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Existence of symmetric capillary surfaces via curvature evolution

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Abstract. The curvature flow is applied to the capillarity problem. The existence of symmetric capillary surfaces is proved rather easily by this method.

1. Introduction and results

Let us consider a capillary tube in a gravity field over a bounded domain $\Omega\subset \boldsymbol{R}^n$. We assume that the fluid surface contained in the tube can be described as a graph $x_{x+1}=u(x_1, \cdots, x_n)$. It is well known that u verifies the equation

$$\text{(1)} \qquad \operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}}=\pm\kappa u \qquad \text{in } \Omega$$

with the boundary condition

$$\text{(2)} \qquad \frac{N\cdot Du}{\sqrt{1+|Du|^2}}=\cos\gamma \qquad \text{on } \partial\Omega,$$

where Du denotes the gradient of u and N is the unit outer normal on $\partial\Omega$. Here $\kappa=\rho g/\sigma$ (>0) represents the capillarity constant with ρ, the density difference across the surface, g, the gravitational acceleration and σ, the surface tension. The contact angle γ is always assumed to satisfy

$$0<\gamma\leq\frac{\pi}{2}.$$

The character of the equation (1) is quite different according to the sign of the term κu. We call $+\kappa u$ case as the positive gravitational field and $-\kappa u$ as the negative one.

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This capillarity problem has been attracted many mathematicians and several existence theorems under various situation are already known. We recall, for instance, results of Concus and Finn [1][2], Finn [5], Finn and Gerhardt [7], Gerhardt [8][9], Giusti [11][12][13], Huisken [14], Simon and Spruck [18] and Ural'tseva [21]. For further references and other information we refer the reader to the excellent text of Finn [6].

All the above works discuss directly the equilibrium problem. Some of them deal with the minimization of the functional

$$J(u)=\int_{\Omega}\sqrt{1+|Du|^2}\,dx\pm\frac{1}{2}\int_{\Omega}\kappa u^2dx-\int_{\partial\Omega}\cos\gamma u ds \tag{3}$$

in the class of bounded variations $BV(\Omega)$. See for instance Massari and Miranda [17; § 3.8].

In this paper, on the other hand, we attack the problem (1)(2) by the curvature flow method; one considers the gradient flow of the functional (3). Although there exist many kind of gradient flows for (3), recent investigation on surfaces moving by their mean curvature suggests that the following type of evolution equation is appropriate:

$$\frac{du}{dt}=\sqrt{1+|Du|^2}\left(\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}}\mp\kappa u\right) \quad \text{in } \Omega\times(0,\ T). \tag{4}$$

Here and hereafter we assume that $\Omega=B_R$ is the ball centered at the origin with radius R. Moreover we only disscus the radial solution of (4).

One advantage of employing the evolution method is that we can obtain the smooth solution directly; if one minimizes (3) in $BV(\Omega)$ one must discuss separately the regularity of the minimizer.

Remark that this normal velocity equation is different from that investigated in Gerhardt [10] or Lichnewsky and Temam [16] by the factor $\sqrt{1+|Du|^2}$. While Huisken [15], Ecker and Huisken [3][4] and Tso [19][20], also partly because of the geometrical reasons, consider the above normal velocity equation. Both evolutions are gradient flows for (3), of course. To our knowledge, however, they all did not seem to discuss the application of this method to the capillarity problem.

We now state our main results. First, as is easily seen, the case of positive gravitational field is rather easy. We have the following

THEOREM A. *Suppose the symmetric* $u_0 \in (B_R)$ *satisfy* $\frac{u_{0r}}{\sqrt{1+(u_{0r})^2}}(R) = \cos\gamma$. *Then the evolution problem*

$$\frac{du}{dt} = \sqrt{1+u_r^2}\left(\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{\sqrt{1+u_r^2}}\right)_r - \kappa u\right) \quad in\ B_R \times (0, \infty) \tag{5}$$

$$\frac{u_r}{\sqrt{1+u_r^2}} = \cos\gamma \quad on\ \partial B_R \times (0, \infty) \tag{6}$$

$$u(\cdot, 0) = u_0 \tag{7}$$

has a smooth solution u and as $t \to \infty$, *u converges to the unique solution of*

$$\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{\sqrt{1+u_r^2}}\right)_r = \kappa u \quad in\ B_R \tag{8}$$

$$\frac{u_r}{\sqrt{1+u_r^2}} = \cos\gamma \quad on\ \partial B_R\,. \tag{9}$$

It is to be noted that the existence theorem for (8)(9) is essentially already known. See for instance Finn [6; Chapter 3], Gerhardt [8]. However, our proof is very elementary and short, compared with the above works.

Next, the case of negative gravitational field needs some consideration. Since it seems to be difficult to handle it generally we content ourselves with finding a solution, using fixed point argument effectively. Let $M > 3R\cot\gamma$, $N > \cot\gamma$ be constants determined later and set

$$E_{M,N} = \left\{ u \in C^1(B_R);\ u(x) = u(r),\ \max|u|M,\ \max|u_r| \leq N,\right.$$
$$\left. \frac{u_r}{\sqrt{1+u_r^2}}(R) = -\frac{\kappa}{R^{n-1}}\int_0^R r^{n-1}u(r)dr = \cos\gamma \right\},$$

where $r = |x|$. If $\kappa > n\cos\gamma/RM$ then there holds $E_{M,N} \neq \phi$. We want to construct a solution in the class $E_{M,N}$. For that purpose choose any $f \in E_{M,N}$ and consider the problem

$$\frac{du}{dt} = \sqrt{1+u_r^2}\left(\frac{1}{r^{n-1}}\left(\frac{r^{n-1}\,u_r}{\sqrt{1+u_r^2}}\right)_r + \kappa f\right) \quad \text{in } B_R \times (0, \infty) \tag{10}$$

(11) $$\frac{u_r}{\sqrt{1+u_r^2}}=\cos\gamma \quad \text{on } \partial B_R\times(0,\ \infty)$$

(12) $$u(\cdot,\ 0)=u_0\ ,$$

where u_0 is assumed to satisfy

$$\frac{u_{0r}}{\sqrt{1+u_{0r}^2}}(R)=\cos\gamma,\ \max|u_{0r}|\leq\frac{M}{3}\ ,\ \max|u_{0r}|\leq N\ .$$

We show that for sufficiently small κ, (10)(11)(12) has a global solution staying in the class $E_{M,N}$, taking M, N appropriately. Then one can define the mapping $T: E_{M,N}\rightarrow E_{M,N}$ which assigns f to $u_\infty:=\lim_{t\rightarrow\infty}u(\cdot,\ t)$. T turns out to be compact and hence it has a fixed point, which is a solution to the equilibrium problem. We thus obtain

THEOREM B. *Let* $\kappa_0=\min\left\{\frac{n\sin\gamma}{3R^2}\ ,\ \frac{n-1}{R^2}\sin\gamma\right\}$. *Then for all* κ *satisfying* $\kappa<\kappa_0$ *we have a solution to the problem*

(13) $$\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{\sqrt{1+u_r^2}}\right)_r=-\kappa u \quad in\ B_R$$

(14) $$\frac{u_r}{\sqrt{1+u_r^2}}=\cos\gamma \quad on\ \partial B_R\ .$$

Note that this kind of fixed point arguments is also used in Huisken [14] for the stationary problem in somewhat different manner and a related result is exhibited. His method of proof, however, is rather complicated than ours and one could not easily apply it to compute the constant κ_0 even in the radially symmetric case. In this context our Theorem B seems to be new and among other things the proof is transparent and short.

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2. Proof of Theorem A

The proof of Theorem A involves standard a priori estimates. First we give a bound for $|u|$. Let u be a smooth solution of (5)(6)(7). Define the comparison function:

$$\delta(x)=-\sqrt{R^2-|x|^2}+L \qquad x\in B_R\,,$$

where we set $L=\sup|u_0|+\frac{n}{\kappa R}+R+1$. Then it is easy to compute

$$\begin{aligned}\frac{d\delta}{dt}-\sqrt{1+|D\delta|^2}\Big(\operatorname{div}\frac{D\delta}{\sqrt{1+|D\delta|^2}}-\kappa\delta\Big)\\ =-\frac{R}{\sqrt{R^2-|x|^2}}\Big(\frac{n}{R}+\kappa\sqrt{R^2-|x|^2}-\kappa L\Big)\\ >0\,.\end{aligned}$$

The parabolic maximum principle implies

$$u\leq\delta \qquad \text{in } B_R\times[0,\,\infty).$$

The estimate for $-u$ will be derived similarly and we establish the desired bound for $|u|$.

Next we prove the estimate for the gradient $|u_r|$. To do so put $v=\sqrt{1+u_r^2}$. Observing that u_r satisfies

$$\begin{aligned}\frac{du_r}{dt}=\frac{u_{rrr}}{1+u_r^2}-\frac{2u_ru_{rr}^2}{(1+u_r^2)^2}-\frac{n-1}{r^2}u_r+\frac{n-1}{r}u_{rr}\\ -\frac{u_ru_{rr}}{\sqrt{1+u_r^2}}\kappa u-\sqrt{1+u_r^2}\,\kappa u_r\,,\end{aligned}$$

we have

$$\begin{aligned}\Big(\frac{d}{dt}-\Big(\frac{1}{1+u_r^2}\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}\Big)\Big)v\\ =-\frac{u_{rr}^2}{v^5}-\frac{2u_r^2u_{rr}^2}{v^5}-\frac{n-1}{r^2}\frac{u_r^2}{v}-\frac{u_r}{v}\kappa u\frac{dv}{dr}-\kappa u_r^2\,,\end{aligned}$$

and so

$$\Big(\frac{d}{dt}-\Big(\frac{1}{1+u_r^2}\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}-\frac{u_r}{v}\kappa u\frac{d}{dr}\Big)\Big)v\leq 0.$$

The parabolic maximum principle then leads

$$\sup v \leq \max\left\{\sup\sqrt{1+u_{0r}^2},\ 1/\sin\gamma\right\},$$

in view of the bound for $|u|$. Hence the desired bound for $|u_r|$ is obtained and standard results imply that (5)(6)(7) has a global in time solution.

Now it remains to show the convergence of u. To show this compute the time derivative of (3):

$$\begin{aligned}\frac{d}{dt}J(u) &= -\int_\Omega\left(\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}}-\kappa u\right)^2\sqrt{1+|Du|^2}\,dx \\ &\leq 0\,.\end{aligned}$$

Therefore in view of uniform estimates for $|u|$ and $|u_r|$ we obtain

$$\begin{aligned}&\int_0^\infty\int_\Omega\left(\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}}-\kappa u\right)^2\sqrt{1+|Du|^2}\,dx \\ &\qquad\qquad = J(u_0)-J(u_\infty)<\infty,\end{aligned}$$

from which we conclude that u converges uniformly to the solution of the equilibrium problem. This completes the proof.

Remark. Observe that the assumption of symmetry makes the discussion transparent. In fact when we deal with the non-symmetric case we have to do complicated computation in deriving the boundary gradient estimate, where no difficulty exists in the symmetric case. See for instance Simon and Spruck [18] in which the equilibrium problem is treated. As to the interior gradient estimate one can give a time independent bound easily by modifying the argument of Ecker and Huisken [4; Theorem 2.3], even in the non-symmetric case.

3. Proof of Theorem B

We prove the Theorem B by slightly modifying the argument of the proof of Theorem A. Let $M>3R\cot\gamma$, $N>\cot\gamma$ be taken and fixed so as to satisfy

$$\frac{n\cos\gamma}{RM}<\kappa<\frac{n}{R}\frac{1}{\sqrt{9R^2+M^2}}\,,\quad \kappa<\frac{n-1}{R^2}\frac{1}{\sqrt{1+N^2}}$$

and define $E_{M,N}$ as in the Introduction. Let u be a solution of (10)(11)

(12). We first give an a priori estimate for $|u|$. We consider the next comparison function:

$$\delta(r)=-\frac{2}{3}M-\kappa\int_0^r\frac{I(s)}{\sqrt{1-\kappa^2I(s)^2}}ds,$$

where $I(r)=r^{1-n}\int_0^r s^{n-1}f(s)$. Then we find

$$\frac{d\delta}{dt}-\sqrt{1+|D\delta|^2}\left(\operatorname{div}\frac{D\delta}{\sqrt{1+|D\delta|^2}}+\kappa f\right)$$
$$=0.$$

Since $|I(r)|\leq RM/n$, $\delta_r/\sqrt{1+\delta_r^2}(R)=\cos\gamma$, and $\max|u_0|\leq M/3$, the maximum principle yields

$$-\frac{2}{3}M-\frac{1}{3}M\leq\delta\leq u\quad\text{in } B_R\times(0,\infty),$$

if there holds $\kappa<\frac{n}{R}\frac{1}{\sqrt{9R^2+M^2}}$. An upper bround for u proceeds similarly using the comparison function

$$\delta(r)=\frac{2}{3}M-\kappa\int_0^r\frac{I(s)}{\sqrt{1-\kappa^2I(s)^2}}ds,$$

and hence we conclude that

$$\sup|u|\leq M\qquad\text{in } B_R\times(0,\infty).$$

Next we give an a priori gradient estimate for u. For this purpose set

$$w=\log v=\log\sqrt{1+u_r^2}.$$

Compute

$$\left(\frac{d}{dt}-\left(\frac{1}{v^2}\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}\right)\right)w$$
$$=-\frac{u_{rr}^2}{v^6}-\frac{2}{v^2}\left(\frac{dw}{dr}\right)^2-\frac{n-1}{r^2}\frac{u_r^2}{v^2}+\frac{1}{v^3}\frac{dw}{dr}$$
$$-\frac{u_r}{v}\frac{dw}{dr}\kappa f-\frac{u_r}{v}\kappa f_r$$

$$\leq -\frac{n-1}{r^2}\frac{u_r^2}{v^2}+\frac{1}{v^3}\frac{dw}{dr}-\frac{u_r}{v}\frac{dw}{dr}\kappa f-\frac{u_r}{v}\kappa f_r.$$

If there holds

$$\left|\frac{u_r}{v}\right| \leq \frac{N}{\sqrt{1+u_r^2}},$$

then we are done. Otherwise

$$\left(\frac{d}{dt}-\left(\frac{1}{v^2}\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{1}{v^3}\frac{d}{dr}-\frac{u_r}{v}\kappa f\right)\right)w$$

$$\leq -\frac{n-1}{r^2}\frac{u_r^2}{v^2}-\frac{u_r}{v}\kappa f_r<0$$

if we have $\kappa<\frac{n-1}{R^2}\frac{1}{\sqrt{1+N^2}}$, and again the maximum principle implies that

$$|u_r| \leq N \qquad \text{in } B_R\times(0,\ \infty).$$

In any case we establish the desired gradient bound.

Now we can define $T: E_{M,N}\rightarrow E_{M,N}$ by $Tf:=u_\infty$. Here we put $u_\infty=\lim_{t\to\infty} u(\cdot,t)$, since we see easily $u(\cdot,t)$ converges to the solution of the stationary problem, as in the proof of Theorem A. Standard parabolic regularity theory shows that T is compact and hence it has a fixed point. It is an easy matter to see that this fixed point is the solution of (13) (14). This completes the proof of Theorem B.

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