers are given by

$$Lk((\iota_+)_\sharp(\bar{\alpha}_0), (\bar{\alpha}_0) = l,\ Lk((\iota_+)_\sharp(\bar{\alpha}_0), (\bar{\beta}_0) = q+1,$$
$$Lk((\iota_+)_\sharp(\bar{\beta}_0), (\bar{\alpha}_0) = (-1)^{n+1}q,\ Lk((\iota_+)_\sharp(\bar{\beta}_0), (\bar{\beta}_0) = 0,$$

where $\bar{\alpha}_0$, $\bar{\beta}_0$ denote cycles in V representing $\bar{\alpha}, \bar{\beta} \in H_n(V)$. Therefore the Seifert matrix A is $\begin{pmatrix} l & q+1 \\ (-1)^{n+1}q & 0 \end{pmatrix}$ and the Alexander polynomial $\varDelta$ is given by

$$\varDelta = \det(tA + (-1)^n A^T) = \det\begin{pmatrix} l(t+(-1)^n) & (q+1)t-q \\ (-1)^{n+1}qt + (-1)^n(q+1) & 0 \end{pmatrix}$$
$$= ((q+1)t-q)((-1)^n qt + (-1)^{n+1}(q+1)).$$

This shows that if K is unknotted, then q should be 0 or -1.

Conversely, if $q=0$ or -1, then, by the above relation, we have $A_n(K; S^{2n+1}) = 0$ which implies that K is unknotted by Theorem 4 or the unknotting theorem of Levine [11]. Thus (i) is proved.

Now, in the following, we assume that $m \neq 0$. By Lemmas 8 and 9, we have

$$(\iota_+)_*(m\bar{\beta}) = (\iota_-)_*(m\bar{\beta}) = 0.$$

The homology class $\bar{\beta}$ is represented by the subset S_2^n in V. Thus there exist $(n+1)$-chains c_+, c_- of Y such that $\partial c_+ = m(\iota_+)(S_2^n)$, $\partial c_- = m(\iota_-)(S_2^n)$. We construct explicitly c_+ and c_- as follows. We may assume that there exists a point $z_0' \in S^n$ such that

$$N(S_1^n) \cap (\{z_0'\} \times S^{n+1}) = (\bigcup_{i=0}^{s} D^{n+1}(z_0', u_i)) \cup (\bigcup_{i=1}^{t} D^{n+1}(z_0', v_i)),$$
$$N(S_2^n) \cap (\{z_0'\} \times S^{n+1}) = \varnothing,$$

where $D^{n+1}(z_0', u_i)$ and $D^{n+1}(z_0', v_i)$ are $(n+1)$-disks in $\{z_0'\} \times S^{n+1}$ with centers (z_0', u_i) and (z_0', v_i) respectively. Thus we may suppose that $\partial(Y \cap (\{z_0'\} \times S^{n+1}))$ is $(\bigcup_{i=0}^{s} \partial D^{n+1}(z_0', u_i)) \cup (\bigcup_{i=1}^{t} \partial D^{n+1}(z_0', v_i))$ and that $Y \cap (\{z_0'\} \times S^{n+1})$ gives the relation $m\beta = 0$, since $\partial D^{n+1}(z_0', u_i)$ and $\partial D^{n+1}(z_0', v_i)$ represent homology classes $-\beta$ and β respectively.

On the other hand, as was observed in the proof of Lemma 9, $\hat{D} \cap Y$

gives the relation $(\iota_+)_*(\bar{\beta})=-q\beta$, where $-q\beta$ is expressed by cycles in $\partial N(S_1^n)$. Therefore we can take as c_+ the union of $m(\hat{D}\cap Y)$, $-q(Y\cap(\{z_0'\}\times S^{n+1}))$ and a subset of $\partial N(S_1^n)$ giving the homologous relation between $-m((\bigcup_{i=1}^{s'}\partial D^{n+1}(z_0, u_i))\cup(\bigcup_{i=1}^{t'}\partial D^{n+1}(z_0, v_i)))$ and $-q((\bigcup_{i=0}^{s}\partial D^{n+1}(z_0', u_i))\cup(\bigcup_{i=1}^{t}\partial D^{n+1}(z_0', v_i)))$, where $m(\hat{D}\cap Y)$ and $q(Y\cap(\{z_0'\}\times S^{n+1}))$ are m copies of $\hat{D}\cap Y$ and q copies of $\{z_0'\}\times S^{n+1}$ having orientations consistent with S^{n+1} and $\hat{D}$ respectively.

Similarly we can take as c_- the union of $m((\hat{D}\cap Y)\cup D^{n+1}(z_0, u_0))$, $-(q+1)(Y\cap(\{z_0'\}\times S^{n+1}))$ and a subset of $\partial N(S_1^n)$, where $D^{n+1}(z_0, u_0)$ is an $(n+1)$-disk in $\{z_0\}\times S^{n+1}$ with center (z_0, u_0) which is a connected component of $Y\cap(\{z_0\}\times S^{n+1})$.

The union of c_+, $-c_-$ and the $(n+1)$-dimensional submanifold of $N(S_2^n)$ with the boundary $\iota_+(S_2^n)\cup\iota_-(S_2^n)$ forms a homology class of $H_{n+1}(S^n\times S^{n+1})$, say $\bar{\delta}$. It is obvious that the intersection number $I(\bar{\alpha}, \bar{\delta})$ in $S^n\times S^{n+1}$ is m. Thus, since $\bar{\alpha}$ represent $m[S^n]\in H_n(S^n\times S^{n+1})$, $\bar{\delta}$ represents a generator of $H_{n+1}(S^n\times S^{n+1})$.

Let c_+' and c_-' be the lifts of c_+ and c_- such that $c_+', c_-'\in C_{n+1}(Y^{(0)})$. Then the union of c_+' and $-t^{-1}c_-'$ forms a homology class of $H_{n+1}(X)$, say δ.

The following two lemmas hold.

LEMMA 11. *The knot module $A_{n+1}(K; S^n\times S^{n+1})$ is a Λ free module generated by δ.*

PROOF. Consider the exact sequence

$$0\longrightarrow A_{n+1}(K; S^n\times S^{n+1})\xrightarrow{\partial}\sum_{i\in Z}H_n(V^{(i)})\xrightarrow{\phi}\sum_{i\in Z}H_n(Y^{(i)})\longrightarrow\cdots$$

in § 4. Let $\bar{\eta}$ be an element contained in the kernel of ϕ. By Lemma 9, we may suppose that

$$\bar{\eta}=\sum_{i=0}^{h}m_i t^{j+i}\bar{\beta},$$

where $t^{j+i}\bar{\beta}\in H_n(V^{(j+i)})$. Then, by the relation $[R]$, we have

$$\begin{aligned}(q+1)m_0&\equiv 0 \mod m,\\ (q+1)m_i&\equiv qm_{i-1} \mod m \qquad (i=1, 2, \cdots, h),\\ 0&\equiv qm_h,\end{aligned}$$

which imply that

$$(q+1)^i m_i \equiv 0 \quad \bmod m \quad (i=0, 1, 2, \cdots, h).$$

It follows from the equations $(q+1)^h m_h \equiv 0$ and $qm_h \equiv 0$ that

$$m_h \equiv 0 \quad \bmod m.$$

Thus, by the induction on h, $\bar{\eta}$ can be written by the linear combination of $\partial t^{j+i}\delta$ $(i=0, 1, 2, \cdots, h)$. This proves the lemma.

LEMMA 12. *The intersection numbers* $I(\alpha, t^i\delta)$ *and* $I(\gamma, t^i\delta)$ *in* $\tilde{X}$ *are as follows*:

(i) $I(\alpha, \delta)=m$, $I(\alpha, t\delta)=-m$,
$I(\alpha, t^i\delta)=0$ $(i\neq 0, 1)$.

(ii) $I(\gamma, \delta)=-q$, $I(\gamma, t\delta)=q+1$,
$I(\gamma, t^i\delta)=0$ $(i\neq 0, 1)$.

PROOF. As is easily verified, the difference of intersection numbers of chains $I((\iota_+)_\# \bar{\alpha}_0, c_+)$ and $I((\iota_-)_\# \bar{\alpha}_0, c_+)$ is m. Thus we have

$$I(\alpha, \delta)=I((\iota_+)_* \bar{\alpha}-(\iota_-)_* \bar{\alpha}, \delta)=m.$$

Similarly we have $I(\alpha, t\delta)=-m$. This proves (i).

The homology class γ is represented by $S^n\times\{x_0\}$ $(x_0\in\hat{D}'\cap Y)$. $S^n\times\{x_0\}$ and c_+ intersect at q points with sign $-$, and $S^n\times\{x_0\}$ and $-c_-$ intersect at $q+1$ points with sign $+$. This proves (ii).

The following lemma is a direct consequence of Lemma 12.

LEMMA 13. *Let* $f(t)\alpha+g(t)\gamma$ *be an element of* $A_n(K; S^n\times S^{n+1})$, *where* $f(t)=\sum_{j=e}^{e'} a_j t^j$ *and* $g(t)=\sum_{j=e''}^{e'''} b_j t^j$ *are Laurent polynomials. Then the intersection number of* $f(t)\alpha+g(t)\gamma$ *and* $t^i\delta$ *is given by*

$$I(f(t)\alpha+g(t)\gamma, t^i\delta)=ma_i-ma_{i-1}-qb_i+(q+1)b_{i-1}.$$

Now suppose that a genus 1 knot K of type $(m, q, l, 0)$ $(m\neq 0)$ is local. Then, by Theorem 3, (II), there exists an element $\xi\in A_n(K; S^n\times S^{n+1})$ such that

(1) $I(\xi, \delta)=1$, $I(\xi, t^i\delta)=0$ $(i\neq 0)$.

Let $\xi=f(t)\alpha+g(t)\gamma$ be as in Lemma 13, then, it follows from the equation (1) and Lemma 13 that

$$(2)\quad \begin{cases} ma_0-ma_{-1}-qb_0+(q+1)b_{-1}=1, \\ ma_i-ma_{i-1}-qb_i+(q+1)b_{i-1}=0 \qquad (i\neq 0). \end{cases}$$

This implies that

$$(3)\quad (-mt+m)f(t)+((q+1)t-q)g(t)=1.$$

Let $-k=\min\{e, e'', 0\}$ and let $F(t)=t^k f(t)$, $G(t)=t^k g(t)$, then $F(t)$, $G(t)$ are integral polynomials in t and they satisfy the equation

$$(-mt+m)F(t)+((q+1)t-q)G(t)=t^k.$$

Therefore, by Lemma 10, m and q must satisfy the condition (*) in Theorem 5, (ii). Thus the "only if" part of the first statement in Theorem 5, (ii) is proved.

Conversely, in the following, we assume that m and q satisfy the condition (*) in Theorem 5, (ii). By Lemma 10, there exist Laurent polynomials $f(t)$ and $g(t)$ satisfying the equation (3), which implies that the element $\xi=f(t)\alpha+g(t)\gamma\in A_n(K; S^n\times S^{n+1})$ has intersection numbers as in (1) above.

Then we have the following lemma.

LEMMA 14. *$A_n(K; S^n\times S^{n+1})$ is a Λ free module generated by $f(t)\alpha+g(t)\gamma$:*

$$A_n(K; S^n\times S^{n+1})\cong\Lambda[f(t)\alpha+g(t)\gamma].$$

PROOF. Consider the exact sequence

$$\cdots\longrightarrow\sum_{i\in Z}H_n(V^{(i)})\xrightarrow{\phi}\sum_{i\in Z}H_n(Y^{(i)})\xrightarrow{\Sigma\iota_*}A_n(K; S^n\times S^{n+1})\longrightarrow\cdots$$

in § 4. Then, for the coefficients b_i of $g(t)$ and $\bar\beta\in H_n(V^{(0)})$, it holds from Lemma 9 and the equation (2) that

$$\phi((\sum_i b_{-i}t^i)\bar\beta)=t\beta \qquad \bmod m.$$

This shows that

$$(\Sigma\iota_*)(\beta)=0.$$

Suppose that there exists a Laurent polynomial $h(t)=\sum_i c_i t^i$ such that

$$h(t)(f(t)\alpha+g(t)\gamma)=0.$$

Then it follows from the intersection numbers as in (1) that

$$\begin{aligned}0&=I(h(t)(f(t)\alpha+g(t)\gamma),t^j\delta)\\&=\sum_i c_i I(f(t)\alpha+g(t)\gamma,t^{j-i}\delta)\\&=c_j.\end{aligned}$$

Thus Λ module generated by $f(t)\alpha+g(t)\gamma$ is a free Λ module.

Let $\hat{p}$: $A_n(K;S^n\times S^{n+1})\longrightarrow A_n(K;S^n\times S^{n+1})/\Lambda[f(t)\alpha+g(t)\gamma]$ be the projection and denote $\hat{p}((\Sigma\iota_*)(\alpha))=[\alpha]$ and $\hat{p}((\Sigma\iota_*)(\gamma))=[\gamma]$. Then, by the relation $[R]$ and the fact $(\Sigma\iota_*)(\beta)=0$, $A_n(K;S^n\times S^{n+1})/\Lambda[f(t)\alpha+g(t)\gamma]$ is isomorphic to a Λ-module with generators $t^i[\alpha]$, $t^i[\gamma]$ $(i\in \boldsymbol{Z})$ and relations

$$\begin{aligned}&q[\alpha]+m[\gamma]=(q+1)t[\alpha]+mt[\gamma],\\&f(t)[\alpha]+g(t)[\gamma]=0.\end{aligned}$$

It follows from the above relations that

$$(-mt+m)f(t)[\alpha]+((q+1)t-q)g(t)[\alpha]=0.$$

On the other hand, by the equation (3), it holds that

$$(-mt+m)f(t)[\alpha]+((q+1)t-q)g(t)[\alpha]=[\alpha].$$

Thus we have

$$[\alpha]=0.$$

The equation (3) implies that

$$(-mt+m)f(t)[\gamma]+((q+1)t-q)g(t)[\gamma]=[\gamma].$$

Therefore, by the above relations and $[\alpha]=0$, we have $[\gamma]=0$. Thus $A_n(K;S^n\times S^{n+1})/\Lambda[f(t)\alpha+g(t)\gamma]=0$. This proves the lemma.

According to Lemmas 11, 14 and the intersection numbers of (1), Theorem 5, (ii) follows from Theorem 4, (II). Theorem 5, (iii) is a direct consequence of $A_{n+1}(K;S^n\times S^{n+1})\cong\Lambda$ (see § 4).

Thus Theorem 5 is completely proved.

REMARK. We can prove the conclusion of Lemma 14 by using only the relation $[R]$. For general study of knots modules in global knot theory, see Tamura-Nakamura [19].

The following corollary is a direct consequence of Proposition 3 and Theorem 5.

COROLLARY. *A genus* 1 *knot* $K_{m,q,l,0}$ *of type* $(m, q, l, 0)$ *is inessential and not local in* $S^n\times S^{n+1}$ *if* $m\neq 0$, *and* m *and* q *do not satisfy the condition* (*) *in Theorem* 5.

§ 9. Knot cobordisms.

Two knots K_0 and K_1 in an m-dimensional smooth manifold M^m are said to be *cobordant* if there exists an $(m-1)$-dimensional submanifold W of $M^m\times[0,1]$ which satisfies the following conditions:

(i) W is diffeomorphic to $S^{m-2}\times I$.

(ii) $\partial W=W\cap((M^m\times\{0\})\cup(M^m\times\{1\}))$
$=(K_0\times\{0\})\cup(K_1\times\{1\})$.

A knot K cobordant to the trivial knot in M^m is said to be *null cobordant*. A knot K is null cobordant if and only if there exists a property imbedded $(m-1)$-disk D^{m-1} in $M^m\times[0,1]$ such that $\partial D^{m-1}=K\times\{0\}$.

It is obvious that homotopy classes in M^m represented by cobordant knots are the same. In particular a null cobordant knot is inessential.

The cobordance is an equivalence relation in the set of knots in M^m. The set of the equivalence classes is called the *knot cobordism* in M^m and is denoted by $C_{m-2}(M^m)$. In case M^m is the m-sphere, $C_{m-2}(S^m)$ is simply denoted by C_{m-2}. By introducing orientations on knots and S^m, the knot cobordism C_{m-2} admits an abelian group structure by the connected sum.

Kervaire proved that $C_{2n}=0$ $(n\geqq 2)$ ([6]). By similar method we can prove the following theorem.

THEOREM 6. $C_{2n}(S^{n+1}\times S^{n+1})=0$ $(n\geqq 2)$.

PROOF. Let K be a knot in $S^{n+1}\times S^{n+1}$ $(n\geqq 2)$. Then, by Proposition 1, there exists a Seifert surface V for K. Since $S^{n+1}\times S^{n+1}$ is stably parallelizable, V is stably parallelizable. Thus, it follows from the result of Kervaire-Milnor [7], that V can be modified to the $(2n+1)$-disk by applying surgeries on V considering V itself as a smooth manifold:

$$V^{(0)}=V,\quad V^{(n)}=D^{2n+1},$$

$$V^{(i)}=\partial((V^{(i-1)}\times I)\cup(D_1^{i+1}\times D_1^{2n-i+1})\cup(D_2^{i+1}\times D_2^{2n-i+1})$$
$$\cup\cdots\cup(D_{q(i)}^{i+1}\times D_{q(i)}^{2n-i+1}))-(V^{(i-1)}\times\{0\})-(\partial V^{(i-1)}\times[0,1))$$
$$(i=1,2,3,\cdots,n).$$

We realize this process in $S^{n+1}\times S^{n+1}\times I$ as follows. Consider $V=V^{(0)}$ in $S^{n+1}\times S^{n+1}\times I$. The submanifolds ∂D_j^2 $(j=1,2,\cdots,q(1))$ of $V^{(0)}$ are inessential in $S^{n+1}\times S^{n+1}\times I$. Thus there exist 2-disks D_j^2 $(j=1,2,\cdots,q(1))$ imbedded in $S^{n+1}\times S^{n+1}\times I$ such that

$$\partial D_j^2=\partial D_j^2\times\{0\},\ D_j^2\cap D_{j'}^2=0 \qquad (j\neq j').$$

We fix a trivialization of a tubular neighborhood of D_j^2 in $S^{n+1}\times S^{n+1}\times I$. Then the obstruction to realize the handle $D_j^2\times D_j^{2n}$ by extending $D_j^2=D_j^2\times\{0\}$ is represented by the element of $\pi_1(SO(2n+1))$ defined by the attaching map of the handle. Since $V^{(0)}$ is stably parallelizable and $\pi_1(SO(2n+1))$ is in stable range, this element is zero. Therefore we can realize the trace $T^{(1)}$ of the surgery on $V^{(0)}$ as a submanifold in $S^{n+1}\times S^{n+1}\times I$:

$$T^{(1)}=(V^{(0)}\times I)\cup(D_1^2\times D_1^{2n})\cup\cdots\cup(D_{q(1)}^2\times D_{q(1)}^{2n}),\ \partial T^{(1)}\supset V^{(1)}.$$

Let us denote the closure of $S^{n+1}\times S^{n+1}\times I-T^{(1)}$ by $E^{(1)}$. Then $V^{(1)}\subset\partial E^{(1)}$ and, as is easily verified, $E^{(1)}$ is n-connected.

Now suppose that $V^{(i-1)}$ and the trace $T^{(i)}$ of the surgery on $V^{(i-1)}$ are similarly realized as submanifolds in $S^{n+1}\times S^{n+1}\times I$ for $i=1,2,3,\cdots,k$, where $k<n$. Let $E^{(k)}$ denote the closure of $S^{n+1}\times S^{n+1}\times I-\bigcup_{i=1}^{k}T^{(i)}$. Then $V^{(k)}\subset\partial E^{(k)}$ and, as is easily verified, $E^{(k)}$ is n-connected. Thus we can realize $(V^{(k)}\times I)\cup(D_1^{k+2}\times\{0\})\cup(D_2^{k+2}\times\{0\})\cup\cdots\cup(D_{q(k+1)}^{k+2}\times\{0\})$ in $E^{(k)}$. Then the obstruction to realize handles $D_i^{k+2}\times D_i^{2n-k}$ by extending $D_i^{k+2}\times\{0\}$ $(i=1,2,\cdots,q(k+1))$ are represented by elements of $\pi_{k+1}(SO(2n+1-k))$ defined by the attaching maps of the handles. Since $V^{(k)}$ is stably parallelizable and $\pi_{k+1}(SO(2n+1-k))$ is in stable range for $k<n$, they are zero. Thus we can realize the trace of the surgery on $V^{(k)}$ in $E^{(k)}$.

Therefore, by the induction on k, the surgeries from $V^{(0)}$ to $V^{(n)}=D^{2n+1}$ are realized in $S^{n+1}\times S^{n+1}\times I$, which implies that K is null cobordant. This completes the proof.

The following corollary is a direct consequence of Theorem 6.

COROLLARY. *An element of the homotopy group $\pi_{2n}(S^{n+1}\times S^{n+1})$ $(n\geqq 2)$ is realizable by an imbedded $2n$-sphere in $S^{n+1}\times S^{n+1}$ if and only if it* *is the zero element.*

This result can be seen as a higher dimensional analogue of the famous problem of realization of 2-dimensional homotopy (homology) classes of $S^2\times S^2$ by imbedded 2-spheres from the viewpoint of codimension 2.

References

[1] Blanchfield, R. C,, Intersection theory of manifolds with operators with applications to knot theory, Ann. of Math. **65** (1957), 340-356.

[2] Haefliger, A., Plongements différentiables de variétés dans variétés, Comm. Math. Helv. **36** (1961), 47-82.

[3] Hirsch, M. W. and B. Mazur, Smoothings of piecewise linear manifolds, Ann. of Math. Studies 80, 1974, Princeton Univ. Press.

[4] Hirsch, M. W. and E. C. Zeeman, Engulfing, Bull. Amer. Math. Soc. **72** (1966), 113-115.

[5] Irwin, M. C., Embeddings of polyhedral manifolds, Ann. of Math. **82** (1965), 1-14.

[6] Kervaire, M., Les noeuds de dimensions supérieurs, Bull. Soc. Math. France **93** (1965), 225-271.

[7] Kervaire, M. and J. Milnor, Groups of homotopy spheres, I, Ann. of Math. **77** (1963), 504-537.

[8] Kervaire, M. and C. Weber, A survey of multidimensional knots, Lecture Notes in Math., 685, Springer-Verlag, 1978.

[9] Lee, Y. W., Contractibly embedded 2-spheres in $S^2\times S^2$, Proc. Amer. Math. Soc. **85** (1982), 280-282.

[10] Levine, J., Imbedding and isotopy of spheres in manifolds, Proc. Cambridge. Philos. Soc. **60** (1964), 433-437.

[11] Levine, J., Unknotting spheres in codimension two, Topology **4** (1965), 9-16.

[12] Matsumoto, Y., Hauptvermutung for $\pi_1=Z$, J. Fac. Sci. Univ. Tokyo **16** (1969), 165-178.

[13] Milnor, J., Some consequences of a theorem of Bott, Ann. of Math. **68** (1958), 444-449.

[14] Papakyriakopoulos, C. D., On Dehn's lemma and the asphericity of knots, Ann. of Math. **66** (1957), 1-26.

[15] Suzuki, Y., Global knots in simple closed 3-manifolds (to appear).

[16] Tamura, I., Foliations and spinnable structures on manifolds, Ann. Inst. Fourier **23** (1973), 197-214.

[17] Tamura, I., Specially spinnable manifolds, Manifolds Tokyo 1973, 181-188, Univ. of Tokyo Press, 1975.

[18] Tamura, I., Unknotted codimension one spheres in smooth manifolds, Topology **23** (1984), 127-132.

[19] Tamura, I. and T. Nakamura, Knot modules in global knot theory (in preparation).

[20] Wall, C. T. C., Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math. **75** (1962), 163-189.

[21] Yano, K., The support of global graph links, J. Math. Soc. Japan **37** (1985), 683-702.

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