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On some alternating sum of dimensions of Siegel cusp forms of general degree and cusp configurations

By Tomoyoshi IBUKIYAMA

Abstract. We fix a prime p, and consider discrete subgroups of the symplectic group of degree n over the rational numbers such that the completion at p is a standard parahoric subgroup of the p-adic symplectic group and those at the other primes are the standard maximal compact subgroups. Firstly, we show the vanishing of the contribution of central unipotent elements to an alternating sum of the dimensions of holomorphic cusp forms belonging to the above discrete groups under the assumption that the weight is bigger than $2n$. Secondly, we describe the configuration of cusps of the Satake compactification associated with discrete subgroups of the above type and relations of cusps under the covering maps between various discrete subgroups. To prove these, we shall generalize slightly the relation between zeta functions of prehomogeneous vector spaces and dimension formula first developped by Morita and Shintani, and also give some combinatorial results on affine weyl groups.

As a generalization of the Eichler-Jacquet-Langlands correspondence between automorphic forms on SL_2 and $SU(2)$, it has been conjectured that there should exist good global correspondence between automorphic forms on $Sp(n, R)$ (the real symplectic group of size $2n$) and its compact twist $Sp(n)$. The most general conjecture on such correspondence were given by R. P. Langlands [21] for any reductive groups as the functoriality on L-groups. (See also Y. Ihara [17]). Many representation theoretical study has been done by several mathematicians. For example, J. Arthur [1] gave the simplified trace formula and R. E. Kottwitz [19] used it to prove the conjecture on Tamagawa numbers. Now, our approach to the theory is rather classical. For those holomorphic forms on $Sp(n, R)$ and spherical functions on $Sp(n)$ belonging to typical and explicitly defined discrete subgroups (parahoric subgroups), some precise conjecture on the above type correspondence has been formulated in [15],

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[16], and when $n=2$, the equality of the dimensions of such automorphic forms has been proved in [13]. In this paper, intending to generalize this dimensional equality, we shall show the vanishing of the contribution of "central" unipotent conjugacy classes to the dimension of holomorphic Siegel new cusp forms belonging to Iwahori (i.e. minimal parahoric) subgroup of $Sp(n, Q)$ for arbitrary n (§1-4. Main Theorem). Secondly, we shall give explicit structures of the cusp configurations of the Satake compactifications of the Siegel upper half space of any degree divided by any parahoric subgroups of $Sp(n, Q)$ and relations between cusps belonging to different parahoric subgroups (§5. Proposition 5.2, 5.3, 5.4). We include these results on cusps in this paper, because the proof is closely connected to the one for our Main Theorem, as will be explained later. On the vanishing of the contribution of unipotent elements, the papers J. Arthur [1] and R. E. Kottwitz [19] seem very close to the content of this paper, but it is assumed in [19] that some q-adic component of the representation treated there is absolutely cuspidal. In our case, this assumption is not satisfied. In any way, the styles are fairly different. In our approach, we use Godement's formula [9] and Shintani's theory of the zeta functions of prehomogeneous vector spaces, and the results in this paper contains explicit description of global central unipotent conjugacy classes and inequivalent cusps. Böcherer and Schulze-Pillot obtained the description of cusps for the group so called $\Gamma_0(p)$ (one of the parahoric subgroups treated in this paper) independently.

Now, the philosophical back ground of our Main Theorem is the Langlands conjecture on stable conjugacy classes: as $Sp(n)$ consists of semisimple elements, unipotent elements of $Sp(n, R)$ should have no contribution to dimensions of "new forms" (, although they usually have some contribution to the whole dimension, unless we take "new part"). Our Main Theorem is of arithmetic nature. In fact, the dimension of the new forms is expressed as some kind of alternating sum of dimensions of Siegel cusp forms belonging to various discrete subgroups (, hence our title), and the contribution of central unipotent conjugacy classes to each dimension in this sum does not vanish in general, although the whole contribution vanishes. To prove this, we need some combinatorial theory of Bruhat-Tits type (§4. Theorem 4.1).

To prove our Main Theorem, we proceed as follows. By Godements's trace formula, the dimension of Siegel cusp forms belonging to a discrete group Γ can be expressed as a sum of contributions of several Γ-conjugacy classes (Godement [9]). The contribution of central unipotent conjugacy classes has been known to be equal to special values of some zeta functions attached to prehomogenious vector spaces. (Morita [24], Shintani [27]). Although we do not know explicit values of the above special values except for a few cases, we can see, by the above Morita-Shintani's formula, which kind of data on conjugacy classes determine the contribution of the class to the dimension. We compare these data attached to families of central unipotent Γ-conjugacy classes in various Γ, and deduce the vanishing of the contribution by some combinatorial argument.

To execute these programs actually, first we have to modify Morita-Shintani's formula slightly, because some discrete subgroups we need have not been treated in their formulation. Secondly, we must classify central conjugacy classes in various Γ explicitly. Such conjugacy classes are divided into several families each of which correspond naturally to equivalence classes of cusps explicitly with the aid of the Bruhat-Tits theory. The contribution of such families to the dimension can be determined by the stabilizers in Γ of the corresponding cusps. Finally, we compare these stabilizers for various cusps and various Γ, complete the proof of our Main Theorem, and of the results on cusp configurations. Some part of our results can be generalized to the case of other algebraic groups, but omitted here.

Now, we outline the content of each section. In § 1, after some preliminary review, we shall state Main Theorem. In § 2, we shall give a modification of Morita-Shintani's formula. In § 3, we shall classify equivalence classes of cusps and central unipotent conjugacy classes. In § 4, we shall give some combinatorial theory on parahoric groups and prove Main Theorem. In § 5, we shall give the results on cusp configurations.

§ 1. Main Theorem

In this section, after a brief review on Siegel modular forms, first we shall review Godement's general dimension formula of cusp forms ([9]), and define the contribution of central unipotent elements. Secondly, we shall explain our choice of discrete subgroups and the exact meaning of "new forms". (This part is a review of [13], [14], [15].) Finally, we shall state Main Theorem.

1.1. Review on Siegel modular forms.

We fix a natural number n once and for all throughout this paper. For any commutative ring K with unit, we denote by $Sp(n, K)$ the (split) symplectic group over K of size $2n$ as usual:

$$Sp(n, K)=\{g \in M_{2n}(K);\ gJ{}^tg=J\},$$

where $J=\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, and 1_n is the unit matrix of size n. We shall often write elements g of $Sp(n, K)$ in the form

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in terms of submatirices $A, B, C, D \in M_n(K)$, and we shall say that g is composed by A, B, C, D for the sake of simplicity. Denote by H_n the Siegel upper half space of degree n:

$$H_n=\{Z=X+iY \in M_n(C);\ X, Y \in M_n(R), X={}^tX,\ Y={}^tY, \\ Y \text{ is positive definite}\}.$$

The group $Sp(n, R)$ acts on H_n by

$$gZ=(AZ+B)(CZ+D)^{-1}, \left(g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, R),\ Z \in H_n\right),$$

where g is composed by A, B, C, D. Let Γ be a discrete subgroup of $Sp(n, R)$ with $\mathrm{vol}(Sp(n, R)/\Gamma)<\infty$. For the sake of simplicity, we always assume that $\Gamma \subset Sp(n, Q)$ throughout this paper, although many parts of this paper can be easily generalized to the other cases. For each integer $k \geq 1$ and each discrete groug Γ, we denote by $S_k(\Gamma)$ the space of Siegel cusp forms on H_n of weight k belonging to Γ. Namely, by definition, $S_k(\Gamma)$ is the C-linear space of all holomorphic functions $f(Z)$ on H_n which satisfy

(i) $f(\gamma Z)=f(Z)J(\gamma, Z)^k$ for all $\gamma \in \Gamma$, and
(ii) $f(Z)\det(\mathrm{Im}(Z))^{k/2}$ is bounded on H_n,

where we put $J(\gamma, Z)=\det(CZ+D)$ for γ composed by A, B, C, D.

1.2. Central quasi unipotent elements.

Now, we shall review Godement's Dimension formula of $S_k(\Gamma)$. give the definition of the contribution of central unipotent elements, and also some related conjectures.

For any integer $k>2n$, the following dimension formula of $S_k(\Gamma)$ has been known (Godement [8] Exp. 10-29).

$$\dim S_k(\Gamma)$$
$$=\frac{\gamma_n\left(k-\frac{n+1}{2}\right)}{2^n(2\pi)^{n(n+1)/2}\gamma_n(k-n-1)}\int_{\Gamma\backslash H_n}\sum_{\gamma\in\Gamma}J(\gamma, Z)^{-k}\det\left(\frac{\gamma Z-\bar{Z}}{2i}\right)^{-k}(\det Y)^k dZ,$$

where $dZ=(\det Y)^{-n-1}\prod_{1\leq i,j\leq n} dX_{ij}dY_{ij}$ for $Z=X+iY\in H_n$,

$$X=(X_{ij}),\ \ Y=(Y_{ij})\in M_n(R)\quad(1\leq i, j\leq n),$$

$\gamma_n(s)=\prod_{i=1}^{n-1}\Gamma\left(s+1+\frac{i}{2}\right)$ as a meromorphic function on $s\in C$, and $\Gamma(s)$ is the usual gamma function.

For the sake of simplicity, we put

$$a_n(k)=\frac{\gamma_n\left(\frac{n+1}{2}\right)}{2^n(2\pi)^{n(n+1)/2}\gamma_n(k-n-1)},$$

and

$$H_\gamma(Z)=\int_{\Gamma\backslash H}\sum_{\gamma\in\Gamma}J(\gamma, Z)^{-k}\det\left(\frac{\gamma Z-\bar{Z}}{2i}\right)^{-k}(\det Y)^k dZ.$$

For any subset C of Γ, set (formally)

$$I_n(C, k)=a_n(k)\int_{\Gamma\backslash H_n}\left(\sum_{\gamma\in C}H_\gamma(Z)\right)dZ.$$

To calculate $\dim S_k(\Gamma)$, we usually divide Γ into a disjoint union of suitable subsets C of Γ such that $I_n(C, k)$ converges, and express $\dim S_k(\Gamma)$ as the sum of those $I_n(C, k)$. For any subset C of Γ such that $I_n(C, k)$ converges, we call $I_n(C, k)$ the contribution of C to $\dim S_k(\Gamma)$.

In this paper, we shall study on $I_n(C, k)$ for various Γ and various subsets C of Γ which consist of "central" unipotent (, or quasi-unipotent) elements of Γ. (The definition will soon be given below.) But, first, we shall review some general conjectures on the dimension formula. *Although our Main Theorem is independent of these conjectures,* these will make our situation clearer. We say here that an element $\gamma\in\Gamma$ is weakly hyperbolic, if $|\alpha|>1$ for some eigenvalue α of γ. The following Conjecture 1 has been known as (a part of) the Selberg principle.

CONJECTURE 1. *Let C_h be the subset of Γ of all weakly hyperbolic elements of Γ. Then, $I_n(C_h, k)=0$ for any $k>2n$.*

If $\gamma\in\Gamma$ is not weakly hyperbolic in the above sense, then it can be easily shown that γ is either torsion, or quasi-unipotent. Here we say that $\gamma\in\Gamma$ is quasi-unipotent, if some power of γ is unipotent. (This was called potentially unipotent in [11], [13], [16].) In this paper, we shall be concerned with quasiunipotent elements. To describe our Conjecture 2, we need some more definitions. We shall say that a quasi-unipotent element $\gamma\in\Gamma$ is central, if there exists a maximal parabolic subgroup P of $Sp(n, Q)$ such that some power of γ is contained in the center U of the unipotent radical of P. If γ is unipotent besides, we shall say that γ is central unipotent. Several experts seem to have had the following Conjecture 2 in mind.

CONJECTURE 2. *Let C_{nc}^{q} be the subset of Γ of all noncentral quasi-unipotent elements of Γ. Then, $I_n(C_{nc}^{q}, k)=0$.*

The conjectures 1 and 2 are known to be true for $n=2$ by Morita [24], Christian [6], and Hashimoto [11].

Taking this conjecture 2 into account, we shall treat only central quasi-unipotent elements in this paper. In the rest of this sections, we shall define the rank of central quasi-unipotent elements. This concept is useful to describe the contribution of such elements (cf. Shintani [27]). First, we review on maximal parabolic subgroups of $Sp(n, Q)$. For each natural number r such that $1\leq r\leq n$, denote by P_r the subgroup of $Sp(n, Q)$ of all elements g of the following form:

$$g=\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & B_{22} \\ 0 & 0 & D_{11} & 0 \\ 0 & C_{22} & D_{21} & D_{22} \end{pmatrix},$$

where A_{11}, B_{11}, $D_{11}\in M_r(Q)$, A_{22}, B_{22}, C_{22}, $D_{22}\in M_{n-r}(Q)$, and A_{12}, B_{12}, ${}^tB_{21}$, ${}^tD_{21}\in M_{r,n-r}(Q)$.

These P_r $(1\leq r\leq n)$ form a complete set of representatives of $Sp(n, Q)$-conjugacy classes of maximal parabolic subgroups of $Sp(n, Q)$. Denote by U_r the center of the unipotent radical of P_r. It is easy to see that

$$U_r=\left\{\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}\in Sp(n, Q);\ X={}^tX\in M_n(Q),\ X=\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\text{ for some } x\in M_r(Q)\right\}$$

and that $\{1_n\}=U_0\subset U_1\subset U_2\subset\cdots\subset U_n$. So, a unipotent element γ is central, if and only if some $Sp(n, Q)$-conjugate is contained in U_n. We can also show that a quasi-unipotent element γ is central, if and only if the unipotent part γ_u of γ is central, where $\gamma=\gamma_s\gamma_u$ is the Jordan decomposition of γ (γ_s: semisimple, γ_u: unipotent, and $\gamma_s\gamma_u=\gamma_u\gamma_s$). In fact, assume that γ is central and that $\gamma_1^m\in U_n$ for some $Sp(n, Q)$-conjugate γ_1 of γ_u. As U_n is a divisible group, there exists $u\in U$ such that $u^m=\gamma_u^m$. As log and exp is well defined for unipotent elements, we get $\gamma_u=u$.

Now, for each r $(1\leq r\leq n)$, define a subset U_r' by

$$U_r'=\left\{\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}\in U_r;\ X={}^tX\in M_n(Q),\ X=\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\text{ for some } x\in GL_r(Q)\right\}.$$

We shall say that a central quasi-unipotent element γ is of rank r, if some $Sp(n, Q)$-conjugate of γ_u is contained in U_r'. This rank is uniquely determined by γ, because the usual rank of the matrix $1_{2n}-\gamma_u$ is invariant by conjugation.

Incidentally, we can easily generalize the above definition of the rank to the case of any Q-form G of semisimple algebraic groups attached to bounded symmetric domains, replacing P_r $(1\leq r\leq n)$ by standard maximal parabolic subgroups of G over Q, because, in such casses, it has been known that the centers of the unipotent radicals of these parabolic subgroups are linearly ordered with respect to inclusion.

1.3. Choice of discrete subgroups.

We shall define discrete subgroups which we shall be concerned in this paper. We fix a prime number p once and for all throughout this paper. There exists the unique Iwahori (i.e. minimal paraboric) subgroup of $Sp(n, Q_p)$ up to $Sp(n, Q_p)$-conjugation (Bruhat-Tits [5]).

We denote by $B_n(p)_p=B(p)_p$ the following representative of Iwahori subgroup of $Sp(n, Q_p)$:

$$B(p)_p=\left\{g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(n, Q_p);\ \begin{array}{l}A, B, C, D\in M_n(Z_p),\\ C\equiv 0 \text{ mod.} p, \text{ and } A \text{ mod.} p\\ \text{and } {}^tD \text{ mod.} p \text{ are upper triangular}\end{array}\right\}$$

Put $B(p)=B(p)_p \cap Sp\left(n, Z\left[\frac{1}{p}\right]\right)$. We shall consider the cusp forms belonging to $B(p)$. But, we are interested only in *new* forms, and in order to extract *old* forms from $S_k(B(p))$, we must consider all cusp forms belonging to each (proper) subgroup of $Sp(n, Q)$ which contains $B(p)$. To write down all these subgroups, we review briefly on Bruhat-Tits theory [5]. Denote by T the maximal split torus consisting of all diagonal matrices in $Sp(n, Q_p)$, and denote by N the normalizer of T. Then, the affine Weyl group W_{aff} of $Sp(n, Q_p)$ can be identified with the group $N/N \cap B(p)_p$, and there exists a subset S_{aff} of $N/N \cap B(p)_p = W_{\mathrm{aff}}$ which forms the set of generators of W_{aff} as the Coxeter system. It is well known that the set of all proper subgroups of $Sp(n, Q_p)$ which contain $B(p)_p$ correspond bijectively to the family of all proper subsets of S_{aff} (Bruhat-Tits, loc. cit.). More precisely, for each subset $\theta \supset S_{\mathrm{aff}}$, define a subgroup U_θ of $Sp(n, Q_p)$ by:

U_θ = the group generated by all double cosets $B(p)_p w B(p)_p$
such that $\bar{w} \in \theta$,

where we take a representative w in N for each $\bar{w} \in \theta$. (It is clear that the group U_θ does not depend on the choice of the representative of w.) This correspondence

$$S_{\mathrm{aff}} \supset \theta \longrightarrow U_\theta \subset Sp(n, Q_p)$$

gives the above bijection.

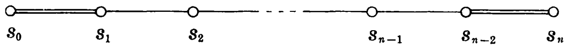
For example, $U_\phi = B(p)_p$, where ϕ is the empty set. Besides, if $\theta \subset \theta' \subset S_{\mathrm{aff}}$, then $U_\theta \subset U_{\theta'}$, and vice versa. We put $\Gamma_\theta = U_\theta \cap Sp\left(n, Z\left[\frac{1}{p}\right]\right)$. We define C-linear vector V of $S_k(B(p))$ by:

$$V = \sum_{\substack{\theta \subset S_{\mathrm{aff}} \\ \sharp(\theta)=1}} S_k(\Gamma_\theta),$$

where the summation means (not necessarily direct) sum as linear vector spaces in $S_k(B(p))$. We define a space $S_k^0(B(p))$ of new forms belonging to $B(p)$ to be the orthogonal complement of V in $S_k(B(p))$ with respect to the Petersson inner product of $S_k(B(p))$. The dimension of $S_k^0(B(p))$ is given by

$$\dim S_k^0(B(p)) = \sum_{\theta \subsetneqq S_{\mathrm{aff}}} (-1)^{\sharp(\theta)} \dim S_k(\Gamma_\theta).$$

(See [13] p. 38 footnote.) We shall consider the contribution of conjugacy classes to this alternating sum of dimensions. For the sake of convenience of the readers, we shall describe Γ_θ more explicitly here. The Coxeter diagram of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is given as follows, where each vertex corresponds to each element of $S_{\mathrm{aff}}=\{s_0, s_1, \cdots, s_n\}$:



In our case, the representatives of $s_i \in S_{\mathrm{aff}}$ in N (which will be also denoted by s_i $(0\leq i\leq n)$) can be taken as follows:

$s_0=\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$, where $A_0=D_0=\mathrm{diag}(0, 1, \cdots, 1)$, $B_0=\mathrm{diag}(p^{-1}, 0, \cdots, 0)$, $C_0=\mathrm{diag}(p, 0, \cdots, 0)\in M_n(Q)$, and $\mathrm{diag}(a_1, \cdots, a_n)$ is the diagonal matrix whose (i, i)-component is a_i.

$s_i=\begin{pmatrix} V_i & 0 \\ 0 & V_i \end{pmatrix}$, for each i such that $1\leq i\leq n-1$, where $V_i=(v_{1k}^{(i)})_{(1\leq 1,k\leq n)}\in M_n(Q)$, $v_{i,i+1}^{(i)}=v_{i+1,i}^{(i)}=1$, and $v_{1k}^{(i)}=0$ for all the other $1, k$.

$s_n=\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$, where $A_n=D_n=\mathrm{diag}(1, \cdots, 1, 0)$, $B_n=\mathrm{diag}(0, \cdots, 0, -1)$, and $C_n=\mathrm{diag}(0, \cdots, 0, 1)$.

For each $\theta\subsetneqq S_{\mathrm{aff}}$, let $t=t(\theta)=n+1-\sharp(\theta)$, and let $i_1(\theta), \cdots, i_t(\theta)$ be all the mutually different indices i $(0\leq i\leq n)$ such that $s_i\in S_{\mathrm{aff}}$. Changing the notations if necessary, we assume that $0\leq i_1(\theta)<i_2(\theta)<\cdots<i_t(\theta)\leq n$. For each $1\leq\nu\leq t+1$, put $n_\nu(\theta)=i_\nu(\theta)-i_{\nu-1}(\theta)$, where we regard $i_0(\theta)=0$ and $i_{t+1}(\theta)=n$. Now, for the sake of simplicity, we introduce the following terminology. A partition of any matrix $A\in M_n(Q)$ into blocks A_{ij} is called θ-partition, if it is of the following form:

$$A=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,t+1} \\ A_{21} & A_{22} & \cdots & A_{2,t+1} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ A_{t+1,1} & A_{t+1,2} & \cdots & A_{t+1,t+1} \end{pmatrix}$$

where $t=\sharp(S_{\mathrm{aff}})-\sharp(\theta)$ and each A_{ij} is a $n_i(\theta)\times n_j(\theta)$ matrix. (If $n_1(\theta)=0$ (resp. $n_{t+1}(\theta)=0$), then we regard the blocks A_{ij} as void for $i=1$ or

$j=1$ (resp. $i=t+1$, or $j=t+1$.)) Now, for any subset $\theta\subsetneq S_{\mathrm{aff}}$ and an element

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(n, Q) \quad (A, B, C, D\in M_n(Q)),$$

denote by $A=(A_{ij})$, $B=(B_{ij})$, $C=(C_{ij})$, or $D=(D_{ij})$ $(1\leq i, j\leq t+1)$ the θ-partition of each A, B, C, D. Then, Γ_θ consists of all those elements $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(n, Q)$ which satisfy the following three conditions (i)$\sim$(iii):

(i) A_{ij}, B_{ij}, C_{ij}, D_{ij} are matrices with integral coefficients except for B_{11}, and each coefficient of B_{11} is contained in $p^{-1}Z$.

(ii) If $1\leq j<i\leq t$, then $A_{ij}\equiv D_{ij}\equiv 0 \bmod p$.

(iii) For each i, j, we have $C_{ij}\equiv 0 \bmod p$, unless $i=j=t+1$.

1.4. Main Theorem.

Notations being as before, for each $\theta\subsetneq S_{\mathrm{aff}}$ and each integer r such that $1\leq r\leq n$, denote by $C_r^u(\theta)$ the set of all central unipotent elements of Γ_θ of rank r, and by $I_n(C_r(\theta), k)$ the contribution of $C_r^u(\theta)$ to $\dim S_k(\Gamma_\theta)$.

MAIN THEOREM. *Let k and r be any integers such that $k\geq 2n+1$ and $1\leq r\leq n$. Notation being as above, the integral $I_n(C_r^u(\theta), k)$ converges for each subset θ of S and each r, and we have*

$$\sum_{\theta\subsetneq S_{\mathrm{aff}}}(-1)^{\#(\theta)}I_n(C_r^u(\theta), k)=0$$

for each r. In other words, the contribution of central uniponent element to $\dim S_k^0(B(p))$ vanishes.

Note here that each $I_n(C_r(\theta), k)$ does not vanish in many cases where its explicit value is known. We note also that similar theorem can be obtained also for central quasi-unipotent elements under some assumption on convergence. (See §2).

§2. Generalization of Morita-Shintani's formula

We denote by Γ a subgroup of $Sp(n, Q)$ with $\mathrm{vol}(Sp(n, R)/\Gamma)<\infty$ as before.

In this section, we shall give a formula for the contribution of central unipotent elements of Γ to $\dim S_k(\Gamma)$. This is a generalization of Morita-Shintani's formula for $\Gamma=\Gamma(N)$. (cf. [27] Prop. 8).

2.1. Classification.

First, we classify central quasi-unipotent elements in terms of cusps of Γ. For each integer r such that $1\leq r\leq n$, consider the following double coset decomposition:

$$Sp(n, Q)=\coprod_{i=1}^{d_r} \Gamma w_i^{(r)} P_r, \quad \text{(disjoint)}. \tag{2.1}$$

Each double coset in the right hand side corresponds to an Γ-equivalence class of $(n-r)(n-r+1)/2$-dimensional cusps of Γ. Denote by C_r^q (resp. C_r^u) the set of all central quasi-unipotent (resp. unipotent) elements of $Sp(n, Q)$ of rank r. For each $w=w_i^{(r)}$ $(1\leq r\leq n, 1\leq i\leq d_r)$, put

$$D_r^q(w)=\{\gamma\in\Gamma\cap wP_r w^{-1}\cap C_r^q;\ w^{-1}\gamma_u w\in U_r\},$$

where we denote by γ_u the unipotent part of the Jordan decomposition of $\gamma=\gamma_s\gamma_u$. Set

$$C_r^q(w)=\{\delta^{-1}\gamma\delta;\ \gamma\in D_r^q(w), \delta\in\Gamma\}.$$

We put also

$$D_r^u(w)=D_r^q(w)\cap C_r^u, \text{ and } C_r^u(w)=C_r^q(w)\cap C_r^u.$$

LEMMA 2.2. *Notation being as above,*

(i) $C_r^q(\Gamma)=C_r^q\cap\Gamma=\coprod_{i=1}^{d_r} C_r^q(w_i^{(r)})$ *(disjoint union), and*

$C_r^u(\Gamma)=C_r^u\cap\Gamma=\coprod_{i=1}^{d_r} C_r^u(w_i^{(r)})$ *(disjoint union).*

(ii) *Let $\mathfrak{S}(w_i^{(r)})$ be a complete set of representatives of $(\Gamma\cap w_i^{(r)}P_r w_i^{(r)-1})\backslash\Gamma$. Then, for each i and r, the following two mapping are bijections:*

$$\mathfrak{S}(w_i^{(r)})\times D_r^q(w_i^{(r)})\ni(\delta,\gamma)\longrightarrow\delta^{-1}\gamma\delta\in C_r^q(w_i^{(r)}),$$
$$\mathfrak{S}(w_i^{(r)})\times D_r^u(w_i^{(r)})\ni(\delta,\gamma)\longrightarrow\delta^{-1}\gamma\delta\in C_r^u(w_i^{(r)}).$$

PROOF. If $\gamma\in C_r^q(w_i^{(r)})\cap C_r^q(w_j^{(r)})$, then $\gamma=\delta_1^{-1}\gamma_1\delta_1=\delta_2^{-1}\gamma_2\delta_2$ for some $\gamma_1\in D_r^q(w_i^{(r)})$, $\gamma_2\in D_r(w_j^{(r)})$, and δ_1, $\delta_2\in\Gamma$. Then, by the uniqueness of the Jordan decomposition, we have $\delta_1^{-1}\gamma_{1,u}\delta_1=\delta_2^{-1}\gamma_{2,u}\delta_2$, where $\gamma_{1,u}$, or $\gamma_{2,u}$ is the unipotent part of γ_1, or γ_2, respectively. As $\gamma_{1,u}$ and $\gamma_{2,u}$ are of rank r, we can show by direct calculation that $w_j^{(r)-1}\delta_2\delta_1^{-1}w_i^{(r)} \in P_r$. Hence, $w_j^{(r)} \in \Gamma w_i^{(r)} P_r$ and we get $i=j$. This proves the disjointness of the right hand sides of (i). If $\gamma \in C_r^q(\Gamma)$, then, by definition, $g^{-1}\gamma_u g \in U_r$ for some $g \in Sp(n, Q)$. By (2.1), $g=\delta w_i^{(r)} p$ for some 1 ($1 \le i \le d_r$), and $\delta \in \Gamma$, $p \in P_r$. As U_r is a normal subgroup of P_r, we get $\delta^{-1}\gamma\delta \in D_r^q(w_i^{(r)})$. Hence, (i) was proved. We can prove (ii) very easily, noting that, if $g^{-1}\gamma g \in U_r'$ for an element $\gamma \in U_r'$, then $g \in P_r$. We omit the details here. q.e.d.

2.2. Integral formula.

Now, we write down the integral $I_n(C_r^q(w_i^{(r)}), k)$ for each r, i and k. Put $U_n = \prod_{k=1}^{n} \frac{2\pi^k}{\Gamma(k)}$ and denote by dk the ivariant measure of $U(n)$ such that $\mathrm{vol}(U(n)) = 2^{-n}U_n$, where we denote by $U(n)$ the stabilizer in $Sp(n, R)$ of $i1_n \in H_n$ which is isomorphic to the usual unitary group of size n. Denote by $\delta_n g$ the invariant measure of $Sp(n, \boldsymbol{R})$ such that $\delta_n g = dk \times dZ$. As in Shintani [27] p. 63, for each $w = w_i^{(r)}$, we have

$$\begin{aligned} &I_n(C_r^q(w), k) \\ &= a_n(k) 2^n/U_n \int_{\Gamma \backslash Sp(n,R)} \sum_{\gamma \in C_r^q(w)} J(g^{-1}\gamma g, i)^{-k} \det\left(\frac{g^{-1}\gamma g i + i}{2i}\right)^{-k} \delta_n g \\ &= a_n(k) 2^n/U_n \int_{w^{-1}\Gamma w \cap P_r \backslash Sp(n,R)} \sum_{\gamma \in D_r^q(w)} J(g^{-1}\gamma g, i)^{-k} \det\left(\frac{g^{-1}\gamma g i + i}{2i}\right)^{-k} \delta_n g, \end{aligned}$$

and also

$$\begin{aligned} &I_n(C_r^u(w), k) \\ &= a_n(k) 2^n/U_n \int_{w^{-1}\Gamma w \cap P_r \backslash Sp(n,R)} \sum_{\gamma \in D_r^u(w)} J(g^{-1}\gamma g, i)^{-k} \det\left(\frac{g^{-1}\gamma g i + i}{2i}\right)^{-k} \delta_n g, \end{aligned}$$

as far as the integrals in the right hand sides converge absolutely. All we need in this paper is the fact that this integral $I_n(C_r^u(w), k)$ actually converges under some conditions on Γ, w and depends only on n, k, and $w^{-1}\Gamma w \cap P_r$ and is independent of Γ itself. But, for the sake of completeness, we shall give here a formula to describe this integral by some special values of zeta functions as in Shintani [27]. To explain the above formula, first we introduce some zeta functions as in [27].

Put

$$V_r = \{x \in M_r(R); x = {}^t x\}.$$

Let ρ be the representation of $GL_r(R)$ on V_r such that $\rho(g)x = gx{}^tg$ $(g \in GL_r(R), x \in V_r)$. Then, the triple $(GL_r(R), V_r, \rho)$ is a prehomogeneous vector space. Let L be a lattice of V_r and Δ_1 be a subgroup of $SL_r(R)$ such that L is stable by the action of Δ_1. For each such pair Δ_1 and L, we define the zeta function $\zeta_r(s, \Delta_1, L)$ as follows:

$$\zeta_r(s, \Delta_1, L) = \sum_{x \in L_r/\sim} \frac{1}{\varepsilon(x)|\det x|^s},$$

where L_r is the set of all positive definite elements of L, $\varepsilon(x)$ is the order of the stabilizer of x in Δ_1, and the summation is over Δ_1-orbbits of L_r. This kind of zeta functions are treated in Shintani [27] under the assumption that $\Delta_1 = SL_r(Z)$. We shall extend his results for more general groups which we need later. Assume that Δ_1 is commensurable to $SL_r(Z)$. Then, we can show that $\zeta_r(s, \Delta_1, L)$ converges absolutely for $Re(s) > \dfrac{r+1}{2}$. In fact, let $\mathcal{L}$ be the module in V_r spanned by $\rho(\delta)x$ $(x \in L, \delta \in SL_r(Z))$. As Δ_1 is commensurable to $SL_r(Z)$, $\mathcal{L}$ is also a lattice in V_r. It is well known that $\zeta_r(s, SL_r(Z), \mathcal{L})$ converges absolutely for $s > \dfrac{r+1}{2}$ (Siegel [28], Shintani [27] p. 50). Now, the convergence of $\zeta_r(s, \Delta_1, L)$ follows from the following relations:

$$\zeta_r(s, SL_r(Z), \mathcal{L}) = \zeta_r(s, \Delta_1 \cap SL_r(Z), \mathcal{L})/[SL_r(Z) : SL_r(Z) \cap \Delta_1],$$

and

$$\zeta_r(s, \Delta_1, L) = \zeta_r(s, \Delta_1 \cap SL_r(Z), \mathcal{L})/[\Delta_1 : \Delta_1 \cap SL_r(Z)].$$

Next, to get the functional equation, we impose some conditions on Δ_1, which we shall explain below. For each s such that $1 \le s \le r$, define subgroups $P_1^{(s)}$, $P_2^{(s)}$ and $P^{(s)}$ of $GL_r(R)$ as follows:

$$P_1^{(s)} = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix};\ a \in GL_s(R),\ b \in GL_{r-s}(R)\right\}$$

$$P_2^{(s)} = \left\{\begin{pmatrix} 1_s & x \\ 0 & 1_{n-s} \end{pmatrix};\ x \in M_{s,r-s}(R)\right\}, \text{ and}$$

$$P^{(s)} = P_1^{(s)} \times P_2^{(s)}.$$

Let Δ' be a subgroup of $GL_r(R)$. For each Δ' and s, we consider the following condition:

CONDITION G1. *The following mapping is a bijection*:

$$(\Delta'\cap P_1^{(s)})\times(\Delta'\cap P_2^{(s)})\ni(\delta_1,\delta_2)\longrightarrow\delta_1\delta_2\in\Delta'\cap P^{(s)}.$$

Now, consider another subgroup Δ_1 of $GL_r(G)$ and the following double coset decomposition:

$$GL_r(Q)=\coprod_{i=1}^{e_s}\Delta_1 h_i^{(s)}P^{(s)}. \tag{2.2}$$

We shall say that Δ_1 is *good*, if Condition G1 is satisfied for every $\Delta'=h_i^{(s)-1}\Delta_1 h_i^{(s)}$ $(1\leq i\leq e_s,\ 1\leq s\leq r)$.

Next, to describe the functional equation, we introduce several notations. Denote by $V_r^{(r)}$ the set of all positive definite elements of V_r. For each $\lambda\in C$, define functions $f_r(x,\lambda)$ and $f_r^*(x,\lambda)$ on $x\in V_r$ by:

$$f_r(x,\lambda)=\begin{cases}(\det x)^{\lambda-(r+1)/2}\exp(-2\pi tr(x)) & (x\in V_r^{(r)})\\ 0 & (x\in V_r^{(r)}),\end{cases}$$

$$f_r^*(x,\lambda)=\ \det(1-ix)^{-\lambda}.$$

We define two kinds of zeta functions as follows:

$$Z(f_r(x,\lambda),L,s)=\int_{GL_r^+(R)/\Delta_1}(\det x)^{2s}\sum_{x\in L_n}f_r(\rho(g)x,\lambda)dg$$

and

$$Z(f_r^*(x,\lambda),L,s)=\int_{GL_r^+(R)/\Delta_1}(\det x)^{2s}\sum_{x\in L_n}f_r^*(\rho(g)x,\lambda)dg,$$

where $dg=(\det g)^{-r}\prod_{1\leq i,j\leq r}dg_{ij}$.

As in Shintani [27] p. 54, 60, it is easy to see that

$$Z(f_r(x,\lambda),L,s)=\pi^{r(r-1)/4}(2\pi)^{-(\lambda+s-(r+1)/2)r}\gamma_r(\lambda+s-r-1)2^{-(r+1)}C_r\zeta_r(s,\Delta_1,L)$$

for sufficiently big s, where we put

$$C_r=\prod_{i=1}^{r}\frac{2\pi^{k/2}}{\Gamma(K/2)}.$$

We define the dual L^* of L as follows:

$$L^*=\{x\in V_r;\ tr(xy)\in Z\quad\text{for all }\ y\in L\}.$$

The following Lemma 2.3 is a modification of Shintani [27] Corollary to Lemma 21 (p. 58).

LEMMA 2.3. *Assume that Δ_1 is commensurable to $SL_r(Z)$ and good. Then, both $Z(f_r(x,\lambda), L, s)$ and $Z(f_r^*(x,\lambda), L^*, s)$ are meromorphic functions of (λ, s) on C^2, and satisfy the functional equation*

$$\begin{aligned} &Z(f_r(x,\lambda), L, s) \\ &=\mathrm{vol}(L)^{-1}\pi^{r(r-1)/4}(2\pi)^{-\lambda r}\gamma_r\Big(\lambda-\frac{r+1}{2}\Big)Z\Big(f_r^*(x,\lambda), L^*, \frac{r+1}{2}-s\Big), \end{aligned}$$

where $\mathrm{vol}(L)=\int_{V_r/L}dg.$

PROOF. We must modify Shintani's proof only at the following point. In our case, there might exist several cusps of Δ_1. In other words, the set $\Delta_1\backslash GL_r(Q)/P^{(s)}$ might contain more than one elements. So, the equation in Shintani [27] p. 55 line 4 from below dose not maintain its validity in our case. In our case, it becomes as follows: we take a double coset decomposition of $GL_r(Q)$ as in (2.2).

Denote by L^{*s} the set of rank s matrices in L^*. Regard V_s as the subspace of V_r by the embedding $V_s\ni x\longrightarrow\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\in V_r$.

Put $\Delta_1(h_i^{(s)})=\Delta_1\cap h_i^{(s)}\Delta_1 h_i^{(s)-1}$. Then

$$\sum_{x\in L^{*s}}f_r^*(\rho(g)x,\lambda)=\sum_{i=1}^{e_s}\sum_{\gamma\in\Delta_1(h_i^{(s)})}\sum_{x\in L^{*s}\cap V_s}f_r^*(\rho(g\gamma)x,\lambda).$$

As in Shintani [27] p. 55, we get

$$\begin{aligned} &\int_{\substack{GL_r^+(R)/\Delta_1\\ \det g\leq 1}}(\det g)^{2s}\sum_{x\in L^{*s}}f_r^*(\rho(g\gamma)x,\lambda)dg \\ &\qquad=\sum_{i=1}^{e_s}\int_{\substack{GL_r^+(R)/\Delta_0(h_i^{(s)})\\ \det g\leq 1}}(\det g)^{2s}\sum_{x\in L^{*s}\cap V_s}f_r^*((\rho(g)x,\lambda)dg. \end{aligned}$$

By the assumption that Δ_1 is good, we can show easily that the last integral over $GL_r^+(R)/\Delta_1(h_i^{(s)})$ $(\det g\leq 1)$ is equal to

$$v_1^*v_2^*\times\frac{C_r\omega_{r-s}}{C_sC_{r-s}}\Big(s-\frac{r}{2}\Big)^{-1}Z\Big(f_s^*(x,\lambda), L^{*s}\cap V_s, \frac{r}{2}\Big),$$

where v_1^*, v_2^* and ω_{r-s} are defined as follows:

$$v_1^* = \mathrm{vol}(GL_s^*(R)/(\Delta_1(h_i^{(s)}) \cap P_2^{(s)})),$$
$$v_2^* = \mathrm{vol}(\Omega^{(s)}/(\Delta_1(h_i^{(s)}) \cap \Omega^{(s)})), \quad \text{and}$$
$$\omega_{r-s} = \begin{cases} \zeta(2)\zeta(3)\cdots\zeta(r-s)/2 & (r-s \geq 2) \\ 1/2 & (r-s=1) \end{cases}.$$

The rest of the proof is completely the same as in Shintani [27], and we omit the details here. q.e.d.

Next, to describe the contribution of central unipotent elements by the above zeta functions, we introduce several notations. Denote by $P_r(R)$ the set of all R-valued points of the algebraic group R_r. Define subgroups $\Xi_1, \Xi_2, \Omega_1, \Omega_2$ of $P_r(R)$ as follows:

$$\Xi_1 = \left\{ \begin{pmatrix} 1_n & \begin{matrix} y_{13} & y_{14} \\ {}^t y_{14} & 0 \end{matrix} \\ 0 & 1_n \end{pmatrix} ; \ y_{13} = {}^t y_{13} \in M_r(R),\ y_{14} \in M_{r,n-r}(R) \right\}$$

$$\Xi_2 = \left\{ \begin{pmatrix} \begin{matrix} 1_r & x_{12} \\ 0 & 1_{n-r} \end{matrix} & 0 \\ 0 & \begin{matrix} 1_r & 0 \\ {}^t x_{12} & 1_{n-r} \end{matrix} \end{pmatrix} ; \ x_{12} \in M_{r,n-r}(R) \right\}$$

$$\Omega_1 = \left\{ \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & 1_r & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix} ; \ h = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix} \in Sp(n-r, (R) \right\}.$$

$$\Omega_2 = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 0 & 0 & {}^t a_{11}^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} ; \ a_{11} \in GL_r(R). \right\}$$

It is easy to show that the following mapping is a bijection:

$$\Xi_1 \times \Xi_2 \times \Omega_1 \times \Omega_2 \ni (y, x, h, a_{11}) \longrightarrow yxha_{11} \in P_r(R). \tag{2.4}$$

Let Δ be a subgroup of $Sp(n, R)$. For each Δ and r, we shall consider the following condition:

CONDITION G2. *The following mapping is a bijection:*

$$(\Xi_1 \cap \Delta) \times (\Xi_2 \cap \Delta) \times (\Omega_1 \cap \Delta) \times (\Omega_2 \cap \Delta) \in (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4 \in \Delta \cap P_r.$$

Let Γ, $w_i^{(r)}$ be as in (2.1).

We shall say that Γ is *good*, if every $\Delta = w_i^{(r)-1}\Gamma w_i^{(r)} \cap P_r$ $(1 \le i \le d_r)$ satisfies Condition G2, and besides every subgroup $w_i^{(r)-1}\Gamma w_i^{(r)} \cap \Omega_1$ or Ω_1 is good, where we identify Ω_1 naturally with $GL_r(R)$.

For good Γ, we shall define zeta functions attached to each cusps of Γ. For each r $(1 \le r \le n)$ and $w = w_i^{(r)}$, define a lattice $L(w)$ by:

$$L(w) = \left\{ x \in M_r(R);\ x = {}^t x \text{ and } \begin{pmatrix} 1_n & \begin{matrix} x & 0 \\ 0 & 0 \end{matrix} \\ 0 & 1_n \end{pmatrix} \in w^{-1}\Gamma w \cap U_r \right\}.$$

Denote by $L^*(w)$ the dual of $L(w)$.

Regard the group $w^{-1}\Gamma w \cap \Omega_1$ as a subgroup of $GL_r(R)$ by the natural identification. Denote by $\Delta_1(w)$ the intersection of $w^{-1}\Gamma w \cap \Omega_1$ and $SL_r(R)$ through this identification. For the sake of simplity, we write

$$\zeta_r(\Gamma, w, s) = \zeta_r(\Delta_1(w), L^*(w), s).$$

PROPOSITION 2.5. *Let Γ be as before and assume that Γ is good. Then, for each $w = w_i^{(r)}$ $(1 \le i \le d_r)$ and $k \ge 2n+1$, the integral $I_n(C_r^u(w), k)$ converges, and we have the following formula*:

$$I_n(C_r^u(w), k) = v_1 v_2 v_3 v_4 \frac{2^{r(n-r)-1}}{U_{n-r}(4\pi)^{(n-r)(n-r+1)/2}} \times \zeta_r(\Gamma, w, r-n)$$
$$\times \prod_{i=1}^{n-r} (2k-n-i)(2k-n-i+2)\cdots(2k-n+i-2)$$

for some volumes v_1, v_2, v_3, v_4 defined below.

DEFINITION OF v_i $(1 \le i \le 4)$. Let $\delta_{n,r}$ be the left invariant measure of $P_r(R)$ given by

$$\delta_{n,r} g = |\det a_{11}|^{-(2n-r+1)} d_r a_{11} \circ \delta_{n-r} h \circ dx_{12} \circ dy_{13} \circ dy_{14},$$

where we write $g = yxa_{11}h$ as in (2.4) and

$$d_r a_{11} = |\det a_{11}|^{-r} \prod_{1 \le i,j \le r} d(a_{11})_{ij}.$$

We define v_i $(1 \le i \le 4)$ as follows:

$$v_1 = \mathrm{vol}(V_r / L_r^*(w)) / 2^{r(r-1)/2},$$

$$v_2 = \mathrm{vol}(\Omega_2/w^{-1}\Gamma w \cap \Omega_2),$$
$$v_3 = \mathrm{vol}(\Xi_1/w^{-1}\Gamma w \cap \Xi_1),$$
$$v_4 = \mathrm{vol}(\Xi_2/w^{-1}\Gamma w \cap \Xi_2),$$

where the volume is measured by dy_{13}, $\delta_{n-r}h$, $dy_{13}dy_{14}$, and dx_{12}, respectively.

PROOF. The proof is obtained in a similar way as in Shintani [27] p. 63, 64, using Lemma 2.3. In Shintani [27], he assumed that $k \geq 2n+3$. But in his Lemma 20 ([27] p. 54), we need not assume that m_{ij} are integers. By this fact, we get more accurate estimation than the results in [27] p. 75, which leads $k \geq 2n+1$.

The assumption that Γ is good is needed to evaluate the integral on $P_r/(P_r \cap w^{-1}\Gamma w)$ by the integral on $\Omega_2/(\Omega_2 \cap w^{-1}\Gamma w)$ and volumes $v_1, \cdots, v_4$. We omit the details here. q.e.d.

COROLLARY 2.6. *Notations and assumptions being as above, the integral $I_n(C_r^u(w), k)$ depends only on n, k, and $P_r \cap w^{-1}\Gamma w$.*

PROOF. obvious. q.e.d.

§ 3. Arithmetic on cusps

In this section, we shall classify cusps of Γ_θ explicitly, and introduce the diagrams corresponding to the representatives of these cusps.

3.1. Reviews on the Bruhat-Tits theory on $Sp(n, Q)$

We review here the well known facts on Weyl group etc. (see Borel-Tits [3], Bruhat-Tits [5], Bourbaki [4]) which will be used later. Let K be any field. Denote by T the maximal split torus of $Sp(n, K)$ such that $T(K)$ (the K-valued points of T) consists of diagonal matrices of $Sp(n, K)$, and put $X^*(T) = \mathrm{Hom}(T, G_m) =$ the group of rational characters of T. For each i $(1 \leq i \leq n)$, denote by α_i the element of $X^*(T)$ defined by:

$$\alpha_i(t) = t_i \quad \text{for all } t = \mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) \in T(K).$$

Then, $\alpha_1, \cdots, \alpha_n$ is the basis of the free Z-module $X^*(T)$. The set Φ of all roots of $Sp(n, Q)$ is given by

$$\Phi = \{\alpha_i \pm \alpha_j (i \neq j),\ 2\alpha_i;\ 1 \leq i, j \leq n\}.$$

A set Π of fundamental roots is given by:

$$\Pi=\{\alpha_1-\alpha_2, \alpha_2-\alpha_3, \cdots, \alpha_{n-1}-\alpha_n, 2\alpha_n\}.$$

The set Φ^+ of positive roots with respect to Π is the set consisting of the following elements:

$$\begin{array}{ll} \alpha_i+\alpha_j,\ \alpha_i-\alpha_j & (1\leq i<j\leq n), \\ 2\alpha_i & (1\leq i\leq n). \end{array}$$

Define the inner product of $X^*(T)$ by:

$$(\alpha_i, \alpha_j)=\delta_{ij} \quad (1\leq i, j\leq n, \delta_{ij} : \text{Kronecker's } \delta).$$

For each $\alpha\in\Phi$, define a reflection w_α on $X^*(T)$ by:

$$w_\alpha(\beta)=\beta-\frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha \qquad (\beta\in X^*(T)).$$

By definition, the Weyl group W of $Sp(n, K)$ is the group generated by $\{w_\alpha; \alpha\in\Phi\}$. Denote by $\mathfrak{S}_n$ the symmetric group on n letters. The group W is the semi-direct product of $\mathfrak{S}_n$ and $(Z/2Z)^n$, where the action of $\sigma\in\mathfrak{S}_n$, or $\delta=(\delta_1, \cdots, \delta_n)\in\{\pm1\}^n=(Z/2Z)^n$ is defined by $\sigma(\alpha_i)=\alpha_{\sigma(i)}$ $(1\leq i\leq n)$, or $\delta(\alpha_i)=\delta_i\alpha_i$ $(1\leq i\leq n)$, respectively. Let N be the normalizer of T in $Sp(n, K)$. Then, for each $\alpha\in\Phi$, there exists $n_\alpha\in T(K)$ such that

$$w_\alpha(\beta)(t)=\beta(n_\alpha t n_\alpha^{-1}) \quad \text{for all } \beta\in X^*(T) \text{ and } t\in T.$$

The mapping $W\ni w_\alpha\longrightarrow n_\alpha\in N(K)$ induces the group isomorphism $W=N(K)/T(K)$. Denote $s_1, \cdots, s_n$ the elements of $N(K)$ defined in § 1-3. (Here, we regard 1 in § 1 as the multiplicative unit of K etc. If we denote by $\bar{s}_i$ $(1\leq i\leq n)$ the image of $\bar{s}_i$ in $N(K)/T(K)$, then

$$\begin{array}{ll} \bar{s}_i=w_{\alpha_i-\alpha_{i+1}} & (i=1, \cdots, n-1), \text{ and} \\ \bar{s}_n=w_{2\alpha_n}. & \end{array}$$

For each $\alpha\in\Phi$, there exists the unique unipotent subgroup U_α of $Sp(n, K)$ such that $tut^{-1}=\alpha(t)u$ for all $t\in T(K)$ and $u\in U_\alpha(K)$. This U_α is called the root subgroup of $Sp(n, K)$ with respect to α, and we have $U_\alpha=G_a$. It is easy to see that $n_\beta U_\alpha n_\beta^{-1}=U_{w_\beta(\alpha)}$ for $\alpha, \beta\in\Phi$, where n_β is the element of N defined as before. The groups T and U_α $(\alpha\in\Phi^+)$

generate a Borel subgroup P_ϕ of $Sp(n, K)$. The set of parabolic subgroups of $Sp(n, K)$ containing P_ϕ corresponds bijectively with the family of subsets of Π: if $P_\phi \subset P \subset Sp(n, K)$, then there exists $\theta \subset S = \{s_1, \cdots, s_n\}$ such that

$$P = \text{the group generated by all } P_\phi g P_\phi \text{ for all } g \in \langle s_1, \cdots, s_n \rangle,$$

where $\langle s_1, \cdots, s_n \rangle$ is the subgroup of $Sp(n, Q)$ spanned by $s_1, \cdots, s_n$. We denote this group P by P_θ.

Denote by W_θ the Coxeter subgroup of W generated by $\{\bar{s}_i; s_i \in \theta\}$. For any θ, $\eta \subset S$, there exists the following bijection:

$$P_\theta(K) \backslash Sp(n, K) // P_\eta(K) \cong W_\theta \backslash W / W_\eta.$$

3.2. Representatives of cusps

For each subset $\theta \subset S_{\mathrm{aff}} = \{s_0, \cdots, s_n\}$, denote by Γ_θ the subgroup of $Sp(n, Q)$ defined in § 1-3. First, we treat the case where $\Gamma_\theta \subset Sp(n, Z)$, that is, the case where $\theta \subset S = \{s_1, \cdots, s_n\}$.

LEMMA 3.1. *For each $\theta \subset S$ and each natural integer r with $1 \leq r \leq n$, we get the following bijection*:

$$\Gamma_\theta \backslash Sp(n, Q) / P_r \cong W_\theta \backslash W / W_r.$$

PROOF. It is well known that $Sp(n, Z)/\Gamma(p) = Sp(n, F_p)$, where F_p is the finite field of p elements.

This isomorphism leads to the following bijections:

$$\Gamma_\theta \backslash Sp(n, Q)/P_r \cong \Gamma_\theta \backslash Sp(n, Z)/Sp(n, Z) \cap P_r \cong \bar{\Gamma}_\theta \backslash Sp(n, F_p)/\overline{P_r(Z)},$$

where $\bar{\Gamma}_\theta$, or $\overline{P_r(Z)}$ is the image of Γ_θ, of Γ_θ, or $P_r \cap Sp(n, Z)$ in $Sp(n, F_p)$ by the reduction mod. p, respectively.

It is easy to see that $\bar{\Gamma}_\theta$ is the parabolic subgroup of $Sp(n, F_p)$. But, $\overline{P_r(Z)}$ is not so in general. This is because $GL_r(Z) \bmod. p = SL_r(F_p) \cup SL_r(F_p)\sigma \neq GL_r(F_p)$, where σ is an element of $GL_r(F_p)$ with $\det \sigma = -1$. Denote by $P_r(F_p)$ the maximal parabolic subgroup of $Sp(n, F_p)$ which contains $\overline{P_r(Z)}$, and by $T(F_p)$ the group consisting of diagonal matrices of $Sp(n, F_p)$. Then, it is easy to see that $P_r(F_p) = T(F_p)\overline{P_r(Z)}$ and

$$\bar{\Gamma}_\theta g P_r(F_p) = \bar{\Gamma}_\theta \overline{g R_r(Z)}$$

for any $g \in T(F_p)$. Hence, we have

$$\bar{\Gamma}_\theta \backslash Sp(n, F_p)/P_r(Z) \cong \bar{\Gamma}_\theta \backslash Sp(n, F_p)/P_r(F_p) \cong W_\theta \backslash W/W_r$$

by virtue of the Bruhat-Tits theory. q.e.d.

For the later use, we need more explicit choice of the representatives of $W_\theta \backslash W/W_r$ and $\Gamma_\theta \backslash Sp(n, Q)/P_r$.

To do this, we define the set S_r consisting of subsets of $X^*(T)$ with r elements as follows:

$$S_r = \{\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\};\ \varepsilon_{i_\nu} = \pm 1\ (1 \leq \nu \leq r),\ 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}.$$

We define the action of W on S_r by:

$$w\{\lambda_1, \cdots, \lambda_r\} = \{w(\lambda_1), \cdots, w(\lambda_r)\} \quad (w \in W, \{\lambda_1, \cdots, \lambda_r\} \in S_r).$$

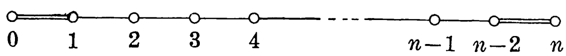
It is easy to see that this action is transitive on S_r. So, if we define the mapping φ of W to S_r by:

$$\varphi(w) = \{w(\alpha_1), \cdots, w(\alpha_r)\} \qquad (w \in W),$$

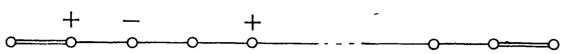
then this mapping is surjective. Besides, it is easy to see that W_r is the subgroup of W consisting of all elements w which stabilize $\{\alpha_1, \cdots, \alpha_r\}$ (i.e. $w(\alpha_i) \in \{\alpha_1, \cdots, \alpha_r\}$ for all i with $1 \leq i \leq r$). Hence, the mapping φ induces the following bijections:

$$(3.2) \qquad W/W_r = S_r \quad \text{and} \quad W_\theta \backslash W/W_r = W_\theta \backslash S_r.$$

It is convenient to express the representatives of W/W_r (or elements of S_r) by the following *picture*: first, we give a number to each vertex of extended Dynkin diagram as follows:



Next, for each $\sigma \in S_r$, we take a representative w of σ and assume $w(\alpha_i) = \varepsilon_{i_\nu}\alpha_{i_\nu}$ for $i = 1, \cdots, n$. For each $i = 1, \cdots, r$, we write $+$ or $-$ sign on i_ν-th vertex of the Dynkin diagram, according that the sign $\varepsilon_{i_\nu} = +1$, or -1:



(So, just r vertices are marked by signs.) It is clear that this picture does not depend on the choice of the representative w and the set of these pictures corresponds bijectively to S_r.

Now, our next problem is to give a complete set of representatives of $W_\theta\backslash S_r$ for each $\theta\subset S$. To do this, for each $\theta\subsetneqq S_{\mathrm{aff}}$, we shall specify some elements of S_r which satisfy some special conditions (Definition 3.3 below). For later use, we *do not* assume here that $\theta\subset S$. Put $X=\{0, 1, 2, \cdots, n\}$.

We call an element i of X a gap of θ, if $s_i\in X$. Denote by gap (θ) the set of all gaps of θ. As in §1, we denote by $i_\nu(\theta)$ $(1\leq\nu\leq t,\ t=\sharp(\mathrm{gap}(\theta)))$ the elements of gap(θ), and changing the indices, if necessary, we assume that $0\leq i_1(\theta)<\cdots<i_t(\theta)\leq n$.

For each ν such that $1\leq\nu\leq t+1$, we define the subset $b_\nu(\theta)$ by:

$$b_\nu(\theta)=\{i\in X;\ i_{\nu-1}(\theta)+1\leq i\leq i_\nu(\theta)\}$$

and call $b_\nu(\theta)$ the i-th sequence of X with respect to θ, where we put $i_{t+1}(\theta)=n$ and $i_0(\theta)=0$. If $i_1(\theta)=0$, then the first sequence $b_1(\theta)=\phi$.

For each element $\sigma=\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\}\in S_r$ and each $\theta\subsetneqq S_{\mathrm{aff}}$, we define a subset $b_\nu(\sigma, \theta)$ of X by:

$$b_\nu(\sigma, \theta)=\{i_1, i_2, \cdots, i_r\}\cap b_\nu(\theta).$$

DEFINITION 3.3. For each pair (σ, θ) of $\sigma\in S_r$ and $\theta\subsetneqq S_{\mathrm{aff}}$, we say that σ is θ-admissible, that θ is σ-admissible, or that (σ, θ) is an admissible pair, if the conditions (1), (2), (3) below are satisfied.

CONDITIONS (1) *For each* ν $(2\leq\nu\leq t(\theta))$ *such that* $b_\nu(\sigma, \theta)\neq\phi$, *there exist* $\mu(\nu)=\mu(\nu, \theta, \sigma)$ *and* $\kappa(\nu)=\kappa(\nu, \theta, \sigma)\in X$ *which satisfies the following conditions* (i), (ii), (iii):

(i) $i_{\nu-1}(\theta)+1\leq\mu(\nu)\leq\kappa(\nu)\leq i_\nu(\theta)$,

(ii) $b_\nu(\sigma, \theta)=\{i\in X;\ i_{\nu-1}(\theta)+1\leq i\leq\mu(\nu),\ \textit{or}\ \kappa(\nu)\leq i\leq i_\nu(\theta)\}$,

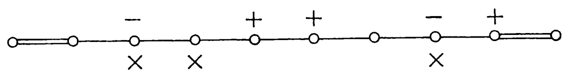
(iii) $\varepsilon_i=\begin{cases}1\cdots\textit{if}\ i_{\nu-1}(\theta)+1\leq i\leq\mu(\nu),\\ -1\cdots\textit{if}\ \kappa(\nu)\leq i\leq i_\nu(\theta).\end{cases}$

(2) *If* $b_{t+1}(\sigma, \theta)\neq\phi$, *then* $\varepsilon_1=1$ *for all* i *with* $i_t(\theta)+1\leq i\leq n$.

(3) *If* $b_1(\sigma, \theta)\neq\phi$ *(which occurs only when* $\theta\not\subset S$*), there exists a natural number* $\kappa=\kappa(1, \sigma, \theta)$ *such that* $1\leq\kappa\leq i_1(\theta)$, $b_1(\sigma, \theta)=\{i\in X;\ 1\leq 1\leq\kappa\}$, *and that* $\varepsilon_i=-1$ *for all* $i\in b_1(\sigma, \theta)$.

Here, it is convenient to illustrate each pair (θ, σ) (which is not necessarily admissible) by the following "picture". We illustrate $\theta\subsetneqq S_{\mathrm{aff}}$

by marking $\times$ below the vertices of theDynkin diagram corresponding to the elements of gap(θ), and illustrate $\sigma\in S_r$ by marking $+$ or $-$ above the r-vertices as before. For example, if $n=9$, gap$(\theta)=\{2, 3, 7\}$, $r=5$, and $\sigma=\{-\alpha_2, \alpha_4, \alpha_5, -\alpha_7, \alpha_8\}$, then the "picture" is as follows:



In this case, by definition, we have $b_1(\theta)=\{0, 1, 2\}$, $b_2(\theta)=\{3\}$, $b_3(\theta)=\{4, 5, 6, 7\}$, $b_4(\theta)=\{8, 9\}$. It is clear that this σ is θ-admissible.

Now, we treat the case where $\Gamma_\theta\subset Sp(n, Z)$.

PROPOSITION 3.4. *For each* $\theta\subset S=\{s_1, \cdots, s_n\}$ *(that is, in the case that* $\Gamma_\theta\subset Sp(n, Z)$*), the set of all* Γ_θ*-equivalence classes of* $(n-r)(n-r+1)/2$ *dimensional cusps of* $\Gamma_\theta\backslash H_n$ *corresponds bijectively to the set of all* θ*-admissible elements of* S_r.

PROOF. By definition, W_θ can be naturally identified through the action on $X^*(T)$ with the direct product of the following groups g_ν $(\nu=1, \cdots, t+1)$:

(i) for each $\nu=1, \cdots, t$,
$g_\nu=$the full permutation group on the set $\{\alpha_i; i\in b_\nu(\theta)\}$

(ii) $g_{t+1}=$the group of all permutations w on $\{\pm\alpha_i; i\in b_{t+1}(\theta)\}$ such that $w(\alpha_i)=\pm\alpha_{\delta(i)}$ for all $i\in b_{t+1}(\theta)$, where δ is a permutation on $b_{t+1}(\theta)$ (determined uniquely by w).

Hence, the proof is obvious. q.e.d.

COROLLARY 3.5. *When* $n\geq 2$, *we get*

$$\sum_{\theta\subset S}(-1)^{\#(\theta)}\#(W_\theta\backslash W/W_r)=0.$$

PROOF. We can prove this by easy combinatorial argument, but we shall omit the proof here, because this fact will not be used in this paper. q.e.d.

Under the assumption that $\theta\subset S$ (i.e. $\Gamma_\theta\subset Sp(n, Z)$), we shall give here each explicit representatives of $\Gamma_\theta\backslash Sp(n, Q)/P_r$ in $Sp(n, Q)$ attached to each θ-admissible element $\sigma\in S_r$. To do this, first, we shall attach to each element $\sigma\in S_r$ an element $w(\sigma)$ of $Sp(n, Z)$.

For each $\sigma=\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\}\in S_r$, we fix a permutation τ on $\{1, 2, \cdots, n\}$ such that $\tau(\nu)=i_\nu$ for each ν $(1\leq\nu\leq r)$. Denote by $V(\sigma)$ the n matrix whose $(i, \tau(i))$-component are 1 for $i=1, \cdots, n$, and all the other components are 0. (In short, $V(\sigma)$ is the permutation matrix attached to τ.) Denote also by $V'(\sigma)$ the following matrix:

$$V'(\sigma)=\begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where $A=\mathrm{diag}(a_1, \cdots, a_n)$, $B=\mathrm{diag}(b_1, \cdots, b_n)$, $a_i=1-b_i$, and

$$b_i=\begin{cases} 1 & \ldots \text{if } \varepsilon_i=-1 \text{ and } 1\leq i\leq r, \\ 0 & \ldots \text{othewise,} \end{cases}$$

Now, put

$$w(\sigma)=\begin{pmatrix} V(\sigma) & 0 \\ 0 & V(\sigma) \end{pmatrix}\times V'(\sigma).$$

Then $w(\sigma)\in N(Q)$. (Note that $w(\sigma)$ is defined indepently of θ.) We denote by $w(\sigma)$ the image of $w(\sigma)$ in $W=N(Q)/T(Q)$.

The subset $\{w(\sigma); \sigma\in S_r$ and σ is θ-admissible$\}$ of $Sp(n, Q)$ gives a complete set of representatives of $\Gamma_\theta\backslash Sp(n, Q)/P_r$, and each $w(\sigma)$ corresponds to the cusp corresponding to σ as in Proposition 3.4.

3.3. Representatives of cusps; in case $\Gamma_\theta\not\subset Sp(n, Z)$

For each $\theta\subsetneqq S_{\mathrm{aff}}$, put $\theta'=\theta\cap S$. It is obvious that $\Gamma_{\theta'}\subset\Gamma_\theta$. So, we can take representatives of $\Gamma_\theta\backslash Sp(n, Q)/P_r$ among representatives of $\Gamma_{\theta'}\backslash Sp(n, Q)/P_r$ obtained in § 3.2.

PROPOSITION 3.6. *For each $\theta\subset S_{\mathrm{aff}}$, the set of Γ_θ-equivalence classes of $(n-r)(n-r+1)/2$ dimensional cusps of $\Gamma_\theta\backslash H_n$ corresponds bijectively to the set of all θ-admissible elements of S_r.*

PROOF. The case where $\Gamma_\theta\subset S$ was proved in Proposition 3.4, so we assume that $\theta\not\subset S$. First, we show that any double coset $\Gamma_\theta wP_r$ $(w\in Sp(n, Q))$ contains $w(\sigma)\in W$ for some θ-admissible $\sigma\in S_r$, where $w(\sigma)$ is defined for each $\sigma\in S_r$ as in § 3-2. As $\Gamma_{\theta'}\subset\Gamma_\theta$, we can assume that $w=w(\sigma')$ for some θ'-admissible $\sigma'\in S_r$. We fix such σ' and put $\sigma'=\{\varepsilon_1\alpha_{i_2}, \varepsilon_2\alpha_{i_2}, \cdots, \varepsilon_r\alpha_{i_r}\}$ $(1\leq i_1<i_2<\cdots<i_r\leq n)$. If $b_2(\sigma', \theta')=\{i_1, \cdots, i_r\}\cap b_2(\theta')=\phi$, or $\varepsilon_i=-1$ for all $i\in b_2(\sigma', \theta')$, then σ' itself is θ-admissible, because $b_1(\sigma', \theta)=b_2(\sigma', \theta')$. So, we assume that $b_2(\sigma', \theta')\neq\phi$ and put $\mu=\mu(2, \theta', \alpha')$. By definition, $\{1, 2, \cdots, \mu\}=\{i\in b_2(\sigma', \theta'); \varepsilon_i=1\}$. It is clear that $w(\sigma')$ is a matrix of the following form:

$$w(\sigma')=\begin{pmatrix} 1_\mu & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_\mu & 0 \\ 0 & C & 0 & D \end{pmatrix},$$

where $A, B, C, D\in M_{n-\mu}(Q)$. Now, put

$$\gamma=\begin{pmatrix} 0 & 0 & -p^{-1}1_\mu & 0 \\ 0 & 1_{n-\mu} & 0 & 0 \\ p1_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-\mu} \end{pmatrix} \text{ and } p=\begin{pmatrix} p^{-1}1_\mu & 0 & 0 & 0 \\ 0 & 1_{n-\mu} & 0 & 0 \\ 5 & 0 & p1_\mu & 0 \\ 0 & 0 & 0 & 1_{n-\mu} \end{pmatrix}$$

Then, $\gamma\in\Gamma_\theta$, $p\in P_r$, and

$$\gamma w(\sigma')p=\begin{pmatrix} 0 & 0 & -1_\mu & 0 \\ 0 & A & 0 & B \\ 1_\mu & 0 & 0 & 0 \\ 0 & C & 0 & D \end{pmatrix}=w(\tau),$$

where $\tau=\{-\alpha_1, \cdots, -\alpha_\mu, \varepsilon_{\mu+1}\alpha_{i_{\mu+1}}, \cdots, \varepsilon_r\alpha_{i_r}\}$. Now, put $\kappa=\kappa(2, \theta, \sigma')$. Then, we have $\kappa=\kappa(1, \theta, \tau)$.

Now, take the permutation ρ on $\{1, 2, \cdots, n\}$ defined by:

$$p(i)=\begin{cases} \mu-i, & \text{if } 1\leq i\leq\mu-1, \\ i, & \text{if } \mu\leq i. \end{cases}$$

Also, put $\sigma=\{-\alpha_{\kappa-\mu}, -\alpha_{\kappa-\mu+1}, \cdots, \alpha_{\kappa-1}, \varepsilon_{i_{\mu+1}}\alpha_{i_{\mu+1}}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\}$.
Then, $\begin{pmatrix} V(\rho) & 0 \\ 0 & V(p) \end{pmatrix}\times\tau=\sigma$ and σ is θ-admissible.
As $\begin{pmatrix} V(\rho) & 0 \\ 0 & V(\rho) \end{pmatrix}.\times W_{\theta'}\subset\Gamma_{\theta'}$, we get $\Gamma_{\theta'}w(\tau)P_r=\Gamma_{\theta'}w(\sigma)P_r$.

Hence, $w(\sigma)\in\Gamma_\theta wP_r$. This proves the first part.
Secondly, assume that $\sigma, \tau\in S_r$ are θ-admissible. We shall show that, if $\Gamma_\theta w(\sigma)P_r=\Gamma_\theta w(\tau)P_r$, then $\sigma=\tau$. To do this, it is enough to show that

$$\Gamma_\theta\cap w(\sigma)P_r w(\tau)^{-1}=\phi, \quad \text{if } \sigma\neq\tau.$$

But, this is included in the more general Lemma 3.8 given below. q.e.d.

Before stating Lemma 3.8, we prepare several notations. We denote by ι_{n-r} the injective homomorphism of $Sp(n-r, Q)$ into $Sp(n, Q)$ defined by:

$$\iota_{n-r} : Sp(n-r, Q) \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_r & 0 \\ 0 & C & 0 & D \end{pmatrix} \in Sp(n, Q).$$

To calculate $w(\sigma)gw(\tau)^{-1}$ for any $g \in Sp(n, Q)$ and any θ-admissible σ and τ, we introduce the following notations: For each $\rho = \{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\} \in S_r$ $(1 \leq i_1 < i_2 < \cdots < i_r \leq n)$, define the mapping f_ρ of $\{1, 2, \cdots, r\} \cup \{n+1, n+2, \cdots, n+r\}$ into $Z/2nZ$ as follows:

$$f_\rho(c) = \begin{cases} i_c & \ldots \text{if } \varepsilon_{i_c} = 1 \text{ and } 1 \leq c \leq r, \\ i_{c-n} & \ldots \text{if } \varepsilon_{i_c} = -1 \text{ and } n+1 \leq c \leq n+r, \\ i_{c-n}+n & \ldots \text{if } \varepsilon_{i_c} = 1 \text{ and } n+1 \leq c \leq n+r, \\ i_c+n & \ldots \text{if } \varepsilon_{i_c} = -1 \text{ and } 1 \leq c \leq r. \end{cases}$$

We shall identify $Z/2nZ$ with the complete set of representatives $\{1, 2, \cdots, 2n\}$ of $Z/2nZ$. For each $g = (g_{ij}) \in Sp(n, Q)$ $(1 \leq i, j \leq 2n)$, each $\sigma \in S_r$ and each $\tau \in S_s$, and each $c \in \{1, \cdots, r\} \cup \{n+1, \cdots, n+r\}$ and $d \in \{1, 2, \cdots, s\} \cup \{n+1, \cdots, n+s\}$, it can be easily shown that the $(f_\tau(c), f_\sigma(d))$-component of $w(\sigma)gw(\tau)^{-1}$ is g_{cd}.

Finally, we define some subsets of $\{1, \cdots, r\}$. For each $\theta \subsetneqq S_{\mathrm{aff}}$, each θ-admissible $\sigma = \{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\}$, and each ν $(1 \leq \nu \leq t(\theta)+1)$, put

$$B_\nu(\theta, \sigma) = \{\kappa \in X;\ 1 \leq \kappa \leq r,\ i_\kappa \in b_\nu(\sigma, \theta) \subset b_\nu(\theta)\},$$

and for each $\varepsilon = 1$ or -1, put

$$B_\nu^\varepsilon(\theta, \sigma) = \{\kappa \in B_\nu(\theta, \sigma);\ \varepsilon_{i_\kappa} = \varepsilon\}.$$

LEMMA 3.8. *Fix an element $\theta \subsetneqq S_{\mathrm{aff}}$ and natural numbers s, r such that $1 \leq r \leq s \leq n$. Let $\sigma \in S_r$ and $\tau \in S_s$ be θ-admissible elements. Then, we have*

$$\iota_{n-r}(Sp(n-r, Q))P_s \cap w(\sigma)^{-1}\Gamma_\theta w(\tau) \neq \phi,$$

if and only if $\sigma \subset \tau$ (as subsets of $\{\pm\alpha_1, \cdots, \pm\alpha_r\}$). Besides, if $s = r$, then $\sigma \subset \tau$ if and only if $\sigma = \tau$.

PROOF. It is easy to show that the condition is sufficient. We shall show that it is necessary. For each element $g=(g_{ij})_{1\leq i,j\leq 2n}\in Sp(n, Q)$, define $r\times s$ submatrices U and V of g as follows:

$$U=(g_{ij})_{1\leq i\leq r,1\leq j\leq s}, \quad \text{and}$$
$$V=(g_{ij})_{1+n\leq i\leq r+n,1+n\leq j\leq n+s}$$

Now, we assume that $g\in \iota_{n-r}(Sp(n-r, Q))P_s$. Then, it is easy to see that $U^tV=1_r$. We shall show that this implies $\sigma\subset\tau$ under the assumption $w(\sigma)gw(\tau)^{-1}\in\Gamma_\theta$. First, we shall show

(a) *If $w(\sigma)gw(\tau)^{-1}\in\Gamma_\theta$, then $U, V\in M_{r,s}(Z)$.*

In fact, by definition of Γ_θ, it is clear that, for c, d with $1\leq c\leq r$, $1\leq d\leq s$, we have $g_{cd}\in Z$, except for the case where both $1\leq f_\sigma(c)\leq i_1(\theta)$ and $n+1\leq f_\tau(d)\leq n+i_1(\theta)$ are satisfied.

Now, we write $\sigma=\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\}$ and $\tau=\{\eta_{j_1}\alpha_{j_1}, \cdots, \eta_{j_s}\alpha_{j_s}\}$ $(1\leq i_1<\cdots<i_r\leq n, 1\leq j_1<\cdots<j_s\leq n)$. As we have assumed that σ and τ are θ-admissible, we see that $\varepsilon_{i_c}=-1$ (resp. $\varepsilon_{j_d}=-1$) for each c with $1\leq c\leq r$ (resp. d with $1\leq d\leq s$) such that $1\leq i_c\leq i_1(\theta)$ (resp. $1\leq j_d\leq i_1(\theta)$). Hence, if $1\leq f_\sigma(c)\leq i_1(\theta)$ and $n+1\leq f_\tau(d)\leq n+i_1(\theta)$, then $n+1\leq c\leq n+r$ and $1\leq d\leq s$. Hence, $U, V\in M_{r,s}(Z)$ and (a) is proved.

Now, we assume that σ is not contained in τ. This means that there exists c with $1\leq c\leq r$, such that $\bar{w}(\sigma)(\alpha_c)\neq\bar{w}(\tau)(\alpha_\nu)$ for any ν with $1\leq\nu\leq s$. Fix such c. For this c, there exists the unique ν with $1\leq n\leq t(\theta)+1$ such that $c\in b_\nu(\theta)$. Now, put $\varepsilon=\varepsilon_{i_c}$ $(=\pm1)$ and $a=\#(B^\varepsilon(\sigma, \theta))$, $b=\#(B^\varepsilon(\tau, \theta))$. By the choice of c and the assumption that σ and τ are θ-admissible, we have $b<a\leq\#(b_\nu(\theta))$. Write U and V by their coefficients as $U=(u_{ij})$ and $V=(v_{ij})$ $(1\leq i\leq r, 1\leq j\leq s)$.

Now, we shall show

(b) *For any $c\in B^\varepsilon(\sigma, \theta)$, we get $u_{cd}v_{cd}\equiv0$* mod. p, *unless $d\in B^\varepsilon(\tau, \theta)$.*

In fact,

(i) if $(\varepsilon_{i_c}, \varepsilon_{j_d})=(1, -1)$, then $(f_\sigma(c+n), f_\tau(d+n))=(i_c+n, j_d)$ and $v_{cd}\equiv0$ mod. p.

(ii) if $(\varepsilon_{i_c}, \varepsilon_{j_d})=(1, 1)$, then $(f_\sigma(c), f_\tau(d))=(i_c, j_d)$, and $(f_\sigma(c+n), f_\sigma(d+n))=(i_c+n, j_d+n)$. Now, if $d\notin B^\varepsilon(\tau, \theta)$, then $j_d\notin b_\nu(\theta)$. Hence $j_d\in b_\mu(\theta)$ for some $\mu\neq n$. If $\mu>\nu$, then $v_{cd}\equiv0$ mod. p, and if $\mu<\nu$, then $u_{cd}\equiv0$ mod. p.

(iii) if $(\varepsilon_{i_c}, \varepsilon_{j_d})=(-1, 1)$, then in the same way as in (i) we get $u_{cd}\equiv0$

mod. p,

(iv) if $(\varepsilon_{i_c}, \varepsilon_{j_d})=(-1, -1)$, and $d \notin B^{\varepsilon}(\tau, \theta)$, then $d \in b_{\mu}(\theta)$ for some $\mu \neq \nu$, and in the same way as in (ii), we get $v_{cd} \equiv 0$ mod. p if $\nu > \mu$, and $u_{cd} \equiv 0$ mod. p if $\nu < \mu$.

So, we get (b).

Now we prove the Lemma. Put $B^{\varepsilon}(\sigma, \theta)=\{c_1, c_2, \cdots, c_a\}$ and $B^{\varepsilon}(\tau, \theta)=\{d_1, \cdots, d_b\}$. Define $a \times b$ matrices U_1 and V_1 by:

$$U_1=(u_{c_i d_j})_{1 \leq i \leq a, 1 \leq j \leq b}, \text{ and } V_1=(v_{c_i d_j})_{1 \leq i \leq a, 1 \leq j \leq b}$$

By the above (b) and the fact that $U^t V=1_r$, we get

$$U_1{}^t V_1 \equiv 1_a \text{ mod. } p.$$

But, rank $(U_1) \leq \min(a, b)=b<a$. This is a contradiction. q.e.d.

COROLLARY 3.9. *For each natural number $n \geq 1$ and each natural number r with $1 \leq r \leq n$, we get*

$$\sum_{\theta \subsetneq S_{\mathrm{aff}}} (-1)^{\sharp(\theta)} \sharp(\Gamma_{\theta} \backslash Sp(n, Q)/P_r)=0.$$

PROOF. We can prove this by some easy combinatorial argument, but we omit the details here, because the Proof of Proposition 4.1 in the next section includes the proof of this corollary. q.e.d.

§ 4. Proof of Main Theorem and combinatorial properties on cusps

In this section, we shall complete the proof of Main Theorem in § 1 by using the following Theorem 4.1 on some combinatorial properties on cusps and their stabilizers.

Fix a natural number r with $1 \leq r \leq n$, and an element $\sigma=\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\} \in S_r$. Denote by $\mathcal{A}(\sigma)$ the following family of subsets of S_{aff}:

$$\mathcal{A}(\sigma)=\{\theta \subset S_{\mathrm{aff}}; \theta \neq S_{\mathrm{aff}} \text{ and } \theta \text{ is } \sigma\text{-admissible}\}.$$

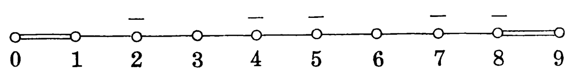
Denote by $w(\sigma)$ the fixed representative in $N(Q)$ of $\sigma \in S_r=W/W_r$ defined as in § 3-1.

THEOREM 4.1. *For each r with $1 \leq r \leq n$ and each $\sigma \in S_{\mathrm{aff}}$, there exists a bijection $\phi=\phi_{\sigma}$ of order two of $\mathcal{A}(\sigma)$ onto itself such that the following two conditions are satisfied:*

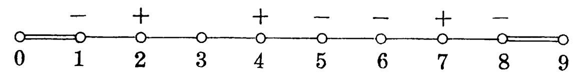
(1) $\#(\phi(\theta))=\#(\theta)+1$, *or* $\#(\theta)-1$,
(2) $P_r \cap w(\sigma)^{-1}\Gamma_\theta w(\sigma)=P_r \cap w(\sigma)^{-1}\Gamma_{\phi_\sigma(\theta)} w(\sigma)$.

REMARK. *Such ϕ_σ is not unique in general, even if we fix r and σ.*

Before proving Theorem 4.1, we give here short explanation how such ϕ_σ is defined by giving examples, using the "picture" of σ. More rigorous definition will be given in the proof.
(1) Assume that the picture of σ contains only minus signs, e.g.



Then, a subset $\theta \subsetneqq S_{\mathrm{aff}}$ is σ-admissible for this σ, if and only if $\mathrm{gap}(\theta)\supset \{2, 5, 8\}$. Now, ϕ_σ is defined for this σ to be the mapping to add, or delete s_1, to θ, or from θ, according as $s_1 \in \theta$ or not, respectively. In general, ϕ_σ adds, or deletes the vertex just before the "first" minus sign.
(2) Assume that the picture of σ contains at least one plus sign, e.g.



Then, a subset $\theta \subsetneqq S_{\mathrm{aff}}$ is σ-admissible for this σ, if and only if $\mathrm{gap}(\theta)\supset \{1, 3, 6, 8\}$. Now, ϕ_σ is defined for this σ to be the mapping to add, or delete s_7, to θ, or from θ, according as $s_7 \in \theta$, or not, respectively. In general, ϕ_σ adds, or deletes the vertex where the "last" plus sign is on.

PROOF OF THEOREM 4.1. First, we shall give more rigorous definition of the bijection $\phi=\phi_\sigma$ of order two on $\mathcal{A}(\sigma)$, and later we shall show that this ϕ satisfies the conditions (1) and (2) in the above Theorem.

To distinguish two cases explained above, we introduce some notations. We divide S_r into the following disjoint union of two subsets S_r^- and S_r' of S_r:

$$S_r=S_r^- \amalg S_r',$$

where S_r^- is defined by:

$$S_r^-=\{\rho=\{\eta_{i_1}\alpha_{i_1}, \cdots, \eta_{i_r}\alpha_{i_r}\}\in S_r; \eta_{i_\nu}=-1 \text{ for all } \nu \text{ with } 1\leq\nu\leq r\},$$

and S_r' is defined to be the compliment of S_r^- in S_r. In other words,

S_r^-, or S_r' corresponds to the first, or the second case of the above examples, respectively. For the sake of simplicity, we shall identify S_{aff} with $X=\{0, 1, \cdots, n\}$ by the bijection: $s_i \longrightarrow i$.

(I) *Definition of* ϕ_σ.

For each $\sigma \in S_r$, we define a mapping ϕ_σ of S_r to S_r as follows:

(a) When $\sigma=\{-\alpha_{i_1}, -\alpha_{i_2}, \cdots, -\alpha_{i_r}\} \in S_r^-$, we define ϕ_σ by:

$$\phi_\sigma(\theta)=\theta \cup \{i_1-1\}, \quad \text{if } i_1-1 \notin \theta, \qquad \text{and}$$
$$\phi_\sigma(\theta)=\theta \setminus \{i_1-1\}, \quad \text{if } i_1-1 \in \theta.$$

(b) When $\sigma=\{\varepsilon_{i_1}\alpha_{i_1}, \cdots, \varepsilon_{i_r}\alpha_{i_r}\} \in S_r'$, then put

$$i_{\max}=\max_{1\leq\nu\leq r}\{i_\nu \in X; \varepsilon_{i_\nu}=+1\}.$$

In this case, we define ϕ_σ by:

$$\phi_\sigma(\theta)=\theta \cup \{i_{\max}\}, \quad \text{if } i_{\max} \notin \theta, \qquad \text{and}$$
$$\phi_\sigma(\theta)=\theta \setminus \{i_{\max}\}, \quad \text{if } i_{\max} \in \theta.$$

(II) ϕ_σ *is a permutation on* $\mathcal{A}(\sigma)$ *of order two.*

To show this, it is sufficient to show that $\phi_\sigma(\theta) \in \mathcal{A}(\sigma)$ for any $\theta \in \mathcal{A}(\sigma)$. First, we show that $\phi_\sigma(\theta) \neq S_{\mathrm{aff}}$. In fact, assume that $\phi_\sigma(\theta)=S_{\mathrm{aff}}$. Then, if $\sigma \in S_r^-$, we get $\theta=\{1, 2, \cdots, n\}$, $b_1(\theta)=\phi$, and $b_2(\theta)=\{1, 2, \cdots, n\}$. So, the condition (2) on admissibility in § 3 is not satisfied for θ, and θ is not σ-admissible.

If $\sigma \in S_r'$, then $\theta=\{0, 1, \cdots, i_{\max}-1, i_{\max}+1, \cdots, n\}$, $b_1(\theta)=\{1, 2, \cdots, i_{\max}\}$, and $b_2(\theta)=\{i_{\max}+1, \cdots, n\}$.

So, the admissibility condition (3) is not satisfied by θ, and θ is not σ-admissible. Hence, $\phi_\sigma(\theta) \neq S_{\mathrm{aff}}$ for any $\theta \in \mathcal{A}(\sigma)$. Now, we show that $\phi_\sigma(\theta)$ is σ-admissible, if θ is σ-admissible. Assume that $\sharp(\phi_\sigma(\theta))=\sharp(\theta)+1$. Then, $\phi_\sigma(\theta)=\theta \cup \{i_\nu(\theta)\}$ for some ν with $1\leq\nu\leq t(\theta)$, where $i_\nu(\theta)$ is the ν-th gap of θ. By definition of ϕ_σ, this $i_\nu(\theta)$ is equal to i_1-1, if $\sigma \in S_r^-$, and to $i_{\max}$, if $\sigma \in S_r'$. In both cases, we get

$$b_\iota(\phi_\sigma(\theta))=b_\iota(\theta), \qquad \text{if } 1\leq\iota\leq\nu-1,$$
$$b_\nu(\phi_\sigma(\theta))=b_\nu(\theta) \cup b_{\nu+1}(\theta), \qquad \text{and}$$
$$b_\iota(\phi_\sigma(\theta))=b_{\iota+1}(\theta), \quad \text{if } \nu<\iota\leq t(\phi_\sigma(\theta))+1=t(\theta).$$

Assume that θ is σ-admissible. Then, by the condition (2) in § 3-2, we get

(a) if $\sigma \in S_r^-$, then

$$b_\nu(\phi_\sigma(\theta)) \cap \{i_1, \cdots, i_r\} = b_{\nu+1}(\theta) \cap \{i_1, \cdots, i_r\} = b_{\nu+1}(\theta),$$
and $\varepsilon_j = -1$ for all $j \in b_{\nu+1}(\theta)$.

(b) if $\sigma \in S'_r$, then

$$b_\nu(\phi_\sigma(\theta)) \cap \{i_1, \cdots, i_r\} = b_\nu(\theta) \cup (b_{\nu+1}(\phi_\sigma(\theta)) \cap \{i_1 \cdots, i_r\})$$
$i_{\max} \in b_\nu(\theta)$, and
$\varepsilon_j = +1$ for all $j \in b_\nu(\theta)$,
$\varepsilon_j = -1$ for all $j \in b_{\nu+1}(\phi_\sigma(\theta)) \cap \{i_1, \cdots, i_r\}$.

Hence, conditions (1), (2) in § 3-2 are also satisfied by $\phi_\sigma(\theta)$ and σ. We shall see the condition (3) now. If $\sigma \in S_r^-$, then by Condition (3) for θ and σ, we get $\nu+1 < t(\theta)+1$, so $\nu < t(\phi_\sigma(\theta))+1$. Hence, Condition (3) is also satisfied by $\phi_\sigma(\theta)$ and σ. If $\sigma \in S'_r$, then by Condition (2), we get $\nu \neq 1$. Hence, Condition (3) is also satisfied. Thus, we proved that $\phi_\sigma(\theta)$ is also σ-admissible. Virtually in the same way, we can show the σ-admissibility of $\phi_\sigma(\theta)$ also when $\sharp(\phi_\sigma(\theta)) = \sharp(\theta)-1$. We omit the details here. Now, it is clear that ϕ_σ is a permutation on $\mathcal{A}(\sigma)$ of order two.

(III) *The mapping* ϕ_σ *satisfies* (1) *and* (2) *in Theorem* 4.1.

In fact, by definition, it satisfies the condition (1). To show (2), we can assume that $\phi_\sigma(\theta) \supset \theta$ without loss of generality. Under this assumption, we get $\Gamma_{\phi_\sigma(\theta)} \supset \Gamma_\theta$, and in order to prove the statement (2) in Theorem 4.1, it is sufficient to prove that any $\gamma \in \Gamma_{\phi_\sigma(\theta)}$ such that $w(\sigma)\gamma w(\sigma)^{-1} \in P_r$ belongs to Γ_θ. To show this, first, we give a description of Γ_θ as the subgroup $\Gamma_{\phi_\sigma(\theta)}$. For any $M \in M_n(Q)$, we denote by

$$M = (M_{ij}) \quad (1 \leq i, j \leq t(\phi(\theta))+1)$$

the $\phi(\theta)$-partition of M, and by

$$M = (M_{ij}) \quad (1 \leq i, j \leq t(\theta)+1)$$

the θ-partition of M. (As for the definition, see § 1-3.) As before, we put $\phi(\theta) = \theta \cup \{i_\nu(\theta)\}$ for some ν with $1 \leq \nu \leq t(\theta)$. We define a mapping f of $\{1, 2, \nu-1, \nu+1, \cdots, t(\theta))\}$ into $\{1, 2, \cdots, t(\theta)\}$ by:

$$f(i) = \begin{cases} i & \ldots \text{if } 1 \leq i \leq \nu-1, \\ i+1 & \quad \text{if } \nu+1 \leq i \leq t(\theta). \end{cases}$$

Then, it is easy to see that

$$\begin{aligned} &M'_{ij}=M_{f(i)f(j)}, && \text{if } i\neq\nu \text{ and } j\neq\nu,\\ &M'_{\nu j}=\begin{pmatrix} M_{\nu,f(j)} \\ M_{\nu+1,f(j)} \end{pmatrix}, && \text{if } j\neq\nu,\\ &M'_{i\nu}=(M_{f(i),\nu}, M_{f(i),\nu+1}), && \text{if } i\neq\nu, \text{ and}\\ &M'_{\nu\nu}=\begin{pmatrix} M_{\nu,\nu} & M_{\nu,\nu+1} \\ M_{\nu+1,\nu} & M_{\nu+1,\nu+1} \end{pmatrix}. \end{aligned}$$

For $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_{\phi(\theta)}$, we denote by (A_{ij}), (B_{ij}), (C_{ij}), or (D_{ij}) the θ-partition of A, B, C, or D, respectively. By the above relation of $\phi(\theta)$-partition and θ-partition, we can easily show that the subgroup Γ_θ of $\Gamma_{\phi(\theta)}$ consists exactly of those elements $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_{\phi(\theta)}$ which satisfies the following three conditions:

(1) $A_{\nu+1,\nu}\equiv D_{\nu,\nu+1}\equiv 0 \mod. p$.
(2) If $\nu=1$, then $B_{12}, B_{21}, B_{22}\equiv 0 \mod. 1$.
(3) If $\nu=t(\theta)$, then $C_{\nu\nu}, C_{\nu,\nu+1}, C_{\nu+1,\nu}\equiv 0 \mod. p$.

Now, we shall show that, if $w(\sigma)\gamma w(\sigma)^{-1}\in P_r$ for $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_{\phi(\theta)}$, then

(4.2) (1) $A_{\nu+1,\nu}=D_{\nu,\nu+1}=0$,
(2) if $\nu=1$ and $\sigma\in S_r^-$, then $B_{12}=B_{21}=B_{22}=0$,
(3) if $\nu=t(\theta)$ and $\sigma\in S'_r$, then $C_{\nu,\nu}=C_{\nu,\nu+1}=C_{\nu+1,\nu}=0$.

This may be proved fairly easily by full use of theory on algebraic groups, but here we try to give rather self-contained proof.

First of all, we note that, by our explicit choice of $w(\sigma)$ in § 3-2, the mapping

$$M_{2n}(Q)\ni X=(x_{ij})_{1\leq i,j\leq 2n}\longrightarrow w(\sigma)Xw(\sigma)^{-1}\in M_{2n}(Q)$$

is just a permutation of the coordinates x_{ij} up to sign. So, there exists a subset $X_r(\sigma)\subset\{1, 2, \cdots, 2n\}^2$, depending on σ and r and not on θ, such that

$$\Gamma_{\phi(\theta)}\cap w(\sigma)^{-1}P_r w(\sigma)=\{g=(g_{ij})_{1\leq i,j\leq 2n}\in\Gamma_{\phi(\theta)};\ g_{ij}=0 \text{ for all } (i,j)\in X_r(\sigma)\}.$$

By definition of $w(\sigma)$, it is obvious that:

(1) if $(i, j) \in X_r(\sigma)$, and $1 \leq i, j \leq n$, then $(j+n, i+n) \in X_r(\sigma)$ and vice versa.

(2) if $(i, j) \in X_r(\sigma)$ and $1 \leq i \leq n < j < 2n$, then $(j-n, i+n) \in X_r(\sigma)$ and vice versa.

So, to obtain $X_r(\sigma)$, we can use the action of $w(\sigma)$ on root subgroups U_α, or on its Lie algebra g_α $(\alpha \in \Phi)$, instead of coordinates. To be more precise, denote by Φ_r the subset of the root system Φ of $Sp(n, Q)$ defined by:

$$\Phi_r = \{\alpha_i - \alpha_j;\ 1 \leq j \leq r < i \leq n\} \cup \{-\alpha_i - \alpha_j;\ 1 \leq i, j \leq n \text{ and } \min(i, j) \leq r\}.$$

If $w(\sigma) g_\alpha w(\sigma)^{-1} = g_{w(\sigma)^{-1}\alpha}$ is not contained in the Lie algebra of P_r, in other words, if $\bar{w}(\sigma)^{-1}\alpha \in \Phi_r$, then the "coordinates" corresponding to g_α must vanish. By using this fact, we shall show (1), (2), (3) of (4.2) as follows:

(1) If $\alpha = \alpha_i - \alpha_j$, $i_\nu(\theta) \leq i \leq i_{\nu+1}(\theta)$, and $i_{\nu-1}(\theta) + 1 \leq j \leq i_\nu(\theta)$, then $\bar{w}(\sigma)^{-1}\alpha \in \Phi_r$. In fact, if $\sigma \in S_r^-$, then $\bar{w}(\sigma)^{-1}(\alpha_i) = -\alpha_\mu$ for some μ with $1 \leq \mu \leq r$, and $\bar{w}(\sigma)^{-1}(\alpha_j) = \pm\alpha_\kappa$ for some $\kappa > r$, hence $\bar{w}(\sigma)^{-1}(\alpha) \in \Phi_r$. If $\sigma \in S_r'$, then $\bar{w}(\sigma)^{-1}(\alpha_j) = \alpha_\kappa$ for some κ with $1 \leq \kappa \leq r$, and $\bar{w}(\sigma)^{-1}(\alpha_i) = +\alpha_\mu$ for some $\mu > r$, or $\bar{w}(\sigma)(\alpha_i) = -\alpha_\mu$ for some μ with $1 \leq \mu \leq r$. Hence $\bar{w}(\sigma)^{-1}(\alpha_i) \in \Phi_r$. This proves (1) of (4.2).

(2) If $\nu = 1$, $\sigma \in S_r^-$, and $\alpha = \alpha_i + \alpha_j$ for some i, j such that $1 \leq i \leq i_2(\theta)$ and $i_1(\theta) + 1 \leq j \leq i_2(\theta)$, then $\bar{w}(\sigma)^{-1}(\alpha) \in \Phi_r$. This is easily seen, because we get

$$\bar{w}(\sigma)^{-1}(\alpha_k) = -\alpha_\mu \ (1 \leq \mu \leq r), \text{ if } i_1(\theta) \leq k \leq i_2(\theta), \text{ and}$$
$$\bar{w}(\sigma)^{-1}(\alpha_k) = \pm\alpha_\kappa \ (\kappa > r), \text{ if } 1 \leq k \leq i_1(\theta).$$

This proves (2) of (4.2).

(3) If $\nu = t(\theta)$, $\sigma \in S_r'$, and $\alpha = -\alpha_i - \alpha_j$ for some i, j such that $i_{\nu-1}(\theta) + 1 \leq i \leq i_{\nu+1}(\theta)$ and $i_{\nu-1}(\theta) + 1 \leq j \leq i_\nu(\theta)$, then $\bar{w}(\sigma)^{-1}(\alpha) \in \Phi_r$. This is easily seen, because

(i) $\bar{w}(\sigma)(\alpha_k) = \alpha_\mu$ $(1 \leq \mu \leq r)$, if $i_{\nu-1}(\theta) + 1 \leq k \leq i_\nu(\theta)$, and

(ii) $\bar{w}(\sigma)(\alpha_k) = -\alpha_\mu$ $(1 \leq \mu \leq r)$, or $\pm\alpha_\mu$ $(\mu > r)$, if $i_\nu(\theta) + 1 \leq k \leq i_{\nu+1}(\theta)$.

This proves (3) of (4.2). Thus, we proved Theorem 4.1. q.e.d.

PROOF OF MAIN THEOREM. Combining Corollary 2.5 and Theorem 4.1, we get our Main Theorem. q.e.d.

§ 5. Relation between cusps

5.1. So far, we regarded cusps just as the double cosets $\Gamma_\theta w P_r$, or their representatives. But, of course, cusps usually mean the boundary components in the Satake compactification $\overline{\Gamma_\theta\backslash H_n}$ of $\Gamma_\theta\backslash H_n$. That is, $(n-r)(n-r+1)/2$-dimensional boundary components of $\overline{\Gamma_\theta\backslash H_n}$ correspond bijectively to the set of all double cosets in $\Gamma_\theta\backslash Sp(n, Q)/P_r$. (cf. [23]) For each θ-admissible $\sigma\in S_r$, we denote by Cusp(σ) the component in $\overline{\Gamma_r\backslash H_n}$ which corresponds to σ. By definition of the Satake compactification, we have

$$\mathrm{Cusp}(\sigma)=\Gamma_\theta(\sigma)\backslash H_{n-r},$$

where we denote by $\Gamma_\theta(\sigma)$ the subgroup of $Sp(n-r, Q)$ defined by:

$$\iota_{n-r}(\Gamma_\theta(\sigma))=w(\sigma)^{-1}\Gamma_\theta w(\sigma)\cap\iota_{n-r}(Sp(n-r, Q)).$$

The group $\Gamma_\theta(\sigma)$ depends on the choice of the representative $w(\sigma)$ of the $\Gamma_\theta-P_r$-double coset, but non-essentially.

In this section, we shall solve the following three problems.

(1) For each $\theta\subsetneqq S_{\mathrm{aff}}$, and each θ-admissible $\sigma\in S_r$, describe $\Gamma_\theta(\sigma)\subset Sp(n-r, Q)$ and hence Cusp(σ).

(2) Let r and s be natural integers with $r\leq s$. For each $\theta\subsetneqq S_{\mathrm{aff}}$ and for each pair (σ, τ) of θ-admissible elements $\sigma\in S_s$ and $\tau\in S_r$, give the condition that Cusp(σ) is on Cusp(τ).

(3) For any $\theta, \theta'\subsetneqq S_{\mathrm{aff}}$ such that $\theta\subset\theta'$ and a θ-admissible $\sigma\in S_r$, give the θ'-admissible $\tau\in S_r$ such that Cusp(τ) is the image of Cusp(σ) by the natural covering $\overline{\Gamma_\theta\backslash H_n}\longrightarrow\overline{\Gamma_{\theta'}\backslash H_n}$.

Complete answers to these questions will be given below. Short intuitive explanation of examples by using *pictures* will be given after Proposition 5.4.

5.2. First, we solve (1). It will be proved that $\Gamma_\theta(\sigma)$ is also a parahopric subgroup of $Sp(n-r, Q)$. That is, denoting $S_{\mathrm{aff}}(n-r)$ the set of vertices of the extended Dynkin diagram of $Sp(n-r, Q)$, it will be proved that $\Gamma_\theta(\sigma)=\Gamma_{\theta'}\subset Sp(n-r, Q)$ for some $\theta'\subsetneqq S_{\mathrm{aff}}(n-r)$. Taking this granted for a while, we shall give a description of θ'. Define the sequence $b_\nu(\theta)$ $(1\leq n\leq t(\theta)+1)$ as in § 3. For each θ-admissible $\sigma\in S_r$, define $B_\nu(\sigma, \theta)$ as in § 3. Denote by $\pi_\sigma(\theta)$ the unique element of $S_{\mathrm{aff}}(n-r)\simeq\{0, 1, \cdots, n-r\}$ such that gap$(\pi_\sigma(\theta))$ consists just of the vertices corresponding to the following numbers:

$$\sum_{\nu=0}^{m}(\#(b_\nu(\theta))-\#(B_\nu(\sigma,\theta))) \qquad (m=1,2,\cdots,t(\theta)).$$

Note that $\#(\mathrm{gap}(\pi_\sigma(\theta)))$ might be smaller than $t(\theta)$. In fact, we have

$$\#(\mathrm{gap}(\pi_\sigma(\theta))=\#\{\nu;\ 1\leq n\leq t(\theta),\ \#(b_\nu(\theta))\neq\#(B_\nu(\sigma,\theta))\}.$$

PROPOSITION 5.1. *For each $\theta\subset S_{\mathrm{aff}}$ and each θ-admissible $\sigma\in S_r$, we have*

$$\mathrm{Cusp}(\sigma)=\Gamma_{\pi_\sigma(\theta)}\backslash H_{n-r}.$$

PROOF. The proof is virtually the same as in the proof of Proposition 4.1. But, here, we must carefully choose the representative $w(\sigma)$, because the answer might depend on the choice.

For $\sigma=\{\varepsilon_{i_1}\alpha_{i_1},\cdots,\varepsilon_{i_r}\alpha_{i_r}\}$, we denote by $\bar{w}(\sigma)$ the unique element of W such that the following conditions (1)(2)(3) are satisfied.

(1) If $1\leq\nu\leq r$, then $\bar{w}(\sigma)(\alpha_\nu)=\varepsilon_{i_\nu}\alpha_{i_\nu}$.

(2) If $r+1\leq\nu\leq n$, then $\bar{w}(\sigma)(\alpha_\nu)=\alpha_c$ for some c $(1\leq c\leq n)$.

(3) If $r+1\leq n, m\leq n$ and $\bar{w}(\sigma)(\alpha_\nu)=\alpha_c$ and $\bar{w}(\sigma)(\alpha_u)=\alpha_d$, then $\nu<\mu$ if and only if $c<d$.

We define $w(\sigma)$ as in § 3 for this $\bar{w}(\sigma)$. The rest is easy and the details are omitted here. q.e.d.

5-3. We shall solve (2).

PROPOSITION 5.2. *Let r, s be integers such that $1\leq r\leq s\leq n$. Fix an element $\theta\subsetneqq S_{\mathrm{aff}}$. Take $\sigma\in S_r$ and $\tau\in S_s$ which are θ-admissible. Then, $(n-s)(n-s+1)/2$-dimensional* $\mathrm{Cusp}(\tau)$ *is on $(n-r)(n-r+1)/2$-dimensional* $\mathrm{Cusp}(\sigma)$, *if and only $\sigma\subset\tau$ (as subsets of $\{\pm\alpha_1,\cdots,\pm\alpha_r\}$).*

PROOF. By definition (cf. [25]), $\mathrm{Cusp}(\tau)$ is on $\mathrm{Cusp}(\sigma)$, if and only if

$$w(\sigma)\iota_{n-r}(g)\in\Gamma_\theta w(\tau)P_s$$

for some $g\in Sp(n-r,Q)$, that is, if and only if

$$\iota_{n-r}(Sp(u-r,Q))P_s\cap w(\sigma)^{-1}\Gamma_\theta w(\tau)\neq\phi.$$

So, the above assertion follows directly from Lemma 3.8. q.e.d.

5-4. Now, we shall solve (3). This is the easiest of the three. When $\theta\subset\theta'\subsetneqq S_{\mathrm{aff}}$, that is, if $\Gamma_\theta\subset\Gamma_{\theta'}$, we must describe just the natural mapping $\Gamma_\theta\backslash Sp(n,p)/P_r\longrightarrow\Gamma_{\theta'}\backslash Sp(n,Q)/P_r$ for each r. To describe this mapping, for each $\theta\subset S_{\mathrm{aff}}$, denote by $\mathcal{A}_r(\theta)$ the subset of S_r defined by:

$$\mathcal{A}_r(\theta)=\{\sigma\in S_r;\ \sigma \text{ is } \theta\text{-admissible}\}.$$

Now, we shall define a mapping $c_{\theta',\theta}$ of $\mathcal{A}_r(\theta)$ onto $\mathcal{A}_r(\theta')$. For each $\sigma\in\mathcal{A}_r(\theta)$, we denote by $c_{\theta',\theta}(\sigma)$ the unique element of $\mathcal{A}_r(\theta')$ such that the following conditions are satisfied:

(1) For fixed ν with $2\leq\nu\leq t(\theta')$, $B_\nu(c_{\theta',\theta}(\sigma),\theta')$ is the disjoint union of all $B_\mu(\sigma,\theta)$, where μ runs over all μ with $1\leq\mu\leq t(\theta)+1$ such that $b_\mu(\theta)\subset b_\nu(\theta')$.

(2) If $b_{t(\theta')+1}(\theta')\neq\phi$, then $B^+_{t(\theta')+1}(c_{\theta',\theta}(\sigma),\theta')$ is the disjoint union of all $B_\mu(\sigma,\theta)$, where μ runs over all μ with $1\leq\mu\leq t(\theta)+1$ such that $b_\mu(\theta)\subset b_{t(\theta')+1}(\theta')$.

(3) If $b_1(\theta')\neq\phi$, then $B^-_1(c_{\theta',\theta}(\sigma),\theta')$ is the disjoint union of all $B_\mu(\sigma,\theta)$, where μ runs over all μ with $1\leq\mu\leq t(\theta)+1$ such that $b_\mu(\theta)\subset b_1(\theta')$.

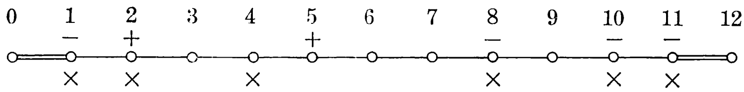
PROPOSITION 5.3. *For each θ, $\theta'\subsetneqq S_{\mathrm{aff}}$ such that $\theta\subset\theta'$, and θ-admissible $\sigma\in S_r$, the image of σ by the natural covering of $\overline{\Gamma_\theta\backslash H_n}$ onto $\overline{\Gamma_{\theta'}\backslash H_n}$ is $c_{\theta',\theta}(\sigma)$.*

PROOF. This is obvious and the proof will be omitted here. q.e.d.

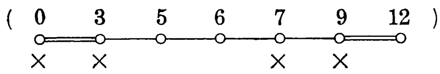
Examples and short intuitive explanation by using picture.

(1) How to get $\pi_\sigma(\theta)$?

Take a following example of a picture of an admissible pair (σ,θ):



In this case, $n=12$, $r=6$. From the picture, delete the vertices where the signs are on. Then, only vertices numbered 0, 3, 4, 6, 7, 9, 12 remain. Each set $\{0\}$, $\{3\}$, $\{5,6,7\}$, $\{9\}$, or $\{12\}$ belonged to the same sequence and the biggest element of each set of these is a gap of $\pi_\sigma(\theta)$, as far as the original sequence contains a gap. Hence, the picture of this $\pi_\sigma(\theta)$ is as follows:

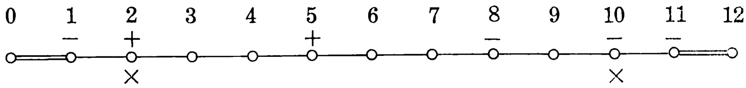


where the numbers in the parentheses are the original numbers.

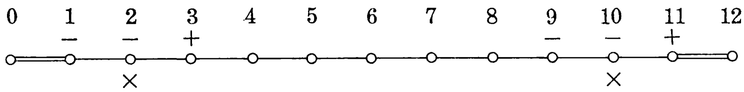
(2) How to get $c_{\theta',\theta}(\sigma)$?

Take a picture as in (1). Assume that $\mathrm{gap}(\theta')=\{2, 10\}$.

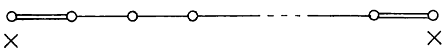
Then, the following picture



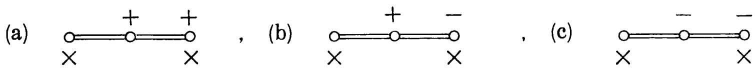
is *not* admissible. Now, to get $c_{\theta',\theta}(\theta)$, we must adjust and convert this to an admissible one. The first sequence must contain only the minus signs, so we convert $+$ to $-$. The plus (resp. minus) signs in the second sequence must be located on the left (resp. right) side of the sequence, so we remove $+$ on the vertex 5 and $-$ on the vertex 8, and mark by $+$ on 3 and $-$ on 9. The last sequence must contain only the plus sign in this case, so we convert $-$ on 11 to $+$. Then, we get



This is a picture of $c_{\theta',\theta}(\sigma)$ in this case.
(3) How to get the Cusp configuration for one fixed Γ_θ?
For example, let θ be given by the following picture:

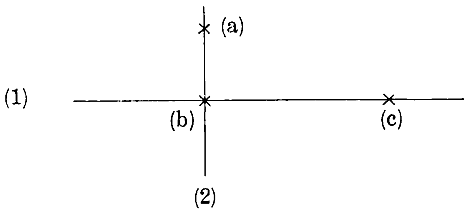
 ,

where $\mathrm{gap}(\theta)=\{0, n\}$. In this case, Γ_θ is usually denoted by $\Gamma_0^{(n)}(p)$. Now, for the sake of simplicity, assume that $n=2$. Then, $\Gamma_0^{(2)}(p)$ has three 0-dimensional cusps (a), (b) and (c):



and two 1-dimensional cusps (1) and (2):

By Proposition 5.2, the cusps (a) and (b) are on (1), and the cusps (b) and (c) are on (2). The whole boundary of $\overline{\Gamma_0^{(2)}(p)\backslash H_2}$ is as follows,



and cusps (1) and (2) are isomorphic to $\Gamma_0^{(1)}(p)\backslash H_1$.

Of course, we can write down all the cusp configurations quite easily by Proposition 5.2, not only for this one, but also for any n and θ.

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Department of mathematics
College of General Education
Osaka University
Machikaneyama-machi 1-1
Toyonaka-City
560 Japan