J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
40 (1993), 229-244.

On the rigidity of PL representations of a surface group

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§ 0. Introduction.

Let Σ_2 be a closed oriented surface of genus 2 with a hyperbolic metric. The geodesic flow of the unit tangent vector bundle $T_1\Sigma_2$ of Σ_2 is an Anosov flow. So, there is an unstable foliation of this flow and this foliation is transverse to the fibres of the projection $T_1\Sigma_2 \to \Sigma_2$. Since Σ_2 has a hyperbolic metric, there exists a total holonomy homomorphism of this foliation

$$\Psi: \quad \pi_1(\Sigma_2) \longrightarrow PSL(2, \mathbf{R}).$$

Ψ is conjugate to a PL-representation

$$\Phi: \quad \pi_1(\Sigma_2) \longrightarrow PL_+(S^1)$$

as follows. Here $PL_+(S^1)$ is the group of orientation preserving piecewise linear homeomorphisms of S^1 (see § 1).

Ghys showed that this flow is obtained from the suspension flow of some hyperbolic toral automorphism

$$\bar{A}: \quad T^2 \longrightarrow T^2$$

by a certain Dehn surgery (see [G] and [H1]). In other words, there exists a homeomorphism between $T_1\Sigma_2 - \{12$ closed orbits$\}$ and the mapping torus of $\bar{A} - \{12$ periodic orbits$\}$ such that it preserves orbits and the unstable foliation. Since the suspension of the unstable linear foliation of the torus by $\bar{A}$ is transversely affine, certain Dehn surgeries along leaf curves with non-trivial holonomy give rise to a transversely PL foliation (see [G] and [H2]). Moreover the unstable foliation of the geodesic flow is transverse to the fibres of the projection $T_1\Sigma_2 \to \Sigma_2$. Hence, this transversely PL foliation can be seen as a PL foliated S^1 bundle. So we obtain Φ which is topologically conjugate to Ψ.

For any homomorphism h: $\pi_1(\Sigma_2) \to Homeo(S^1)$, we defined the Euler number $eu(h)$. Wood shows that $|eu(h)| \leq 2$ in [W]. When the absolute value of the Euler number is maximum, Matsumoto shows the next theorem.

THEOREM ([M]). *Let* ϕ, ψ: $\pi_1(\Sigma_2) \to Diff_+^2(S^1)$ *be homomorphisms such that*

$$eu(\phi) = eu(\psi) = \pm 2.$$

Then ϕ *and* ψ *are topologically conjugate.*

Hence if $\{\Psi_t\}_{|t|<\varepsilon}$ is a smooth perturbation of Ψ, then Ψ_t is topologically conjugate to Ψ. On the other hand we will show if $\{\Phi_t\}_{|t|<\varepsilon}$ is a perturbation of Φ which satisfies some conditions then Φ_t is conjugate to Φ by a rotation.

The presentation of $\pi_1(\Sigma_2)$ we used to describe Φ has a symmetric form (see (1, 2)). Considering a fundamental domain in the Poincaré disk for this presentation, we showed that there is a 6-fold covering

$$p_0: \quad T_1\Sigma_2 \longrightarrow M(3, 6, 6)$$

where $M(3, 6, 6)$ is a Brieskorn manifold $\{(z_1, z_2, z_3) \in \boldsymbol{C}^3;\ z_1^3 + z_2^6 + z_3^6 = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. $M(3, 6, 6)$ can be considered as a Seifert fibred manifold over S^2 with three singular fibres. The covering p_0 induces a transversely PL foliation which is transverse to the fibres of this Seifert fibration. So there exists a homomorphism

$$\tilde{\phi}: \quad \pi_1(M(3, 6, 6)) \longrightarrow PL_+^{\sim}(S^1)$$

such that $\tilde{\phi}(z) = T(1)$ (see [E-H-N]). Here $PL_+^{\sim}(S^1)$ is the universal covering group of $PL_+(S^1)$, $z \in \pi_1(M(3, 6, 6))$ is the class of a general fibre and $T(1)$ is a translation of $\boldsymbol{R}$ by 1. Let Γ be a triangle group $\Gamma(3, 6, 6) = \langle \tau_1, \tau_2, \tau_3;\ (\tau_1)^3 = (\tau_2)^6 = (\tau_3)^6 = \tau_1\tau_2\tau_3 = 1 \rangle$ which is isomorphic to $\pi_1(M(3, 6, 6))/\langle z \rangle$. $\tilde{\phi}$ induces a homomorphism

$$\phi: \quad \Gamma \longrightarrow PL_+(S^1)$$

such that $\Phi = \phi \circ \underline{p_{0*}}$ where

$$\underline{p_{0*}}: \quad \pi_1(\Sigma_2) = \pi_1(T_1\Sigma_2)/\langle z \rangle \longrightarrow \Gamma = \pi_1(M(3, 6, 6))/\langle z \rangle$$

is induced from p_0 and $z \in \pi_1(T_1\Sigma_2)$ is a class of a fibre of the projection $T_1\Sigma_2 \to \Sigma_2$ (see [H2]).

The purpose of this paper is to study the rigidity of ϕ.

THEOREM. ϕ *cannot be perturbed under the condition that the image* *of τ_2 is the rotation by $\frac{1}{6}$ and the number of bending points of the image of τ_1 is* 4.

So Φ cannot be perturbed keeping the symmetric relations.

The organization of this paper is as follows. In § 1, we prepare some notations. We define $PL_+^{\sim}(S^1)$ and $PL_+(S^1)$. And we explain how to describe the elements of PL homeomorphisms. We also review the PL representations ϕ and Φ. In § 2, we prove the Theorem.

Finally, the author would like to express his gratitude to Professor T. Tsuboi for valuable advice and continuous encouragement. He also thanks Professors Y. Mitsumatsu and H. Minakawa for helpful conversation.

§ 1. Preliminaries.

In this section, we prepare some notations which will be used in this paper.

Let $T(\theta)(\theta \in \boldsymbol{R})$ be the shift of $\boldsymbol{R}$ by θ, i.e.,

$$(T(\theta))(x)=x+\theta \quad (x \in \boldsymbol{R}).$$

$Homeo_+^{\sim}(S^1)$ denotes the set of orientation preserving homeomorphisms F of $\boldsymbol{R}$ such that $F \circ T(1)=T(1) \circ F$. Every $F \in Homeo_+^{\sim}(S^1)$ induces a homeomorphism f: $\boldsymbol{R}/\boldsymbol{Z}=S^1 \rightarrow \boldsymbol{R}/\boldsymbol{Z}=S^1$.

$$\pi: \ Homeo_+^{\sim}(S^1) \longrightarrow Homeo_+(S^1)$$

is defined by $\pi(F)=f$. Conversely for every $f \in Homeo_+(S^1)$, there exist countably many $\tilde{f} \in Homeo_+^{\sim}(S^1)$ such that $\pi(\tilde{f})=f$.

DEFINITION. Such an $\tilde{f} \in Homeo_+^{\sim}(S^1)$ is called *a lift* of f.

DEFINITION.
(i) $F \in Homeo_+^{\sim}(S^1)$ belongs to $PL_+^{\sim}(S^1)$ if F is piecewise linear and bending points of F do not accumulate in $\boldsymbol{R}$.
(ii) Let $PL_+(S^1)$ be $\pi(PL_+^{\sim}(S^1))$.

REMARK: In this paper, $Homeo_+^{\sim}(S^1)$ and $Homeo_+(S^1)$ are given the topology induced from the maximal norm. Then π is a covering. $PL_+^{\sim}(S^1) \subset Homeo_+^{\sim}(S^1)$ and $PL_+(S^1) \subset Homeo_+(S^1)$ have the induced topology.

We describe $F \in PL_{+}^{\sim}(S^1)$ by its bending points and its derivative. Since $F \circ T(1) = T(1) \circ F$, it is sufficient to describe the restriction of F on some closed interval $[x_1, x_1+1]$ where x_1 is one of the bending points of F. More precisely, if the bending points of $F|_{[x_1, x_1+1]}$ is $\{x_1, x_2, x_3, \cdots, x_p, x_1+1\}$ and $F|_{[x_1, x_1+1]}$ is given by,

$$F|_{[x_1, x_1+1]}(x) = \begin{cases} \lambda_1(x-x_1)+y_1 & \text{on } [x_1, x_2] \\ \lambda_2(x-x_2)+y_2 & \text{on } [x_2, x_3] \\ \lambda_3(x-x_3)+y_3 & \text{on } [x_3, x_4] \\ \vdots & \\ \lambda_p(x-x_p)+y_p & \text{on } [x_p, x_1+1], \end{cases}$$

where $y_i = F(x_i)$ $(i=1, 2, 3, \cdots p)$, then we write

$$\begin{array}{rl} F\colon & x_1 \longmapsto y_1 \\ & [\lambda_1] \\ & x_2 \longmapsto y_2 \\ & [\lambda_2] \\ & x_3 \longmapsto y_3 \\ & \vdots \\ & x_p \longmapsto y_p \\ & [\lambda_p] \\ & x_1+1 \longmapsto y_1+1, \end{array}$$

or more simply

x_1	x_2	x_3	$x_4 \cdots x_{p-1}$	x_p	x_1+1
λ_1	λ_2	λ_3	$\cdots$	λ_{p-1}	λ_p
y_1	y_2	y_3	$y_4 \cdots y_{p-1}$	y_p	y_1+1

or

λ_1	λ_2	λ_3	$\cdots$	λ_{p-1}	λ_p

.

In order to describe $f \in PL_{+}(S^1)$, we also use the description of some lift of f.

For $F, G \in PL_{+}^{\sim}(S^1)$, the composition $G \circ F$ is described by a two-column table. For example, if

F:

x_1	x_2	x_3	x_4	x_1+1
λ_1	λ_2	λ_3	λ_4	
y_1	y_2	y_3	y_4	y_1+1

G:

x'_1	x'_2	x'_3	x'_4	x'_1+1
λ'_1	λ'_2	λ'_3	λ'_4	
y'_1	y'_2	y'_3	y'_4	y'_1+1

and

$$y_1<x'_1<y_2<x'_2<y_3=x'_3<x'_4<y_4<y_1+1,$$

then $G\circ F$ is described by the following,

x_1 $\cdots$ x_1+1

λ_1	λ_2	λ_3	λ_4	λ_1
λ'_4	λ'_1	λ'_2	λ'_3	λ'_4

.

By a one-column table

x_1 $\cdots$ x_1+1

$G\circ F$:

$\lambda_1\lambda'_4$	$\lambda_1\lambda'_1$	$\lambda_2\lambda'_1$	$\lambda_2\lambda'_2$	$\lambda_3\lambda'_3$	$\lambda_3\lambda'_4$	$\lambda_4\lambda'_4$

.

DEFINITION. For $F\in PL_+^{\sim}(S^1)$, the bending points of F has the period 1. Hence it is possible to consider the bending points of F in R/Z. Let $BP(F)$ be {bending points of F}/Z.

Now we define an important element f of $PL_+(S^1)$ which is needed to describe the homomorphisms Φ and ϕ. f is defined by

(1.1)
$$f:\ \frac{4-\sqrt{3}}{12}\longmapsto\frac{7\sqrt{3}-8}{12}$$

$$[\Lambda^{-1}]$$

$$\frac{10+\sqrt{3}}{12}\longmapsto\frac{10-3\sqrt{3}}{12}$$

$$[1]$$

$$\frac{8+3\sqrt{3}}{12}\longmapsto\frac{8-\sqrt{3}}{12}$$

$$[\Lambda]$$
$$\frac{26-7\sqrt{3}}{12} \longmapsto \frac{14+\sqrt{3}}{12}$$
$$[1]$$
$$\frac{16-\sqrt{3}}{12} \longmapsto \frac{7\sqrt{3}+4}{12},$$

where $\Lambda=7+4\sqrt{3}$. f has the next properties;

(1) $\{\bar{f}\}^3=1$,

(2) $\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^6=1$,

where $\bar{f}=f\circ T\left(-\frac{1}{3}\right)$ and $T(\theta)$ $(\theta\in S^1=\boldsymbol{R}/\boldsymbol{Z})$ is the rotation of S^1 by θ (for convenience, we use the same notation as the shift of $\boldsymbol{R}$).

Σ_2 denotes the orientable closed surface with genus 2 and its fundamental group $\pi_1(\Sigma_2)$ has the next presentation

$$\text{(1.2)} \qquad \begin{aligned}\pi_1(\Sigma_2) &= \langle a_1, a_2, a_3, a_4, a_5, a_6;\ a_1a_3a_5=1,\ a_2a_4a_6=1,\ a_1a_2a_3a_4a_5a_6=1\rangle \\ &= \langle a_1, a_3, a_4, a_6;\ [a_6, a_1^{-1}][a_3, a_4^{-1}]=1\rangle.\end{aligned}$$

Let Γ be the triangle group

$$\text{(1.3)} \qquad \Gamma(3, 6, 6)=\langle \tau_1, \tau_2, \tau_3;\ (\tau_1)^3=(\tau_2)^6=(\tau_3)^6=\tau_1\tau_2\tau_3=1\rangle.$$

In [H2], we obtained a piecewise linear representation Φ of the unstable foliation of the geodesic flow on Σ_2 with a hyperbolic metric.

$$\Phi:\ \pi_1(\Sigma_2) \longrightarrow PL_+(S^1)$$

is defined by

$$\Phi(a_i)=T\left(-\frac{i-1}{6}\right)\circ f\circ T\left(\frac{i-1}{6}\right) \qquad (i=1, 2, 3, 4, 5, 6).$$

Φ is topologically conjugate to a total holonomy homomorphism Ψ of the unstable foliation of the geodesic flow on the unit tangent vector bundle $T_1\Sigma_2$ (see § 0). Moreover we obtained two homomorphisms. One is

$$\phi:\ \Gamma \longrightarrow PL_+(S^1)$$

defined by

$$\phi(\tau_1)=\bar{f}$$

$$\phi(\tau_2)=T\left(\frac{1}{6}\right)$$

which represents a foliation on the Brieskorn manifold $M(3, 6, 6)$ transverse to the Seifert fibration. And the other is

$$\underline{p_{0*}}\colon\ \pi_1(\Sigma_2) \longrightarrow \Gamma$$

defined by

$$\underline{p_{0*}}(a_i)=(\tau_2)^{1-i}\tau_1(\tau_2)^{i+1} \qquad (i=1, 2, 3, 4, 5, 6)$$

which is induced from a covering map p_0: $T_1\Sigma_2 \to M(3, 6, 6)$.

PROPOSITION 1 ([H2, PROPOSITION 7]).

$$\Phi=\phi\circ\underline{p_{0*}}.$$

§ 2. Rigidity of ϕ.

In this section, we prove that ϕ cannot be perturbed under some conditions.

DEFINITION. For $g \in PL_+(S^1)$

$$B_1(g)=\{x\in BP(g);\ g(x)\notin BP(g)\}$$
$$B_2(g)=\{x\in BP(g);\ g(x)\in BP(g) \text{ and } g^2(x)\in BP(g)\}$$
$$B_3(g)=\{x\in BP(g);\ g(x)\in BP(g) \text{ and } g^2(x)\in BP(g)\}.$$

DEFINITION. For a finite set B, let $\sharp B$ be the cardinal number of B.

LEMMA. *Suppose that $g\in PL_+(S^1)$ has no fixed points and $g^3=1$. Then*

(1) *$\sharp B_3(g)$ is a multiple of 3,*
(2) $\sharp BP(g)=2\sharp B_1(g)+\sharp B_3(g)$,
(3) $\sharp B_1(g)=\sharp B_2(g)$.

PROOF: Since $g^3=1$, B_3 is invariant under g. From the assumption, g has no fixed points. So (1) holds. Let $x\in B_2(g)$. Then

$$g\colon\ x \underset{[\mu]}{\overset{[\lambda]}{\longmapsto}} g(x)$$

and

$$g\colon\quad g(x) \underset{[\mu']}{\overset{[\lambda']}{\longmapsto}} g^2(x),$$

where $\lambda \neq \mu$ and $\lambda' \neq \mu'$. If $\lambda\lambda' \neq \mu\mu'$, then $x \in BP(g^2)$. Because $g^2(x) \notin BP(g)$, $x \in BP(g^3) = BP(\text{identity of } S^1) = \emptyset$. This is a contradiction. So x is not a bending point of $g^2 = g^{-1}$. Hence,

$$\begin{aligned}\sharp BP(g^2) &= \sharp BP(g) + \sharp g^{-1}(BP(g)) - 2\sharp B_2(g) - \sharp B_3(g)\\ &= 2\sharp B_1(g) + \sharp B_3(g)\end{aligned}$$

for $\sharp BP(g) = \sharp B_1(g) + \sharp B_2(g) + \sharp B_3(g)$. (2) is proved. Now (3) is obvious. ∎

PROPOSITION 2. *There is a neighborhood U of $\bar{f}$ in $PL_+(S^1)$ such that if $g \in U$ satisfies the next three conditions*

(1) $\sharp BP(g) = 4$,
(2) $g^3 = 1$,
(3) $\left\{g \circ T\left(\frac{1}{6}\right)\right\}^6 = 1$,

then g is conjugate to $\bar{f}$ by some rotation of S^1.

Therefore,

THEOREM.

$$\phi\colon\quad \Gamma \longrightarrow PL_+(S^1)$$

cannot be perturbed under the condition that the image of τ_2 is the rotation $T\left(\frac{1}{6}\right)$ and $\sharp BP$(the image of τ_1) $= 4$.

REMARK: Up to the conjugation by some PL-homeomorphism of S^1, every homomorphism

$$\zeta\colon\quad \Gamma \longrightarrow PL_+(S^1)$$

satisfies $\zeta(\tau_2) = T\left(\frac{1}{6}\right)$.

From proposition 1,

COROLLARY. *Φ cannot be perturbed up to the conjugation by PL-homeomorphisms of S^1 keeping the properties*

(1) *the image of a_i is conjugate to that of a_1 by the rotation*

$$T\left(\frac{i-1}{6}\right) \quad (i=2, 3, \cdots, 6),$$

(2) $\sharp BP(\Phi(a_1))=4$.

Proof of Proposition 2: Suppose there is $g \in PL_+(S^1)$ near $\bar{f}$ such that $\sharp BP(g)=4$, $g^3=1$, $\left\{g \circ T\left(\frac{1}{6}\right)\right\}^6=1$. From the above lemma (2), $4=2\sharp B_1(g)+\sharp B_3(g)$. From lemma (1) and (3), $\sharp B_2(g)=\sharp B_1(g)=2$ and $\sharp B_3(g)=0$.

Step 1. To begin with, we consider the case that the derivative of g are λ^{-1}, 1, λ, 1 ($\lambda>1$), i.e.,

	x_1	x_2	x_3	x_4	x_1+1
g:	λ^{-1}	1	λ	1	.
	y_1	y_2	y_3	y_4	y_1+1

The two-column table of $\{\bar{f}\}^2$ is

Λ^{-1}	1	Λ	1	
1	Λ	1	Λ^{-1}	1

Conjugating by some rotation, we may think that $x_1=0$. Since $g^3=1$ and g is near $\bar{f}$, $B_2(g)=\{x_3, x_4\}$ so $y_3=x_1$ and $y_4=x_2$. Hence if we set $y_1=a$ and $x_2-x_1=\alpha$, then

$$\begin{aligned} g\colon\quad 0 &\longmapsto a \\ &[\lambda^{-1}] \\ \alpha &\longmapsto a+\frac{\alpha}{\lambda} \\ &[1] \\ \alpha-a-\frac{\alpha}{\lambda} &\longmapsto 0 \\ &[\lambda] \\ \alpha-a &\longmapsto \alpha \\ &[1] \\ 1 &\longmapsto a+1. \end{aligned}$$

The two-column table of g^2 is obtained from the above two-column table of $\{\bar{f}\}^2$ by replacing Λ by λ. Hence,

$$
\begin{array}{rl}
g^2:\quad 0 & \longmapsto 2a+\dfrac{\alpha}{\lambda}-\alpha \\
 & [\lambda^{-1}] \\
\alpha & \longmapsto 2a+2\dfrac{\alpha}{\lambda}-\alpha \\
 & [1] \\
2\alpha-2a-2\dfrac{\alpha}{\lambda}-1 & \longmapsto -1 \\
 & [\lambda] \\
2\alpha-2a-\dfrac{\alpha}{\lambda}-1 & \longmapsto \alpha-1 \\
 & [1] \\
1 & \longmapsto 2a+\dfrac{\alpha}{\lambda}-\alpha+1.
\end{array}
$$

Since $g^2=g^{-1}$,

$$\left(2a+\frac{\alpha}{\lambda}-\alpha+1\right)-(\alpha-1)=\left(\alpha-a-\frac{\alpha}{\lambda}\right)-\alpha.$$

So,

$$3a+2\frac{\alpha}{\lambda}-2\alpha+2=0. \tag{2.1}$$

The two-column table $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^2$ is also obtained from that of $\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2$ which is

Λ^{-1}	1	Λ	1	
	Λ^{-1}	1	Λ	1

.

Hence,

$$
\begin{array}{rl}
\left\{g\circ T\left(\dfrac{1}{6}\right)\right\}^2:\quad -\dfrac{1}{6} & \longmapsto \dfrac{1}{6}+2a \\
 & [\lambda^{-1}] \\
-\dfrac{\lambda}{6}-a\lambda-\dfrac{1}{6} & \longmapsto a \\
 & [\lambda^{-2}]
\end{array}
$$

$$\alpha-\frac{1}{6}\longmapsto\frac{a}{\lambda}+\frac{\alpha}{\lambda^2}+\frac{1}{6\lambda}+a$$

$$[\lambda^{-1}]$$

$$\alpha-a-\frac{\alpha}{\lambda}-\frac{1}{6}\longmapsto\frac{1}{6\lambda}+a$$

$$[1]$$

$$\alpha-a-\frac{1}{6\lambda}-\frac{1}{6}\longmapsto a+\frac{\alpha}{\lambda}$$

$$[\lambda]$$

$$\alpha-a-\frac{a}{\lambda}-\frac{\alpha}{\lambda^2}-\frac{1}{6\lambda}-\frac{1}{6}\longmapsto 0$$

$$[\lambda^2]$$

$$\alpha-a-\frac{1}{6}\longmapsto a\lambda+\alpha+\frac{\lambda}{6}$$

$$[\lambda]$$

$$\alpha-2a-\frac{1}{3}\longmapsto\alpha$$

$$[1]$$

$$\frac{5}{6}\longmapsto 2a+\frac{7}{6}.$$

The cyclic order of elements of

$$\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2\left(BP\left(\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2\right)\right)\cup BP\left(\bar{f}\circ T\left(\frac{1}{6}\right)\right)$$

is shown in the center horizontal line of the following two-column table of $\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^3$;

(2.2)

Λ^{-1}	Λ^{-2}	Λ^{-1}	1	Λ	Λ^2	Λ	1	
1			Λ^{-1}			1	Λ	1

b_1 (between Λ^{-1} and 1 of the top row), b_2 (between Λ^2 and Λ of the top row).

Two points $b_1, b_2\in BP\left(\bar{f}\circ T\left(\frac{1}{6}\right)\right)$ belong to $\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2\left(BP\left(\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2\right)\right)$.

Since g is a perturbation of $\bar{f}$, two bending points of g corresponding to b_1, b_2 may move to the left or right of some points of $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^2$ $\left(BP\left(\left\{g\circ T\left(\frac{1}{6}\right)\right\}^2\right)\right)$.

⟨*Case* 1⟩ We assume b_1 and b_2 do not move. Then

$$\begin{cases} \dfrac{1}{6\lambda}+a=-\dfrac{1}{6} \\ a\lambda+\alpha+\dfrac{\lambda}{6}=\alpha-\dfrac{1}{6}, \end{cases}$$

i.e.,

$$\frac{1}{6\lambda}+a=-\frac{1}{6}. \tag{2.3}$$

$$\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3:\ -\frac{1}{6}\longmapsto 2a-\frac{1}{6\lambda}+\frac{1}{6}$$

$$[\lambda^{-1}]$$

$$-\frac{\lambda}{6}-a\lambda-\frac{1}{6}\longmapsto a-\frac{1}{6\lambda}$$

$$[\lambda^{-2}]$$

$$\alpha-\frac{1}{6}\longmapsto a+\frac{a}{\lambda}+\frac{\alpha}{\lambda^2}$$

$$[\lambda^{-1}]$$

$$\alpha-a-\frac{1}{6\lambda}-\frac{1}{6}\longmapsto a+\frac{\alpha}{\lambda^2}-\frac{1}{6\lambda^2}$$

$$[1]$$

$$\alpha-a-\frac{a}{\lambda}-\frac{\alpha}{\lambda^2}-\frac{1}{6\lambda}-\frac{1}{6}\longmapsto a-\frac{a}{\lambda}-\frac{1}{6\lambda^2}$$

$$[\lambda]$$

$$\alpha-a-\frac{a}{\lambda}-\frac{\alpha}{\lambda^2}-\frac{1}{6}\longmapsto 0$$

$$[\lambda^2]$$

$$\alpha-2a-\frac{1}{3}\longmapsto a\lambda+\alpha+\frac{\lambda}{6}$$

$$[\lambda]$$

$$\alpha-3a-\frac{1}{2}\longmapsto\alpha$$

[1]

$$\frac{5}{6}\longmapsto 2a-\frac{1}{6\lambda}+\frac{7}{6}.$$

In order to satisfy $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3=\left\{g\circ T\left(\frac{1}{6}\right)\right\}^{-3}$,

$$\left(\alpha-a-\frac{a}{\lambda}-\frac{\alpha}{\lambda^2}-\frac{1}{6\lambda}\right)-\left(\alpha-a-\frac{1}{6\lambda}-\frac{1}{6}\right)=\left(2a-\frac{1}{6\lambda}+\frac{7}{6}\right)-\alpha.$$

So,

$$2a+\frac{a}{\lambda}+\frac{\alpha}{\lambda^2}-\frac{1}{6\lambda}-\alpha+\frac{7}{6}=0. \tag{2.4}$$

From (2.1) (2.3) and (2.4), we obtain

$$\lambda=7\pm4\sqrt{3}.$$

Since we assume $\lambda>1$, $\lambda=7+4\sqrt{3}=\Lambda$, $a=\frac{2\sqrt{3}-4}{3}$, and $\alpha=\frac{3+\sqrt{3}}{3}$. Therefore,

$$g=T\left(-\frac{8-\sqrt{3}}{12}\right)\circ\bar{f}\circ T\left(\frac{8-\sqrt{3}}{12}\right).$$

This implies that g is conjugate to $\bar{f}$ by some rotation of S^1.

⟨*Case* 2⟩ We assume either b_1 or b_2 moves to the left or right. For example, if both b_1 and b_2 move to the right, then the derivative of $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3$ is

λ^{-1}	λ^{-2}	λ^{-1}	1	λ^{-1}	1	λ	1	λ	λ^2	λ	1

So it contradicts $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3=\left\{g\circ T\left(\frac{1}{6}\right)\right\}^{-3}$ for $\lambda>1$. Similarly, unless both b_1 and b_2 move to the left, it is impossible. In this remainded case that both b_1 and b_2 move to the left, we calculate $BP\left(\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3\right)$ and

$BP\left(\left\{g\circ T\left(\frac{1}{6}\right)\right\}^{-3}\right)$ using the presentation of g as in Case 1. Then we obtain (2.3) which implies that both b_1 and b_2 do not move. This contradicts the assumption.

Step 2. We study the general case. That is to say, the derivative of g are α, β, γ and δ. As before we may assume that g is

$$g:\quad \begin{array}{ccccccccc} 0 & & x_1 & & x_2 & & x_3 & & 1 \\ \hline & \alpha & & \beta & & \gamma & & \delta & \\ \hline y_1 & & y_2 & & 0 & & x_1 & & y_1+1 \end{array}$$

where $0<\alpha<1$, $\gamma>1$ and $\beta, \delta>0$, g^2 is

$$\begin{array}{c|c|c|c|c|c} \hline & \alpha & \beta & \gamma & \delta & \\ \hline \beta & \gamma & \delta & \alpha & \beta & \\ \hline \end{array}\ .$$

Since $g^2=g^{-1}$, there exists a lift $\tilde{g}$ of g such that $\tilde{g}^2(0)=\tilde{g}(y_1)=x_2-2$, $\tilde{g}^2(x_1)=\tilde{g}(y_2)=x_3-2$, $\tilde{g}^{-2}(-1)=\tilde{g}^{-1}(x_2-1)=y_1+1$ and $\tilde{g}^{-2}(x_1-1)=\tilde{g}^{-1}(x_3-1)=y_2+1$. So, we obtain

$$\begin{array}{rl} g^2\colon\ 0 & \longmapsto x_2-2 \\ & [\alpha\beta] \\ x_1 & \longmapsto x_3-2 \\ & [\beta^2] \\ y_1+1 & \longmapsto -1 \\ & [\beta\gamma] \\ y_2+1 & \longmapsto x_1-1 \\ & [\beta\delta] \\ x_2 & \longmapsto y_1 \\ & [\alpha\gamma] \\ x_3 & \longmapsto y_2 \\ & [\beta\delta] \\ 1 & \longmapsto x_2-1. \end{array}$$

Thus we get

	y_1	y_2	0	x_1	y_1+1
g^{-1}:	α^{-1}	β^{-1}	γ^{-1}	δ^{-1}	.
	0	x_1	x_2	x_3	1

It implies that $\alpha\beta=\gamma^{-1}$ and $\beta^2=\delta^{-1}$, i.e.,

$$\begin{cases}\alpha\beta\gamma=1\\ \beta^2\delta=1.\end{cases}$$

So, we get $\beta=\dfrac{1}{\alpha\gamma}$ and $\delta=(\alpha\gamma)^2$. As in Step 1, the two-column table of $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^2$ is also obtained from that of $\left\{\bar{f}\circ T\left(\frac{1}{6}\right)\right\}^2$. Therefore, we obtain

$\left\{g\circ T\left(\frac{1}{6}\right)\right\}^2$:	$\alpha^3\gamma^2$	α^2	γ^{-1}	$\alpha\gamma$	α^{-1}	γ^2	$\alpha^2\gamma^3$	$\alpha^4\gamma^4$

The bending points of $g\circ T\left(\frac{1}{6}\right)$ corresponding to b_1 and b_2 in (2.2) may also move to the left or right. When b_1 and b_2 do not move, $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3$ is

$\alpha^5\gamma^4$	$\alpha^4\gamma^2$	$\alpha^2\gamma$	1	$\alpha\gamma^2$	$\alpha^2\gamma^4$	$\alpha^4\gamma^5$	$\alpha^6\gamma^6$

Since $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3=\left\{g\circ T\left(\frac{1}{6}\right)\right\}^{-3}$, the derivative satisfy the relations; $\alpha^5\gamma^4=(\alpha\gamma^2)^{-1}$, $\alpha^4\gamma^2=(\alpha^2\gamma^4)^{-1}$, $\alpha^2\gamma=(\alpha^4\gamma^5)^{-1}$ and $1=(\alpha^6\gamma^6)^{-1}$, i.e.,

$$\alpha^6\gamma^6=1.$$

Now $\alpha\gamma>0$, so $\alpha\gamma=1$. It is the case that we consider in Step 1.

If either b_1 and b_2 move, it is impossible that $\left\{g\circ T\left(\frac{1}{6}\right)\right\}^3=\left\{g\circ T\left(\frac{1}{6}\right)\right\}^{-3}$ with the same reason of ⟨Case 2⟩ in Step 1. Hence, Step 2 is reduced to Step 1. ■

Acknowledgements. The author would like to thank the referee for careful reading and valuable advice.

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(Received March 1, 1991)
(Revised September 7, 1992)

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