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Stability of compact leaves close to invariant fibered manifolds

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Abstract. A Fuller-type index is defined for a pair $(\mathcal{F}; N)$, where $\mathcal{F}$ is a C^1 foliation on a smooth manifold M and N is a compact submanifold of M saturated by compact leaves of $\mathcal{F}$ as a fibration $L \hookrightarrow N \rightarrow B$, provided that $\pi_1(L)=\mathbf{Z}$ and the associated action of $\pi_1(B)$ on $\pi_1(L)$ is trivial. Using this index, sufficient conditions for persistence of a fiber L close to N as a compact leaf under C^1 small perturbations of $\mathcal{F}$ are given by the non vanishing of the Euler characteristic of B and some hypotheses on the behaviour of the foliation in a neighborhood of N.

Introduction

Let $\mathcal{F}$ be a C^1 foliation on a smooth (C^∞) manifold M and let N be a compact C^1 submanifold saturated by compact leaves of $\mathcal{F}$. We say that $\mathcal{F}$ has the *compact leaf stability property close to* N if, shortly speaking, every foliation sufficiently C^1 close to $\mathcal{F}$ has a compact leaf close to N. In this paper we deal with the case where N is $\mathcal{F}$-saturated as a fibration $L \hookrightarrow N \rightarrow B$ with $\pi_1(L)=\mathbf{Z}$.

For vector fields Fuller [F] has associated to each isolated compact set K of periodic orbits of a C^1 vector field X an index whose non vanishing implies that all C^1 vector fields sufficiently C^0 close to X have a periodic orbit close to K (see also [B_2]).

By adapting Fuller's method to our foliation context, in Theorem 1 (§ 2) we associate to the pair $(\mathcal{F}; N)$ an index (cf. Definition 2.2) whose non vanishing implies the compact leaf stability property close to N. From this stability criterion we obtain Theorem 2 (§ 4) where we treat the case that the restriction of the foliation $\mathcal{F}$ to a neighborhood of N saturates the fibers of some tubular neighborhood of N. In this case stability is ensured by the hypotheses that stabilize the fibers of a

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fibration (i.e. $M=N$) and stabilize also an isolated compact leaf (i.e. B is a point).

One of the difficulties to calculate the index of $(\mathcal{F}; N)$ is the construction of a suitable C^1 perturbation of $\mathcal{F}$ when its dimension is greater than one. To resolve this difficulty we reduce the problem to a stability question for a one dimensional C^1 foliation. For this we construct in § 3 a manifold M' with a one dimensional foliation $\mathcal{F}'$ so that in a neighborhood of N the foliation $\mathcal{F}$ is given by the pull-back of $\mathcal{F}'$ under a map preserving the stability property. This idea is due to Bonatti and Haefliger ([B$_1$], B-H]) in their study of compact leaf stability where $\mathcal{F}$ is given by a fibration $F \hookrightarrow M \rightarrow B$ (i.e., $M=N$) and $H_1(F; \boldsymbol{R})=\boldsymbol{R}$.

In Theorem 3 (§ 5) we prove that if $\mathcal{F}$ is normally elementary along N (cf. Definition 1.2) then stability follows from the non vanishing of the Euler characteristic of B. Here we prove that under a suitable homotopy of $\mathcal{F}$ this theorem reduces to Theorem 2.

§ 1. Definitions and results

Throughout this paper M will be a smooth manifold and $\mathcal{F}$ a C^1 codimension q foliation on M. We shall consider on the spach $\mathrm{Fol}_q^1(M)$ of all C^1 codimension q foliations of M the Epstein compact C^1-topology [E]. We also fix a Riemannian metric on M but the results do not depend on its choice.

We shall say that a C^1 connected compact boundaryless submanifold N of M is an *$\mathcal{F}$-invariant fibered manifold* if N is saturated by $\mathcal{F}$ and the leaves of the restricted foliation $\mathcal{F}|_N$ are the fibers of a fibration $L \hookrightarrow N \xrightarrow{p} B$, whose fiber L and base B are closed connected manifolds. For sufficiently small $\varepsilon>0$, a compact codimension q submanifold L' of M is *ε-close* to N if L' lies in an ε-tubular neighborhood $(\eta(L); \Pi)$ of some fiber L of p and is diffeomorphic to this fiber under Π. In this context the expression "*$\mathcal{F}$ has the compact leaf stability property close to N*" will mean that for arbitrarily small $\varepsilon>0$ all foliations sufficiently C^1 close to $\mathcal{F}$ have a compact leaf ε-close to N. We shall also say that N is *isolated* if the compact leaves of $\mathcal{F}$ sufficiently close to N are exactly the fibers of p.

For $x \in N$, L_x will denote the fiber of p passing through x.

If N reduces to an isolated leaf $L=L_x$ (B=point) and α is a generator of $\pi_1(L)=\boldsymbol{Z}$, then the *index* of $\mathcal{F}$ at L, $I_\alpha(\mathcal{F}; L)$, is defined as the fixed point index $i(H_{\alpha_x}(\mathcal{F}); x)$ at x of the holonomy map $H_{\alpha_x}(\mathcal{F})$ of $\mathcal{F}$ along some loop α_x in L_x based at x and representing α. Of course, $I_\alpha(\mathcal{F}; L)$ does not depend on the choice of $x \in L$, nor on the representing loop α_x, nor on the domain of $H_{\alpha_x}(\mathcal{F})$. Moreover, if $\beta = \alpha^{-1}$ then $I_\alpha(\mathcal{F}; L) = (-1)^q . I_\beta(\mathcal{F}; L)$.

THEOREM 1. *Let* $\mathcal{F} \in \mathrm{Fol}_q^1(M)$ *and let* $L \hookrightarrow N \xrightarrow{p} B$ *be an isolated* $\mathcal{F}$*-invariant fibered manifold with* $\pi_1(L) = \boldsymbol{Z}$ *on which* $\pi_1(B)$ *acts trivially. Given a generator* α *of* $\pi_1(L)$ *one can define an integer number* $I_\alpha(\mathcal{F}; N)$, *the Fuller index of* $\mathcal{F}$ *at* N, *satisfying*:

1. *if* N *is a single leaf* L, *then* $I_\alpha(\mathcal{F}; N) = I_\alpha(\mathcal{F}; L)$,
2. *if* $I_\alpha(\mathcal{F}; N)$ *is nonzero, then* $\mathcal{F}$ *has the compact leaf stability property close to* N,
3. *if* $\{\mathcal{F}_t\}_{t \in [0,1]}$ *is a continuous path of* C^1 *foliations on a neighborhood* U *of* N *aud for sufficiently small* $\varepsilon > 0$ *and for each* $t \in [0, 1]$ *the compact leaves of* $\mathcal{F}_t$ ε*-close to* N *are exactly the fibers of* p, *then* $I_\alpha(\mathcal{F}_t; N)$ *does not depend on* t.

If $(\eta(A); \Pi)$ is a tubular neighborhood of a submanifold $A \subset M$, then its restriction to a subset A' of A will be denoted by $\eta(A')$. In particular the fiber $\Pi^{-1}(x)$, $x \in A$, will be denoted by $\eta(x)$. A tubular subneighborhood $(\eta_0(A); \Pi_0)$ of $(\eta(A); \Pi)$ (i.e. $\eta_0(A) \subset \eta(A)$ and Π_0 is the restriction of Π to $\eta_0(A)$) will be frequently indicated by $\eta_0(A) \subset \eta(A)$.

If N has a tubular neighborhood $(\zeta(N); P)$ such that $\zeta(L_x)$ is saturated by the leaves of $\mathcal{F}|_{\zeta(N)}$ for each $x \in N$, then the foliation $\mathcal{F}|_{\zeta(N)}$ can be viewed as a family of C^1 foliations $\{\mathcal{F}_b\}_{b \in B}$, where $\mathcal{F}_b$ is the restriction of $\mathcal{F}$ to $\zeta(p^{-1}(b))$. Moreover, if N is isolated so is the leaf L_x of the foliation $\mathcal{F}_b$, where $p(x) = b$. In this case $I_\alpha(\mathcal{F}_b; L_x)$ is defined and does not depend on the choice of the point $x \in N$; so it will be denoted by $I_\alpha^*(\mathcal{F}; N)$.

THEOREM 2. *Let* $\mathcal{F}$, $L \hookrightarrow N \xrightarrow{p} B$ *and* α *be as in Theorem* 1. *Suppose that for some tubular neighborhood* $(\zeta(N); P)$ *of* N, *the subset* $\zeta(L_x)$ *is saturated by the leaves of* $\mathcal{F}|_{\zeta(N)}$ *for each* x *in* N. *If* $\chi(B) \neq 0$ *and* $I_\alpha^*(\mathcal{F}; N) \neq 0$ *then* $\mathcal{F}$ *has the compact leaf stability property close to* N.

The hypothesis on the action of $\pi_1(B)$ on $\pi_1(L)$ in Theorem 2 can be dropped by passing to a double cover of the base B. If the action is not trivial, $I_\alpha^*(\mathcal{F}; N)$ is well defined up to sign.

1.1 Now, for each $x\in N$ choose an embedded q-disk D_x centered at x and transverse to $\mathcal{F}$, and identify $D_x\cong \boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$ $(s=\dim B)$ and $x\cong(0,0)$ so that $D_x\cap N\cong\boldsymbol{R}^s\times\{0\}$. Relative to this identification, the holonomy map $H_{\gamma_x}(\mathcal{F})$ is deffned on a neighborhood of $(0,0)$ and coincides with the identity map on $\boldsymbol{R}^s\times\{0\}$ for each loop γ_x in L_x.

In analogy with elementary closed orbits of vector fields, we define;

1.2 DEFINITION. With the same notation and identification as above, an $\mathcal{F}$-invariant fibered manifold N is *normally elementary* if for every $x\in N$ the map $H_{\gamma_x}(\mathcal{F})-Id_{\boldsymbol{R}^q}$ is transversal to $\boldsymbol{R}^s\times\{0\}$ at $(0,0)$, where γ_x represents a generator of $\pi_1(L_x;x)$ and $Id_{\boldsymbol{R}^q}$ denotes the identity map of $\boldsymbol{R}^q$.

Evidently this definition depends neither on the choice of γ_x nor on the identification $D_x\cong\boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$. We can now state;

THEOREM 3. *Let* $\mathcal{F}\in \mathrm{Fol}_q^1(M)$ *and let* $L\hookrightarrow N\xrightarrow{p}B$ *be a normally elementary* $\mathcal{F}$*-invariant manifold with* $\pi_1(L)=\boldsymbol{Z}$. *If* $\chi(B)\neq 0$ *then* $\mathcal{F}$ *has the compact leaf stability property close to* N.

Henceforth, the expression "the holonomy map of $\mathcal{F}$ is defined on a q-disk D" will mean that the perturbed holonomy maps of all foliations sufficiently C^1 close to $\mathcal{F}$ are defined on D and take values in a greater embedded q-disk containing D.

§ 2. Proof of Theorem 1

To prove Theorem 1 we will need Lemma 2.1 below. In order to state this lemma let L be a compact leaf of a codimension q foliation $\mathcal{F}$ on M and let γ be a loop in L based at $x_0\in L$ and representing a generator of $\pi_1(L;x_0)=\boldsymbol{Z}$. Fix two embedded closed disks D^s and D^q $(0\leq s\leq q)$ centered at x_0 with $D^s\subset\mathrm{Int}(D^q)$ and $H_\gamma(\mathcal{F})$ defined on D^q.

2.1 LEMMA. *Suppose that* $\mathcal{F}$ *is trivial on the saturation of* D^s. *Given (possibly intersecting) compact subset* K_i *of* D^s *and given neighborhoods* $\mathcal{R}_i'$ *of the* $\mathcal{F}$*-saturation of* K_i, $i=1,2$, *there exist arbitrarely small neighborhoods* $W_1''\subset W_1'$ *of* K_1 *and* W_2'' *of* K_2 *in* D^q, *and neighborhoods* $\mathcal{R}_i''\subset\mathcal{R}_i'$ *of the* $\mathcal{F}$*-saturation of* K_i *in* M *satisfying the following property: if* $\mathcal{F}'$ *and* $\mathcal{F}''$ *are sufficiently* C^1 *close to* $\mathcal{F}$ *and if* $\{H_t\}_{t\in[0,1]}$ *is* *a continuous path of diffeomorphisms in the* C^1 *topology with* $H_0=H_\gamma(\mathcal{F}')$ *and such that:*

— H_t *is sufficiently* C^1 *close to* $H_\gamma(\mathcal{F})$ *for* $t\in[0, 1]$,

— $\mathcal{F}'\equiv\mathcal{F}''$ *on* $\mathcal{R}_2'$,

— $H_t\equiv H_\gamma(\mathcal{F}')$ *outside of* W_1' *for* $t\in[0, 1]$,

— $H_1\equiv H_\gamma(\mathcal{F}'')$ *on* $W_1''\cup W_2''$,

then, there exist a continuous path $\{\mathcal{F}_t\}_{t\in[0,1]}$ *on* $\mathrm{Fol}_q^1(M)$ *with* $\mathcal{F}_0=\mathcal{F}'$ *such that for all* $t\in[0, 1]$ *we have:*

2.1.1) $\mathcal{F}_t$ *is* C^1 *close to* $\mathcal{F}$,

2.1.2) $H_\gamma(\mathcal{F}_t)\equiv H_t$,

2.1.3) $\mathcal{F}_t\equiv\mathcal{F}'$ *outside of* $\mathcal{R}_1'$,

2.1.4) $\mathcal{F}_1\equiv\mathcal{F}''$ *on* $\mathcal{R}_1''\cup\mathcal{R}_2''$.

This lemma is proved in the Appendix. In its proof we shall use the same arguments as used in § 3 and § 5.

2.2 DEFINITION OF $I_\alpha(\mathcal{F}; N)$. Fix the hypothesis of Theorem 1.

We identify L with some fiber of p.

Since $\pi_1(B)$ acts trivially on $\pi_1(L)$, for each $x\in N$ we can transport α to a loop α_x in L_x representing a generator of $\pi_1(L_x; x)$ and satisfying the following compatibility condition: is an open subset $U\subset N$ trivialized with respect to $p: N\to B$ the family of homotopy classes $\{\bar{\alpha}_x\}_{x\in U}$ is given by transporting a selected loop α_z $(z\in U)$ to each fiber in U using the product structure.

Let $\mathcal{F}'\in\mathrm{Fol}_q^1(M)$ be sufficiently C^1 close to $\mathcal{F}$ with finitely many compact leaves $L_1', \cdots, L_r'$ sufficiently close to N. From the construction above, for each $i=1, \cdots, r$, we can transport α to a generator α_i of $\pi_1(L_i')$. Hence, one defines:

$$I_\alpha(\mathcal{F}; N)=\sum_{i=1}^{r} I_{\alpha_i}(\mathcal{F}'; L_i')$$

Theorem 1 follows immediatly from:

2.3 PROPOSITION. *With the same hypothesis as in Theorem* 1, *we have:*

1. *there exists a foliation* $\mathcal{F}'$ *on* M, *arbitrarily* C^1 *close to* $\mathcal{F}$ *and having finitely many compact leaves close to* N,
2. $I_\alpha(\mathcal{F}; N)$ *does not depend on the foliation* $\mathcal{F}'$ *given in* 1.

PROOF. Fix a closed tubular neighborhood $(\eta(N);Q)$ of N. Let $\{U_i\}_{1\leq i\leq k}$ and $\{U_i'\}_{1\leq i\leq k}$ be closed s-disks ($s=\dim(B)$) in B such that $U_i\subset \mathrm{Int}(U_i')$ and $\bigcup_{i=1}^{k}\mathrm{Int}(U_i)$ covers B.

Consider partial sections $\sigma_i: U_i'\to N$ of p and set $\sigma_i(U_i)=\tilde{U}_i$ and $\sigma_i(U_i')=\tilde{U}_i'$. Fix $x_i\in\mathrm{Int}(\tilde{U}_i)$ and a loop α_{x_i} in L_{x_i} representing α as in 2.2.

Passing to smaller $\eta(N)$ if necessary, we assume that for $\mathcal{F}'$ sufficiently C^1 close to $\mathcal{F}$ the perturbed holonomy map of $\mathcal{F}'$ along the loop α_{x_i}, denoted by $H_i(\mathcal{F}')$, is defined on $\eta(\tilde{U}_i')$. Furthermore, since N is isolated, we can also admit that $\mathrm{Fix}(H_i(\mathcal{F}))=\tilde{U}_i'$.

1. We shall modify $\mathcal{F}$ sucessively in small neighborhoods of $p^{-1}(U_1),\cdots, p^{-1}(U_k)$ so as to obtain foliations $\mathcal{F}_1,\cdots,\mathcal{F}_k$ arbitrarily C^r close to $\mathcal{F}$ such that:

a. $\mathcal{F}_i\equiv\mathcal{F}$ outside of a neighborhood of $p^{-1}(\bigcup_{j=1}^{i}U_j)$ in M.

b. The maps $H_j(\mathcal{F}_i)$ have just a finite number of fixed points in $\eta(V_j)$, where $V_j\subset\tilde{U}_j'$ is a neighborhood of $\tilde{U}_j$ and $1\leq j\leq i$.

Then $\mathcal{F}'=\mathcal{F}_k$ will be the required foliation. We shall argue inductively.

The foliation $\mathcal{F}_1$ is directly obtained from Lemma 2.1. It suffices to choose a sufficiently small neighborhood $V_1\subset\tilde{U}_i'$ of $\tilde{U}_1$ and a sufficiently C^1 small perturbation of $H_1(\mathcal{F})$ with just a finite number of fixed points in $\eta(V_1)$ and supported in a small neighborhood of V_1 in $\eta(\tilde{U}_1')$.

Suppose that we have construct $\mathcal{F}_i$. Thus $H_{i+1}(\mathcal{F}_i)$ has finitely many fixed points in $\eta(W_{i+1}')$, where $W_{i+1}'\subset\tilde{U}_{i+1}'$ is an open neighborhood of $\bigcup_{j=1}^{i}\sigma_{i+1}(U_j\cap U_{i+1}')$.

Fix a compact subneighborhood $W_{i+1}\subset W_{i+1}'$ of $\bigcup_{j=1}^{i}\sigma_{i+1}(U_j\cap U_{i+1}')$ and let $K=\tilde{U}_{i+1}-W_{i+1}'$. Now, select a sufficiently C^1 small perturbation $\tilde{H}_{i+1}$ of $H_{i+1}(\mathcal{F}_i)$ coinciding with $H_{i+1}(\mathcal{F}_i)$ on $\eta(W_{i+1})$, having finitely many fixed points in $\eta(V_{i+1})$, where V_{i+1} is a neighborhood of $\bigcup_{j=1}^{i+1}\sigma_{i+1}(U_j\cap U_{i+1}')$ in $\tilde{U}_{i+1}'$, and supported on a neighborhood of K in $\eta(\tilde{U}_{i+1}')$.

As before, from 2.1 we obtain a foliation $\mathcal{F}_{i+1}$ C^1 close to $\mathcal{F}$ which agrees with $\mathcal{F}_i$ outside of a neighborhood of $p^{-1}(p(K))$ in M and such that $H_{i+1}(\mathcal{F}_{i+1})=\tilde{H}_{i+1}$. Clearly $\mathcal{F}_{i+1}$ satisfies (a) and agrees with $\mathcal{F}_i$ on a neighborhood of $\eta[p^{-1}(\bigcup_{j=1}^{i}U_j)]$. Thus, $\mathcal{F}_{i+1}$ also satisfies (b) and this proves (1).

2. Our argument is adapted from one of Bonatti [B_1, Appendix 2, pp. 243-245].

Let the foliations $\mathcal{F}'$ and $\mathcal{F}''$ satisfy 2.3.1. Up to a diffeomorphism of M arbitrarily C^∞ close to the identity map, we can assume that the compact leaves of $\mathcal{F}'$ and $\mathcal{F}''$ which are close to N do not meet $\bigcup_{i=1}^{k}\eta(\partial\tilde{U}_i')$. Hence, for each $i=1,\cdots,k$ the perturbed holonomy maps $H_i(\mathcal{F}')$ and $H_i(\mathcal{F}'')$ have finitely many fixed points, all of them contained in the interior of $\eta(\tilde{U}_i')$ and very close to $\tilde{U}_i'$.

We shall construct on a tubular subneighborhood $\eta_0(N)\subset\eta(N)$ a continuous path of C^1 foliations $\{\mathcal{F}_t\}_{t\in[0,k]}$ from $\mathcal{F}_0=\mathcal{F}'$ to $\mathcal{F}_k=\mathcal{F}''$ such that $\mathcal{F}_t$ is C^1 close to $\mathcal{F}$ for all $t\in[0,k]$ and satisfying the following properties for each $i=1,\cdots,k$:

a'. $\mathcal{F}_t\equiv\mathcal{F}_{i-1}$ on a neighborhood of $\eta_0[p^{-1}(B-\mathrm{Int}\,U_i')]$ in $\eta_0(N)$ and the compact leaves of $\mathcal{F}_t$ close to N do not intersect $\eta_0(\partial\tilde{U}_i')$ for all $t\in[i-1,i]$,

b'. $\mathcal{F}_i\equiv\mathcal{F}''$ on a neighborhood of $\eta_0[p^{-1}(\bigcup_{j=1}^{i}U_j)]$ in $\eta_0(N)$,

c'. $\mathcal{F}_i$ has finitely many compact leaves close to N which do not intersect $\bigcup_{j=1}^{k}\eta_0(\partial\tilde{U}_j')$,

d'. $I_{i-1}=I_i$, where I_j denotes the sum of the indices of compact leaves of $\mathcal{F}_j$ close to N.

We shall construct this path inductively.

In order to construct $\{\mathcal{F}_t\}_{t\in[0,1]}$ fix neighborhoods V_1 and V_1' of $\tilde{U}_1$ with $Cl(V_1)\subset\mathrm{Int}(V_1')$ and $Cl(V_1')\subset\mathrm{Int}(\tilde{U}_1')$. Since $\mathcal{F}'$ and $\mathcal{F}''$ are C^1 close to $\mathcal{F}$, moving $[H_1(\mathcal{F}')](x)$ along the geodesic arc (in the induced metric on $\eta(\tilde{U}_1')$) to $[H_1(\mathcal{F}'')](x)$ we construct a C^1 isotopy $\{\tilde{H}_t\}_{t\in[0,1]}$ of $\tilde{H}_0=H_1(\mathcal{F}')$ such that:

— $\tilde{H}_t$ is C^1 close to $H_1(\mathcal{F})$, and $\tilde{H}_t\equiv H_1(\mathcal{F}')$ on $\eta(\tilde{U}_1'-V_1')$, for all $t\in[0,1]$,

— $\tilde{H}_1\equiv H_1(\mathcal{F}'')$ on $\eta(V_1)$.

The map $\tilde{H}_1$ has finitely many fixed points on $\eta(V_1)\cup\eta(\tilde{U}_1'-V_1')$.

Let $\mathcal{W}\subset\mathrm{Int}(\tilde{U}_1'-\tilde{U}_1)$ be a small compact neighborhood of $Cl(V_1'-V_1)$ in $\tilde{U}_1'$. By a suitable C^1 small perturbation of $\tilde{H}_1$ with support in $\eta(\mathcal{W})$ we can also assume that:

— $\tilde{H}_1$ has finitely many fixed points (all of them close to $\tilde{U}_1'$).

Since $\mathcal{W}$ intersects V_1 and $\tilde{U}_1'-V_1'$, in order to keep the above properties of $\tilde{H}_t$ and $\tilde{H}_1$ on $\eta(\tilde{U}_1'-V_1')$ and $\eta(V_1)$ respectively, we should pass to a smaller neighborhood V_1 and a greater neighborhood V_1'.

Furthermore, this construction gives $i(H_1(\mathcal{F}'); \Omega_0) = i(\tilde{H}_t; \Omega_t)$ for all $t \in [0, 1]$, where $\Omega_t = \mathrm{Fix}(\tilde{H}_t)$.

Since $\mathcal{F}'$ and $\mathcal{F}''$ can be taken arbitrarily close to $\mathcal{F}$, from Lemma 2.1 we obtain a continuous path of C^1 foliations $\{\mathcal{F}_t\}_{t\in[0,1]}$ on $\eta_1(N) \subset \eta(N)$ with $H_1(\mathcal{F}_t) = \tilde{H}_t$ and satisfying items (a′) and (b′) for $i=1$ (in fact, to apply Lemma 2.1 to the situation above we have to consider a smaller $\eta'(N) \subset \eta(N)$ and take $\tilde{H}_t$ coinciding with $H_1(\mathcal{F}')$ outside of $\eta'(\tilde{U}'_1)$. Moreover, $\mathcal{F}_1$ has finitely many compact leaves close to N.

On the other hand, since $i(H_1(\mathcal{F}'); \Omega_0) = i(\tilde{H}_1; \Omega_1)$ and $\mathcal{F}'$ is close enough to $\mathcal{F}$, we conclude from (a′) that $I_0 = I_1$. Perturbing $\mathcal{F}_1$ by a convenient diffeomorphism of $\eta_1(N)$ C^∞-close to the identity map, we have that (c′) also holds.

Suppose that we have constructed $\mathcal{F}_i$ on $\eta_i(N) \subset \eta(N)$.

Hence, $H_{i+1}(\mathcal{F}_i)$ has just a finite number of fixed points which are in the interior of $\eta_i(\tilde{U}'_{i+1})$ and very close to $\tilde{U}'_{i+1}$. Furthermore, taking smaller $\eta_i(N)$ if necessary, we can assume that $H_{i+1}(\mathcal{F}_i) \equiv H_{i+1}(\mathcal{F}'')$ on $\eta_i(W_{i+1})$, where $W_{i+1} \subset \tilde{U}'_{i+1}$ is an open neighborhood of $\bigcup_{j=1}^{i} \sigma_{i+1}(U_j \cap U'_{i+1})$.

We shall construct $\{\mathcal{F}_t\}_{t\in[i,i+1]}$ in an analogous way we have constructed $\{\mathcal{F}_t\}_{t\in[0,1]}$. Since $\mathcal{F}_i$ and $\mathcal{F}''$ are C^1 close to $\mathcal{F}$, we construct a C^1 isotopy $\{\tilde{H}_t\}_{t\in[i,i+1]}$ of $H_{i+1}(\mathcal{F}_i) = \tilde{H}_i$ by first moving $[H_{i+1}(\mathcal{F}_i)](x)$ along the geodesic arcs (in the induced metric on $\eta_i(\tilde{U}'_{i+1})$) to $[H_{i+1}(\mathcal{F}'')](x)$ and then taking a suitable C^1 small perturbation of $\tilde{H}_{i+1}$ with support in $\eta_i[\tilde{U}'_{i+1} - \bigcup_{j=1}^{i+1} \sigma_{i+1}(U_j \cap U'_{i+1})]$ such that:

— $\tilde{H}_t$ is C^1 close to $H_{i+1}(\mathcal{F})$ for all $t \in [i, i+1]$,

— $\tilde{H}_t \equiv H_{i+1}(\mathcal{F}_i)$ on $\eta_i(\tilde{U}'_{i+1} - V'_{i+1})$ for all $t \in [i, i+1]$, where $V'_{i+1} \subset \tilde{U}'_{i+1}$ is a neighborhood of $\tilde{U}_{i+1}$ with $Cl(V'_{i+1}) \subset \mathrm{Int}(\tilde{U}'_{i+1})$,

— $\tilde{H}_{i+1} \equiv H_{i+1}(\mathcal{F}'')$ on $\eta_i(V_{i+1})$, where $V_{i+1} \subset V'_{i+1}$ is a neighborhood of $\tilde{U}_{i+1}$,

— $\tilde{H}_{i+1} \equiv H_{i+1}(\mathcal{F}'')$ on $\eta_i(W_{i+1})$,

— $\tilde{H}_{i+1}$ has finitely many fixed points (which are close to $\tilde{U}'_{i+1}$).

Once given the isotopy $\{\tilde{H}_t\}_{t\in[i,i+1]}$, applying Lemma 2.1 and the same arguments used above for $i=1$ we obtain the required path of foliations $\{\mathcal{F}_t\}_{t\in[i,i+1]}$. This compietes the proof of (2). ∎

§ 3. Construction of the total space

As explained in the introduction, we reduce the problem of compact leaves close to an $\mathcal{F}$-invariant fibered manifold to the same problem for a one dimensional foliation $\mathcal{G}$ on a manifold M'. The goal of this section is to realize this simplified foliated manifold. In fact, we describe its construction in a more general situation that we really need.

For a disk bundle $\Pi: \xi(A)\to A$ over a manifold A, with a section $\sigma: A\to\xi(A)$, we shall use the same notation as for tubular neighborhoods in § 1, so we write $(\xi(A); \Pi)$ and $\xi(x)$ means the disk $\Pi^{-1}(x)$, $x\in A$, and so on. We shall always identify A with $\sigma(A)$.

3.1 LEMMA. *Let L and G be closed connected manifolds, and let $(\xi(L); \Pi)$ be an r-disk bundle with a section L. Given*

1. *a C^1 family of foliations $\{\mathcal{F}_\lambda; \lambda\in \boldsymbol{R}^s\}$ on $\xi(L)$ such that L is a leaf of each $\mathcal{F}_\lambda$,*
2. *a C^1 family of maps $\{f_\lambda: L\to G; \lambda\in\boldsymbol{R}^s\}$ such that $(f_\lambda)_*: \pi_1(L)\to\pi_1(G)$ is an isomorphism,*

there exists an r-disk bundle $(\tilde{\xi}(G), \tilde{\Pi})$ with a section G and there also exist

1'. *a C^1 family of foliations $\{\mathcal{G}_\lambda\}_{\lambda\in W}$ on $\tilde{\xi}(G)$ such that G is a leaf of $\mathcal{G}_\lambda$ for each λ in a neighborhood W of $0\in\boldsymbol{R}^s$,*

2'. *a C^1 family of fiber preserving maps $\{F_\lambda: \xi_1(L)\to\tilde{\xi}(G)\}_{\lambda\in W}$ extending $\{f_\lambda\}_{\lambda\in W}$ in a tubular subneighborhood $\xi_1(L)\subset\xi(L)$ such that, for all $\lambda\in W$, the map F_λ preserves the foliations $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$, and the restriction of F_λ to the disk $\xi_1(x)$ is a diffeomorphism for each $x\in L$.*

It follows from 3.1.2' that F_λ is transversal to $\mathcal{G}_\lambda$, $\mathcal{F}_\lambda=F_\lambda^*(\mathcal{G}_\lambda)$ and F_λ restricted to $\xi_1(x)$ conjugates the holonomies of $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$. Moreover, in a neighborhood of L the map F_λ is uniquely determined by f_λ and by its restriction to $\xi_1(x)$.

We remark that lemma 3.1 holds for any previously chosen relatively compact neighborhood W of $0\in\boldsymbol{R}^s$. This will be clear in its proof.

PROOF OF LEMMA 3.1. Fix loops $\{\alpha_i\}_{1\le i\le k}$ in L based at x_0 whose homotopy classes generate $\pi_1(L, x_0)$. Let $y_\lambda=f_\lambda(x_0)$ and $\beta_i=f_0(\alpha_i)$.

Since $(f_0)_*$ is an isomorphism, according to Haefliger's construction [H] there exist an r-disk bundle $(\tilde{\xi}(G), \tilde{\Pi})$ with a section G, foliation $\mathcal{G}_0$ on $\tilde{\xi}(G)$ having G as a leaf, and (eventually passing to a smaller $\xi(L)$) a C^1 diffeomorphism $\tilde{f}_0: \xi(x_0)\to\tilde{\xi}(y_0)$ conjugating the holonomies of $\mathcal{F}_0$ and $\mathcal{G}_0$, that is: the relation $H_{\beta_i}(\mathcal{G}_0)\circ\tilde{f}_0=\tilde{f}_0\circ H_{\alpha_i}(\mathcal{F}_0)$ holds in a neighborhood $U\subset\xi(x_0)$ of x_0 for all $i=1, \cdots, k$.

For a smaller U and for λ in a neighborhood W of $0\in\boldsymbol{R}^s$, the C^1

family of diffeomorphisms $\{H_{\lambda,i}:\tilde{f}_0(U)\to\tilde{\xi}(y_0)\}_{1\le i\le k}^{\lambda\in W_i}$ defined by $H_{\lambda,i}\circ\tilde{f}_0=\tilde{f}_0\circ H_{\alpha_i}(\mathcal{F}_\lambda)$ represents a C^1 family of small perturbations of the holonomy of $\mathcal{G}_0$. Applying the Realization Theorem of Bonatti-Haefliger [B-H] we obtain a neighborhood $W'\subset W$ of $0\in\boldsymbol{R}^s$ and a C^1 family of foliations $\{\mathcal{G}_\lambda\}_{\lambda\in W'}$ on $\tilde{\xi}(G)$ (eventually passing to a smaller $\tilde{\xi}(G)$) such that $H_{\beta_i}(\mathcal{G}_\lambda)=H_{\lambda,i}$ on a neighborhood $V\subset\tilde{\xi}(y_0)$ of y_0. Since y_0 is a fixed point of $H_{\lambda,i}$ for all $\lambda\in W'$ and $i=1,\cdots,k$, each $\mathcal{G}_\lambda$ has a compact leaf G_λ passing through y_0 and close to G. Translating $\mathcal{G}_\lambda$ along the fibers of $\tilde{\Pi}$ we can assume that G_λ coincides with G, for all λ in W'.

For λ in $\boldsymbol{R}^s$ small enough, $\tilde{f}_\lambda:(\tilde{f}_0)^{-1}(V)\to\tilde{\xi}(y_\lambda)$ be the C^1 family of embeddings uniquely defined by $\tilde{f}_\lambda=g_\lambda\circ\tilde{f}_0$, where $g_\lambda: V\to\tilde{\xi}(y_\lambda)$ are the C^1 embeddings given by projecting V into $\tilde{\xi}(y_\lambda)$ along the leaves of $\mathcal{G}_\lambda$. Hence, $\tilde{f}_\lambda$ conjugates $H_{\alpha_i}(\mathcal{F}_\lambda)$ and $H_{f_\lambda(\alpha_i)}(\mathcal{G}_\lambda)$ in a neighborhood of x_0 in $\xi(x_0)$, for $i=1,\cdots,k$ and λ close to $0\in\boldsymbol{R}^s$.

Now, the map F_λ is the unique fiber preserving extension of the pair $(f_\lambda;\tilde{f}_\lambda)$ defined on a tubular subneighborhood $\xi_1(L)\subset\xi(L)$ which preserves also the foliations $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$. This finishes the proof. ∎

3.2 THE TOTAL SPACE REALIZATION LEMMA. *Let $(\xi(N);P)$ be an r-disk bundle with a section N and let $\mathcal{F}\in\mathrm{Fol}_q^1(\xi(N))$ be such that $L\hookrightarrow N\xrightarrow{p}B$ is an $\mathcal{F}$-invariant fibered manifold. Suppose that $\xi(L_x)$ is $\mathcal{F}$-saturated for each $x\in N$. Then given a fibration $G\hookrightarrow N_0\xrightarrow{p_0}B$ and a C^1 fiber preserving map $f: N\to N_0$ inducing the identity map on B and an isomorphism $f_*:\pi_1(L)\to\pi_1(G)$, there exist an r-disk bundle $(\tilde{\xi}(N_0);\tilde{P})$ with a section N_0, a C^1 foliation $\mathcal{G}$ on $\tilde{\xi}(N_0)$ and a C^1 fiber preserving map $F:\xi_1(N)\to\tilde{\xi}(N_0)$ extending f on a tubular subneighborhood $\xi_1(N)\subset\xi(N)$ satisfying the conditions:*

1. *$G\hookrightarrow N_0\xrightarrow{p_0}B$ is a $\mathcal{G}$-invariant fibered manifold,*
2. *$\tilde{\xi}(G_y)$ is $\mathcal{G}$-saturated for each $y\in N_0$,*
3. *the restriction of F to $\xi_1(x)$ is a diffeomorphism for all $x\in N$ and F preserves the foliations $\mathcal{F}$ and $\mathcal{G}$.*

In particular, F is transversal to $\mathcal{G}$ and $F^*(\mathcal{G})=\mathcal{F}$.

PROOF. Let $\mathcal{W}=\{W_i\}_{1\le i\le k}$ be an open cover of B trivializing the three fibrations: p, p_0 and $P\circ p$. Hence, the disk bundle $(\xi(N);P)$, the foliation $\mathcal{F}$ and the map f are given by:

— an r-disk bundle $(\xi(L);\Pi)$ with a section L,

— a C^1 family of foliations $\{\mathcal{F}_{\lambda,i}\}_{1\leq i\leq k}^{\lambda\in W_i}$ on $\xi(L)$ such that L is a leaf of each $\mathcal{F}_{\lambda,i}$,

— a C^1 family of maps $\{f_{\lambda,i}: L\to G\}_{1\leq i\leq k}^{\lambda\in W_i}$ such that each $(f_{\lambda,i})_*: \pi_1(L)\to\pi_1(G)$ is an isomorphism,

— two C^1 cocycles $\Psi_{i,j}: W_i\cap W_j\to \mathrm{Diff}^1(\xi(L))$ and $\varphi_{i,j}: W_i\cap W_j\to\mathrm{Diff}^1(G)$ associate to the cover $\mathcal{W}$ satisfying the following conditions:

— $\Psi_{i,j}(\lambda)(L)=L$

— $\Pi\circ\Psi_{i,j}(\lambda)=\Psi_{i,j}(\lambda)\circ\Pi$

— $\Psi_{i,j}(\lambda)(\mathcal{F}_{\lambda,i})=\mathcal{F}_{\lambda,j}$

— $\varphi_{i,j}(\lambda)\circ f_{\lambda,i}=f_{\lambda,j}\circ\Psi_{i,j}(\lambda)$ on L.

We choose the open sets of the cover $\mathcal{W}$ small enough so that Lemma 3.1 applies for each $i=1,\cdots,k$. So, let $(\tilde{\xi}(G);\tilde{\Pi})$, $\{\mathcal{G}_{\lambda,i}\}_{1\leq i\leq k}^{\lambda\in W_i}$, and $\{F_{\lambda,i}:\xi_1(L)\to\tilde{\xi}(G)\}_{1\leq i\leq k}^{\lambda\in W_i}$ be given by 3.1.

To obtain the foliation $\mathcal{G}$ and the map F it suffices to construct a cocycle $\Phi_{i,j}: W_i\cap W_j\to\mathrm{Emb}^1(\tilde{\xi}_1(G);\tilde{\xi}(G))$, $\tilde{\xi}_1(G)\subset\tilde{\xi}(G)$ satisfying the following properties for each $\lambda\in W_i\cap W_j$:

a. $\Phi_{i,j}(\lambda)\equiv\varphi_{i,j}(\lambda)$ on G,

b. $\tilde{\Pi}\circ\Phi_{i,j}(\lambda)=\Phi_{i,j}(\lambda)\circ\tilde{\Pi}$,

c. $\Phi_{i,j}(\lambda)(\mathcal{G}_{\lambda,i})=\mathcal{G}_{\lambda,j}$,

d. $\Phi_{i,j}(\lambda)\circ F_{\lambda,i}=F_{\lambda,j}\circ\Psi_{i,j}(\lambda)$

Fix $z\in L$. Since $F_{\lambda,i}$ restricted to $\xi_1(z)$ is a diffeomorphism, then for $\lambda\in W_i\cap W_j$ the maps

$$\tilde{\varphi}_{i,j}(\lambda): F_{\lambda,i}(\xi(z))\to\tilde{\xi}[(\varphi_{i,j}(\lambda))(f_{\lambda,i}(z))]$$

defined by

$$\tilde{\varphi}_{i,j}(\lambda)\circ F_{\lambda,i}=F_{\lambda,j}\circ\Psi_{i,j}(\lambda)$$

conjugate the holonomies of $\mathcal{G}_{\lambda,i}$ and $\mathcal{G}_{\lambda,j}$.

Let $\Phi_{i,j}(\lambda):\tilde{\xi}_1(G)\to\tilde{\xi}(G)$ be the unique extension of the pair $(\varphi_{i,j}(\lambda)$, $\tilde{\varphi}_{i,j}(\lambda))$ satisfying properties (a), (b) and (c) above in a tubular subneighborhood $\tilde{\xi}_1(G)\subset\tilde{\xi}(G)$. Clearly $\Phi_{i,j}(\lambda)$ is an embedding and from the unicity of this extension it is easy to check that $\Phi_{i,j}(\lambda)$ satisfies also (d) and does not depend on the choice of $z\in L$.

Finally, to see that $\{\Phi_{i,j}\}_{i,j}$ is a cocycle, we use the unicity of $\Phi_{i,j}(\lambda)$ as extension of the pair $(\varphi_{i,j}(\lambda)$, $\tilde{\varphi}_{i,j}(\lambda))$ and the fact that $\Phi_{i,j}(\lambda)$ does not depend on the choice of $z\in L$. ∎

§4. Proof of Theorem 2

Theorem 2 follows from Theorem 1 and the following proposition.

4.1 PROPOSITION. *On the hypothesis of Theorem* 2, $I_\alpha(\mathcal{F}; N) = \chi(B).I_\alpha^*(\mathcal{F}; N)$.

We shall apply the result established in the previous section to reduce the proof of this proposition to the case where $\mathcal{F}$ is an oriented one dimensional C^1 foliation. Thus, we shall first prove it for this special case.

So, let $\mathcal{F}$ be a one dimensional oriented C^1 foliation and $S^1 \hookrightarrow N \xrightarrow{p} B$ an $\mathcal{F}$-invariant fibered manifold with $\dim(B) = s$.

4.2 LEMMA. *Suppose that for some tubular neighborhood* $(\zeta(N); P)$ *of* N *in* M, *the leaves of* $\mathcal{F}|_{\zeta(N)}$ *project under* P *into the fibers of* p. *Then there exists a sequence of foliations* $\{\mathcal{F}_n\}_{n \in N}$ *on* M, *converging to* $\mathcal{F}$ *in the* C^1 *topology with the following properties for all* $n \in N$:

1. N *is an invariant submanifold under* $\mathcal{F}_n$,
2. $\mathcal{F}_n$ *has finitely many compact leaves on* N *which coincide with fibers of* p,
3. *the leaves of* $\mathcal{F}_n|_{\zeta_1(N)}$ *project under* P *into the leaves of* $\mathcal{F}_n|_N$, *where* $\zeta_1(N)$ *is a subneighborhood of* $\zeta(N)$,
4. *if* $\gamma \subset N$ *is a compact leaf of* $\mathcal{F}_n$ *then* $\mathcal{F}_n$ *agrees with* $\mathcal{F}$ *on* $\zeta(\gamma)$.

We point out that from a Hart's result [Ha], up to a C^1 diffeomorphism the foliation $\mathcal{F}$ is given by a C^1 vector field X on $\zeta(N)$. However, the classical argument of perturbing X by a lift (under p and P) of a suitable C^∞ vector field on B does not apply readly. Since the projections p and P are of class C^1 the perturbation so obtained would be only of class C^0 which is not necessarily integrable. That is why the proof of lemma 4.2 is some more technical than it would be if we were dealing with the C^2 class.

PROOF OF LEMMA 4.2. Choose a convenient smooth triangulation $\Delta = \{\Delta_1, \cdots, \Delta_r\}$ of B so that for each $i = 1, \cdots, r$ there is an s-simplex $\tilde{\Delta}_i \subset N$ projecting diffeomorphicaly onto Δ_i and satisfying $\tilde{\Delta}_i \cap \tilde{\Delta}_j = \emptyset$ if $1 \leq i \neq j \leq r$. Furthermore, denote by $\{\delta_i(\Delta); r+1 \leq i \leq r+k\}$ the set of all $(s-1)$-simplexes of Δ, and, for each $i = r+1, \cdots, r+k$, choose a small neighborhood V_i of $\partial_i(\Delta)$ diffeomorphic to a closed s-disk. Let $\tilde{V}_i \subset N$ projecting diffeomorphicaly onto V_i be such that $\tilde{V}_i \cap \tilde{V}_j = \emptyset$ and $\tilde{V}_i \cap \tilde{\Delta}_l = \emptyset$ for $r+1 \leq i \neq j \leq r+k$ and $1 \leq l \leq r$. Setting $T = \tilde{\Delta}_1 \cup \cdots \cup \tilde{\Delta}_r \cup \tilde{V}_{r+1} \cup \cdots \cup \tilde{V}_{r+k}$, we shall use the notation $T_i = \tilde{\Delta}_i$ if $1 \leq i \leq r$ and $T_i = \tilde{V}_i$ if $r+1 \leq i \leq r+k$.

Passing to a smaller and compact $(\zeta(N); P)$ we have that $\zeta(T)$ is transversal to $\mathcal{F}$. Hence, the saturation of $\zeta(T)$ by small arcs of leaves of $\mathcal{F}$ is a disjoint union $W = W_1 \cup \cdots \cup W_{r+k}$ of foliated compact sets, each of them identified with $T_i \times D^{q-s} \times [-2, 1]$, where D^{q-s} is the closed unit disk in $\boldsymbol{R}^{q-s}$ centered at the origin and the leaves of $\mathcal{F}$ are identified with the intervals $\{x\} \times \{y\} \times [-2, 1]$ with orientation coming from the usual orientation of $[-2, 1]$. Relative to the identification $W_i \cong T_i \times D^{q-s} \times [-2, 1]$ above, $T_i \cong T_i \times \{0\} \times \{0\}$ and the projection P is expressed by $P(x, y, t) = (x, t)$.

The foliations $\mathcal{F}_n$ will be obtained modifying $\mathcal{F}$ inside of W. For this, let Z be a smooth vector field on B without periodic orbits and with finitely many singularities, all of them contained in $B - \bigcup_{r+1}^{r+k} V_i$.

Let $g : B \to [0, 1]$ and $\{g_i : V_i \to [0, 1]\}_{i=r+1}^{i=r+k}$ be smooth maps such that:

— $g_i \equiv 0$ on a small collar neighborhood of ∂V_i and $g_i > 0$ on a neighborhood $V_i' \subset V_i$ of $\partial_i(\Delta)$,

— $g \equiv 0$ on a neighborhood of the $(s-1)$-skeleton of Δ and $g > 0$ on $B - \bigcup_{r+1}^{r+k} V_i'$.

On each T_i consider the smooth vector field

$$X^i = \begin{cases} p^*(gZ) & \text{if } 1 \leq i \leq r \\ p^*(g_i Z) & \text{if } r+1 \leq i \leq r+k. \end{cases}$$

Let $\varphi : [0, 1] \to [0, 1]$ be a C^∞ map such that $\varphi \equiv 1$ on a neighborhood of zero and $\varphi \equiv 0$ on a neighborhood of 1. Now we define on $T_i \times D^{q-s}$ the vector field $Y^i(x, y) = (\varphi(\|y\|).X^i(x), 0)$.

Given $\varepsilon > 0$ we choose smooth isotopies $\{f^{\varepsilon}_{i,t}\}_{t\in[-2,1]}^{i=1,\dots,r+k}$ defined on $T_i \times D^{q-s}$ of the form $f^{\varepsilon}_{i,t}(x, y) = (f^{\varepsilon}_{i,t,y}(x), y)$, supported outside of a collar neighborhood of $\partial(T_i \times D^{q-s})$ and such that:

— $f^{\varepsilon}_{i,t} = Id$ for $t \in [-2, -1]$,

— $f^{\varepsilon}_{i,t} = (Y^i)_\varepsilon$ for $t \in [0, 1]$, where $(Y^i)_\varepsilon$ is the time ε map of the flow of Y^i,

— $f^{\varepsilon}_{i,t}(x, y) = (x, y)$ if $p(x)$ is a singularity of the vector field Z, for all $t \in [-2, 1]$,

— the isotopies $\{f_{i,t}^{\varepsilon}\}_{t\in[-2,1]}$ converge to the trivial isotopy $\{h_t=Id_{T_i\times D^{q-s}}\}_{t\in[-2,1]}$ in the C^{∞} topology as ε goes to zero.

These isotopies define foliations $\mathcal{F}_{\varepsilon}$ on M coinciding with $\mathcal{F}$ on $M-W$, and on each W_i the leaves of $\mathcal{F}_{\varepsilon}$ are given by $\{(f_{i,t}^{\varepsilon}(z), t); t\in[-2, 1]\}_{z\in T_i\times D^{q-s}}$.

By construction N is invariant under $\mathcal{F}_{\varepsilon}$ and $\mathcal{F}_{\varepsilon}$ converges to $\mathcal{F}$ in the C^1 topology as ε goes to zero. Moreover, the foliations $\mathcal{F}_{\varepsilon}$ satisfy condition (3) in a small subneighborhood $\zeta_1(N)\subset\zeta(N)$.

Fix $\varepsilon>0$ and let F be a leaf of $\mathcal{F}_{\varepsilon}$ lying in N. Denote $[z_{\lambda}, z_{\lambda}']$ each connected arc with extremities z_{λ} and z_{λ}' of $F\cap W$, where $z_{\lambda}<z_{\lambda}'$ following the orientation of $\mathcal{F}_{\varepsilon}$ and $\lambda\in\Lambda$. First observe that, for all $\lambda\in\Lambda$, the points $p(z_{\lambda})$ and $p(z_{\lambda}')$ are on the same orbit of the vector field Z, and $p(z_{\lambda}')$ belongs to the positive semi-orbit of $p(z_{\lambda})$, i.e. $p(z_{\lambda}')=Z_t(p(z_{\lambda}))$ for some $t\geq 0$.

If $Z(p(z_{\delta}))=0$ for some $\delta\in\Lambda$ then $p(z_{\lambda})=p(z_{\lambda}')$ for all $\lambda\in\Lambda$ and it follows that F coincides with the fiber passing through z_{δ}. Otherwise, for some $\lambda\in\Lambda$ and some $i\in\{1, \cdots, r+k\}$, we have $z_{\lambda}\in W_i$ and $X^i(z_{\lambda})\neq 0$. In this case $p(z_{\lambda}')=Z_t(p(z_{\lambda}))$ for $t>0$, and therefore, $p(F)$ is not a closed curve in B. Consequently, F is not compact. Now we can easily conclude that $\mathcal{F}_{\varepsilon}$ also satisfy (4) and the proof is finished. ■

We shall need the following elementary lemma.

4.3 LEMMA. *Let $\Phi: U\rightarrow\boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$ be a continuous map of the form $\Phi(x, y)=(g(x), f_x(y))$ defined in a neighborhood $U\subset\boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$ of the origin. If* $\mathrm{Fix}(\Phi)=\{(0, 0)\}$ *and* $\mathrm{Fix}(g)=\{0\}$ *then* $i(\Phi; (0, 0))=i(g; 0).i(f_0; 0)$.

PROOF. It suffices to verify that there exist a C^0 homotopy $\{\Phi_t\}_{t\in[0,1]}$ between Φ and the product map (g, f_0) such that $\mathrm{Fix}(\Phi_t)=\{(0, 0)\}$, for all $t\in[0, 1]$. Such a homotopy is given by $\Phi_t(x, y)=(g(x), (1-t).f(y)+t.f_0(y))$, for $t\in[0, 1]$. ■

4.4 PROPOSITION. *The Proposition* 4.1 *holds if $\mathcal{F}$ is a C^1 one dimensional oriented foliation and α is defined by the orientation of $\mathcal{F}$.*

PROOF. Let $\mathcal{F}'$ be a foliation C^1 close to $\mathcal{F}$ as described in Lemma 4.2, and let $\gamma_i, \cdots, \gamma_r$ be its compact leaves lying in N. Since N is isolated, the compact leaves of $\mathcal{F}'$ close to N are exactly $\gamma_1, \cdots, \gamma_r$. Therefore,

$$I_{\alpha}(\mathcal{F}; N)=\sum_{j=1}^{r} I_{\alpha}(\mathcal{F}'; \gamma_j).$$

Fix $j\in\{1, \cdots, r\}$, $x_j\in\gamma_j$ and a small s-disk D contained in N, centered at x_j and transversal to $\mathcal{F}|_N$. Identifying $\zeta(D)\cong \boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$ and $x_j\cong(0,0)$ so that $D\cong\boldsymbol{R}^s\times\{0\}$ and $P(x,y)=x$, the perturbed first return map $H_{\gamma_j}(\mathcal{F}')$ is defined on a neighborhood of $(0,0)$. According to 4.2.3, we have $[H_{\gamma_j}(\mathcal{F}')](x,y)=(g(x), f_x(y))$. On the other hand, from Lemma 4.3 it follows that $i(H_{\gamma_j}(\mathcal{F}'); x_j)=i(g; 0).\ i(f_0; 0)$. Furthermore, since $\mathcal{F}'$ agrees with $\mathcal{F}$ on $\zeta(\gamma_j)$, f_0 is the first return map of $\mathcal{F}$ restricted to $\zeta(\gamma_j)$. Thus, $i(f_0; 0)=I_\alpha^*(\mathcal{F}; N)$ and $i(g;0)$ coincides with the index $i(Z; p(x_j))$ of Z at $p(x_j)$. Therefore $I_\alpha(\mathcal{F}; N)=\sum_{j=1}^{r} i(Z; p(x_j)).\ I_\alpha^*(\mathcal{F}; N)=\chi(B).\ I_\alpha^*(\mathcal{F}; N)$. ∎

4.6 Proof of Proposition 4.1. Since $\pi_1(L)=\boldsymbol{Z}$, there exist an S^1-bundle $S^1\hookrightarrow N_0\xrightarrow{p_0}B$ and a C^1 fiber preserving map $f: N\to N_0$ inducing the identity map on B and such that $f_*:\pi_1(L)\to\pi_1(S^1)$ is an isomorphism (cf. [B-H, § 4]).

Let $\mathcal{G}$, $(\tilde{\zeta}(N_0); \tilde{P})$ and $F:\zeta_1(N)\to\tilde{\zeta}(N_0)$ be given by applying Lemma 3.2 to $\mathcal{F}$, $L\hookrightarrow N\xrightarrow{p}B$, $S'\hookrightarrow N_0\xrightarrow{p_0}B$ and f. Since $\pi_1(B)$ acts trivially on $\pi_1(L)$, the same is true for the action on $\pi_1(S^1)$. Consequently, $\beta=f_*(\alpha)$ defines an orientation on $S^1\hookrightarrow N_0\to B$.

We recall that F maps small disks transversal to $\mathcal{F}$ diffeomorphically onto disks transversal to $\mathcal{G}$. Therefore, it conjugates the holonomies of $\mathcal{F}$ and $\mathcal{G}$, i.e. the relation $F\circ[H_{\gamma_x}(\mathcal{F})]=[H_{f(\gamma_x)}(\mathcal{G})]\circ F$ holds on small disks transversal to L_x, where γ_x is a loop in L_x based at $x\in N$. Thus, since N is isolated so is N_0, and then $I_\beta^*(\mathcal{G}; N_0)$ is defined. Moreover, we have that $I_\alpha^*(\mathcal{F}; N)=I_\beta^*(\mathcal{G}; N_0)$.

From 4.4 we have $I_\beta(\mathcal{G}; N_0)=\chi(B).\ I_\beta^*(\mathcal{G}; N_0)$. Then, it remains to prove that $I_\alpha(\mathcal{F}; N)=I_\beta(\mathcal{G}; N_0)$.

For this, let $\mathcal{G}'$ be a small C^1 perturbation of $\mathcal{G}$ with support in a small neighborhood of N_0 and having just a finite number of compact leaves $G_1', \cdots, G_r'$ close to N_0. Thus, F is transversal to $\mathcal{G}'$ and $\mathcal{F}'=F^*(\mathcal{G}')$ is a small C^1 perturbation of $\mathcal{F}$ with support in a small neighborhood of N. The map F defines a bijection between the set of compact leaves of $\mathcal{F}'$ close to N and $\{G_1', \cdots, G_r'\}$. Moreover, if L' and G' are compact leaves of $\mathcal{F}'$ and $\mathcal{G}'$ close to N and N_0 respectively, and $F(L')\subset G'$ then their holonomies are conjugate under F. Consequently, $I(\mathcal{F}'; L')=I(\mathcal{G}'; G')$ and the proposition is proved. ∎

§ 5. Proof of Theorem 3

The proof of Theorem 3 will proceed, as we have said, by reducing it to Theorem 2. This is done in Proposition 5.2 below.

The following remarks will be useful.

5.1 REMARKS. Let $[H_{\gamma_x}(\mathcal{F})[(u, y)=(g(u, y), f_u(y))$ be the holonomy map as in 1.1.

1. The map $H_{\gamma_x}(\mathcal{F})-Id_{R^q}$ is transversal to $\boldsymbol{R}^s\times\{0\}$ at $(0, 0)$ if and only if $f_0-Id_{R^{q-s}}$ is a diffeomorphism in a neighborhood of the origin.
   Consequently we have:
2. If N is normally elementary then it is isolated.
3. If in Proposition 4.1 we add the hypothesis that N is normally elementary then $I_\alpha(\mathcal{F}; N)=\pm\chi(B)$, since in this case $I_\alpha^*(\mathcal{F}; N)=\pm 1$.

5.2 PROPOSITION. *Let $\mathcal{F}\in \mathrm{Fol}_q^1(M)$ and $L\hookrightarrow N\xrightarrow{p} B$ be an $\mathcal{F}$-invariant fibered manifold with $\pi_1(L)=\boldsymbol{Z}$. Given a tubular neighborhood $(\eta(N); Q)$ of N there exists a continuous path of C^1 foliations $\{\mathcal{F}_t\}_{t\in[0,1]}$ on a neighborhood of N with $\mathcal{F}_0=\mathcal{F}$ and satisfying the following conditions:*

1. *$L\hookrightarrow N\xrightarrow{p} B$ is an $\mathcal{F}_t$-invariant fibered manifold for all $t\in[0, 1]$.*
2. *The leaves of $\mathcal{F}_1$ are mapped by Q into fibers of p.*

Furthermore, if N is $\mathcal{F}$-normally elementary then the path $\{\mathcal{F}_t\}_{t\in[0,1]}$ can be choosen so that N is $\mathcal{F}_t$-normally elementary for all $t\in[0, 1]$.

PROOF. The construction of $\{\mathcal{F}_t\}_{t\in[0,1]}$ will be carried out in three steps. Step 2 deal with an auxiliary one dimensional foliation.

Step 1. Reduction to a local construction problem.

Step 2. Local construction for the one dimensional case.

Step 3. Return to dimension of $\mathcal{F}$ greater than one.

We fix, once for all, closed s-disks ($s=\dim B$) U and U' in B with $U\subset\mathrm{Int}(U')$ which we identify with standard disks in the euclidean space $\boldsymbol{R}^s$.

Step 1. Reduction to a local construction problem.

Suppose we have constructed a path $\{\mathcal{F}_t\}_{t\in[0,1]}$ on a tubular subneighborhood $\eta_1(N)\subset\eta(N)$ satisfying:

i. $\mathcal{F}_0=\mathcal{F}|_{\eta_1(N)}$.

ii. $L\hookrightarrow N\xrightarrow{p} B$ is an $\mathcal{F}_t$-invariant fibered manifold.

iii. $\mathcal{F}_t\equiv\mathcal{F}_0$ outside of $\eta_1(p^{-1}(U'))$.

iv. $\eta_1(L_x)$ is $\mathcal{F}_1$-invariant for all $x \in p^{-1}(U)$.

v. If $x \in p^{-1}(U')$ and $\eta_1(L_x)$ is $\mathcal{F}|_{\eta_1(N)}$-invariant, then $\eta_1(L_x)$ is $\mathcal{F}_t$-invariant.

Now, to obtain $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfying (1) and (2) its suffices apply this construction to each U_i of a cover $\{U_i\}_{1\le i\le k}$ of B by closed s-disks.

Step 2. Local construction for the one dimensional case.

In this step we assume that $\mathcal{G}$ is a 1-dimensional foliation satisfying the hyphothesis of the proposition. In this case the local situation is described as follows: $\mathcal{G}$ is a foliation on the $(q-s)$-disk bundle $(U'\times\eta(S^1); Q)$ with section $U'\times S'$, where $(\eta(S^1); \bar{Q})$ is a $(q-s)$-disk bundle with section S^1 such that $Q(u, z)=(u, \bar{Q}(z))$. Moreover $S^1 \hookrightarrow U'\times S^1 \xrightarrow{p} U'$ is a $\mathcal{G}$-invariant fibered manifold.

The construction of the path $\{\mathcal{G}_t\}_{t\in[0,1]}$ required in step 1 will be carried out in two steps and we shall use two auxiliary foliations $\mathcal{H}$ and $\mathcal{K}$. First we fix a point $y_0 \in S^1$ and deform $\mathcal{G}\equiv\mathcal{G}_0$ in a neighborhood W of $U\times\eta(y_0)$ to obtain a path $\{\mathcal{G}_t\}_{t\in[0,1/2]}$ such that $\mathcal{G}_{1/2}=\mathcal{H}$ on W. Second we deform $\mathcal{G}_{1/2}$ outside of a neighborhood W' of $U\times\eta_1(y_0)$ to construct a path $\{\mathcal{G}_t\}_{t\in[1/2,1]}$ such that $\mathcal{G}_1\equiv\mathcal{K}$ on a neighborhood of $U\times\eta_1(S^1)$.

Fix $(u_0, y_0)\in \mathrm{Int}(U)\times S^1 \subset U'\times\eta(S')$. Let $[y_0^-, y_0^+]\subset S^1$ be a closed segment containing y_0 in its interior, small enough so that saturating $\{u_0\}\times\eta(y_0)$ by small arcs of leaves of $\mathcal{G}$ and projecting these arcs onto $\{u_0\}\times\eta([y_0^-, y_0^+])$ under the map $(u, z)\to(u_0, z)$, we obtain a C^1 foliation $\mathcal{H}_0$ on $\{u_0\}\times\eta([y_0^-, y_0^+])$. Denote by $\mathcal{H}$ the C^1 foliation on $U'\times\eta([y_0^-, y_0^+])$ obtained by transporting $\mathcal{H}_0$ under the product structure to each factor $\{u\}\times\eta([y_0^-, y_0^+])$. Clearly $\mathcal{G}\equiv\mathcal{H}$ on $U'\times[y_0^-, y_0^+]$.

For each $y\in[y_0^-, y_0^+]$ denote by $H_{[y_0,y]}(\mathcal{G})$ and $H_{[y_0,y]}(\mathcal{H})$ the holonomy maps defined on $U'\times\eta(y_0)$, where $[y_0, y]\subset[y_0^-, y_0^+]$ is the arc from y_0 to y. Fix in $U'\times\eta(y_0)$ a product metric and choose a C^1 isotopy supported on a small neighborhood of $U\times\eta(y_0)$, constructed by moving $[H_{[y_0,y]}(\mathcal{G})](u, z)$ along the geodesic arc in $U'\times\eta(y)$ to $[H_{[y_0,y]}(\mathcal{H})](u, z)$ for $y\in[y_0^-, y_0^+]$. In this way we produce a continuous path of C^1 foliations $\{\mathcal{G}_t\}_{t\in[0,1/2]}$ on $U'\times\eta(S^1)$ with $\mathcal{G}_0=\mathcal{G}$ such that for $t\in[0, 1/2]$ the following conditions hold:

— $S^1 \hookrightarrow U'\times S^1 \xrightarrow{p} U'$ is a $\mathcal{G}_t$-invariant fibered manifold.

— $\mathcal{G}_t\equiv\mathcal{G}$ outside of a small neighborhood of $U\times\eta(y_0)$.

— $\mathcal{G}_{1/2}\equiv\mathcal{H}$ on $\tilde{U}\times\eta([y_1^-, y_1^+])$, where $\tilde{U}\subset U'$ is a neighborhood of U and $[y_1^-, y_1^+]\subset S^1$ is a small neighborhood of y_0.

— If $u\in U'$ and $\{u\}\times\eta(S^1)$ is $\mathcal{G}$-invariant, then it is also $\mathcal{G}_t$-invariant.

From now on we fix an orientation on S^1 and to each $y\in S^1$, $y\neq y_0$, $[y_0, y]\subset S^1$ will denote the simple oriented arc from y_0 to y.

In a small enough $\eta(y_0)$ the maps

$$k_y(u, z)=(u, [p_2\circ H_{[y_0,y]}(\mathcal{G}_{1/2})](u, z));\ (u, z)\in\tilde{U}\times\eta(y_0)$$

are C^1 diffeomorphisms for each $y\in S^1$, where $p_2(u, z)=z$. Therefore, they define on $\tilde{V}\times\eta_1(S^1)$ a foliation $\mathcal{K}$ whose holonomy maps $H_{[y_0,y]}(\mathcal{K})$ coincide with k_y, where $\tilde{V}\subset\tilde{U}$ is a neighborhood of U and $\eta_1(S^1)\subset\eta(S^1)$. Since $\mathcal{G}_{1/2}\equiv\mathcal{H}$ on $\tilde{U}\times\eta([y_1^-, y_1^+])$, in view of the construction of $\mathcal{H}$ it follows that $\mathcal{K}=\mathcal{H}$ on $\tilde{V}\times\eta_1([y_1^-, y_1^+])$ and $\mathcal{K}$ is a C^1 foliation.

At this point we remark that $\{u\}\times\eta(S^1)$ is $\mathcal{K}$-invariant for $u\in\tilde{V}$.

Now recall that we have identified U with a standard disk in $\boldsymbol{R}^s$ and notice that $p_2\circ H_{[y_0,y]}(\mathcal{G}_{1/2})\equiv p_2\circ H_{[y_0,y]}(\mathcal{K})$ on $\tilde{V}\times\eta(y_0)$. Therefore, moving $[H_{[y_0,y]}(\mathcal{G}_{1/2})](u, z)$ along the euclidean geodesic arc to $[H_{[y_0,y]}(\mathcal{K})](u, z)$ using a suitable linear interpolation (depending only on $u\in\tilde{V}$), provides a continuous path of C^1 foliations $\{\mathcal{G}_t\}_{t\in[1/2,1]}$ on $U'\times\eta_1(S^1)$ (passing to a small enough $\eta_1(S^1)$) satisfying the following properties for all $t\in[1/2, 1]$:

— $S^1\hookrightarrow U'\times S^1\xrightarrow{p}U'$ is a $\mathcal{G}_t$-invariant fibered manifold.
— $\mathcal{G}_t\equiv\mathcal{G}_{1/2}$ outside of a small neighborhood of $U\times\eta_1(S^1)$.
— $\mathcal{G}_1\equiv\mathcal{K}$ on a neighborhood of $U\times\eta_1(S^1)$,
— If $\{u\}\times\eta(S^1)$ is $\mathcal{G}_{1/2}$-invariant then it is also $\mathcal{G}_t$-invariant.

We point out that the C^1 differentiability of $\mathcal{G}_t$ is ensured by the fact that $\mathcal{G}_t\equiv\mathcal{H}$ on $\tilde{U}\times\eta([y_1^-, y_1^+])$, for all $t\in[1/2, 1]$.

Hence, $\{\mathcal{G}_t\}_{t\in[0,1]}$ satisfies the conditions of step 1 for the one dimensional case.

Step 3. Return to dimension of $\mathcal{F}$ greather than one.

Fix a fiber L_{y_0} with $p(y_0)\in U$ and identity $\eta(p^{-1}(U'))$ with $U'\times\eta(L_{y_0})$ so that the corresponding projections, also denoted by p and Q, are expressed as: $p(u, y)=u$ and $Q(u, z)=(u, \bar{Q}(z))$, where $\bar{Q}$ is the restriction of Q to $\eta(L_{y_0})$.

Let $\alpha: S^1\to L_{y_0}$ be a C^1 embedding passing through y_0 and representing a generator of $\pi_1(L_{y_0})$. Let $\mathcal{G}$ be the one dimensional foliation induced by $\mathcal{F}$ on $U'\times\eta(S^1)$.

Now, we choose a C^1 map $f:(L_{y_0}, y_0)\to(S^1, y_0)$ inducing an isomorphism $f^*:\pi_1(L_{y_0})\to\pi_1(S^1)$. As in §3 we extend the pair $(f, Id_{U'\times\eta(y_0)})$ to a unique C^1 map $F:\tilde{U}\times\eta_1(L_{y_0})\to U'\times\eta(S^1)$ preserving the foliations $\mathcal{F}$

and $\mathcal{G}$ and mapping diffeomorphically $\tilde{U}\times\eta(y)$ into $U'\times\eta(f(y))$ for each $y\in L_{y_0}$, where $\tilde{U}\subset U'$ is a neighborhood of U and $\eta_1(L_{y_0})\subset\eta(L_{y_0})$. As remarked in § 3, the map F is transversal to $\mathcal{G}$ and $F^*(\mathcal{G})=\mathcal{F}$. Moreover, if $\{u\}\times\eta(L_{y_0})$ is invariant under $\mathcal{F}_{U'\times\eta(L_{y_0})}$ then $F(\{u\}\times\eta_1(L_{y_0}))$ is contained in $\{u\}\times\eta(S^1)$. Therefore, if $\{\mathcal{G}_t\}_{t\in[0,1]}$ is the path constructed in step 2 then $\mathcal{F}_t=F^*(\mathcal{G}_t)$, $t\in[0,1]$, satisfies the conditions (i), (ii), (iii) and (v) of step 1.

Let Q^* be the projection on $U'\times L_{y_0}$ whose fibers are $F^{-1}(\{u\}\times\eta(f(y)))$, $(u, y)\in U'\times L_{y_0}$. One easily check that in a neighborhood of $U\times\eta_1(L_{y_0})$ the leaves of $\mathcal{F}_1$ project under Q^* into the fibers of p.

To obtain (iv), passing to a small enough $\eta_1(L_{y_0})$ we deform Q^* to Q in $U\times\eta_1(L_{y_0})$ by a C^1 isotopy $\{h_t\}_{t\in[0,1]}$ preserving (i), (ii), (iii) and (v) and supported in a neighborhood of $U\times\eta_1(L_{y_0})$. Once more, this isotopy is constructed by moving $(u', z)\in(Q^*)^{-1}(u, y)$ along the geodesic arc (in a product metric) to $(u, z)\in Q^{-1}(u, y)$. Now, the path $\{(\mathcal{F}_t)\}_{t\in[0,1]}$ followed by the path $\{h_t(\mathcal{F}_1)\}_{t\in[0,1]}$ satisfies the five required conditions in step 1.

To finish the proof we observe that in step 2 the foliations $\mathcal{G}$ and $\mathcal{G}_{1/2}$ are diffeomorphic, so if $U'\times S^1$ is $\mathcal{G}$-normally elementary then it is also $\mathcal{G}_{1/2}$-normally elementary. Moreover, in view of the construction of $\{\mathcal{G}_t\}_{t\in[1/2,1]}$ the first return maps $H(\mathcal{G}_t)$ have the form $H(\mathcal{G}_t)(u, z)=(g_t(u, z), f_u(z))$ for $t\in[1/2, 1]$ and $(u, z)\in U'\times\eta(y_0)$. From Remark 5.1.1 it follows that since $U'\times S^1$ is $\mathcal{G}_{1/2}$-normally elementary then it is also $\mathcal{G}_t$-normally elementary for all $t\in[1/2, 1]$. ■

5.3 Proof of Theorem 3. Without lost of generality we suppose that the action of $\pi_1(B)$ on $\pi_1(L)$ is trivial, by passing to a double cover of the base B. Furthermore, in view of Remark 5.1.2, N is an isolated $\mathcal{F}$-invariant fibered manifold thus $I_\alpha(\mathcal{F}; N)$ is defined.

Let $\{\mathcal{F}_t\}_{t\in[0,1]}$ be given by Proposition 5.2 so that N is $\mathcal{F}_t$-normally elementary. From the continuity of $\{\mathcal{F}_t\}_{t\in[0,1]}$ in the C^1 topology, the compactness of $[0, 1]$ and Remark 5.1.2, it follows that there exists a neighborhood U of N such that for all $t\in[0, 1]$ the compact leaves of $\mathcal{F}_t$ close to N and lying in U are exactly the fibers of p. Hence, Theorem 1 and Remark 5.1.3 imply that $I_\alpha(\mathcal{F}; N)=I_\alpha(\mathcal{F}_1; N)=\pm\chi(B)$ and the proof is finished. ■

Remark. Let $\mathcal{F}\in \mathrm{Fol}_q^1(M)$ and let $L\hookrightarrow N\xrightarrow{p}B$ be an $\mathcal{F}$-invariant fibered manifold with $\pi_1(L)=\mathbf{Z}$. From the technics described in § 3 and

§ 5 we can construct a manifold M' with a C^1 one dimensional foliation $\mathcal{G}$ having an S^1-bundle $S^1 \to N' \xrightarrow{p'} B$ as a $\mathcal{G}$-invariant fibered manifold, and a C^1 map $F: \eta(N) \to M'$ defined in a neighborhood $\eta(N)$ of N, satisfying the following properties:

— $F(N)=N'$, $p' \circ F = p$ on N, and $F_*: \pi_1(L) \to \pi_1(S^1)$ is an isomorphism,

— F is transversal to $\mathcal{G}$ and $F^*(\mathcal{G})=\mathcal{F}$.

Consequently, if the action of $\pi_1(B)$ on $\pi_1(L)$ is trivial and N is isolated, then it follows from the arguments given in the proof of Proposition 4.1 that $I_\alpha(\mathcal{F}; N)=I_\beta(\mathcal{G}; N')$, where $\beta=F_*(\alpha)$ and α is a generator of $\pi_1(L)$. ∎

Appendix: Proof of Lemma 2.1

For one dimensional foliations it is easy to construct $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfying 2.1.1, 2.1.2 and 2.1.3. We shall reduce the general case to the one dimensional case.

We identify a neighborhood of the $\mathcal{F}$-saturation of D^s with $\boldsymbol{R}^s \times \eta(L)$ viewed as a neighborhood of $D^s \times L$, where $(\eta(L); Q)$ is a deleted $(q-s)$-disk bundle with a section identified to L. Relative to this identification the $\mathcal{F}$-saturation of D^s is $D^s \times L$. We suppose (passing to a smaller q-disk D^q) that $D^q \subset \boldsymbol{R}^s \times \eta(x_0)$ and the q-disks $\boldsymbol{R}^s \times \eta(x)$ are transversal to $\mathcal{F}$ for all $x \in L$.

Let γ be represented by an embedded circle $S^1 \subset L$ generating $\pi_1(L; x_0)$.

Now, we fix a C^1 map $f: (L, x_0) \to (S^1, x_0)$ inducing an isomorphism $f_*: \pi_1(L; x_0) \to \pi_1(S^1; x_0)$.

Let $\mathcal{G}$ (resp. $\mathcal{G}'$) be the one dimensional foliation induced by $\mathcal{F}$ (resp. $\mathcal{F}'$) on $\boldsymbol{R}^s \times \eta(S^1)$. Then $\mathcal{F}'$ and $\mathcal{G}'$ have the same holonomy. So, we can choose a small neighborhood $\mathcal{V}$ of $D^s \times L$ such that for all foliations $\mathcal{F}'$ which are C^1 close to $\mathcal{F}$ the identity map of D^q extends to a C^1 map $F': \mathcal{V} \to \boldsymbol{R}^s \times \eta(S^1)$ (depending on $\mathcal{F}'$) satisfying:

— for all $x \in L$ the restriction of F' to $\mathcal{V} \cap (\boldsymbol{R}^s \times \eta(x))$ is an embedding with $F'(\mathcal{V} \cap (\boldsymbol{R}^s \times \eta(x))) \subset \boldsymbol{R}^s \times \eta(f(x))$,

— $(F')^*(\mathcal{G}') \equiv \mathcal{F}'$ on $\mathcal{V}$,

— F' depends continuously on $\mathcal{F}'$ in the C^1 topology.

In this way taking the pull-back under F' we reduce the construction of $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfying 2.1.1, 2.1.2 and 2.1.3 to the one dimensional case.

From now on we assume that the family $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfying 2.1.1, 2.1.2 and 2.1.3 is already constructed. In order to make $\{\mathcal{F}_t\}_{t\in[0,1]}$ satisfy 2.1.4 we shall modify each $\mathcal{F}_t$ under diffeomorphisms supported in a small neighborhood of the $\mathcal{F}$-saturation of K_1.

We fix neighborhood $W_1''\subset W_1'$ of K_1 and W_2'' of K_2 in D^q and assume that $H_1\equiv H_\gamma(\mathcal{F}'')$ on $W_1''\cup W_2''$. Thus, by construction $\mathcal{F}''$ and $\mathcal{F}_1$ have the same holonomy on $W_1''\cup W_2''$.

Let $\mathcal{U}\subset\mathcal{R}_1'\cup\mathcal{R}_2'$ be a small neighborhood of the $\mathcal{F}$-saturation of $K_1\cup K_2$ such that for all foliations $\mathcal{F}''$ and $\mathcal{F}_1$ which are C^1 close to $\mathcal{F}$, we can extend the identity map of D^q to a C^1 embedding $F'':\mathcal{U}\to \boldsymbol{R}^s\times\eta(L)$ (depending on $\mathcal{F}''$ and $\mathcal{F}_1$) such that:

— $F''(\mathcal{U}\cap(\boldsymbol{R}^s\times\eta(x)))\subset\boldsymbol{R}^s\times\eta(x)$ for all $x\in L$.

— $(F'')^*(\mathcal{F}_1)\equiv\mathcal{F}''$ on $\mathcal{U}$.

— F'' is C^1 close to the identity map of $\mathcal{U}$.

Now, moving $F''(y)$ along the geodesic segment defined by y and $F''(y)$ (using the metric induced on each factor $\boldsymbol{R}^s\times\eta(x)$, $x\in L$ by the metric on M) and passing to a convenient small subneighborhood $\mathcal{W}\subset\mathcal{U}$ of the $\mathcal{F}$-saturation of $K_1\cup K_2$, we can modify F'' just outside of $\mathcal{W}$ to construct a diffeomorphism of M, denoted also by F'', satisfying the following properties:

— $(F'')^*(\mathcal{F}_1)\equiv\mathcal{F}''$ on $\mathcal{W}$,

— $F''\equiv Id_M$ outside of $\mathcal{U}$.

— $F''(\mathcal{U}\cap(\boldsymbol{R}^s\times\eta(x)))\subset\boldsymbol{R}^s\times\eta(x)$ for all $x\in L$,

— F'' is C^1 close to the identity map of M.

Since $\mathcal{F}_1\equiv\mathcal{F}'$ outside of a small neighborhood of the $\mathcal{F}$-saturation of K_1 and $\mathcal{F}'\equiv\mathcal{F}''$ on $\mathcal{R}_2'$, it follows from the construction of F'' that in fact $F''\equiv Id_M$ outside of $\mathcal{R}_1'$.

Finally, the desired path is given by $\{(F_t'')^*(\mathcal{F}_t)\}_{t\in[0,1]}$ where in each disk $\boldsymbol{R}^s\times\eta(x)$, $x\in L$, $\{F_t''\}_{t\in[0,1]}$ is given by the baricentric isotopy from $F_0''\equiv Id_M$ to $F_1''\equiv F^*$. ∎

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