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Stability of stationary interfaces in a generalized mean curvature flow

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Abstract. The dynamics of hypersurfaces whose normal velocities depend on mean curvatures and positions are considered. It is shown that the stability of stationary hypersurfaces is determined by an eigenvalue problem on the hypersurfaces and in consequence, the stability of stationary hypersurfaces with various shapes is discussed. For example, it is shown that any bounded stationary hypersurface is unstable if the normal velocity does not depend on positions.

§ 1. Introduction.

Let $N\geq 2$ be an integer, and let $\Gamma(t)$ be a family of $(N-1)$-dimensional hypersurfaces that are boundaries of open sets. Because $\Gamma(t)$ separates the open set from its complement, we will call such $\Gamma(t)$ an interface. We assume that $\Gamma(t)$ evolves continuously depending on its curvature, normal vector and position. Then the dynamics of $\Gamma(t)$ can be written as

(1.1) $$\begin{aligned}&V(x)=F(\kappa(x), \nu(x), x) \qquad x\in\Gamma(t), \qquad t>0,\\ &\Gamma(0)=\Gamma^0.\end{aligned}$$

Here $\kappa(x)$ and $\nu(x)$ are the mean curvature and the outward unit normal vector of $\Gamma(t)$ at $x\in\Gamma(t)$, $V(x)$ the normal velocity in the outward direction, $F(\kappa, \nu, x)$ a function of $(\kappa, \nu, x)\in \boldsymbol{R}\times S^{N-1}\times \boldsymbol{R}^N$. In order to ensure the existence and uniqueness of a (generalized) solution of (1.1), we assume that F is continuously differentiable with respect to (κ, ν, x) and satisfies

(1.2) $$\frac{\partial}{\partial\kappa}F<0 \qquad \text{for all} \quad (\kappa, \nu, x).$$

(See Chen-Giga-Goto [1] and Evans-Spruck [4].)

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We say that $\Gamma=\Gamma^e$ is a *stationary interface* for (1.1) if and only if $V(x)=0$ on Γ^e, i.e. Γ^e is a stationary interface if and only if Γ^e is a C^2-hypersurface that is a boundary of an open set and satisfies

$$F(\kappa(x), \nu(x), x)=0, \qquad x\in\Gamma^e.$$

The purpose of this paper is to study the stability of stationary interfaces.

The dynamics of interfaces of the form (1.1) has been extensively studied. When

$$F=-(N-1)\kappa,$$

(1.1) is called a mean curvature flow equation in $\boldsymbol{R}^N$. It was shown in [1] that, for any bounded Γ^0, $\Gamma(t)$ disappears in a finite time (see also [5, 6, 7]). Hence there exists no stationary interface in this case. If $N=2$ and Γ is in a bounded domain Ω and has endpoints on $\partial\Omega$, then the stationary interface exists and the interface with minimal length is stable (see [10]). The more general case that interfaces have boundaries will be treated in the forthcoming paper [3]. On the other hand, Ei-Iida-Yanagida [2] introduced

$$F=\{-(N-1)d(x)\kappa(x)-\langle\nabla d(x), \nu(x)\rangle+c\}d(x)$$

as the dynamics of internal layers of a certain spatially inhomogeneous reaction-diffusion equation, where $d(x)$ is a given positive function, c is a constant, and $\langle\cdot,\cdot\rangle$ denotes the inner product of two N-dimensional vectors. As we will see in a simple example, there exists a stationary interface by taking F suitably.

In Section 2, we describe a precise definition of the stability, and then consider some special forms of F. First we consider the case where F is invariant in some direction. In this case, we will show that any bounded stationary interface is unstable. In particular, if F does not depend on x, then any stationary interface is unstable. Next we consider the case where F is invariant with respect to a certain rotation. In this case, we will show that any stationary interface is unstable if it is not invariant under the rotation. In particular, if F is radially symmetric with respect to origin, then any non-radial interface is unstable. We also give a simple example which admits a stable radial interface.

In order to prove these results, we consider a linearized equation of (1.1) at Γ^e, and introduce an associated eigenvalue problem. We show in Section 3 that the stability is closely related with the sign of the maximal eigenvalue. By using this property, we will give proofs of our main results in Sections 4 and 5.

In this paper, we only consider the interfaces without boundary. The dynamics of interfaces with boundaries will be studied in a forthcoming paper [3].

§ 2. Definitions and main results.

Let $D(\Gamma)$ denote an open set surrounded by an interface Γ, and let $\operatorname{dist}(x, \Gamma)$ be a signed distance defined by

$$\operatorname{dist}(x, \Gamma)=\begin{cases} +\inf_{y\in\Gamma}|y-x| & \text{if} \quad x\notin D(\Gamma), \\ -\inf_{y\in\Gamma}|y-x| & \text{if} \quad x\in D(\Gamma). \end{cases}$$

Let ε be a small parameter, and let $D(\Gamma, \varepsilon)$ be an open set in $\boldsymbol{R}^N$ defined by

$$D(\Gamma, \varepsilon):=\{x\in \boldsymbol{R}^N: -\infty<\operatorname{dist}(x, \Gamma)<\varepsilon\}.$$

Let Γ^e be a stationary interface of (1.1). We define the stability of Γ^e (in the sense of Liapunov) as follows. The stationary interface Γ^e is said to be *stable* if, given any $\varepsilon>0$, there exists $\delta>0$ such that, for any Γ^0 with

$$D(\Gamma^e, -\delta)\subset D(\Gamma^0)\subset D(\Gamma^e, +\delta),$$

the solution of (1.1) subject to the initial condition $\Gamma(0)=\Gamma^0$ satisfies

$$D(\Gamma^e, -\varepsilon)\subset D(\Gamma(t))\subset D(\Gamma^e, +\varepsilon) \qquad \text{for all} \quad t\geqq 0.$$

More strongly, the stationary interface Γ^e is said to be *exponentially stable* if it is stable and satisfies

$$D(\Gamma^e, -C_1\exp(-C_2t))\subset D(\Gamma(t))\subset D(\Gamma^e, C_1\exp(-C_2t)) \qquad \text{for all} \quad t\geqq 0$$

for some constants $C_1>0$ and $C_2>0$. The stationary interface Γ^e is said said to be *unstable* if it is not stable.

Let p be an $(N-1)$-dimensional vector. We say that F is invariant in the direction of p if, for any $(\kappa, \nu, x)\in \boldsymbol{R}\times S^{N-1}\times \boldsymbol{R}^N$, F satisfies $F(\kappa, \nu, x+Cp)\equiv F(\kappa, \nu, x)$ for all $C\in \boldsymbol{R}$.

Our first result is as follows.

THEOREM 2.1. *Suppose that F is invariant in some direction. Then any bounded stationary interface is unstable.*

The following corollary is a direct consequence of Theorem 2.1.

COROLLARY 2.1. *Suppose that F does not depend on x, that is $F=F(\kappa, \nu)$. Then any bounded stationary interface is unstable.*

Next, let T_θ be an $N\times N$ matrix given by

$$T_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Let U be an element of $SO(N)$, and let $G(U)$ be a one-parameter group of transformations

$$G(U) := \{U^{-1}T_\theta U\}_{\theta\in[0,2\pi)}.$$

We say that F is $G(U)$-invariant if, for any $(\kappa, \nu, x)\in \boldsymbol{R}\times S^{N-1}\times \boldsymbol{R}^N$, F satisfies

$$F(\kappa, \nu, x)\equiv F(\kappa, \nu, U^{-1}T_\theta Ux)$$

for all $\theta\in[0, 2\pi)$. We also say that Γ^e is $G(U)$-invariant if, for any $x\in\Gamma^e$,

$$U^{-1}T_\theta Ux\in\Gamma^e$$

holds for all $\theta\in[0, 2\pi)$.

THEOREM 2.2. *Suppose that F is $G(U)$-invariant for some U. Then any bounded stationary interface is unstable if it is not $G(U)$-invariant.*

We note that F is radially symmetric with respect to $x=0$, that is $F=F(\kappa, \nu, |x|)$ if and only if F is $G(U)$-invariant for all $U\in SO(N)$. Similarly Γ^e is radially symmetric with respect to $x=0$ if and only if Γ^e is $G(U)$-invariant for all $U\in SO(N)$. Thus the following corollary immediately follows from Theorem 2.2.

COROLLARY 2.2. *Suppose that $F=F(\kappa, \nu, |x|)$. Then any bounded* *stationary interface is unstable if it is not radially symmetric with respect to* $x=0$.

If Γ^e is radially symmetric, then it may be stable. In the next section, we will give a simple example which admits a stable stationary interface.

§ 3. Linearization.

In this section, we write the outward normal unit vector, the mean curvature and the principal curvatures at $x \in \Gamma^e$ of Γ^e by $\nu(x)$, $\kappa(x)$ and $\kappa_i(x)$, respectively.

Let $\Gamma(t)$ be a perturbation of Γ^e, and assume that $\Gamma(t)$ can be written as

$$\Gamma(t) = \{y := x + \varepsilon s(t, x)\nu(x);\ x \in \Gamma^e\},$$

where $\varepsilon > 0$ is a sufficiently small parameter and $s(t, x)$ is a function of t and $x \in \Gamma^e$. Then we have

$$\begin{aligned} V(y) &= \varepsilon s_t(t, x) + 0(\varepsilon^2), \\ \kappa(y) &= \kappa(x) - \varepsilon(N-1)^{-1}\left\{\Delta s(t, x) + s(t, x)\sum_{i=1}^{N-1}\kappa_1(x)^2\right\} + 0(\varepsilon^2), \\ \nu(y) &= \nu(x) - \varepsilon\nabla s(t, x) + 0(\varepsilon^2), \end{aligned}$$

where Δ and ∇ are the Laplacian and the gradient on Γ^e. On above calculations, we refer, e.g., Lemma 3.6 in [9]. We put

$$\begin{aligned} F_\kappa &:= \frac{\partial F}{\partial \kappa}(\kappa(x), \nu(x), x), \\ F_\nu &:= \frac{\partial F}{\partial \nu}(\kappa(x), \nu(x), x), \\ F_x &:= \frac{\partial F}{\partial x}(\kappa(x), \nu(x), x). \end{aligned}$$

From the above consideration, we see that $s(t, x)$ satisfies

$$s_t(t, x) = L[s(t, x)] + 0(\varepsilon).$$

where L is a linear differential operator on Γ^e given by

$$L[s(t, x)] := -(N-1)^{-1}F_\kappa\left\{\Delta s(t, x) + s(t, x)\sum_{i=1}^{N-1}\kappa_1(x)^2\right\}$$

$$-\langle F_\nu, \nabla s(t, x)\rangle + (F_x, \nu\rangle s(t, x).$$

If we neglect higher order terms of ε and seek $s(t, x)$ of the form $s(t, x) = \exp(\lambda t)\varphi(x)$, then we are led to the following eigenvalue problem on Γ^e:

$$\lambda\varphi(x) = L[\varphi(x)], \qquad x \in \Gamma^e. \tag{3.1}$$

LEMMA 3.1. *Let $\sigma(L)$ be the spectrum of L and put*

$$\lambda_0 := \sup(\mathrm{Re}\,\lambda;\ \lambda \in \sigma(L)\}.$$

Then λ_0 is a simple eigenvalue of L and its associated eigenfunction can be taken positive on Γ^e.

PROOF. Let $\tilde{L}$ be an operator defined by $\tilde{L} = L - \omega I$, where ω is a real parameter and I is an identity operator. Let $\sigma(\tilde{L})$ be the spectrum of $\tilde{L}$. It is clear that

$$\sigma(\tilde{L}) = \{\lambda - \omega;\ \lambda \in \sigma(L)\}.$$

In view of (1.2), if we take $\omega > 0$ large enough, then-$\tilde{L}$ becomes a positive operator. Hence it follows from Krein-Rutman's theorem [8] that

$$\tilde{\lambda}_0 := \sup\{\mathrm{Re}\,\tilde{\lambda} : \tilde{\lambda} \in \sigma(\tilde{L})\} = \lambda_0 - \omega.$$

is a simple eigenvalue of $\tilde{L}$, and its associated eigenfunction can be taken positive on Γ^e.

Let $\varphi_0(x)$ be an eigenfunction of $\tilde{L}$ associated with $\tilde{\lambda}_0$. Then $\varphi_0(x)$ is also an eigenfunction of L associated with the eigenvalue λ_0. This completes the proof. ■

The following theorem implies that the linearized stability (resp. instability) implies the Liapunov stability (resp. instability).

THEOREM 3.1. *Let Γ^e be a bounded stationary interface. If $\lambda_0 < 0$, then Γ^e is exponentially stable. Conversely, if $\lambda_0 > 0$, then Γ^e is unstable.*

PROOF. First we assume that $\lambda_0 < 0$. Note that the eigenfunction associated with λ_0 satisfies $\varphi_0(x) > 0$ on Γ^e. Let $\mu > 0$ be a small parameter and let $\Gamma^+(t)$, $t \geq 0$, be a family of interfaces given by

$$\Gamma^+(t) := \{y = x + \mu \exp(\lambda_0 t/2)\varphi_0(x)\nu(x);\ x \in \Gamma^e\}.$$

We denote the outward normal velocity, the mean curvature and outward normal unit vector of $\Gamma^+(t)$ by $V^+(y)$, $\kappa^+(y)$ and $\nu^+(y)$, respectively.

Then we have

$$V^{+}(y)=(\lambda_0/2)\mu\exp(\lambda_0 t/2)\varphi_0(x)+0(\mu^2),$$

and

$$\begin{aligned}F(\kappa^{+}(y),\nu^{+}(y),y)&=L[\mu\exp(\lambda_0 t/2)\varphi_0(x)]+0(\mu^2)\\&=\lambda_0\mu\exp(\lambda_0 t/2)\varphi_0(x)+0(\mu^2).\end{aligned}$$

Hence, if $\mu>0$ is sufficiently small, then

$$V^{+}(y)>F(\kappa^{+}(y),\nu^{+}(y),y)\qquad\text{for all}\quad t\geqq 0.$$

Thus, by applying Theorem 4.1 of [1], we see that if $\Gamma^0\subset\Gamma^{+}(0)$, then the solution $\Gamma(t)$ of (1.1) with $\Gamma(0)=\Gamma^0$ satisfies

$$D(\Gamma(t))\subset D(\Gamma^{+}(t))\qquad\text{for all}\quad t\geqq 0. \tag{3.2}$$

Similarly, let $\Gamma^{-}(t)$, $t>0$, be a family of interfaces given by

$$\Gamma^{-}(t):=\{y=x-\mu\exp(\lambda_0 t/2)\varphi_0(x)\nu(x);\ \ x\in\Gamma^e\},$$

and let $V^{-}(y)$, $\kappa^{-}(y)$ and $\nu^{-}(y)$ denote the outward normal velocity, the mean curvature and the outward normal unit vector of $\Gamma^{-}(t)$, respectively. Then we have

$$V^{-}(y)<F(\kappa^{-}(y),\nu^{-}(y),y).$$

Hence, by Theorem 4.1 of [1], we see that if $\Gamma^{-}(0)\subset\Gamma^0$, then

$$D(\Gamma^{-}(t))\subset D(\Gamma(t))\qquad\text{for all}\quad t\geqq 0. \tag{3.3}$$

Now, for any $\varepsilon>0$, we take $\mu>0$ so small that

$$\max_{x\in\Gamma^e}\mu\varphi_0(x)<\varepsilon,$$

and then take $\delta>0$ so small that

$$0<\delta<\min_{x\in\Gamma^e}\mu\varphi_0(x).$$

By (3.2) and (3.3), if

$$D(\Gamma^e,-\delta)\subset D(\Gamma^0)\subset D(\Gamma^e,+\delta),$$

then $\Gamma(t)$ satisfies

$$D(\Gamma^e,-\varepsilon)\subset D(\Gamma^{-}(t))\subset D(\Gamma(t))\subset D(\Gamma^{+}(t))\subset D(\Gamma^e,+\varepsilon)$$

for all $t \geqq 0$. This implies that the stationary interface is stable. Moreover it follows from (3.2) and (3.3) that, by taking $C_1=\delta$ and $C_2=-\lambda_0/2>0$, we have

$$D(\Gamma^e, C_1\exp(-C_2t))\subset D(\Gamma^-(t))\subset D(\Gamma(t))\subset D(\Gamma^+(t))\subset D(\Gamma^e,\ C_1\exp(-C_2t)$$

for all $t \geqq 0$. Thus it is shown that Γ^e is exponentially stable.

Next we assume that $\lambda_0>0$. Let $\varepsilon>0$ be sufficiently small and be fixed. For any $\delta>0$, we take $\mu>0$ so small that

$$\max_{x\in\Gamma^e} \mu\varphi_0(x)<\delta.$$

Now put

$$\Gamma^0=\{y=x+\mu\varphi_0(x)\nu(x);\ x\in\Gamma^e\}.$$

We note that, if Γ^e is stable, then the solution $\Gamma(t)$ of (1.1) with $\Gamma(0)=\Gamma^0$ must satisfy

$$D(\Gamma^e,\ -\varepsilon)\subset D(\Gamma(t))\subset D(\Gamma^e,\ +\varepsilon)$$

by taking δ sufficiently small.

Let

$$\Gamma^+(t)=\{y=x+\mu\exp(\lambda_0t/2)\varphi_0(x)\nu(x);\ x\in\Gamma^e\}.$$

Then we can show in a similar manner to (3.2) that

$$V^+(y)<F(\kappa^+(y), \nu^+(y), y)$$

as long as

$$|\mu\exp(\lambda_0t/2)\varphi_0(x)|\leqq\varepsilon.$$

Hence we have

$$D(\Gamma^+(t))\subset D(\Gamma(t))$$

as long as $D(\Gamma^+(t))\subset D(\Gamma^e,\ \varepsilon)$. This leads to

$$D(\Gamma(t))\not\subset D(\Gamma^e,\ \varepsilon)$$

for some $t>0$. Hence Γ^e is unstable. ∎

Here we apply the above theorem to a simple example. Assume that F is given by

$$F=-\kappa+g(|x|),$$

where g is a C^1-function satisfying

$$g(1)=1/(N-1).$$

Then

$$\Gamma^e:=\{x\in \boldsymbol{R}^N;\ |x|=1\}$$

is a stationary interface. The eigenvalue problem (3.1) can be written as

$$\lambda\varphi(x)=(N-1)^{-1}\Delta\varphi(x)+\{1+g'(1)\}\varphi(x), \qquad x\in\Gamma^e.$$

We note that $\varphi(x)\equiv 1$ is an eigenfunction associated with $\lambda_0=1+g'(1)$. Hence, by Theorem 3.1. Γ^e is exponentially stable if $g'(1)<-1$ and is unstable if $g'(1)>-1$.

§ 4. Proof of Theorem 2.1.

In this section, we complete the proof of Theorem 2.1.

Assume that F is invariant in the direction of p, and let Γ^e be a bounded stationary interface. Let Γ be an interface defined by

$$\Gamma:=\{x+\mu p;\ x\in\Gamma^e\},$$

where μ is a real parameter. Then, by assumption, Γ also is a stationary interface for every μ. Since Γ^e is bounded, we have $\Gamma\neq\Gamma^e$ for $\mu\neq 0$. If $|\mu|$ is sufficiently small, then Γ can be written as

$$\Gamma:=\{x+v(x;\ \mu)\nu(x);\ x\in\Gamma^e\}$$

by using some function $v(x;\ \mu)$ on Γ^e.

Let $g(x)$ be a function on Γ^e defined by

$$g(x):=\frac{\partial v}{\partial\mu}(x;\ 0).$$

We note that $g(x)$ is written as

$$g(x)=\langle p, \nu(x)\rangle.$$

It is clear from the definition of L that $g(x)$ satisfies $L[g(x)]=0$. Hence $g(x)$ is an eigenfunction of L associated with $\lambda=0$. Since Γ^e is bounded, $g(x)$ is not identically zero. Moreover, $g(x)$ must change its sign because the volume of $D(\Gamma)$ is equal to that of $D(\Gamma^e)$. By Lemma 3.1, this implies that $\lambda_0>0$. Thus, by Theorem 3.1. Γ^e is unstable. ■

§ 5. Proof of Theorem 2.2.

In this section, we complete the proof of Theorem 2.2.

Assume that F is $G(U)$-invariant for some $U\in SO(N)$, and let Γ^e be a bounded stationary interface that is not $G(U)$-invariant. Let Γ be an interface defined by

$$\Gamma:=\{x'=U^{-1}T_\theta Ux;\ x\in\Gamma^e\},$$

where $\theta\in[0,2\pi)$ is a parameter. Then, by assumption, Γ also is a stationary interface for any θ. Assume that θ is sufficiently small. Then $\Gamma\neq\Gamma^e$ and Γ can be rewritten as

$$\Gamma:=\{x+w(x;\ \theta)\nu(x);\ x\in\Gamma^e\}$$

by using some function $w(x;\ \theta)$ on Γ^e.

Let $h(x)$ be a function on Γ^e defined by

$$h(x):=\frac{\partial w}{\partial\theta}(x;\ 0).$$

We note that $h(x)$ is written as

$$h(x)=\langle q(x),\nu(x)\rangle,$$

where $q(x)$ is an N-dimensional vector given by

$$q(x):=\lim_{\theta\to 0}(U^{-1}T_\theta Ux-x)/\theta.$$

It is clear from the definition of L that $h(x)$ satisfies $L[h(x)]=0$. Hence $h(x)$ is an eigenfunction of L associated with $\lambda=0$. Since Γ^e is not $G(U)$-invariant, $h(x)$ is not identically zero. Moreover, since the volume of $D(\Gamma)$ is equal to that of $D(\Gamma^e)$, $h(x)$ must change its sign. By Lemma 3.1, this implies $\lambda_0>0$. Thus, by Theorem 3.1· Γ^e is unstable. ∎

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