

## ***$L^2$ -theory of Singular Perturbation of Hyperbolic Equations II Asymptotic expansions of dissipative type***

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**Abstract.** We consider the Cauchy problem for linear strictly hyperbolic equations of order  $m+1$  with a small parameter  $\varepsilon$ :

$$(1.1) \quad (i\varepsilon)L(t, x, D_t, D_x) + M(t, x, D_t, D_x)u(t, x; \varepsilon) = f(t, x) \quad (t, x) \in (0, T) \times \mathbb{R}^d$$

$$(1.2) \quad D_t^j u(0, x; \varepsilon) = g_j(x) \quad j=0, 1, 2, \dots, m$$

where  $L$  and  $M$  are linear strictly hyperbolic operators of order  $m+1$  and  $m$ . The aim of this paper is to give asymptotic expansions of solutions to this singular perturbation problem, when the characteristic roots of  $L$  and  $M$  separate each other. The points are to construct formal solutions (Proposition 2.3), consisting of regular terms and singular ones (correction terms of dissipative type), and to give their estimates by Sobolev norms of higher order (Theorem 3.1). Finally, these results and an a priori estimate in [U] give the error estimates in order to obtain asymptotic expansions in the sense of arbitrarily higher order Sobolev norms (Theorem 4.1).

### § 1. Introduction.

We consider the following initial value problem with a positive small parameter  $\varepsilon \in (0, \varepsilon_0]$ :

$$(1.1) \quad \begin{aligned} P(t, x, D_t, D_x; \varepsilon)u(t, x; \varepsilon) = \\ (i\varepsilon)L(t, x, D_t, D_x; \varepsilon) + M(t, x, D_t, D_x; \varepsilon)u(t, x; \varepsilon) = f(t, x; \varepsilon) \end{aligned}$$

$$\text{for } (t, x) \in (0, +\infty) \times \mathbb{R}^d,$$

$$(1.2) \quad D_t^j u(0, x; \varepsilon) = g_j(x; \varepsilon), \quad j=0, 1, 2, \dots, m.$$

(In this paper, we denote the dimension of the  $x$ -space by  $d$  and reserve  $n$  for the suffix of general terms of various expansions.)

We assume

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$$(1.3) \quad L(t, x, D_t, D_x; \varepsilon) = D_t^{m+1} + \sum_{j=1}^{m+1} L_j(t, x, D_x; \varepsilon) D_t^{m+1-j}$$

$$(1.4) \quad M(t, x, D_t, D_x; \varepsilon) = m_0(t, x; \varepsilon) D_t^m + \sum_{j=1}^m M_j(t, x, D_x; \varepsilon) D_t^{m-j},$$

where  $L_j$  and  $M_j$  are differential operator of order  $j$  with smooth coefficients in  $(t, x, \varepsilon)$ .  $D_t$  denotes  $\partial/\partial t$  and  $D_x$  denotes  $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ . The coefficients are assumed to have bounded derivatives, that is, for any  $j, k, \beta$ , there exists a constant  $C_{j,k,\beta}$  such that

$$(1.5) \quad \sup \{ |\partial_t^j \partial_x^k \partial_x^\beta a(t, x; \varepsilon)|; 0 \leq \varepsilon \leq \varepsilon_0, t \geq 0, x \in \mathbf{R}^d \} \leq C_{j,k,\beta}.$$

The principal symbols of  $L$  and  $M$  are

$$l(t, x, \tau, \xi; \varepsilon) = \tau^{m+1} + \sum_{j=1}^{m+1} l_j(t, x, \xi; \varepsilon) \tau^{m+1-j}$$

and

$$m(t, x, \tau, \xi; \varepsilon) = m_0(t, x; \varepsilon) \tau^m + \sum_{j=1}^m m_j(t, x, \xi; \varepsilon) \tau^{m-j}.$$

We assume the following assumptions (H0), (E1), (H1) and (S0).

(H0) Regular Hyperbolicity of  $L$ :  $l(t, x, \tau, \xi; \varepsilon)$  has the decomposition

$$l(t, x, \tau, \xi; \varepsilon) = \prod_{j=1}^{m+1} (\tau - \varphi_j(t, x, \xi; \varepsilon))$$

where  $\varphi_j(t, x, \xi; \varepsilon)$  are real distinct elements such that

$$(1.6) \quad \varphi_1(t, x, \xi; \varepsilon) < \varphi_2(t, x, \xi; \varepsilon) < \dots < \varphi_{m+1}(t, x, \xi; \varepsilon) \text{ uniformly,}$$

that is,  $\varphi_{j+1}(t, x, \xi; \varepsilon) - \varphi_j(t, x, \xi; \varepsilon)$  is uniformly positive with respect to  $(t, x, \xi; \varepsilon)$  for  $j=1, \dots, m$ .

(E1) Strong ellipticity:

$$(1.7) \quad \operatorname{Re} m_0(t, x; \varepsilon) \geq \delta > 0.$$

This assumption is a special case of (E) in [U].

(H1) Regular Hyperbolicity of  $M$ :  $m(t, x, \tau, \xi; \varepsilon)$  has the decomposition

$$m(t, x, \tau, \xi; \varepsilon) = m_0(t, x; \varepsilon) \prod_{j=1}^m (\tau - \phi_j(t, x, \xi; \varepsilon))$$

where  $\phi_j(t, x, \xi; \varepsilon)$  are real distinct elements such that

$$\phi_1(t, x, \xi; \varepsilon) < \phi_2(t, x, \xi; \varepsilon) < \dots < \phi_m(t, x, \xi; \varepsilon) \text{ uniformly.}$$

(S0) Separation: The roots  $\{\phi_j(t, x, \xi; \varepsilon)\}$  uniformly separate  $\{\varphi_j(t, x, \xi; \varepsilon)\}$ :

$$(1.8) \quad \varphi_1 < \phi_1 < \cdots < \phi_m < \varphi_{m+1}, \text{ uniformly.}$$

We assume the data  $f(t, x; \varepsilon) \in C_0^\infty([0, \infty) \times \mathbf{R}^d \times [0, \varepsilon_0])$  and  $g_j(x; \varepsilon) \in C_0^\infty(\mathbf{R}^d \times [0, \varepsilon_0])$ . They have asymptotic expansions with respect to  $\varepsilon$ :

$$(1.9) \quad f(t, x; \varepsilon) = \sum_{n=0}^N \varepsilon^n f_n(t, x) + R_{N+1}(f; \varepsilon),$$

$$(1.10) \quad g_j(x; \varepsilon) = \sum_{n=0}^N \varepsilon^n g_{j,n}(x) + R_{N+1}(g_j; \varepsilon).$$

We consider in this paper an asymptotic expansion of the solution  $u(t, x; \varepsilon)$  to (1.1) and (1.2). We postulate that the solution has an expansion

$$(1.11) \quad u(t, x; \varepsilon) \sim v(t, x; \varepsilon) + w(t, x; \varepsilon),$$

$$(1.12) \quad v(t, x; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n v_n(t, x) \quad (\text{regular part}),$$

$$(1.13) \quad w(t, x; \varepsilon) = \sum_{n=m}^{\infty} \varepsilon^n w_n(t, x; \varepsilon) \quad (\text{singular part}),$$

where  $v$  and  $w$  mean formal sums (See § 2) such that

$$(1.14) \quad Pv \sim f$$

$$(1.15) \quad Pw \sim 0$$

$$(1.16) \quad D_i^j(v+w) \sim g_j \text{ at } t=0.$$

We quote an a priori estimate for the problem (1.1) and (1.2) established in [U].

**THEOREM 1.1** *Assume the same conditions as above. For any natural number  $p$ , there exist positive constants  $C$  and  $\gamma_0$  such that for any positive  $\varepsilon \leq \varepsilon_0$ , any  $\gamma \geq \gamma_0$  and for any solution  $u(t) \in C^\infty([0, T]; C_0^\infty(\mathbf{R}_x^d))$  to (1.1)–(1.2), we have*

$$(1.17) \quad C \left\{ \int_0^T e^{-2\gamma t} \varepsilon^{2p} \|D^p f(t)\|^2 dt \right. \\ + \varepsilon \sum_{j=0}^p \varepsilon^{2j} \|D^m u(0)\|_j^2 + \gamma \sum_{j=1}^p \varepsilon^{2j} \|D^m u(0)\|_{j-1}^2 \\ \left. + \varepsilon \sum_{j=0}^{p-1} \varepsilon^{2j} \|D^j f(0)\|^2 + \gamma \sum_{j=1}^{p-1} \varepsilon^{2j} \|D^{j-1} f(0)\|^2 \right\}$$

$$\begin{aligned} &\geq \gamma \int_0^T e^{-2\gamma t} \varepsilon^{2p} (\varepsilon \|D^{m+p} u(t)\|^2 + \gamma \|D^{m+p-1} u(t)\|^2) dt \\ &+ e^{-2\gamma T} \varepsilon^{2p} (\varepsilon \|D^{m+p} u(T)\|^2 + \gamma \|D^{m+p-1} u(T)\|^2). \end{aligned}$$

We put

$$\begin{aligned} u_N(t, x; \varepsilon) &= \sum_{n=0}^N \varepsilon^n v_n(t, x) + \sum_{n=N+1}^{N+m} \varepsilon^n w_n(t, x; \varepsilon), \\ R_{N+1}(u; \varepsilon) &= u(t, x; \varepsilon) - u_N(t, x; \varepsilon). \end{aligned}$$

We will see (in Proposition 4.1)

$$(1.18) \quad \begin{cases} ((i\varepsilon)L + M)R_{N+1}(u; \varepsilon) = R_{N+1}(f; \varepsilon) + \varepsilon^{N+1} \left\{ \rho(t, x; \varepsilon) + \tilde{\chi}\left(\frac{t}{\varepsilon}, x; \varepsilon\right) \right\}, \\ D_t^j R_{N+1}(u; \varepsilon) = R_{N+1}(g_j; \varepsilon) - \varepsilon^{N+1} \gamma_j(x; \varepsilon). \end{cases}$$

We apply Theorem 1.1 to (1.18) in order to obtain estimates of  $R_{N+1}(u; \varepsilon)$ . Thus, we have our main result Theorem 4.1. For an arbitrarily higher order Sobolev norm and  $\nu \in N$ , there exists a large number  $N$ , such that

$$\begin{aligned} (i) \quad &\begin{cases} ((i\varepsilon)L + M)u_N = f + O(\varepsilon^\nu) \\ D_t^j u_N = g_j + O(\varepsilon^\nu) \end{cases} \quad \text{and} \\ (ii) \quad &u = u_N(t, x; \varepsilon) + O(\varepsilon^\nu), \end{aligned}$$

where  $O(\varepsilon^\nu)$ 's are measured by the given Sobolev norm.

In § 2, we give the formal expansions (1.12) and (1.13) in Proposition 2.3. Each term of the singular part (1.13) has exponential decay off the initial plane, when  $\varepsilon$  tends to  $+0$ . This is the reason why we call the asymptotic expansion dissipative type. The main part of this paper is § 3 where we give successive estimates of  $v_n$  and  $w_n$  in Theorem 3.1. In § 4, we estimate the error term of the truncated expansion of the solution, using the a priori estimate (1.17) established in [U]. We have the conclusion of Theorem 4.1, which give asymptotic estimates in the sense of arbitrarily higher order Sobolev norm.

In the 2-dimensional  $(t, x)$ -space, E.M. De Jager constructed asymptotic solutions to (1.4) when  $m=1$  ([DeJ]). We followed his formal construction. In order to establish higher order Sobolev norm estimates of the remainder term of asymptotic solution in the general  $(d+1)$ -dimensional  $(t, x)$ -space, we use higher order estimates prepared in [U]. The separation condition (S0) is essential for the  $L^2$ -estimates in [U]. Asymptotic problems of singular perturbations of linear and nonlinear hyperbolic

equations of 2nd order in 2-dimensional space are systematically studied in [Ge]. In the  $(d+1)$ -dimensional  $(t, x)$ -space, R. Gao considered (1.4) with zero Cauchy data when  $m_0(t, x; \varepsilon) = 1$  ([Ga]). The singular part of formal solution in [Ga] has a little different form:  $w_n(t, x; \varepsilon) = \beta_n(t, x)e^{-t/\varepsilon}$ . There, the classical Leray-Gårding inequality of hyperbolic equation was used to estimate the remainder term. Since we prepared in [U] higher order estimates (in a framework of pseudo-differential operators in  $D_x$ ), we have also  $C^k$  estimates of the remainder terms. In case of linear ordinary differential equations, related problems are treated by A. Yoshikawa ([Yo1], [Yo2]).

Asymptotic expansion for the dispersive case in [U] will be discussed in a forthcoming paper. We hope our results are able to be foundation for singular perturbation of hyperbolic initial-boundary value problems.

*Corrections to the previous article "L<sup>2</sup>-theory of Singular Perturbation of Hyperbolic Equations I"*

(i) In the statements in § 3 of [U], the phrases meaning "for any natural number  $p$ " should be placed before the phrases meaning existence of  $C$  and  $\gamma_0$ , since the constants in the inequalities generally depend on higher order differentiability  $p$ .

(ii)  $\varepsilon_0$  should be arbitrarily fixed. Consequently,  $C$  and  $\gamma_0$  depend on  $\varepsilon_0$ .

## § 2. Formal construction of asymptotic solutions.

For any  $n \in N$ , we have the Taylor expansion of  $L$ :

$$(2.1) \quad L(t, x, D_t, D_x; \varepsilon) = \sum_{n=0}^N \varepsilon^n L^{(n)}(t, x, D_t, D_x) + R_{N+1}(L; \varepsilon),$$

where  $L^{(n)}(t, x, D_t, D_x)$ 's and  $R_{N+1}(L; \varepsilon)$  are differential operators of order  $m+1$ . We have also

$$(2.2) \quad M(t, x, D_t, D_x; \varepsilon) = \sum_{n=0}^N \varepsilon^n M^{(n)}(t, x, D_t, D_x) + R_{N+1}(M; \varepsilon),$$

where  $M^{(n)}(t, x, D_t, D_x)$ 's and  $R_{N+1}(M; \varepsilon)$  are differential operators of order  $m$ .  $R_{N+1}(L; \varepsilon) = O(\varepsilon^{N+1})$  and  $R_{N+1}(M; \varepsilon) = O(\varepsilon^{N+1})$  in the sense that for any  $j, \alpha$  there exists a positive constant  $C$  such that each of their coefficients  $a(t, x; \varepsilon)$  satisfies

$$(2.3) \quad \sup \{ |\partial_t^j \partial_x^\alpha a(t, x; \varepsilon)|; (t, x) \in [0, \infty) \times \mathbf{R}^d \} \leq C \varepsilon^{N+1}$$

for  $\varepsilon \in [0, \varepsilon_0]$ .

Introducing a stretched variable  $s=t/\varepsilon$ , we define

$$(2.4) \quad \begin{aligned} \tilde{P}(s, x, D_s, D_x; \varepsilon) &= (-i)\varepsilon^m P(t, x, D_t, D_x; \varepsilon) \\ &= \varepsilon^{m+1} L(\varepsilon s, x, \varepsilon^{-1} D_s, D_x; \varepsilon) - i\varepsilon^m M(\varepsilon s, x, \varepsilon^{-1} D_s, D_x; \varepsilon). \end{aligned}$$

$\tilde{P}$  has the Taylor expansion with respect to  $\varepsilon$ :

$$(2.5) \quad \tilde{P}(s, x, D_s, D_x; \varepsilon) = \sum_{n=0}^N \varepsilon^n \tilde{P}^{(n)}(s, x, D_s, D_x) + R_{N+1}(\tilde{P}; \varepsilon).$$

Each term  $\tilde{P}^{(n)}(s, x, D_s, D_x)$  is a differential operator of order at most  $m+1$ . Its coefficients are polynomials in  $s$ . The sum of degree in  $s$  and order in  $D_x$  is at most  $n$ . The remainder term  $R_{N+1}(\tilde{P}; \varepsilon)$  is also a differential operator of order  $m+1$ . Its coefficients  $\tilde{a}(s, x; \varepsilon)$ 's satisfy

$$(2.6) \quad \begin{aligned} \sup_{x \in R^n} |D_s^j D_x^\alpha \tilde{a}(s, x; \varepsilon)| &\leq C \varepsilon^{N+1} (1+s)^{\max\{N+1-j, 0\}} \\ \text{for } (s, \varepsilon) &\in [0, \infty) \times [0, \varepsilon_0], \end{aligned}$$

where  $C$  is a constant independent of  $\varepsilon, s$ .

We notice that

$$(2.7) \quad \tilde{P}^{(0)}(s, x, D_s, D_x) = D_s^{m+1} - i\mu(x)D_s^m,$$

where

$$(2.8) \quad \mu(x) = m_0(0, x; 0).$$

As we planned in § 1, we construct a formal expansion of the solution  $u(t, x; \varepsilon)$  to the problem (1.4).

From (1.14), we have a sequence of equations for the regular part  $v = \sum \varepsilon^n v^{(n)}$ :

$$(2.9) \quad M^{(0)}v_0(t, x) = f_0(t, x)$$

$$(2.10) \quad M^{(0)}v_n(t, x) = f_n(t, x) - \sum_{k=0}^{n-1} (iL^{(k)} + M^{(k+1)})v_{n-1-k}(t, x)$$

for  $n \geq 1$ .

We assume for the singular part  $w = \sum \varepsilon^n w^{(n)}$ ,

$$w(t, x; \varepsilon) = \tilde{w}\left(\frac{t}{\varepsilon}, x; \varepsilon\right) = \sum_{n=0}^{\infty} \varepsilon^n \tilde{w}_n\left(\frac{t}{\varepsilon}, x\right).$$

Then, (1.15) means  $\tilde{P}\left(\sum_{n=0}^{\infty} \varepsilon^n \tilde{w}_n(s, x)\right) \sim 0$ . We have a sequence of equations

$$(2.11) \quad (D_s^{m+1} - i\mu(x)D_s^m)\tilde{w}_0(s, x) = 0,$$

$$(2.12) \quad (D_s^{m+1} - i\mu(x)D_s^m)\tilde{w}_n(s, x) = - \sum_{k=1}^n \tilde{P}^{(k)}(s, x, D_s, D_x)\tilde{w}_{n-k}(s, x),$$

for  $n \geq 1$ .

From the initial conditions (1.16), we have

$$(\varepsilon D_t)^j v(0, x; \varepsilon) + D_s^j \tilde{w}(0, x; \varepsilon) \sim \varepsilon^j g_j(x; \varepsilon)$$

for  $0 \leq j \leq m$ ,

that is, a sequence of conditions

$$(2.13) \quad D_t^j v_n(0, x) + D_s^j \tilde{w}_{n+j}(0, x) = g_{j,n}(x)$$

for  $n=0, 1, \dots$  and  $j=0, 1, \dots, m$ .

We choose  $\{\tilde{w}_n; 0 \leq n \leq m-1\}$  as

$$(2.14) \quad \tilde{w}_0(s, x) = \dots = \tilde{w}_{m-1}(s, x) \equiv 0.$$

Hence,

$$(2.15) \quad D_t^j v_0(0, x) = g_{j,0}(x), \quad 0 \leq j \leq m-1.$$

REMARK 2.1. This choice (2.14) is not unique to solve the system of successive equations (2.9), (2.10), (2.11) and (2.12) with (2.13). When  $\varepsilon=0$ , the equation (1.1) reduces to (2.9). Since we are interested in  $v_0(t, x)$  approximating the exact solution  $u(t, x; \varepsilon)$ , we use the conditions (2.15).

We will show that  $\{v_n\}$  and  $\{\tilde{w}_n\}$  are determined successively by the coefficients of asymptotic expansions of  $f, g_j$ 's.

PROPOSITION 2.1. Under the assumption (H1),  $v_0(t, x) \in C^\infty([0, \infty); C_0^\infty(\mathbb{R}^d))$  is determined by (2.9), (2.15). In short,  $v_0(t, x)$  is determined by  $f_0(t, x)$  and  $\{g_{j,0}(x); 0 \leq j \leq m-1\}$ .  $D_t^{m+r} v_0(0, x)$  is determined by  $\{D_t^k f_0(0, x); 0 \leq k \leq p\}$  and  $\{g_{j,0}(x); 0 \leq j \leq m-1\}$ .

PROOF. The former part concerning  $v_0(t, x)$  is classical. The latter part concerning the traces follows easily from the equation (2.9) with (2.15). Q.E.D.

PROPOSITION 2.2.

$$(2.16) \quad \begin{cases} (D_s - i\mu(x))D_s^m \tilde{w}_m(s, x) = 0 \\ D_s^m \tilde{w}_m(0, x) = g_{m,0}(x) - D_t^m v_0(0, x) \end{cases}$$

has a unique exponential solution

$$\tilde{w}_m(s, x) = e^{-\mu(x)s} (i\mu(x))^{-m} \{g_{m,0}(x) - D_t^m v_0(0, x)\}.$$

In short,  $\tilde{w}_m(s, x)$  is determined by  $f_0(0, x)$  and  $\{g_{j,0}(x); 0 \leq j \leq m\}$ .

PROOF. Easy.

Q.E.D.

In order to construct  $\{v_n\}$ ,  $\{\tilde{w}_{m+n}\}$  successively, we recall some elementary facts on exponential polynomial solutions in our context. Let

$$B(s, x) = \sum_{j=0}^p b_{p-j}(x) (is)^j$$

a polynomial in  $s$  of degree  $p$ .  $b_j(x)$ 's are smooth functions of  $x$ .

LEMMA 2.1. *The equation*

$$(2.17) \quad \begin{cases} (D_s - i\mu(x)) D_s^m \tilde{w}(s, x) = e^{-\mu(x)s} B(s, x) \\ D_s^m \tilde{w}(0, x) = c(x) \end{cases}$$

has a unique exponential polynomial solution of degree  $p+1$  of the form

$$(2.18) \quad \tilde{w}(s, x) = e^{-\mu(x)s} \sum_{j=0}^{p+1} h_{p+1-j}(x) (is)^j.$$

Here,  $h_j(x)$  is a linear combination of  $\{b_0(x), \dots, b_j(x)\}$  for  $j=0, 1, \dots, p$ .  $h_{p+1}(x)$  is a linear combination of  $\{b_0(x), \dots, b_p(x), c(x)\}$  with coefficients in the ring  $C[\mu(x), \mu(x)^{-1}]$ .

PROOF. Since  $x$  is an idle parameter, we often suppress it. It is clear that

$$(2.19) \quad D_s^m \tilde{w}(s) = e^{-\mu s} \left\{ \sum_{j=1}^{p+1} i b_{p-j+1} \frac{(is)^j}{j} + c \right\}.$$

Therefore, we have a unique exponential polynomial solution  $\tilde{w}(s) = e^{-\mu s} \sum_{j=0}^{p+1} h_{p+1-j}(is)^j$ , where  $h_j (0 \leq j \leq p)$  is a linear combination of  $\{b_0, \dots, b_j\}$  and  $h_{p+1}$  is that of  $\{b_0, \dots, b_p, c\}$  with coefficients in  $C[\mu, \mu^{-1}]$ . (See Appendix.)

Q.E.D.

PROPOSITION 2.3. (i) *Under the assumption (H1) and (2.14), there exist uniquely  $\{v_n(t, x); n \geq 1\}$ , and exponential polynomials of degree  $2n$  in  $s$   $\{\tilde{w}_{m+n}(s, x); n \geq 1\}$  such that  $v_n(t, x)$  satisfies (2.10) and (2.13) for  $0 \leq j \leq m-1$  and that  $\tilde{w}_n(s, x)$  satisfies (2.12) and (2.13) for  $j=m$ .*



(ii) Moreover,  $v_n(t, x) \in C^\infty([0, +\infty); C^\infty(\mathbf{R}^d))$  is determined by  $\{f_k(t, x); 0 \leq k \leq n\}$ ,  $\{g_{j,k}(x); 0 \leq j \leq m, 0 \leq k \leq n-1\}$  and  $\{g_{j,n}(x); 0 \leq j \leq m-1\}$ .  $D_t^{m+p}v_n(0, x)$  is determined by  $\{D_t^l f_q(0, x); 0 \leq q \leq n, 0 \leq l \leq p+n-q\}$ ,  $\{g_{j,k}(x); 0 \leq j \leq m, 0 \leq k \leq n-1\}$  and  $\{g_{j,n}(x); 0 \leq j \leq m-1\}$ .

(iii) Define  $h_j^{(n)}(x)$  by

$$(2.20) \quad \tilde{w}_{m+n}(s, x) = e^{-\mu(x)s} \sum_{j=0}^{2n} h_{2n-j}^{(n)}(x) (is)^j \quad (n \geq 0).$$

Then,  $h_j^{(n)}(x)$  is determined by  $\{D_t^l f_q(0, x); 0 \leq q \leq n, 0 \leq l \leq n-q\}$  and  $g_{j,k}(x); 0 \leq j \leq m, 0 \leq k \leq n\}$ .

REMARK. In § 3, we give precise estimates for  $v_n(t, x)$  and  $\{h_j^{(n)}(x); 0 \leq j \leq 2n\}$ . We show, here, only functional dependence to understand the construction procedure.

PROOF OF PROPOSITION. Let  $n=1$ .  $v_1(t)$  is uniquely determined by (2.10) and (2.13):

$$\begin{cases} M^{(0)}v_1(t) = f_1(t) - (iL^{(0)} + M^{(1)})v_0(t) \\ D_t^j v_1(0) = g_{j,1}, \quad 0 \leq j \leq m-2, \\ D_t^{m-1}v_1(0) = g_{m-1,1} - D_s^{m-1}\tilde{w}_m(0). \end{cases}$$

By Propositions 2.1 and 2.2,  $v_1(t)$  is determined by  $\{f_0(t), f_1(t)\}$ ,  $\{g_{j,0}; 0 \leq j \leq m\}$  and  $\{g_{j,1}; 0 \leq j \leq m-1\}$ . Then,  $D_t^m v_1(0)$  is given by  $\{f_0(0), D_t f_0(0), f_1(0)\}$ ,  $\{g_{j,0}; 0 \leq j \leq m\}$  and  $\{g_{j,1}; 0 \leq j \leq m-1\}$ . By induction on  $p$ , we have the assertion for  $D_t^{m+p}v_1(0, x)$  for  $p \geq 1$ .

Since  $\tilde{P}^{(1)}(s, x, D_s, D_x)\tilde{w}_m$  is an exponential polynomial in  $s$  of degree at most 1,  $\tilde{w}_{m+1}(s)$  is of degree at most 2 by Lemma 2.1. The coefficients  $h_j^{(1)}$ 's are determined by  $g_{m,0}$ ,  $D_t^m v_0(0)$ ,  $g_{m,1}$  and  $D_t^m v_1(0)$ . Hence, they are determined by  $\{f_0(0), D_t f_0(0), f_1(0)\}$  and  $\{g_{j,k}; 0 \leq j \leq m, k=0, 1\}$ .

When  $n=N \geq 1$ , we assume the proposition is valid for  $n=1, 2, \dots, N-1$ .  $v_N(t)$  is given by  $f_N(t)$ ,  $\{v_j(t); 0 \leq j \leq N-1\}$ ,  $\{g_{j,N}; 0 \leq j \leq m-1\}$  and  $\{D_t^j \tilde{w}_{N+j}(0); 0 \leq j \leq m-1\}$  through the equation (2.10) with (2.13). By the assumption of induction,  $\{v_j(t); 0 \leq j \leq N-1\}$  are determined by  $\{f_j(t); 0 \leq j \leq N-1\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-2\}$  and  $\{g_{j,N-1}; 0 \leq j \leq m-1\}$ . On the other hand,  $\{D_s^j \tilde{w}_{N+j}(0); 0 \leq j \leq m-1\} = \{D_s^{m+k-N} \tilde{w}_{m+k}; \max\{0, N-m\} \leq k \leq N-1\}$ , whose elements are, by induction, determined by  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$  and  $\{D_t^l f_q(0); \max\{0, N-m\} \leq k \leq N-1, 0 \leq q \leq k, 0 \leq l \leq k-q\}$ . The last family is  $\{D_t^l f_q(0); 0 \leq l+q \leq N-1\}$ . Hence, we have proved the assertions (ii) concerning  $v_N(t)$ .

We now prove the assertion concerning the trace  $D_t^{m+p}v_N(0)$ . First of all,  $\{D_t^j v_N(0); 0 \leq j \leq m-1\}$  are given by  $\{g_{j,N}; 0 \leq k \leq m-1\}$  and  $\{D_s^j \tilde{w}_{N+j}(0); 0 \leq j \leq m-1\}$ . Hence, they are given by  $\{D_t^l f_q(0); 0 \leq l+q \leq N-1\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$  and  $\{g_{j,N}; 0 \leq j \leq m-1\}$ . Next, from (2.10),  $D_t^m v_N(0)$  is given by  $f_N(0)$ ,  $\{D_t^j v_N(0); 0 \leq j \leq m-1\}$  and  $\{D_t^j v_k(0); 0 \leq j \leq m+1, 0 \leq k \leq N-1\}$ . By induction, it is given by  $\{D_t^l f_q(0); 0 \leq l+q \leq N\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$  and  $\{g_{j,N}; 0 \leq j \leq m-1\}$ . Therefore, the desired assertion is proved for  $p=0$ . When  $p \geq 1$ , from (2.10),  $D_t^{m+p}v_N(0)$  is given by  $D_t^p f_N(0)$ ,  $I = \{D_t^j v_N(0); 0 \leq j \leq m+p-1\}$  and  $J = \{D_t^j v_k(0); 0 \leq j \leq m+p+1, 0 \leq k \leq N-1\}$ . By induction on  $p$ ,  $I$  is given by  $D_t^p f_N(0)$ ,  $\{D_t^l f_q(0); 0 \leq q \leq N, 0 \leq l \leq p-1+N-q\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$  and  $\{g_{j,N}; 0 \leq j \leq m-1\}$ .  $J$  is given by  $\{D_t^l f_q(0); 0 \leq q \leq N-1, 0 \leq l \leq p+N-q\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-2\}$  and  $\{g_{j,N-1}; 0 \leq j \leq m-1\}$ . Hence, we obtain the desired result. Thus, (ii) is proved.

$\tilde{w}_{m+N}(s)$  satisfies

$$\begin{cases} (D_s - i\mu) D_s^m \tilde{w}_{m+N}(s) = e^{-\mu s} \sum_{j=0}^{2N-1} b_{2N-1-j}(is)^j, \\ D_s^m \tilde{w}_{m+N}(0) = g_{m,N} - D_t^m v_N(0). \end{cases}$$

By induction,  $\{b_{2N-1-j}; 0 \leq j \leq 2N-1\}$  are determined by  $\{D_t^l f_q(0); 0 \leq q \leq N-1, 0 \leq l \leq N-1-q\}$  and  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$ . Since  $D_t^m v_N$  is given by  $\{D_t^l f_q(0); 0 \leq q+l \leq N\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$  and  $\{g_{j,N}; 0 \leq j \leq m-1\}$ , we obtain the desired result. Q.E.D.

### § 3. Higher order estimates of $v_n$ and $w_n$ .

In § 2, we have constructed a formal series solution. We will give in this section precise estimates of each term. We use the same notation of norms as in [U]. (Review of notation:  $\|\cdot\|_k$  is the  $H^k(\mathbf{R}_x^d)$ -norm, and for  $H^p(\mathbf{R}_x^d)$ -valued functions  $f(t)$ ,  $\|D_t^p f(t)\|^2 = \sum_{k=0}^p \|D_t^{p-k} f(t)\|_{k*}^2$ .)

We always assume (H1). We consider

$$(3.1) \quad \begin{cases} M^{(0)}(t, x, D_t, D_x) v(t, x) = f(t, x), \\ D_t^j v(0, x) = g_j(t, x), \quad 0 \leq j \leq m-1, \end{cases}$$

where  $f(t, x) \in C_0^\infty([0, \infty) \times \mathbf{R}^d)$ ,  $g_j \in C_0^\infty(\mathbf{R}^d)$ . Then, we have

**PROPOSITION 3.1.** *For any nonnegative integer  $p$ , any real  $r$ , there*

exist positive constants  $c_0, C, \gamma_0$  such that for any positive  $T$  and any  $\gamma \geq \gamma_0$ , we have

$$(3.2) \quad \int_0^T e^{-2\gamma t} \|D^p f(t)\|_r^2 dt + C\gamma \|D^{m+p-1}v(0)\|_r^2 \\ \geq c_0\gamma^2 \int_0^T e^{-2\gamma t} \|D^{m+p-1}v(t)\|_r^2 dt + c_0\gamma e^{-2\gamma T} \|D^{m+p-1}v(T)\|_r^2$$

and

$$(3.3) \quad \|D^{m+p-1}v(0)\|_r^2 \leq C \left( \sum_{j=0}^{m-1} \|g_j\|_{m-1-j+p+r}^2 + \|D^{p-1}f(0)\|_r^2 \right).$$

PROOF. A known estimate (e.g. Corollary 2.1 in [U]) gives (3.2) for  $p=0$ . By induction on  $p$ , we obtain (3.2) easily for general  $p$ . In fact, we assume (3.2) for  $p-1$ . From the equations

$$M^{(0)}(D_t v) = D_t f - [D_t, M^{(0)}]v,$$

$$M^{(0)}(\Lambda v) = \Lambda f - [\Lambda, M^{(0)}]v,$$

$$(\text{where } \Lambda = (1 - \Delta)^{1/2},)$$

we have

$$\int_0^T e^{-2\gamma t} \{ \|D^{p-1}(D_t f(t) - [D_t, M^{(0)}]v(t))\|_r^2 \\ + \|D^{p-1}(\Lambda f(t) - [\Lambda, M^{(0)}]v(t))\|_r^2 \} dt \\ + C\gamma [\|D^{m+p-2}D_t v(0)\|_r^2 + \|D^{m+p-2}\Lambda v(0)\|_r^2] \\ \geq c_0\gamma^2 \int_0^T e^{-2\gamma t} \{ \|D^{m+p-2}D_t v(t)\|_r^2 + \|D^{m+p-2}\Lambda v(t)\|_r^2 \} dt \\ + c_0\gamma e^{-2\gamma T} \{ \|D^{m+p-2}D_t v(T)\|_r^2 + \|D^{m+p-2}\Lambda v(T)\|_r^2 \}.$$

Since the commutator terms are absorbed in the right hand side, we have (3.2) for  $p$ .

Differentiating the equation (3.1), we have (3.3). In fact, we can apply Lemma 3.1 in [U] to this case replacing  $m$  by  $m-1$ ,  $\varepsilon$  by 1 and  $q$  by  $r$ . Q.E.D.

We prepare  $L^2$ -estimates of exponential polynomials.  $\mu(x)$  denotes  $m_0(0, x; 0)$  as we defined in § 2. From (E1),  $\operatorname{Re} \mu(x) \geq \delta > 0$ . We have an elementary

LEMMA 3.1. Let  $w(t) = \left(\frac{it}{\varepsilon}\right)^j e^{-\mu(x)t/\varepsilon}$  for  $j \geq 0$ . Then,

$$(3.4) \quad \int_0^T e^{-2\gamma t} |w(t)|^2 dt \leq \frac{\Gamma(2j+1)}{(2\delta)^{2j+1}} \varepsilon.$$

$$\begin{aligned} \text{PROOF.} \quad \int_0^T e^{-2\gamma t} |w(t)|^2 dt &\leq \int_0^T e^{-2(\delta/\varepsilon + \gamma)t} \left(\frac{t}{\varepsilon}\right)^{2j} dt \\ &\leq \frac{\varepsilon^{-2j}}{\left(2\left(\frac{\delta}{\varepsilon} + \gamma\right)\right)^{2j+1}} \int_0^\infty e^{-\sigma} \sigma^{2j} d\sigma. \end{aligned} \quad \text{Q.E.D.}$$

LEMMA 3.2. Let  $\tilde{w}(s, x) = e^{-\mu(x)s} \sum_{j=0}^J h_{J-j}(x) (is)^j$ . Then, for any  $k, l, \alpha$ , there exists a positive constant  $C$  independent from  $\varepsilon, T, \gamma$  such that

$$(3.5) \quad \int_0^T e^{-2\gamma t} \|[s^k D_s^l D_x^\alpha \tilde{w}(s)]_{s=t/\varepsilon}\|^2 dt \leq C \varepsilon \sum_{j=0}^J \|h_j\|_{|\alpha|}^2,$$

and

$$(3.6) \quad \|D_s^l D_x^\alpha \tilde{w}(0)\|_r^2 \leq C \sum_{j=0}^{\min\{l, J\}} \|h_{J-j}\|_{|\alpha|+r}^2.$$

$$\begin{aligned} \text{PROOF.} \quad \|(s^k D_s^l D_x^\alpha \tilde{w}(s))\|^2 &\leq \sum_{j=0}^J \|s^k D_x^\alpha (h_{J-j} D_s^l (is)^j e^{-\mu s})\|^2 \\ &\leq \sum_{j=0}^J \sum_{a=0}^l \binom{l}{a}^2 \|s^k (D_s^{l-a} (is)^j) D_x^\alpha (h_{J-j} (i\mu)^a e^{-\mu s})\|^2 \\ &\leq \sum_{j=0}^J \sum_{a=0}^l \sum_{q=0}^{|\alpha|} C_{j,a,q} (s^{k+\max\{j-l+a, 0\}+|\alpha|-q} e^{-\delta s})^2 \|h_{J-j}\|_q^2, \end{aligned}$$

since  $\mu(x)$  has bounded derivatives on  $R^d$ . Hence, by Lemma 3.1, we have (3.5). (3.6) follows in a similar way. Q.E.D.

We give estimates of  $\tilde{w}_m$ . We put

$$h_0^{(0)}(x) = \{g_{m,0}(x) - D_t^m v_0(0, x)\} (i\mu(x))^{-m}.$$

Then,  $\tilde{w}_m(s, x) = e^{-\mu(x)s} h_0^{(0)}(x)$  by Proposition 2.2.

PROPOSITION 3.2. For any real  $r$ , there exists a positive constant  $C$  such that

$$(3.7) \quad \|h_0^{(0)}\|_r^2 \leq C \left\{ \sum_{j=0}^m \|g_{j,0}\|_{m-j+r}^2 + \|f_0(0)\|_r^2 \right\}.$$

PROOF. It follows easily from Proposition 3.1 applied to (2.9) and (2.15). Q.E.D.

COROLLARY. Put  $w_m(t, x; \varepsilon) = \tilde{w}\left(\frac{t}{\varepsilon}, x\right)$ . Then, for any non negative

integer  $p$  and any real  $r$ , there exist a positive constant  $C$  independent from  $T, \gamma, \varepsilon$  such that

$$\int_0^T e^{-2\gamma t} \|D^p w_m(t; \varepsilon)\|_r^2 dt \leq C \varepsilon^{-2p+1} \sum_{k=0}^p \varepsilon^{2k} \left\{ \sum_{j=0}^m \|g_{j,0}\|_{m-j+k+r}^2 + \|f_0(0)\|_{k+r}^2 \right\},$$

and

$$\|D^p w_m(0; \varepsilon)\|_r^2 \leq C \varepsilon^{-2p} \sum_{k=0}^p \varepsilon^{2k} \left\{ \sum_{j=0}^m \|g_{j,0}\|_{m-j+k+r}^2 + \|f_0(0)\|_{k+r}^2 \right\}.$$

REMARK. The negative powers of  $\varepsilon$  come only from differentiation with respect to  $t$ , since  $D_x(e^{-\mu(x)t/\varepsilon})$  is bounded, when  $\varepsilon \rightarrow +0$  for  $t \in [0, \infty)$ .

We estimate  $\{v_n\}_{n \geq 1}$  and  $\{\tilde{w}_{m+n}\}_{n \geq 1}$  by induction on  $n$ .

THEOREM 3.1. Let  $n \geq 1$ . Suppose  $v_n(t, x)$  and  $\tilde{w}_{m+n}(s, x) = e^{-\mu(x)s} \sum_{j=0}^{2n} h_{2n-j}^{(n)}(x)(is)^j$  are the functions constructed in § 2.

(i) For any integer  $p$ , any real  $r$ , there exist positive constants  $C, c_0$  and  $\gamma_0$  such that for any small  $\varepsilon, \gamma \geq \gamma_0, T > 0$

$$\begin{aligned} (3.8) \quad & \int_0^T e^{-2\gamma t} \left\{ \|D^p f_n(t)\|_r^2 + \sum_{k=1}^n \sum_{j=1}^k \gamma^{-2j} \|D^{p+2j} f_{n-k}(t)\|_r^2 \right\} dt \\ & + C \sum_{k=0}^n \gamma^{1-2k} \left\{ \sum_{j=0}^{m-1} \|g_{j,n-k}\|_{m-1-j+p+2k+r}^2 + \sum_{j=0}^{n-k} \|D^{p+j-1+2k} f_{n-j-k}(0)\|_r^2 \right. \\ & \left. + \sum_{q=1}^{n-k} \sum_{j=0}^m \|g_{j,n-q-k}\|_{m-j+2k+q-1+p+r}^2 \right\} \\ & \geq c_0 \gamma^2 \int_0^T e^{-2\gamma t} \|D^{m+p-1} v_n(t)\|_r^2 dt + c_0 \gamma e^{-2\gamma T} \|D^{m+p-1} v_n(T)\|_r^2 \end{aligned}$$

and

$$\begin{aligned} (3.9) \quad & \|D^{m+p-1} v_n(0)\|_r^2 \leq C \left\{ \sum_{j=0}^{m-1} \|g_{j,n}\|_{m-1-j+p+r}^2 + \sum_{j=0}^n \|D^{p-1+j} f_{n-j}(0)\|_r^2 \right. \\ & \left. + \sum_{q=1}^n \sum_{j=0}^m \|g_{j,n-q}\|_{m-j+q-1+p+r}^2 \right\}. \end{aligned}$$

(ii) For any  $r \geq 0$ , there exists a positive constant  $C$  such that

$$\begin{aligned} (3.10) \quad & \max_{0 \leq j \leq 2n} \|h_j^{(n)}\|_r^2 \\ & \leq C \sum_{q=0}^n \left\{ \sum_{j=0}^m \|g_{j,n-q}\|_{m-j+q+r}^2 + \sum_{j=0}^{n-q} \|D^j f_{n-q-j}(0)\|_{q+r}^2 \right\}. \end{aligned}$$

PROOF. First of all, we prove (i) when  $n=1$ .  $v_1(t, x)$  is defined by

(2.10) and (2.13):

$$\begin{cases} M^{(0)}v_1(t) = f_1(t) - (iL^{(0)} + M^{(1)})v_0(t) \\ D_t^j v_1(0) = g_{j,1}, \quad 0 \leq j \leq m-2 \\ D_t^{m-1} v_1(0) = g_{m-1,1} - D_s^{m-1} \tilde{w}_m(0). \end{cases}$$

From Proposition 3.1, we have

$$(3.11) \quad \begin{aligned} & \int_0^T e^{-2\gamma t} \|D^p(f_1 - (iL^{(0)} + M^{(1)})v_0(t))\|_r^2 dt + C\gamma \|D^{m+p-1}v_1(0)\|_r^2 \\ & \leq c_0\gamma^2 \int_0^T e^{-2\gamma t} \|D^{m+p-1}v_1(t)\|_r^2 dt + c_0\gamma e^{-2\gamma T} \|D^{m+p-1}v_1(T)\|_r^2, \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} \|D^{m+p-1}v_1(0)\|_r^2 & \leq C \left( \sum_{j=0}^{m-1} \|g_{j,1}\|_{m-1-j+p+r}^2 \right. \\ & \left. + \|D_s^{m-1}\tilde{w}_m(0)\|_{p+r}^2 + \|D^{p-1}f_1(0)\|^2 + \|D^{p+m}v_0(0)\|_r^2 \right). \end{aligned}$$

We have the desired estimates from Proposition 3.1, Lemma 3.2 and Proposition 3.2.

Next, assuming that the assertion (i) is true up to a natural number  $n$  and that the assertion (ii) is true up to  $n-1$ , we prove at first the assertion (ii) for  $n$ . By (2.12) and (2.13),  $\tilde{w}_{m+n}(s, x) = e^{-\mu(x)s} \sum_{j=0}^{2n} h_{2(n)-j}^{(n)}(x) (is)^j$  satisfies

$$(3.13) \quad \begin{cases} (D_s - i\mu) D_s^m \tilde{w}_{m+n}(s, x) = - \sum_{k=1}^n \tilde{P}^{(k)}(s, x, D_s, D_x) \tilde{w}_{m+n-k}(s, x) \\ D_s^m \tilde{w}_{m+n}(0, x) = g_{m,n}(x) - D_t^m v_n(0, x). \end{cases}$$

We have

$$\begin{aligned} & - \sum_{k=1}^n \tilde{P}^{(k)} w_{m+n-k}(s) = - \sum_{k=1}^n \sum_{j=0}^{2(n-k)} \tilde{P}^{(k)}(h_{2(n-k)-j}^{(n-k)}(is)^j e^{-\mu s}) \\ & = e^{-\mu s} \sum_{j=0}^{2n-1} b_{2n-1-j}^{(n)} (is)^j. \end{aligned}$$

Here,  $b_j^{(n)} = \sum_{k=1}^n \sum_{l=0}^{2(n-k)} Q_{j,l}^{(k)}(h_l^{(n-k)})$ ,

and  $Q_{j,l}^{(k)}$  is a differential operator in  $x$  of order at most  $\min\{k, m+1\}$ .

By Lemma 2.1 and the assumption of induction, we have

$$\max_{0 \leq j \leq 2n} \|h_j^{(n)}\|_r^2$$

$$\begin{aligned}
&\leq C \left\{ \sum_{k=1}^n \max_{0 \leq j \leq 2(n-1)} \|h_j^{(n-k)}\|_{k+r}^2 + \|g_{m,n} - D_j^m v_n(0)\|_r^2 \right\} \\
&\leq C \sum_{k=1}^n \sum_{q=0}^{n-k} \left\{ \sum_{j=0}^m \|g_{j,n-k-q}\|_{m-j+q+k+r}^2 \right. \\
&\quad \left. + \sum_{j=0}^{n-k-q} \|D^j f_{n-k-q-j}(0)\|_{q+k+r}^2 \right\} + \|g_{m,n}\|_r^2 \\
&\quad + C \left\{ \sum_{j=0}^{m-1} \|g_{j,n}\|_{m-j+r}^2 + \sum_{j=0}^n \|D^j f_{n-j}(0)\|_r^2 \right. \\
&\quad \left. + \sum_{q=1}^n \sum_{j=0}^m \|g_{j,n-q}\|_{m-j+q+r}^2 \right\}.
\end{aligned}$$

In the right hand side, the first multiple sum  $\sum \sum \{\dots\}$  can be absorbed in the rest, which is estimated by

$$C_2 \sum_{q=0}^n \left\{ \sum_{j=0}^m \|g_{j,n-q}\|_{m-j+q+r}^2 + \sum_{j=0}^{n-q} \|D^j f_{n-j}(0)\|_r^2 \right\}.$$

Thus, the assertion (ii) for  $n$  is proved.

Finally, we prove the assertion (i) for  $n+1$ . Then,  $v_{n+1}(t, x)$  satisfies

$$\begin{aligned}
(3.14) \quad &\begin{cases} M^{(0)} v_{n+1}(t) = f_{n+1}(t) - \sum_{a=0}^n (iL^{(a)} + M^{(a+1)}) v_{n-a}(t), \\ D_t^j v_{n+1}(0) = g_{j,n+1} - D_s^j \tilde{w}_{n+1+j}(0), \quad j=0, 1, \dots, m-1. \end{cases}
\end{aligned}$$

The desired estimates will be derived from Proposition 3.1. To do that, we have to estimate the quantity

$$\begin{aligned}
&\int_0^T e^{-2\gamma t} \|D^p f_{n+1}(t) - D^p \sum_{a=0}^n (iL^{(a)} + M^{(a+1)}) v_{n-a}(t)\|_r^2 dt \\
&\quad + C\gamma \|D^{m+p-1} v_{n+1}(0)\|_r^2,
\end{aligned}$$

which is estimated, from (3.3) applied to (3.14), by

$$\begin{aligned}
(3.15) \quad &\int_0^T e^{-2\gamma t} \|D^p f_{n+1}(t)\|_r^2 dt + C \sum_{a=0}^n \int_0^T e^{-2\gamma t} \|D^{p+m+1} v_{n-a}(t)\|_r^2 dt \\
&\quad + C\gamma \left\{ \sum_{j=0}^{m-1} \|g_{j,n+1}\|_{m-1-j+p+r}^2 + \sum_{j=0}^{m-1} \|D_s^j \tilde{w}_{n+1+j}(0)\|_{m-1-j+p+r}^2 \right. \\
&\quad \left. + \|D^{p-1} f_{n+1}(0)\|_r^2 + \sum_{a=0}^n \|D^{m+p} v_{n-a}(0)\|_r^2 \right\}.
\end{aligned}$$

Firstly, putting

$$I_1 = \sum_{a=0}^n \int_0^T e^{-2\gamma t} \|D^{p+m+1} v_{n-a}(t)\|_r^2 dt,$$

we have by induction

$$\begin{aligned}
 (3.16) \quad I_1 \leq & C \gamma^{-2} \sum_{a=0}^n \left[ \int_0^T e^{-2\gamma t} \left\{ \|D^{p+2} f_{n-a}(t)\|_r^2 \right. \right. \\
 & + \sum_{k=1}^{n-a} \sum_{j=1}^k \gamma^{-2j} \|D^{p+2+2j} f_{n-a-k}(t)\|_r^2 \Big\} dt \\
 & + \sum_{k=0}^{n-a} \gamma^{1-2k} \left\{ \sum_{j=0}^{m-1} \|g_{j, n-a-k}\|_{m-1-j+p+2(k+1)+r}^2 \right. \\
 & + \sum_{j=0}^{n-a-k} \|D^{p+j+1+2k} f_{n-a-j-k}(0)\|_r^2 \\
 & \left. \left. + \sum_{q=1}^{n-a-k} \sum_{j=0}^m \|g_{j, n-a-q-k}\|_{m-j+p+2(k+1)+q-1+r}^2 \right\} \right].
 \end{aligned}$$

We notice that the terms with  $a > 0$ , except the integrals  $\int_0^T e^{-2\gamma t} \|D^{p+2} f_{n-a}(t)\|_r^2 dt$ 's, are absorbed by the sum of the terms with  $a = 0$ . Thus, the sums of the integrals in (3.16) are reduced to

$$\begin{aligned}
 I_{11} &= C \int_0^T e^{-2\gamma t} \left\{ \sum_{a=0}^n \gamma^{-2} \|D^{p+2} f_{n-a}(t)\|_r^2 \right. \\
 &\quad \left. + \sum_{k=1}^n \sum_{j=1}^k \gamma^{-2(j+1)} \|D^{p+2(j+1)} f_{n+1-(k+1)}(t)\|_r^2 \right\} dt \\
 &= C \int_0^T e^{-2\gamma t} \sum_{k=1}^{n+1} \sum_{j=1}^k \gamma^{-2j} \|D^{p+2j} f_{n+1-k}(t)\|_r^2 dt.
 \end{aligned}$$

The rest of the R.H.S. of (3.16) is reduced to

$$\begin{aligned}
 I_{12} &= C \sum_{k=0}^n \gamma^{1-2(k+1)} \left\{ \sum_{j=0}^{m-1} \|g_{j, n-k}\|_{m-1-j+p+2(k+1)+r}^2 \right. \\
 &\quad + \sum_{j=0}^{n-k} \|D^{p+j-1+2(k+1)} f_{n-j-k}(0)\|_r^2 \\
 &\quad \left. + \sum_{q=1}^{n-k} \sum_{j=0}^m \|g_{j, n-q-k}\|_{m-1-j+p+2(k+1)+q+r}^2 \right\} \\
 &= C \sum_{k=1}^{n+1} \gamma^{1-2k} \left\{ \sum_{j=0}^{m-1} \|g_{j, n+1-k}\|_{m-1-j+p+2k+r}^2 \right. \\
 &\quad + \sum_{j=0}^{n+1-k} \|D^{p+j-1+2k} f_{n+1-j-k}(0)\|_r^2 \\
 &\quad \left. + \sum_{q=1}^{n+1-k} \sum_{j=0}^m \|g_{j, n+1-q-k}\|_{m-1-j+p+2k+q+r}^2 \right\}.
 \end{aligned}$$

$I_1$  is thus estimated by  $I_{11} + I_{12}$ .

Secondly, we estimate



$$I_2 = \sum_{j=0}^{m-1} \|D_s^j \tilde{w}_{n+1+j}(0)\|_{m-1-j+p+r}^2.$$

When  $n \leq m-1$ ,

$$\begin{aligned} I_2 &= \sum_{j=0}^n \|D_s^{m-n-1+j} \tilde{w}_{m+j}(0)\|_{n+p-j+r}^2 \\ &\leq C \sum_{j=0}^n \max_{0 \leq l \leq 2j} \|h_l^{(j)}\|_{n+p-j+r}^2 \\ &\leq C \sum_{j=0}^n \sum_{q=0}^j \left\{ \sum_{a=0}^m \|g_{a,j-q}\|_{m-a+q+n+p-j+r}^2 + \sum_{a=0}^{j-q} \|D^a f_{j-q-a}(0)\|_{q+n+p-j+r}^2 \right\}. \end{aligned}$$

The sum of terms with  $j=n$  absorbs the rest. Hence,

$$I_2 \leq C_1 \sum_{q=0}^n \left\{ \sum_{a=0}^m \|g_{a,n-q}\|_{m-a+q+p+r}^2 + \sum_{a=0}^{n-q} \|D^a f_{n-q-a}(0)\|_{q+p+r}^2 \right\}.$$

Putting  $q_1 = q+1$ , we have

$$\begin{aligned} (3.17) \quad I_2 &\leq C_2 \sum_{q=1}^{n+1} \left\{ \sum_{a=0}^m \|g_{a,n+1-q}\|_{m-a+q-1+p+r}^2 \right. \\ &\quad \left. + \sum_{a=0}^{n+1-q} \|D^a f_{n+1-q-a}(0)\|_{q-1+p+r}^2 \right\}, \end{aligned}$$

where we replace  $q_1$  by  $q$ .

When  $n > m-1$ ,

$$\begin{aligned} I_2 &= \sum_{j=0}^{m-1} \|D_s^j \tilde{w}_{n+1+j}(0)\|_{m-1-j+p+r}^2 \\ &= \sum_{j=0}^{m-1} \|D_s^j \tilde{w}_{m+(n-m+1+j)}(0)\|_{m-1-j+p+r}^2. \end{aligned}$$

We have in a similar way the same estimate as (3.17).

Lastly, we estimate

$$I_3 = \sum_{a=0}^n \|D^{m+p} v_{n-a}(0)\|_r^2.$$

By induction,

$$\begin{aligned} I_3 &\leq C \left\{ \sum_{a=0}^n \sum_{j=0}^{m-1} \|g_{j,n-a}\|_{m-j+p+r}^2 + \sum_{a=0}^n \sum_{j=0}^{n-a} \|D^{p+j} f_{n-a-j}(0)\|_r^2 \right. \\ &\quad \left. + \sum_{a=0}^{n-1} \sum_{q=1}^{n-a} \sum_{j=0}^m \|g_{j,n-a-q}\|_{m-j+q+r}^2 \right\}. \end{aligned}$$

The sum of terms with  $a=0$  absorbs the rest except the first multiple sum. Hence,

$$I_3 \leq C \sum_{a=0}^n \sum_{j=0}^{m-1} \|g_{j,n-a}\|_{m-j+p+r}^2$$

$$\begin{aligned}
& + C_1 \left\{ \sum_{j=0}^n \|D^{p+j} f_{n-j}(0)\|_r^2 + \sum_{q=1}^n \sum_{j=0}^m \|g_{j, n-q}\|_{m-j+p+q+r}^2 \right\} \\
& \leq C \sum_{q=1}^{n+1} \sum_{j=0}^{m-1} \|g_{j, n+1-q}\|_{m-j+p+r}^2 \\
& + C_1 \left\{ \sum_{j=1}^{n+1} \|D^{p-1+j} f_{n+1-j}(0)\|_r^2 + \sum_{q=2}^{n+1} \sum_{j=0}^m \|g_{j, n+1-q}\|_{m-j+p+q-1+r}^2 \right\}.
\end{aligned}$$

Combining the estimates of  $I_1, I_2, I_3$  to (3.15), we obtain (3.8) for  $n+1$ .

In this reasoning, we already obtain (3.9), since from (3.3) applied to (3.14),

$$\begin{aligned}
& \|D^{m+p-1} v_{n+1}(0)\|^2 \leq I_2 \\
& + I_3 + \sum_{j=0}^{m-1} \|g_{j, n+1}\|_{m-1-j+p+r}^2 + \|D^{p-1} f_{n+1}(0)\|_r^2. \quad \text{Q.E.D.}
\end{aligned}$$

#### § 4. Remainder estimates of asymptotic solutions.

We recall our problem. Under the assumptions (H0), (E1), (H1) and (S0) in § 1,  $u(t, x; \varepsilon)$  is the unique solution to the problem

$$(4.1) \quad \begin{cases} ((i\varepsilon)L + M)u(t, x; \varepsilon) = f(t, x; \varepsilon), \\ D_t^j u(0, x; \varepsilon) = g_j(x; \varepsilon), \quad 0 \leq j \leq m, \end{cases}$$

where  $f(t, x; \varepsilon) \in C_0^\infty([0, \infty) \times \mathbf{R}^d \times [0, \varepsilon_0])$  and  $g_j(x; \varepsilon) \in C_0^\infty(\mathbf{R}^d \times [0, \varepsilon_0])$ . Using  $\{v_n\}_{n \geq 0}$ ,  $\{\tilde{w}_{m+n}\}_{n \geq 0}$  constructed in § 2, we define the partial sum

$$u_N(t, x; \varepsilon) = \sum_{n=0}^N \varepsilon^n v_n(t, x) + \sum_{n=m}^{N+m} \varepsilon^n \tilde{w}_n\left(\frac{t}{\varepsilon}, x\right)$$

and its remainder term

$$R_{N+1}(u; \varepsilon) = u(t, x; \varepsilon) - u_N(t, x; \varepsilon).$$

Our main result is

**THEOREM 4.1.** *For any  $p, N \in \mathbf{N}$ , there exist positive constants  $\gamma_0$  and  $C_{p,N}$  such that for any  $\gamma \geq \gamma_0$  and for any positive  $\varepsilon \leq \varepsilon_0$ ,*

$$\begin{aligned}
(4.2) \quad & C_{p,N} (1 + \varepsilon \gamma) \varepsilon^{2(N+1)+1} \\
& \geq \gamma \int_0^T e^{-2\gamma t} \varepsilon^{2p} (\varepsilon \|D^{m+p} R_{N+1}(u; \varepsilon)(t)\|^2 \\
& + \gamma \|D^{m+p-1} R_{N+1}(u; \varepsilon)(t)\|^2) dt \\
& + e^{-2\gamma T} \varepsilon^{2p} (\varepsilon \|D^{m+p} R_{N+1}(u; \varepsilon)(T)\|^2 + \gamma \|D^{m+p-1} R_{N+1}(u; \varepsilon)(T)\|^2).
\end{aligned}$$

REMARK. The constant  $C_{p,N}$  depends on data  $\{f_j; 0 \leq j \leq N\}$ ,  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ . Dependence can be analyzed by the results in § 3. (See the proof below.)

COROLLARY. For any  $k, N_0 \in \mathbb{N}$  and any positive  $T$ , there exists  $N_1 \in \mathbb{N}$  such that for any  $N \geq N_1$  there exists a positive constant  $C_N$  independent of  $\varepsilon$ ,

$$(4.3) \quad \sup_{\substack{0 \leq t \leq T \\ x \in \mathbb{R}^n}} \sum_{j+|\alpha| \leq k} |D_t^j D_x^\alpha R_{N+1}(u; \varepsilon)(t, x)| \leq C_N \varepsilon^{N_0}.$$

In order to estimate  $R_{N+1}(u; \varepsilon)$  by Theorem 1.1, we prepare

PROPOSITION 4.1.  $R_{N+1}(u; \varepsilon)$  satisfies

$$(4.4) \quad \begin{aligned} & ((i\varepsilon)L + M)R_{N+1}(u; \varepsilon) \\ &= R_{N+1}(f; \varepsilon) + \varepsilon^{N+1}\rho(t, x; \varepsilon) + \varepsilon^{N+1}\tilde{\chi}\left(\frac{t}{\varepsilon}, x; \varepsilon\right), \end{aligned}$$

$$(4.5) \quad \begin{aligned} & D_t^j R_{N+1}(u)(0, x; \varepsilon) = R_{N+1}(g_j; \varepsilon) - \varepsilon^{N+1}\eta_j(x; \varepsilon), \\ & 0 \leq j \leq m, \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} \rho(t, x; \varepsilon) &= \sum_{\substack{q+r \geq N \\ 0 \leq q \leq N-1 \\ 1 \leq r \leq N}} \varepsilon^{q+r-N} iL^{(q)} v_r + \sum_{\substack{q+r \geq N+1 \\ 1 \leq q, r \leq N}} \varepsilon^{q+r-N-1} M^{(q)} v_r \\ &\quad - \sum_{r=0}^N \varepsilon^r (i\varepsilon^{-N} R_N(L; \varepsilon) + \varepsilon^{-N-1} R_{N+1}(M; \varepsilon)) v_r, \end{aligned}$$

$$(4.7) \quad \begin{aligned} \tilde{\chi}(s, x; \varepsilon) &= i \left\{ \sum_{\substack{q+r \geq N+1 \\ 1 \leq q, r \leq N}} \varepsilon^{q+r-N-1} \tilde{P}^{(q)}(s, x, D_s, D_x) \tilde{w}_{m+r}(s, x) \right. \\ &\quad \left. + \sum_{r=0}^N \varepsilon^{r-N-1} R_{N+1}(\tilde{P}; \varepsilon) \tilde{w}_{m+r}(s, x) \right\}. \end{aligned}$$

$$(4.8) \quad \begin{cases} \eta_j(x; \varepsilon) = \sum_{n=N+1}^{N+m-j} \varepsilon^{n-N-1} D_s^j \tilde{w}_{n+j}(0, x), \\ \quad \quad \quad 0 \leq j \leq m-1 \\ \eta_m(x; \varepsilon) = 0. \end{cases}$$

PROOF. We obtain (4.4) and (4.5) by straight computation from construction of  $\{v_n\}$  and  $\{\tilde{w}_{m+n}\}$  defined by (2.9), (2.10) and (2.13). Q.E.D.

## PROOF OF THEOREM 4.1.

(i). By the assumptions on  $f(t, x; \varepsilon)$ , we have

$$\int_0^T e^{-2\gamma t} \|D^p R_{N+1}(f; \varepsilon)(t)\|^2 dt \leq \frac{C_{p,N}(f)}{\gamma} \varepsilon^{2(N+1)},$$

and

$$\|D^j R_{2(N+1)}(f; \varepsilon)(0)\|^2 \leq C_{j,N}(f) \varepsilon^{2(N+1)}.$$

$$(ii). \quad \int_0^T e^{-2\gamma t} \|D^p \rho(t)\|^2 dt \leq C_N \int_0^T e^{-2\gamma t} \sum_{q=0}^N D^{m+1+p} v_q(t) \|^2 dt.$$

By (3.8) in Theorem 3.1, we have

$$\begin{aligned} & \int_0^T e^{-2\gamma t} \|D^p \rho(t)\|^2 dt \leq C_N \sum_{q=0}^N \left[ \gamma^{-2} \int_0^T e^{-2\gamma t} \left\{ D^{p+2} f_q(t) \|^2 \right. \right. \\ & \quad + \sum_{k=1}^q \sum_{j=1}^k \gamma^{-2j} \|D^{p+2+2j} f_{q-k}(t)\|^2 \Big\} dt \\ & \quad + C \sum_{k=0}^q \gamma^{-1-2k} \left\{ \sum_{j=0}^{m-1} \|g_{j,q-k}\|_{m-1-j+p+2+2k}^2 \right. \\ & \quad + \sum_{j=0}^{q-k} \|D^{p+1+j+2k} f_{q-j-k}(0)\|^2 \\ & \quad \left. \left. + \sum_{r=1}^{q-k} \sum_{j=0}^m \|g_{j,q-r-k}\|_{m-j+p+2k+r+1}^2 \right\} \right] \\ & \leq \frac{C_{p,N}}{\gamma}. \end{aligned}$$

$C_{p,N}$  depends on  $\{f_j; 0 \leq j \leq N\}$ ,  $\{g_{j,k}; 0 \leq j \leq m-1, 0 \leq k \leq N\}$  and  $\{g_{m,k}; 0 \leq k \leq N-1\}$ , but does not on  $\varepsilon$ .

(3.9) in Theorem 3.1 shows

$$\begin{aligned} & \|D^j \rho(0; \varepsilon)\|^2 \leq C \sum_{q=j}^N \|D^{m+1+j} v_q(0)\|^2 \\ & \leq C \sum_{q=1}^N \left\{ \sum_{k=0}^{m-1} \|g_{k,q}\|_{m-k+j+1}^2 + \sum_{k=0}^q D^{j+1+k} f_{q-k}(0) \|^2 \right. \\ & \quad \left. + \sum_{r=1}^q \sum_{k=0}^m \|g_{k,q-r}\|_{m-k+j+r+1}^2 \right\} \\ & \leq C_{j,N}. \end{aligned}$$

$C_{j,N}$  depends on  $\{f_j; 0 \leq j \leq N\}$ ,  $\{g_{j,k}; 0 \leq j \leq m-1, 0 \leq k \leq N\}$  and  $\{g_{m,k}; 0 \leq k \leq N-1\}$ , but does not on  $\varepsilon$ .

(iii). Putting  $\chi(t, x; \varepsilon) = \tilde{\chi}\left(\frac{t}{\varepsilon}, x; \varepsilon\right)$ , we have

$$\begin{aligned}
\varepsilon^{2p} \|D^p \chi(t, \cdot; \varepsilon)\|^2 &= \sum_{k=0}^p \varepsilon^{2k} \|D_s^{p-k} \tilde{\chi}(s, \cdot; \varepsilon)\|_k^2 \\
&\leq \sum_{k=0}^p \varepsilon^{2k} \left\{ \sum_{\substack{q+r \geq N+1 \\ 1 \leq q, r \leq N}} \varepsilon^{2(q+r-N-1)} \|D_s^{p-k} \tilde{P}^{(q)} \tilde{w}_{m+r}(s)\|_k^2 \right. \\
&\quad \left. + \sum_{r=0}^N \varepsilon^{2(r-N-1)} \|D_s^{p-k} R_{N+1}(\tilde{P}; \varepsilon) \tilde{w}_{m+r}(s)\|_k^2 \right\}.
\end{aligned}$$

By Lemma 3.2 and Theorem 3.1,

$$\begin{aligned}
&\varepsilon^{2p} \int_0^T e^{-2\gamma t} \|D^p \chi(t, x; \varepsilon)\|^2 dt \\
&\leq C_{p,N} \varepsilon \sum_{r=1}^N \sum_{j=0}^{2r} \|h_j^{(r)}\|_{m+1+p}^2 \\
&\leq C_{p,N} \varepsilon \sum_{r=1}^N \sum_{q=0}^r \left\{ \sum_{k=0}^m \|g_{k,r-q}\|_{m-j+q+m+1+p}^2 \right. \\
&\quad \left. + \sum_{j=0}^{r-q} \|D^j f_{r-q-j}(0)\|_{q+m+1+p}^2 \right\} \\
&\leq C'_{p,N} \varepsilon.
\end{aligned}$$

$C'_{p,N}$  depends on  $\{f_j; 0 \leq j \leq N\}$  and  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ , but does not depend on  $\varepsilon$ .

Similarly,

$$\begin{aligned}
\varepsilon^{2j} \|D^j \chi(0; \varepsilon)\|^2 &= \sum_{k=0}^j \varepsilon^{2k} \|D_s^{j-k} \tilde{\chi}(0; \varepsilon)\|_k^2 \\
&\leq C_{j,N} \sum_{k=0}^j \varepsilon^{2k} \sum_{r=1}^N \sum_{q=0}^{2r} \|h_q^{(r)}\|_{k+m+1}^2 \\
&\leq C_{j,N}^{(1)} \sum_{r=1}^N \sum_{q=0}^{2r} \|h_q^{(r)}\|_{j+m+1}^2 \\
&\leq C_{j,N}^{(2)} \sum_{r=1}^N \sum_{q=0}^r \left\{ \sum_{k=0}^m \|g_{k,r-q}\|_{m-k+q+j+m+1}^2 \right. \\
&\quad \left. + \sum_{k=0}^{r-q} \|D^k f_{r-q-k}(0)\|_{q+j+m+1}^2 \right\}, \\
&\leq C_{j,N}^{(3)}.
\end{aligned}$$

$C_{j,N}^{(3)}$  depends on  $\{f_j; 0 \leq j \leq N\}$  and  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ , but does not depend on  $\varepsilon$ .

$$\begin{aligned}
\text{(iv).} \quad &\|D^m R_{N+1}(u; \varepsilon)(0)\|_j^2 = \sum_{k=0}^m \|D_t^{m-k} R_{N+1}(u; \varepsilon)(0)\|_{j+k}^2 \\
&\leq \sum_{k=0}^m \left\{ \|R_{N+1}(g_{m-k}; \varepsilon)\|_{j+k}^2 + \varepsilon^{2(N+1)} \|\eta_{m-k}(\cdot; \varepsilon)\|_{j+k}^2 \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon^{2(N+1)} \left\{ C_{j,N}^{(1)} + \sum_{k=1}^m \sum_{n=N+1}^{N+k} \varepsilon^{2(n-N-1)} \|D_s^{m-k} \tilde{w}_{m+n-k}(0)\|_{j+k}^2 \right\} \\
&\leq \varepsilon^{2(N+1)} \left\{ C_{j,N}^{(1)} + C_{j,N}^{(2)} \sum_{k=1}^m \sum_{n=\max\{k, N+1\}}^{N+k} \sum_{r=0}^{\min\{m-k, 2(n-k)\}} \|h_{2(n-k)-r}^{(n-k)}\|_{j+k}^2 \right\} \\
&\leq \varepsilon^{2(N+1)} \left[ C_{j,N}^{(1)} + C_{j,N}^{(2)} \sum_{k=1}^m \sum_{n=\max\{k, N+1\}}^{N+k} \sum_{q=0}^{n-k} \left\{ \sum_{\nu=0}^m \|g_{\nu, n-k-q}\|^2_{m-\nu+q+j+k} \right. \right. \\
&\quad \left. \left. + \sum_{\nu=0}^{n-k-q} \|D^\nu f_{n-k-\nu-q}(0)\|_{q+j+k}^2 \right\} \right] \\
&\leq C_{j,N} \varepsilon^{2(N+1)}.
\end{aligned}$$

$C_{j,N}$  depends on  $\{f_j; 0 \leq j \leq N\}$  and  $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ , but does not depend on  $\varepsilon$ .

From (i), (ii), (iii) and (iv), we obtain (4.2) by Theorem 1.1. Q.E.D.

### Appendix

We perform elementary computation to obtain  $h_j$ 's in (2.18) from (2.19). For convenience, we define  $\binom{a}{b} = 0$ , if  $a < b$ . Differentiating (2.18), we have

$$\begin{aligned}
D_s^m \tilde{w} &= \sum_{j=0}^{p+1} h_{p+1-j} D_s^m (e^{-\mu s} (is)^j) \\
&= \sum_{j=0}^{p+1} h_{p+1-j} \sum_{k=0}^{p+1} \binom{m}{k} (i\mu)^{m-k} (is)^{j-k} e^{-\mu s} \frac{j!}{(j-k)!} \\
&= \sum_{l=0}^{p+1} \frac{(is)^l}{l!} \left\{ \sum_{j=l}^{p+1} h_{p+1-j} \binom{m}{j-l} j! (i\mu)^{m-j+l} \right\} e^{-\mu s} \\
&= e^{-\mu s} \left\{ \sum_{l=0}^p \frac{i}{l+1} b_{p-l} (is)^{l+1} + c \right\}.
\end{aligned}$$

Therefore, we have

$$(A1) \quad (i\mu)^m h_0 = \frac{ib_0}{p+1}.$$

When  $0 < l < p+1$ , we have

$$(A2) \quad (i\mu)^m h_{p+1-l} = \frac{i}{l} b_{p+1-l} - \frac{1}{l!} \sum_{k=0}^{p-l} (p+1-k)! \binom{m}{p+1-k-l} h_k (i\mu)^{m+k-p-1+l}.$$

When  $l=0$ , we have

$$(A3) \quad (i\mu)^m h_{p+1} = c - \sum_{j=1}^{p+1} j! \binom{m}{j} h_{p+1-j} (i\mu)^{m-j}.$$

Hence, by induction, we have the desired conclusion.

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