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An extension of Komatsu's second structure theorem for ultradistributions

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Abstract. It is shown that each ultradistribution can locally be represented by an ultradifferential operator applied to an ultradifferentiable function which is only slightly less regular than the test functions associated to the original ultradistribution.

1 Introduction. It is a classical result that, for all k, any distribution can locally be written as a differential operator applied to a function of class C^k. Komatsu's second structure theorem treats the same question in the case of ultradistributions: For a space $\mathcal{D}$ of ultradifferentiable test functions he shows that any ultradistribution $\mu\in\mathcal{D}'$ is locally given by an ultradifferential operator applied to a measure. Here, this result is extended in two ways: First, it is shown that μ can locally be represented by an ultradifferential operator applied to an ultradifferentiable function belonging to some class larger than $\mathcal{D}$. Second, the requirement that $\mathcal{D}$ be strictly non-quasianalytic is replaced by plain non-quasianalyticity. To be precise, we are dealing with classes of ultradifferentiable functions in the sense of Braun, Meise, and Taylor [2]. A comparison of some theories of ultradistributions is given in Section 8 of [2]. The main tool of the present paper is Theorem 7, where a result of Rubel and Taylor is applied to construct an entire function with certain growth properties.

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2 Weight functions. An increasing function $\omega\colon [0,\infty[\to[0,\infty[$ is called a weight function if it satisfies

$$(\alpha)\quad \omega(2t)=O(\omega(t)),\qquad (\beta)\quad \int_1^\infty \frac{\omega(t)}{t^2}dt<\infty,$$

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(γ) $\log t = o(\omega(t))$, (δ) $\varphi = \omega \circ \exp$ is convex.

We extend ω to $\boldsymbol{C}^N$ by requiring $\omega(z) = \omega(|z|)$.

3 Ultradistributions. For φ as in 2(δ) we set

$$\varphi^*(y) = \sup_{x \geq 0} xy - \varphi(x), \quad y \geq 0.$$

For open $K \subset \boldsymbol{R}^N$ and $l > 0$ we define

$$\mathcal{E}_\omega^l(K) = \left\{ f \in C^\infty(K) \,\middle|\, \sup_{x \in K} \sup_{\alpha \in N_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{l} \varphi^*(l|\alpha|) \right) < \infty \right\}.$$

For open $\Omega \subset \boldsymbol{R}^N$ we set

$$\mathcal{E}_{\{\omega\}}(\Omega) = \operatorname*{proj}_{K \subset\subset \Omega} \operatorname*{ind}_{l \to \infty} \mathcal{E}_\omega^l(K), \quad \mathcal{E}_{(\omega)}(\Omega) = \operatorname*{proj}_{K \subset\subset \Omega} \operatorname*{proj}_{\varepsilon \to 0} \mathcal{E}_\omega^\varepsilon(K),$$
$$\mathcal{D}_{\{\omega\}}(\Omega) = \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N) \cap \mathcal{D}(\Omega), \quad \mathcal{D}_{(\omega)}(\Omega) = \mathcal{E}_{(\omega)}(\boldsymbol{R}^N) \cap \mathcal{D}(\Omega).$$

The elements of $\mathcal{D}_{\{\omega\}}(\Omega)$ and $\mathcal{D}_{(\omega)}(\Omega)$, respectively, are called ultradifferentiable functions of Roumieu type and of Beurling type, respectively. The elements of the corresponding dual spaces are called ultradistributions. All spaces are equipped with their natural topologies, dual spaces carry the strong topology, denoted by X_b'. For details we refer to Braun, Meise, and Taylor [2]. There it is shown that the condition of non-quasianalyticity, 2(β), implies that $\mathcal{D}_{\{\omega\}}(\Omega)$ and $\mathcal{D}_{(\omega)}(\Omega)$ are non trivial. For $\omega(t) = t^d$, $0 < d < 1$, the class $\mathcal{E}_{\{\omega\}}$ is the classical Gevrey class of exponent $1/d$.

4 Ultradifferential operators. Suppose that $G \in H(\boldsymbol{C}^N)$ satisfies for some $k > 0$

$$\log|G(z)| \leq k(1 + \omega(z)) \quad \text{for all } z \in \boldsymbol{C}^N,$$

then we define an ultradistribution T_G by

$$\langle T_G, f \rangle = \frac{1}{(2\pi)^N} \int_{\boldsymbol{R}^N} G(t) \hat{f}(t) \, dm_N(t) = \sum_{\alpha \in N_0^N} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(0), \quad f \in \mathcal{D}_{(\omega)}(\boldsymbol{R}^N).$$

By [2], 3.5, the first representation shows that $T_G \in \mathcal{D}_{(\omega)}(\boldsymbol{R}^N)'$, the second one shows that T_G is supported by the origin, thus by [2], 5.3, T_G is in fact in $\mathcal{E}_{(\omega)}(\boldsymbol{R}^N)'$. We define

$$G(D) \colon \mathcal{D}_{(\omega)}(\boldsymbol{R}^N)' \to \mathcal{D}_{(\omega)}(\boldsymbol{R}^N)', \quad G(D)\mu = T_G * \mu.$$

$G(D)$ is continuous by [2], 6.3. We say that $G(D)$ is an ultradifferential operator of class (ω). If $G \in H(C^N)$ satisfies $\log|G| = o(\omega)$, then we define an ultradifferential operator $G(D)$ of class $\{\omega\}$ in an analogous way. Following Chou [3], we say that an ultradifferential operator is elliptic if there is $\varepsilon > 0$ with $|\mathrm{Im}\, z| \geq \varepsilon |z|$ for all $|z| > 1/\varepsilon$ with $G(z) = 0$.

5 Notation. Let $(z_j)_j$ be a sequence in $\boldsymbol{C}\backslash\{0\}$ with $|z_j| \leq |z_{j+1}| \to \infty$. Then we let

$$n(r) = \#\{j \mid |z_j| \leq r\}, \quad N(r) = \int_0^r \frac{n(t)}{t} dt.$$

6 THEOREM (Rubel and Taylor [6], 5.2). *Let ω be a weight function. A sequence $(z_j)_j$ in $\boldsymbol{C}\backslash 0$ is the precise sequence of zeros of an entire function h with $\log|h(z)| = O(\omega(z))$ if and only if*

(*a*) $N(r) = O(\omega(r))$

and

(*b*) *there is $A > 0$ such that for all $k \in \boldsymbol{N}$, $0 < r < R$*

$$\frac{1}{k} \left| \sum_{r < |z_j| \leq R} \frac{1}{z_j^k} \right| \leq A \left(\frac{\omega(r)}{r^k} + \frac{\omega(R)}{R^k} \right).$$

7 THEOREM. *For each weight function ω there are an even entire function $f \in H(\boldsymbol{C})$ and a constant C satisfying*

$$\log|f(z)| \leq C\omega(z) + C \quad \textit{for all } z \in \boldsymbol{C}, \tag{1}$$
$$\log|f(x)| \geq \omega(x) \quad \textit{for all } x \in \boldsymbol{R},$$
$$|\mathrm{Im}\, z| > \frac{|z|}{C} \quad \textit{for all } z \textit{ with } f(z) = 0.$$

PROOF. We may assume that ω is C^1. For K with $\omega(2t) \leq K\omega(t)$, $t > 0$, we set

$$p = \min\{k \in N \mid k \text{ even}, \ 2^k > K\}.$$

For $j \in N$ and $l = 0, \cdots, p-1$ we set

$$a_j = \min\{r > 0 \mid r\omega'(r) = pj\}, \quad \lambda_l = \exp\left((2l+1)\frac{\pi i}{p}\right), \quad z_{j,l} = \lambda_l a_j$$

The convexity of $\omega\circ\exp$ implies that $r\mapsto r\omega'(r)$ is non decreasing. Thus for $a_j\le r<a_{j+1}$

$$pj=a_j\omega'(a_j)\le r\omega'(r)\le a_{j+1}\omega'(a_{j+1})=p(j+1).$$

On the other hand, $pj=n(r)$, thus

$$n(r)\le r\omega'(r)\le n(r)+p.$$

This implies

$$N(r)=\int_0^r\frac{n(t)}{t}dt\le\int_0^r\frac{t\omega'(t)}{t}dt\le\omega(r), \tag{2}$$

$$N(r)\ge\int_1^r\frac{t\omega'(t)-p}{t}dt=\omega(r)-\omega(1)-p\log r, \tag{3}$$

$$n(t)\le\int_t^{et}\frac{n(s)}{s}ds\le N(et)\le\omega(et)\le K^2\omega(t). \tag{4}$$

For $k=1,\cdots,p-1$ we have $\sum_{l=0}^{p-1}\lambda_l^{-k}=0$, and thus

$$\sum_{r<|z_{j,l}|\le R}\frac{1}{z_{j,l}^k}=0\quad\text{for } 0<r<R,\, k<p. \tag{5}$$

By (4), we have $n(t)=O(\omega(t))$. By the monotonicity of ω and 2(β) this imples $n(t)=o(t)$, thus the sequence $(z_{j,l})_{j,l}$ is of genus zero or one (see Boas [1], 2.5.5). Hence the canonical product

$$f(z)=\prod_{j=1}^\infty\prod_{l=0}^{p-1}\left(1-\frac{z}{z_{j,l}}\right)\exp\left(\frac{z}{z_{j,l}}\right)=\prod_{j=1}^\infty\left(1+\frac{z^p}{a_j^p}\right)$$

exists; here the second identity is a consequence of (5) and of the fact that the λ_l are the pth roots of -1. It is clear that f is even, has the required zeros, and satisfies

$$\max_{|z|=r}|f(z)|=f(r)=f(-r).$$

From this, Jensen's formula, and (3) we derive

$$\log f(\pm r)\ge\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{it})|dt=N(r)\ge\omega(r)-\omega(1)-p\log r.$$

We use Theorem 6 to show that there is an entire function h with zeros $(z_{j,l})_{j,l}$ that satisfies (1). Once we know the existence of at least one such function, it is not difficult to see that (1) holds for f, too. Hypothesis (a) of Theorem 6 is satisfied because of (2), while (5) implies (b) for $k=1, \cdots, p-1$. The case $k \geq p$ will be treated like in the proof of Rubel and Taylor [6], 3.5. The estimate (4) is applied several times.

$$\begin{aligned}
\frac{1}{k} \sum_{r<|z_j|\leq R} \frac{1}{|z_j|^k} &= \frac{1}{k}\int_r^R \frac{dn(t)}{t^k} = \frac{n(t)}{kt^k}\Big|_r^R + \int_r^R \frac{n(t)}{t^{k+1}} dt \\
&\leq \frac{K^2}{k}\frac{\omega(R)}{R^k} + K^2 \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} \frac{\omega(t)}{t^{k+1}} dt \\
&\leq \frac{K^2}{k}\frac{\omega(R)}{R^k} + K^2 \sum_{j=0}^{\infty} \omega(2^{j+1} r) \int_{2^j r}^{2^{j+1} r} \frac{dt}{t^{k+1}} \\
&= \frac{K^2}{k}\frac{\omega(R)}{R^k} + K^2 \sum_{j=0}^{\infty} \omega(2^{j+1} r) \frac{2^k - 1}{kr^k 2^{k(j+1)}} \\
&\leq \frac{K^2}{k}\frac{\omega(R)}{R^k} + K^2 \sum_{j=0}^{\infty} K^{j+1} \omega(r) \frac{1}{kr^k 2^{kj}} \\
&= \frac{K^2}{k}\frac{\omega(R)}{R^k} + \frac{K^3}{k}\frac{\omega(r)}{r^k}\frac{1}{1-K/2^k}.
\end{aligned}$$

Thus, by 6 there is an entire function h with $\log|h(z)| = O(\omega(z))$ and precise sequence of zeros given by $(z_{j,l})_{j,l}$. If we assume $h(0)=1$, then Hadamard's factorization theorem (Boas [1], 2.7.1) implies the existence of $b \in \boldsymbol{C}$ with $h(z) = e^{bz} f(z)$. This yields the estimate from above for f since

$$\log|f(z)| = \log\sqrt{|h(z)h(-z)|} = O(\omega(z)).$$

The next theorem corresponds to Komatsu's second structure theorem [4], 10.3, for the case of weight functions in the sense of 2.

8 THEOREM. *For any $\mu \in \mathcal{E}_{(\omega)}(\boldsymbol{R}^N)'$ and any $l>0$ there are an elliptic ultradifferential operator $G(D)$ of class (ω) and $f \in \mathcal{E}_\omega^l(\boldsymbol{R}^N)$ with $\mu = G(D)f$.*

PROOF. By an appropriate version of the Paley-Wiener theorem (Braun, Meise, and Taylor [2], 7.4), there are C and k with

$$|\hat{\mu}(t)| \leq C \exp(k\omega(t)) \quad \text{for all } t \in \boldsymbol{R}^N. \tag{6}$$

By Theorem 7, there is $F \in H(\boldsymbol{C})$ even, with no zeros in a conic neighborhood of $\boldsymbol{R}\backslash\{0\}$, satisfying

$$(1+k+l)\omega(x) \leq \log|F(x)| \quad \text{for all } x \in \boldsymbol{R}. \tag{7}$$

Since F is even, we can define $G \in H(\boldsymbol{C}^N)$ by

$$G(z) = F(\sqrt{z_1^2 + \cdots + z_N^2}).$$

It is clear that $G(D)$ is an elliptic ultradifferential operator of class (ω). For $t \in \boldsymbol{R}^N$ we have $\log|G(t)| = \log|F(|t|)| \geq (1+k+l)\omega(t)$, thus for $\lambda \in \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)'$

$$\int_{R^N} \left| \frac{\hat{\mu}(-t)}{G(t)} \hat{\lambda}(t) \right| dm_N(t) \leq C \int_{R^N} \exp(-(1+l)\omega(t)) |\hat{\lambda}(t)| dm_N(t).$$

By [2], 7.4, this shows that the following defines an element f of $(\mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)_b')'$, which coincides with $\mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)$ by [2], 4.9:

$$\langle f, \lambda \rangle = \frac{1}{(2\pi)^N} \int_{R^N} \frac{\hat{\mu}(-t)}{G(t)} \hat{\lambda}(t) dm_N(t), \quad \lambda \in \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)'.$$

For $\phi \in \mathcal{D}_{(\omega)}(\boldsymbol{R}^N)$ we have by [2], 7.2 (with $\langle \check{T}, \phi \rangle = \langle T, \phi(-\cdot) \rangle$)

$$\begin{aligned}
\langle \mu, \phi \rangle &= \frac{1}{(2\pi)^N} \int_{R^N} \hat{\mu}(-t) \hat{\phi}(t) dm_N(t) \\
&= \frac{1}{(2\pi)^N} \int_{R^N} \frac{\hat{\mu}(-t)}{G(t)} G(t) \hat{\phi}(t) dm_N(t) \\
&= \frac{1}{(2\pi)^N} \int_{R^N} \frac{\hat{\mu}(-t)}{G(t)} \widehat{\check{T}_G * \phi}(t) dm_N(t) \\
&= \langle f, \check{T}_G * \phi \rangle = \langle G(D) f, \phi \rangle.
\end{aligned}$$

It remains to prove $f \in \mathcal{E}_\omega^l$

$$\begin{aligned}
|f^{(\alpha)}(x)| &= \frac{1}{(2\pi)^N} \left| \int_{R^N} \frac{\hat{\mu}(-t)}{G(t)} (-it)^\alpha e^{-i\langle x, t \rangle} dm_N(t) \right| \\
&\leq \frac{C}{(2\pi)^N} \int_{R^N} \exp(-(l+1)\omega(t)) |t|^{|\alpha|} dm_N(t) \\
&\leq \frac{C}{(2\pi)^N} \sup \exp(-l\omega(\gamma) + |\alpha| \log \gamma) \int_{R^N} \exp(-\omega(t)) dm_N(t) \\
&= C' \exp\left(l \sup_{x \in R} \left(\frac{|\alpha|}{l} x - \varphi(x) \right) \right) \\
&\leq C' \exp\left(l \max\left(\varphi^*\left(\frac{|\alpha|}{l} \right), 0 \right) \right)
\end{aligned}$$

$$\leq C' \exp\left(l\varphi(0) + l\varphi^*\left(\frac{|\alpha|}{l}\right)\right).$$

9 COROLLARY. *Let $\Omega \subset \boldsymbol{R}^N$ open, $K \subset \Omega$ compact, $\nu \in \mathcal{D}_{(\omega)}(\Omega)'$, and $l > 0$ be given. There are an ultradifferential operator $G(D)$ of class (ω) and $h \in \mathcal{E}_\omega^l$ such that*

$$\operatorname{Supp} h \subset \Omega \quad \textit{and} \quad \operatorname{Supp}(\nu - G(D)h) \subset \Omega \backslash K.$$

PROOF. Choose $\chi \in \mathcal{D}_{(\omega)}(\Omega)$ with $\chi(t) = 1$ for all $t \in K$, apply Theorem 8 to $\mu = \chi\nu$ and suitable l' depending on χ, and set $h = \chi f$.

10 COROLLARY. *For any $\mu \in \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)'$ and each weight $\sigma = o(\omega)$, there are an elliptic ultradifferential operator $G(D)$ of class $\{\omega\}$ and $f \in \mathcal{E}_\sigma^1$ with $\mu = G(D)f$.*

PROOF. By Braun, Meise, and Taylor [2], 7.6, there is a weight ρ with $\rho = o(\omega)$ and $\mu \in \mathcal{E}_{(\rho)}(\boldsymbol{R}^N)' \subset \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)'$. Enlarging σ if necessary, we may assume because of [2], 1.9, that $\rho = o(\sigma)$. We apply 8 with σ in the place of ω. We get $T_G \in \mathcal{E}_{(\sigma)}(\boldsymbol{R}^N)' \subset \mathcal{E}_{\{\omega\}}(\boldsymbol{R}^N)'$. Thus $G(D)$ is of class $\{\omega\}$.

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