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Critical exponent of blowup for semilinear heat equation on a product domain

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Abstract. We show that for many regions of product type $D=D_1\times D_2$ the critical exponent of blowup for the Dirichlet mixed problem of the semilinear heat equation $\partial_t u=\Delta u+u^p$ is determined from those of the factors by the formula $1/(p^*(D)-1)=1/(p^*(D_1)-1)+1/(p^*(D_2)-1)$. As an application we obtain a formula for the first Dirichlet eigenvalue of the Laplace-Beltrami operator on spherical slice domains.

0. Introduction.

Let D be a domain of $\boldsymbol{R}^N$. It is generally shown by Meier [M2] that for the Dirichlet mixed problem of the semilinear heat equation

$$\text{(P)}\qquad \begin{aligned} &\frac{\partial u}{\partial t}=\Delta u+u^p, \quad \text{in }]0,\infty[\times D,\\ &u(t,x)=0 \quad \text{on } \partial D,\\ &u(0,x)=u_0(x), \quad \text{in } D, \end{aligned}$$

there exists a constant $p^*=p^*(D)$, called the critical exponent of blowup for the domain D, and characterized by the following property:
If $p>p^*$, then there exists a non-trivial non-negative Cauchy data u_0 for which the solution is global in t;
If $1<p<p^*$, then the solution blows up in finite time whatever the non-trivial non-negative Cauchy data u_0 may be.

Here and in the sequel "non-trivial" means not identically zero. Also, we consider only bounded continuous initial data for the sake of simplicity (and without essential loss of generality). Since H. Fujita gave in [F1] the first result of this type:

$$p^*(\boldsymbol{R}^N)=1+\frac{2}{N}, \tag{0.1}$$

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there appeared many results by various researchers. We recall here especially that of Meier [M1]:

(0.2) $$p^*(\boldsymbol{R}^{N-k}\times\boldsymbol{R}_+^k)=1+\frac{2}{N+k}.$$

More geuerally, let $\Gamma\subset\boldsymbol{R}^N$ be a (not necessarily convex) cone. Then Levine and Meier [LM1] proved

(0.3) $$p^*(\Gamma)=1+\frac{2}{N+\gamma},$$

where γ is the non-negative root of $\gamma(\gamma+(N-2))=\omega_1$, ω_1 being the first eigenvalue of the Dirichlet problem of the Laplace-Beltrami operator on $\Gamma\cap\boldsymbol{S}^{N-1}$. (For further references on this subject see the survey article by Levine [L].)

In this paper we prove the following formula for the critical exponent of blowup for the product domain:

(0.4) $$\frac{1}{p^*(D_1\times D_2)-1}=\frac{1}{p^*(D_1)-1}+\frac{1}{p^*(D_2)-1},$$

under some regularity assumption on the domains. Indeed this formula holds well for domains in (0.2) cited above and was inferred from that result.

In the final section we give some examples of conclusions of our formula, including a geometric application.

2. Preliminaries.

We collect here the basic techniques used in the sequel. The most essential tool for us is Lemma 1.1 below essentially taken from [M2], where it is used to prove the abstract existence of critical exponent of blowup. We reproduced it here with detailed proof for the sake of self-containedness.

$$(\tilde{\mathrm{P}})\qquad \begin{aligned} &\frac{\partial u}{\partial t}=\Delta u, \quad \text{in }]0,\infty[\times D,\\ &u(t,x)=0 \quad \text{on } \partial D,\\ &u(0,x)=u_0(x), \quad \text{in } D \end{aligned}$$

be the linear problem to be compared with (P). In the sequel, we only consider non-negative solutions of (P) and ($\tilde{\mathrm{P}}$) even if it is not explicitly mentioned.

LEMMA 1.1. (i) *If Problem* ($\tilde{\mathrm{P}}$) *has a non-trivial* (*super-*) *solution* W *such that*

$$\int_0^\infty \| W(t, \cdot)\|_\infty^{p-1} dt < +\infty \tag{1.1}$$

then (P) *admits a non-trivial solution which is global in time.*

(ii) *If every non-trivial solution* W *of* ($\tilde{\mathrm{P}}$) *satisfies*

$$\limsup_{t\to\infty} \| W(t, \cdot)\|_\infty^{p-1} t = +\infty \tag{1.2}$$

then every non-trivial solution of (P) *blows up in a finite time.*

PROOF. (i) Let $\beta(t)$ be the solution of the following Cauchy problem for the ordinary differential equation:

$$\beta'(t) = \beta(t)^p \| W(t, \cdot)\|_\infty^{p-1}, \quad \beta(0) = \beta_0. \tag{1.3}$$

Then $\bar{u}(t, x) = \beta(t) W(t, x)$ is a super-solution of (P) for the Cauchy data $u_0(x) = \beta_0 W(0, x)$. In fact,

$$\bar{u}_t - \Delta \bar{u} \geq \beta' W \geq \beta^p W^p = \bar{u}^p.$$

Thus it suffices to show that (1.3) admits a global solution for some β_0. This equation is explicitly integrated as

$$\beta(t) = \left(\beta_0^{-(p-1)} - (p-1)\int_0^t \| W(s, \cdot)\|_\infty^{p-1} ds\right)^{-1/(p-1)}.$$

Therefore (1.1) assures this to be global for sufficiently small β_0.

(ii) Let $z(v : w)$ be the solution of the Cauchy problem

$$\frac{dz}{dv} = z^p, \quad z(0 : w) = w. \tag{1.4}$$

If $w = W(t, x)$ is a solution of ($\tilde{\mathrm{P}}$) with the Cauchy data $u_0(x)$, then $\underline{u}(t, x) = z(t : W(t, x))$ is a sub-solution of (P) with the same Cauchy data. In fact, we have from (1.4)

$$\underline{u}_t - \Delta \underline{u} = \underline{u}^p - \frac{d^2 z}{dw^2} |\nabla W|^2 + z_w (W_t - \Delta W)$$

$$=\underline{u}^p-\frac{d^2z}{dw^2}|\nabla W|^2.$$

Hence if we show $d^2z/dw^2\geq 0$, $\underline{u}$ will be a sub-solution. Integrating (1.4) we have

$$z(v:w)=\left(\frac{1}{w^{p-1}}-(p-1)v\right)^{-1/(p-1)} \quad (w\neq 0), \tag{1.5}$$
$$=0 \quad (w=0).$$

Differentiating this twice, we have

$$\frac{d^2z}{dw^2}=p\frac{z}{w^2}\left(\left(\frac{z}{w}\right)^{2p-2}-1\right).$$

Since by (1.4) we have $0\leq w\leq z$, we conclude $d^2z/dw^2\geq 0$.

Thus it suffices to show that $\underline{u}$ blows up. In view of (1.5), this happens if there exists (t, x) such that $W(t, x)^{p-1}t\geq 1/(p-1)$. This is assured by (1.2).

COROLLARY 1.2. *The greatest lower bouud for p satisfying* (1.1) *and the least upper bound for p satisfying* (1.2) *both agree with the critical exponent of blowup.*

PROOF. Set

$$A=\{p>1;\ \text{(1.1) holds for some non-trivial solution } W \text{ of } (\tilde{\mathrm{P}})\},$$
$$B=\{p>1;\ \text{(1.2) holds for any non-trivial solution } W \text{ of } (\tilde{\mathrm{P}})\},$$

and let

$$\alpha=\inf\{p; p\in A\}, \quad \beta=\sup\{p; p\in B\}.$$

We shall show that $\{p>\alpha\}\subset A$, $\{1<p<\beta\}\subset B$ and that $\alpha=\beta$. Then Lemma 1.1 will immediately imply $p^*=\alpha=\beta$, including the existence of the critical exponent p^*.

Let $p>\alpha$. Then there exists q such that $\alpha<q<p$ and a non-trivial solution W snch that

$$\int_0^\infty \| W(t, \cdot)\|_\infty^{q-1}dt<\infty.$$

Hence in view of the boundedness of $\| W(t, \cdot)\|_\infty$ in t, we have

$$\int_0^\infty \| W(t, \cdot)\|_\infty^{p-1}dt\leq \sup_t \| W(t, \cdot)\|_\infty^{p-q}\int_0^\infty \| W(t, \cdot)\|_\infty^{q-1}dt<\infty.$$

Thus $p \in A$. Next let $p<\beta$. Then there exists q such that $p<q<\beta$ and that for every non-trivial solution W we have

$$\limsup_{t\to\infty} \|W(t, \cdot)\|_\infty^{q-1} t = \infty,$$

hence

$$\limsup_{t\to\infty} \|W(t, \cdot)\|_\infty^{p-1} t = \limsup_{t\to\infty} \{\|W(t, \cdot)\|_\infty^{q-1} t\}^{(p-1)/(q-1)} \cdot t^{1-(p-1)/(q-1)} = \infty.$$

Thus $p \in B$. Now we compare α with β. Assume $p>\beta$. Then for any q such that $\beta<q<p$ we can find a non-trivial solution W such that

$$\limsup_{t\to\infty} \|W(t, \cdot)\|_\infty^{q-1} t < \infty,$$

hence

$$\|W(t, \cdot)\|_\infty^{q-1} \leq Ct^{-1}.$$

Thus we will have

$$\|W(t, \cdot)\|_\infty^{p-1} \leq C' t^{-\gamma}, \quad \gamma = \frac{p-1}{q-1} > 1,$$

hence

$$\int_0^\infty \|W(t, \cdot)\|_\infty^{p-1} dt < \infty,$$

implying $p \geq \alpha$. Since $p>\beta$ was arbitrary, we conclude that $\alpha \leq \beta$. If $\alpha<\beta$, then $\alpha<p<\beta$ would imply $p \in A \cap B$. But this is absurd in view of Lemma 1.1.

It is easy to observe the following monotonicity property of the critical exponent. It seems to us, however, that it is not explicitly stated in the literature. So we give a proof.

LEMMA 1.3. *Let $D_1 \subset D_2$. Then we have $p^*(D_1) \leq p^*(D_2)$.*

PROOF. This follows directly from the comparison theorem for the non-linear heat equation applied on the smaller domain D_1. In view of Lemma 1.1, this follows also from the comparison theorem for linear heat equation as follows: Let $1<p<p^*(D_1)$. We shall show that any non-trivial solution of (P) for D_2 with exponent p blows up in a finite time. Assume that there exists a non-trivial time-global solution. Then by Lemma 1.1 (ii) there exists a non-trivial solution $W(t, x)$ of $(\tilde{\mathrm{P}})$ for D_2 satisfying

$$\limsup_{t\to\infty} \|W(t, \cdot)\|_\infty^{p-1} t < +\infty.$$

Choosing a smaller initial data, we then find that $(\tilde{\mathrm{P}})$ for D_1 also possesses a non-trivial solution satisfying the same estimate as above, hence satisfying

$$\|W(t,\cdot)\|_\infty \leq Ct^{-1/(p-1)}.$$

Thus we have, for any $\varepsilon>0$

$$\int_0^\infty \|W(t,\cdot)\|_\infty^{p+\varepsilon-1} dt < +\infty.$$

In view of Lemma 1.1 (i) this implies that $p+\varepsilon \geq p^*(D_1)$. But this contradicts with $p<p^*(D_1)$. Thus we proved $p\leq p^*(D_2)$. Since p is arbitrary, we obtain $p^*(D_1)\leq p^*(D_2)$.

2. Main Theorem and its Proof.

First we introduce a class of domains which are favorable to our discussion:

DEFINITION 2.1. We say that a domain $D\subset \boldsymbol{R}^N$ is *asymptotically regular at infinity* if for some non-negative non-trivial Cauchy data $u_0(x)$ with compact support the solution $u(t,x)$ of the Dirichlet mixed problem for the linear heat equation $(\tilde{\mathrm{P}})$ satisfies either of the following asymptotic:

(i) $\|u(t,\cdot)\|_\infty^\lambda \in L_1(\boldsymbol{R}_+)$ for any $\lambda>0$.

(ii) There exists $\lambda>0$ such that for every $\varepsilon>0$ we have, with some $C=C(\varepsilon)>0$,

$$\frac{1}{C}t^{-\lambda-\varepsilon} \leq \|u(t,\cdot)\|_\infty \leq Ct^{-\lambda+\varepsilon} \quad \text{for} \quad t\geq C. \tag{2.1}$$

A bounded domain satisfies (i) because $\|u(t,\cdot)\|_\infty \leq Ce^{-at}$ for some $a>0$. We conjecture that most (eventually all) domains are asymptotically regular at infinity in our sense. We shall show later that conical domains are examples of such.

We need the following variant of this definition:

LEMMA 2.2. *Assume that the asymptotic* (i) (*resp.* (ii)) *of Definition* 2.1 *holds for the solution for some non-negative non-trivial Cauchy data with compact support. Then* (i) (*resp.* (ii)) *holds for the solution for every such Cauchy data.*

PROOF. Let $v(t,x)$ be the solution corresponding to another non-trivial Cauchy data $v_0(x)$ with compact support. Since the solutions have strictly positive value for $t>0, x\in D$, we have $u(t_0, x)\geq Cv_0(x)$ for any fixed $t_0>0$. Thus by the comparison theorem we have

$$\|u(t, \cdot)\|_\infty\geq C\|v(t-t_0, \cdot)\|_\infty.$$

Obviously we have a similar estimate with u and v interchanged. Finally it suffices to notice that the translation by t_0 does not change the form of the asymptotics.

If D has a regular boundary such that it assures the strong maximum principle, then we can compare the solution u with the fundamental solution $E(t, x, y)$ of the Dirichlet mixed problem ($\tilde{\mathrm{P}}$) with the intitial data $\delta(x-y)$ for any fixed $y\in D$, and thus we can replace $\|u(t, \cdot)\|_\infty$ in Definition 2.1 by $\|E(t, \cdot, y)\|_\infty$ or $\sup_{x,y\in D} E(t, x, y)$ or by the one for any bounded initial data. This would be more elegant as a definition. But what we actually need is only the behavior of solutions with compact support initial data.

Now we present our main theorem.

THEOREM 2.3. *Let $D_1\subset \boldsymbol{R}^{N-k}, D_2\subset \boldsymbol{R}^k$ be Euclidean domains one of which is asymptotically regular at infinity. Then for the critical exponent of blowup of the* Dirichlet *mixed problem* (P) *we have the following relation*:

$$\frac{1}{p^*(D_1\times D_2)-1}=\frac{1}{p^*(D_1)-1}+\frac{1}{p^*(D_2)-1}. \tag{0.4bis}$$

PROOF. Let p^* denote the value given as $p^*(D_1\times D_2)$ in the above formula, namely,

$$\frac{1}{p^*-1}=\frac{1}{p^*(D_1)-1}+\frac{1}{p^*(D_2)-1}.$$

Assume first that $p>p^*$. Then we can find $p_1>p^*(D_1)$, $p_2>p^*(D_2)$ such that

$$\frac{1}{p-1}<\frac{1}{p_1-1}+\frac{1}{p_2-1}.$$

The Dirichlet problem (P) for D_1, D_2 with the exponent p_1, p_2 respectively, admit time-global non-trivial solutions. Hence by Lemma 1.1 (ii), there exist solutions W_j of the Dirichlet problem ($\tilde{\mathrm{P}}$) for $D_j, j=1,2$, respectively, such that

$$\limsup_{t\to\infty}\|W_j(t,\cdot)\|_\infty^{p_j-1}t<+\infty,\quad j=1,2,$$

hence

$$\|W_j(t,\cdot)\|_\infty\leq Ct^{-1/(p_j-1)},\quad j=1,2,$$

for some constant C. Then $W(t,x)=W(t,x')W(t,x'')$, where x', x'' denote the variables of $\boldsymbol{R}^{N-k}, \boldsymbol{R}^k$ respectively, is a solution of $(\tilde{\mathrm{P}})$ in $D_1\times D_2$ which satisfies

$$\int_0^\infty\|W(t,\cdot)\|_\infty^{p-1}dt\leq\int_0^\infty Ct^{-q}dt,$$

with

$$q=\left(\frac{1}{p_1-1}+\frac{1}{p_2-1}\right)(p-1)>1. \tag{2.2}$$

Thus in view of Lemma 1.1 (i) we conclde that (P) admits a time global non-trivial solution for $D_1\times D_2$ with the exponent p, and therefore $p>p^*(D_1\times D_2)$. Since $p>p^*$ is arbitrary, we conclude that $p^*\geq p^*(D_1\times D_2)$ and we proved one direction of the inequality:

$$\frac{1}{p^*(D_1\times D_2)-1}\geq\frac{1}{p^*(D_1)-1}+\frac{1}{p^*(D_2)-1}. \tag{2.3}$$

Note that for this part no assumption on the domains is used.

Now we prove the opposite direction. Note that if either of $p^*(D_j)$ is equal to 1, then the above inequality implies $p^*(D_1\times D_2)=1$, hence (0.4) holds in this sense. Assume therefore that $p^*(D_j)>1$, $j=1,2$ and let $1<p<p^*$. Then we can find $p_1<p^*(D_1)$, $p_2<p^*(D_2)$ such that

$$\frac{1}{p-1}>\frac{1}{p_1-1}+\frac{1}{p_2-1}.$$

We shall show that every non-trivial solution of (P) with the exponent p blows up in a finite time. In view Lemma 1.1 (ii) it suffices to show that for every solution $W(t,x)$ of $(\tilde{\mathrm{P}})$ on $D_1\times D_2$ we have

$$\limsup_{t\to\infty}\|W(t,\cdot)\|_\infty^{p-1}t=+\infty.$$

In view of the comparison theorem, it suffices to do so for a Cauchy data of product type $w_1(x')w_2(x'')$, because we can find such a data (of compact support) supporting any given non-trivial non-negative data from below. By the uniqueness of solutions the solution itself is then of product type $W(t,x)=W_1(t,x')W_2(t,x'')$, where W_j is a solution of the problem ($\tilde{\mathrm{P}}$) on the domain D_j with the Cauchy data w_j, $j=1,2$, respectively. By the choice of p_j and Lemma 1.1 (i) we have

$$\int_0^\infty \| W_j(t,\cdot)\|_\infty^{p_j-1}dt=+\infty,\quad j=1,2 \tag{2.4}$$

This implies that for any $\varepsilon>0$ there exist sequences $\{t_{jn}\}_{n=1}^\infty$, $j=1,2$, tending to $+\infty$ such that

$$\| W_j(t_{jn},\cdot)\|_\infty^{p_j-1}\geq Ct_{jn}^{-(1+\varepsilon)},\quad n=1,2,\cdots,\quad j=1,2.$$

Since we cannot expect in general that we can choose a sequence common to $j=1,2$, this implies formally nothing. In view of Lemma 2.2, however, our assumption on the asymptotic regularity of the domain, say D_1, at infinity implies that condition (ii) of Definition 2.1 holds for D_1 with some λ_1. We have $\lambda_1\leq 1/(p_1-1)$ since otherwise the integral in (2.4) would be finite. Thus for any $\varepsilon>0$ we have

$$\begin{aligned}\| W(t_{2n},\cdot)\|_\infty^{p-1}t_{2n}&=(\| W_1(t_{2n},\cdot)\|_\infty\| W_2(t_{2n},\cdot)\|_\infty)^{p-1}t_{2n}\\ &\leq C({}_{2n}^{-\lambda_1-\varepsilon}\cdot t_{2n}^{-(1+\varepsilon)/(p_2-1)})^{p-1}t_{2n}\\ &\leq Ct_{2n}^{1-q-\varepsilon(p-1)p_2/(p_2-1)},\end{aligned}$$

where q, given by (2.2), is now less than 1. Thus if ε is sufficiently small, the power of t_{2n} becomes positive and hence the last term tends to infinity as $t_{2n}\to\infty$. This proves $p<p^*(D_1\times D_2)$. Since $p<p^*$ is arbitrary, we conclude that $p^*\leq p^*(D_1\times D_2)$, that is, the opposite inequality in (0.4).

3. Examples and Applications.

We first prove the following

PROPOSITION 3.1. *A cone Γ is asymptotically regular at infinity in the sense of Definition* 2.1.

PROOF. Let $u_0(x)\in C_0^\infty(\Gamma)$. It can be expanded via the eigenfunctions as follows:

$$u_0(x)=u_0(r,\theta)=\sum_{n=1}^\infty \phi_n(\theta)r^{-(N-1)/2}\int_0^\infty \sqrt{r\lambda}J_{\nu_n}(r\lambda)\chi_n(\lambda)d\lambda, \tag{3.1}$$

where $\phi_n(\theta)$ is the eigenfunction corresponding to the n-th Dirichlet eigenvalue ω_n of the spherical (positive) Laplacian Δ_θ for the domain $\Gamma\cap S^{N-1}$,

$$\nu_n = \gamma_n + \frac{1}{2}(N-2) = \left[\omega_n + \frac{1}{4}(N-1)^2\right]^{1/2}, \tag{3.2}$$

and J_{ν_n} is the Bessel function. The integral is what is known as the Hankel transform. The spectral density $\chi_n(\lambda)$ is given as the Hankel transform of $r^{(N-1)/2}\int_{\Gamma\cap S^{N-1}} u_0(r,\theta)\phi_n(\theta)d\theta$ and is a smooth function as described in Lemma 3.2 below. Then the solution of $(\tilde{\mathrm{P}})$ for this initial data is given by

$$u(x,t) = u(r,\theta,t) = \sum_{n=1}^{\infty} \phi_n(\theta) r^{-(N-1)/2} \int_0^\infty e^{-\lambda^2 t} \sqrt{r\lambda} J_{\nu_n}(r\lambda)\chi_n(\lambda)d\lambda. \tag{3.3}$$

For this solution we shall show that the asymptotic (2.1) of Definition 2.1 holds in the form

$$\frac{1}{C} t^{-(N+\gamma)/2} \leq \sup_x |u(x,t)| \leq C t^{-(N+\gamma)/2}, \tag{3.4}$$

where $\gamma = \gamma_1$ is the quantity appearing in the critical exponent of blowup (0.3) of the domain Γ. To show (3.4), it suffices to show that the same holds for the first term of the sum (3.3), and that the other terms are of $O(t^{-(N+\gamma_n)/2})$, $\gamma_n > \gamma$ being the quantity determined from ω_n by the same formula (3.2) as for γ. For this purpose we review some properties of the Hankel transform. General reference for them is [Z].

LEMMA 3.2. *Let* $\nu \geq -1/2$. (i) *The inverse of the Hankel transform*

$$F(\lambda) = (\mathcal{H}_\nu f)(\lambda) = \int_0^\infty \sqrt{r\lambda} J_\nu(r\lambda) f(r) dr$$

is given by the same formula:

$$f(r) = (\mathcal{H}_\nu^{-1} F)(r) = \int_0^\infty \sqrt{r\lambda} J_\nu(r\lambda) F(\lambda) d\lambda.$$

(ii) *Let* N_ν *be a differential operator with rational coefficients defined by*

$$N_\nu f := r^{\nu+1/2} D_r r^{-\nu-1/2} f = \frac{df}{dr} - \frac{\nu + \frac{1}{2}}{r} f.$$

Then

$$\mathcal{H}_{\nu+1}(-rf) = N_\nu \mathcal{H}_\nu f, \quad \mathcal{H}_{\nu+1}(N_\nu f) = -\lambda \mathcal{H}_\nu f.$$

(iii) *If* $f(r)$ *is in* $C_0^\infty(0,\infty)$, *then* $F(\lambda)$ *is multi-valued analytic on* $C\setminus\{0\}$.

The principal branch satisfies the following estimate on the real axis:

$$(3.5)\qquad |F(\lambda)|\le C\lambda^{\nu+1/2},\ \textit{for}\ \ 0<\lambda<1,$$
$$\le C_k\lambda^{-k}\ \textit{for any}\ k,\ \ \textit{for}\ \ \lambda\ge 1.$$

Moreover, as $\lambda\to 0$ *we have*

$$(8.6)\qquad F(\lambda)\sim K\lambda^{\nu+1/2},\quad \textit{where}\quad K=\frac{1}{2^{\nu}\Gamma(\nu+1)}\int_0^\infty r^{\nu+1/2}f(r)dr.$$

The asserted asymptotic can be deduced by a standard argument from the above properties applied to $\chi_n(\lambda)$: Let $n=1$ and omit the suffix for the sake of simplicity. We have

$$r^{-(N-1)/2}\int_0^\infty e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda$$
$$=r^{-N-1)/2}\int_0^1 e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda+r^{-(N-1)/2}\int_1^\infty e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda$$
$$=\mathrm{I}+\mathrm{II};$$
$$|\mathrm{I}|\le Cr^{-(N-1)/2}\int_0^1 e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\lambda^{\nu+1/2}d\lambda,$$
$$|\mathrm{II}|\le Cr^{-(N-1)/2}\int_1^\infty e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)^{-k}d\lambda.$$

Assume $0\le r\le 1$. Employ the change of variable

$$(3.7)\qquad \lambda=s/\sqrt{t}.$$

In view of the asymptotic $J_\nu(z)\sim z^\nu/2^\nu\Gamma(\nu+1)$ at the origin, we obtain

$$|\mathrm{I}|\le Cr^{\nu+1-N/2}t^{-\nu-1}\int_0^{\sqrt{t}} e^{-s^2}s^{2\nu+1}ds.$$

Since the power of r is positive, this quantity is of $O(t^{-\nu-1})$ and decays faster than the main asymptotic term. We further divide the quantity II as follows:

$$|\mathrm{II}|=r^{-(N-1)/2}\int_1^{1/r} e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda+r^{-(N-1)/2}\int_{1/r}^\infty e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda$$
$$=\mathrm{IIa}+\mathrm{IIb}.$$

Then by the same change of variable we have

$$|\mathrm{IIa}|\le Cr^{-(N-1)/2}\int_1^{1/r} e^{-\lambda^2 t}\sqrt{r\lambda}(r\lambda)^\nu\lambda^{-k}d\lambda$$

$$\leq Cr^{\nu+1-N/2}t^{-(\nu-k+3/2)/2}\int_{\sqrt{t}}^{\infty}e^{-s^2}s^{\nu+1-k}ds$$

and this is even of exponential decay in t. Similarly, noticing that for $r\lambda\geq 1$, $\sqrt{r\lambda}J_\nu(r\lambda)$ is bounded, we have

$$|\mathrm{IIb}|\leq Cr^{-(N-1)/2}\int_{1/r}^{\infty}e^{-\lambda^2 t}\lambda^{-k}d\lambda$$
$$\leq Ct^{(k-1-(N-1)/2)/2}\left(\frac{\sqrt{t}}{r}\right)^{(N-1)/2}\int_{\sqrt{t}/r}^{\infty}e^{-s^2}s^{-k}ds.$$

Note that there exists $\delta>0$ such that

$$x^{(N-1)/2}\int_x^{\infty}e^{-s^2}s^{-k}ds=O(e^{-\delta x^2}),\quad \text{as}\quad x\to\infty.$$

Since $x=\sqrt{t}/r\geq\sqrt{t}$ now, we see that this term is also of exponential decay in t.

Next assume $r\geq 1$. This time we divide the integral I as follows:

$$\mathrm{I}=r^{-(N-1)/2}\int_0^{1/r}e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda+r^{-(N-1)/2}\int_{1/r}^{1}e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda$$
$$=\mathrm{Ia}+\mathrm{Ib}.$$

By the same change of variable we have

$$|\mathrm{Ia}|\leq t^{-(\nu+1+N/2)/2}\left(\frac{\sqrt{t}}{r}\right)^{N/2-\nu-1}\int_0^{\sqrt{t}/r}e^{-s^2}s^{2\nu+1}ds.$$

Note that the power of $\sqrt{t}/r$ is equal to $-\gamma<0$. The function

$$x^{-\gamma}\int_0^x e^{-s^2}s^{2\nu+1}ds$$

of x is obviously bounded. Hence the above quantity is of $O(t^{-(\nu+1+N/2)/2})=O(t^{-(N+\gamma)/2})$ uniformly in r in this region. Similarly, we have

$$|\mathrm{Ib}|\leq t^{-(\nu+1+N/2)/2}\left(\frac{\sqrt{t}}{r}\right)^{(N-1)/2}\int_{\sqrt{t}/r}^{\sqrt{t}}e^{-s^2}s^{\nu+1/2}ds=O(t^{-(N+\gamma)/2}).$$

The estimate of II is similar to this and even easier (of faster decay). The estimate from above for terms with $n\geq 2$ is quite the same.

To prove the converse estimate, let α be a positive constant less than the first positive zero of $J_\nu(\lambda)$ and of $\chi(\lambda)$. Note that $(u_0(r,\theta),\phi_1(\theta))_\theta\not\equiv 0$, hence (3.6) holds for χ_1 with $K\neq 0$. Thus specializing $r=\sqrt{t}$ we have, with some $c>0$,

$$\begin{aligned}
&\sup_r\left|r^{-(N-1)/2}\int_0^\infty e^{-\lambda^2 t}\sqrt{r\lambda}J_\nu(r\lambda)\chi(\lambda)d\lambda\right|\\
&\geq t^{-(N-2)/4}\left|\int_0^\infty e^{-\lambda^2 t}\sqrt{\lambda}J_\nu(\sqrt{t}\lambda)\chi(\lambda)d\lambda\right|\\
&\geq t^{-(N-2)/4}\left|\int_0^\alpha e^{-\lambda^2 t}\sqrt{\lambda}J_\nu(\sqrt{t}\lambda)\chi(\lambda)d\lambda\right|-t^{-(N-2)/4}\left|\int_\alpha^\infty e^{-\lambda^2 t}\sqrt{\lambda}J_\nu(\sqrt{t}\lambda)\chi(\lambda)d\lambda\right|\\
&\geq ct^{-(N-2)/4}\int_0^\alpha e^{-\lambda^2 t}\sqrt{\lambda}(\sqrt{t}\lambda)^\nu\lambda^{\nu+1/2}d\lambda-Ct^{-(N-1)/4}\int_\alpha^\infty e^{-\lambda^2 t}\lambda^{-k}d\lambda.
\end{aligned}$$

Here the first term of the last side has an asymptotic expansion with the main term of the form $Kt^{-(\nu+1+N/2)/2}=Kt^{-(N+\gamma)/2}$ (with another constant K). The second term decays exponentially in t. Thus we proved the desired asymptotic (3.4).

REMARK. In the above proof we did not essentially use the second estimate of (3.5): It may be replaced by $O(\lambda^M)$ for some $M>0$, which is the case if $F(\lambda)$ is the Hankel transform of distributions in $\mathcal{E}'(0,\infty)$. Thus for this domain the same asymptotic holds e.g. for the fundamental solution of the heat equation.

Now we discuss some examples. Note that formula (0.4) can be generalized to

$$\frac{1}{p(D_1\times\cdots\times D_k)-1}=\frac{1}{p(D_1)-1}+\cdots+\frac{1}{p(D_k)-1},$$

provided that each factor D_j (except one) is asymptotically regular at infinity.

Example 3.3. (i) We can interpret Fujita's result (0.1) as

$$\frac{1}{p^*(\boldsymbol{R}^N)-1}=\frac{1}{p^*(\boldsymbol{R})-1}+\cdots+\frac{1}{p^*(\boldsymbol{R})-1}\quad (N\text{ times}),$$

based on the particular case $p^*(\boldsymbol{R})=1+2/1=3$.
(ii) The domain $D=\{x\in\boldsymbol{R}^N; x_1\geq 0,\cdots,x_k\geq 0\}$ considered by [M1] can be regarded as the product $D=\boldsymbol{R}^{N-k}\times\boldsymbol{R}_+^k$. According to [M1] we have

(0.2bis) $$p^*(D)=1+\frac{2}{N+k},\quad\text{especially}\quad p^*(\boldsymbol{R}_+)=1+\frac{2}{1+1}=2.$$

Our formula then implies that

$$\frac{1}{p^*(\boldsymbol{R}^{N-k}\times\boldsymbol{R}_+^k)-1}=\frac{1}{p^*(\boldsymbol{R}^{N-k})-1}+k\frac{1}{p^*(\boldsymbol{R}_+)-1}.$$

This is consistent because $N+k=(N-k)+2k$.
(iii) In particular, we have

$$p^*(\boldsymbol{R}\times D)=\frac{3p^*(D)-1}{p^*(D)+1},\quad p^*(\boldsymbol{R}_+\times D)=\frac{2p^*(D)-1}{p^*(D)}. \tag{3.8}$$

(iv) Let D_1 be a bounded domain and let D_2 be arbitrary. In this case we have $p^*(D_1)=1$, as was first proved by Itô [I]. Hence our formula implies $p^*(D_1\times D_2)=1$, irrespective of the asymptotic regularity of D_2 at infinity, as remarked in the proof of Theorem 2.3. Thus in particular the critical exponent of blowup of any cylinder or half-cylinder with bounded base is equal to 1.

If we combine these with Lemma 1.3 on the comparison of domains in inclusion relation, we can determine the critical exponent of blowup for wider class of domains:

Example 3.4. Let D be the paraboloid $x_n\geq x_1^2+\cdots+x_{n-1}^2$ in $\boldsymbol{R}^N$. Then we have $p^*(D)=1$ and $p^*(\boldsymbol{R}^N\setminus\bar{D})=1+2/N$. In fact, since D contains a half-cylinder, we have $p^*(D)\geq 1$. On the other hand, since D is contained in a (translated) cone Γ of arbitrarily narrow base $\Omega\subset\boldsymbol{S}^{N-1}$, we have

$$p^*(D)\leq 1+\frac{2}{N+\gamma},$$

where $\gamma\to+\infty$ as Ω becomes smaller. Thus we obtain $p^*(D)=1$. We can obtain the value of $p^*(\boldsymbol{R}^N\setminus\bar{D})$ quite similarly, by comparing $\boldsymbol{R}^N\setminus\bar{D}$ with $\boldsymbol{R}^N\setminus\bar{\Gamma}$ and $\boldsymbol{R}^N$,

Finally, we shall distinguish the following geometric application of our formula which seems to be more interesting than the other examples. This complicated formula would never have been remarked through properly geometric considerations:

PROPOSITION 3.5. i) *Let Γ_j, $j=1,2$ be two cones in $\boldsymbol{R}^{N_j}$, $j=1,2$, respectively. Put*

$$\Omega_j=\Gamma_j\cap\boldsymbol{S}^{N_j-1},\ j=1,2,\quad \Omega=(\Gamma_1\times\Gamma_2)\cap\boldsymbol{S}^{N_1+N_2-1}.$$

Then the first Dirichlet eigenvalue ω of the Laplace-Beltrami operator *of the spherical domain Ω is determined from those ω_j, $j=1,2$ of Ω_j, $j=1,2$ by the formula*

$$\omega=\omega_1+\omega_2+2(\sqrt{\omega_1+(N_1/2-1)^2}+1)(\sqrt{\omega_2+(N_2/2-1)^2}+1)-\frac{N_1N_2}{2}. \tag{3.9}$$

ii) *As the limiting case, let $\Omega'\subset S^{N-2}$ be an $(N-2)$-dimensional spherical domain considered as situated in the equator of S^{N-1}. Let Ω denote the slice-formed domain of S^{N-1} defined as the pull-back $\pi^{-1}\Omega'$ of Ω' by the projection from the poles to the equator, and let Ω_+ be its part in the northern hemisphere. Then the first Dirichlet eigenvalue ω resp. ω_+ of these domains can be expressed by the corresponding value ω' of Ω' as follows*:

$$\omega=\omega'+\sqrt{\frac{(N-3)^2}{4}+\omega'}-\frac{N-3}{2},$$

$$\omega_+=\omega'+3\sqrt{\frac{(N-3)^2}{4}+\omega'}-\frac{N-7}{2}.$$

PROOF. By the result of [LM1] we have

$$p^*(\Gamma_j)=1+\frac{2}{\gamma_j+N_j},\quad j=1,2$$

and similar formula for $p^*(\Gamma_1\times\Gamma_2)$. Combining these with (0.4) which can be applied in view of Proposition 3.1, and eliminating γ,γ_j from

$$\omega=\gamma(\gamma+N_1+N_2-2),\quad \gamma_j(\gamma_j+N_j-2)-\omega_j=0,\ j=1,2,\quad \gamma=\gamma_1+\gamma_2.$$

we obtain (3.9). The second assertion can be proved just similarly with use of (3.8). (This is not a special case of the first one.)

Note that Ω is not the product (of Ω_j, $j=1,2$. This is why the formula is much more complicated than the simple sum of the eigenvalues of the factors).

Example 3.6. The first Dirichlet eigenvalue $\pi^2/(\beta-\alpha)^2$ of the arc $\alpha\le\varphi\le\beta$ in S^1 for the Laplace-Beltrami operator $-\partial^2/\partial\varphi^2$ is obtained by an elementary calculation. Thus, the first Dirichlet eigenvalue of the 2-dimensional spherical slice domain $\alpha\le\varphi\le\beta$, $0\le\theta\le\pi$ is equal to

$$\frac{\pi^2}{(\beta-\alpha)^2}+\frac{\pi}{\beta-\alpha},$$

and that of $\alpha\leq\varphi\leq\beta$, $0\leq\theta\leq\pi/2$ is equal to

$$\frac{\pi^2}{(\beta-\alpha)^2}+3\frac{\pi}{\beta-\alpha}+2.$$

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