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Moishezon-Fano Threefolds of Index Three

By Iku NAKAMURA

Abstract. We consider an analogue of a Fano threefold of index three in the category of Moishezon spaces. This is by definition a (compact complex) Moishezon threefold with the first Betti number $b_1=0$ whose anticanonical line bundle c_1 is effective and divisible by three. We prove that if moreover its linear system $|c_1/3|$ is free from fixed components, it is isomorphic to either a smooth quadric hypersurface in P_C^4 or a certain P_C^2-bundle over P_C^1.

§ 0. Introduction.

This is a continuation of [N1], where we study threefolds with their first Chern class c_1 divisible by three and the second Betti number $b_2(X)$ equal to one. The purpose of the present article is to study threefolds with c_1 divisible by three and possibly with $b_2(X)\geq 2$ under certain mild conditions. This class of threefolds is an analogue of Fano threefolds of index three in the category of Moishezon spaces. A smooth quadric hypersurface Q_C^3 in P_C^4 is, up to isomorphism, the unique projective Fano threefold of index three [Is]. However there are many Moishezon threefolds with their first Chern class equal to $3c_1(D)$ for some nonample effective divisor D. Our consequence is summarized as follows.

THEOREM. *Any Moishezon threefold with the first Betti number b_1 equal to zero and with the first Chern class c_1 (the anti-canonical line bundle) divisible by three is isomorphic to either Q_C^3 or a P_C^2-bundle over P_C^1 if the linear system $|c_1/3|$ has no fixed components.*

We recall that any Moishezon threefold with c_1 divisible by at least four is isomorphic to P_C^3 under a similar assumption. See (2.1) and (5.1) in this article. The remaining case in the above theorem where the linear system $|c_1/3|$ has fixed components will be studied in [N4] under the stronger condition that the threefold is a global deformation of a P_C^2-bundle over P_C^1.

Part of our consequence in the present article was announced in [N3].

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Notation.

$\mathrm{Bs}\|L\|$	the scheme-theoretic base locus of $\|L\|$
$c_i(X)$	the i-th Chern class of X
$\mathcal{F}(a, b, c)$	$O_{P^1}(a)\oplus O_{P^1}(b)\oplus O_{P^1}(c)$
F_b	$\mathrm{Proj}(O_{P^1}(b)\oplus O_{P^1})$
$g^*\|L\|$	$\{g^*D;\ D\in \|L\|\}$
$h^q(X, F)$	$\dim H^q(X, F)$ for a coherent sheaf F
$\kappa(X, L)$	L-dimension of X [Ii]
$N_{C/N}$	the normal bundle of C in X
O_X, O_S, O_Z	the structure sheaf of X, S, Z respectively
$\hat{O}_X$	the formal completion of O_X
ω_X (or K_X)	the dualising sheaf (canonical line bundle) of X
$\omega_S, \omega_l, \omega_C$	the dualising sheaf of S, l, C respectively
$\omega_{X/\Delta}$	the relative dualising sheaf of X over Δ
$P(\mathcal{F}(a, b, c))$	$\mathrm{Proj}(\mathcal{F}(a, b, c))$
$\chi(X, F)$	$\sum_{q\in Z}(-1)^q h^q(X, F)$
$(\ )_S, (\ )_X$	the intersection numbers on S, X

§ 1. Some P^2-bundles over P^1.

In this section we work over an algebraically closed field k of any characteristic.

First we start with recalling some algebraic 3-folds with $c_1(X)=3c_1(L)$, that is, P^2-bundles over P^1. Choose integers $a\geq b\geq 0$ such that $a+b-2$ is divisible by 3. Let $3n=a+b-2\geq 0$. Let $\mathcal{F}:=\mathcal{F}(a, b, 0)=O_{P^1}(a)\oplus O_{P^1}(b)\oplus O_{P^1}$, $X=P(\mathcal{F})$ and let $\pi: X\to P^1$ be the natural projection. Let H be a tautological line bundle of X with $\pi_*H\simeq\mathcal{F}$. Then the canonical sheaf K_X of X is given by the formula,

$$K_X=-3H+\pi^*(\det\mathcal{F}+K_{P^1})=-3H+(a+b-2)F$$

where F is a fiber of π. Letting $L:=L(\mathcal{F})=H-nF$, we have $K_X=-3L$,

$L^3=\deg \pi_* L=2$. Since $\pi_* L \simeq \mathcal{F}\otimes O_{P1}(-n)$, and $R^q\pi_* L=0(q\geq 1)$, we have

$$H^q(X, L)\simeq H^q(\mathcal{F}\otimes O_{P1}(-n)) \quad (q\geq 0).$$

We see that $R^q\pi_*(-pL)=0(q\geq 0, p=1, 2)$, whence $H^q(X, -pL)=0$ for the same values of q and p. There are 3 cases.

Case 1. $n=0,\ a\geq b\geq 0.$
Case 2. $a\geq b\geq n\geq 1.$
Case 3. $a\geq n>b\geq 0.$

Case 1-1. Assume that $a=b=1$. Then $h^0(X, L)=5$ and $\mathrm{Bs}|L|=\phi$. The morphism $h_0: X\to \boldsymbol{P}^4$ associated with $|L|$ has a hyperquadric W with Hessian-rank 4 as its image. In fact, we can choose elements x_0, x_1(resp. x_2, x_3) from $H^0(O_{P1}(a-n)\oplus 0\oplus 0)$ (resp. $H^0(0\oplus O_{P1}(b-n)\oplus 0)$) such that $x_0x_3=x_1x_2$. h_0 is a small resolution of W whose exceptional set is $\boldsymbol{P}(0\oplus 0\oplus O_{P1})\simeq \boldsymbol{P}^1$ with normal bundle $\simeq O_{P1}(-1)\oplus O_{P1}(-1)$.

Case 1-2. Assume that $a=2$, $b=0$. Then $h^0(X, L)=5$ and $\mathrm{Bs}|L|=\phi$. The morphism $h_0: X\to \boldsymbol{P}^4$ associated with $|L|$ has a hyperquadric W with Hessian-rank 3 as its image. In fact, we can choose elements x_0, x_1 and x_2 from $H^0(O_{P1}(a)\oplus 0\oplus 0)$ such that $x_1^2=x_0x_2$. h_0 is a divisorial contraction whose exceptional set is $E:=\boldsymbol{P}(0\oplus O_{P1}(b)\oplus O_{P1})\simeq \boldsymbol{P}^1\times \boldsymbol{P}^1$. The restriction map $h_{0|E}: E\to \boldsymbol{P}^1$ is a $\boldsymbol{P}^1$-bundle whose fiber C has the normal bundle $N_{C/X}\simeq O_{P1}\oplus O_{P1}(-2)$.

Case 2. In this case, $h^0(X, L)=n+4$, $B:=\mathrm{Bs}|L|\simeq \boldsymbol{P}(0\oplus 0\oplus O_{P1})\simeq \boldsymbol{P}^1$. Since $\pi_* L\simeq \mathcal{F}\otimes O_{P1}(-n)$, any element of $H^0(X, L)$ is written as $s_0(x)y_0+s_1(x)y_1+s_2(x)y_2$ for some $(s_0, s_1, s_2)\in H^0(\mathcal{F}\otimes O_{P1}(-n))$ and suitable homogeneous coordinates y_i. In particular, $h^0(X, L)=a+b-2n+2=n+4$. Since $s_2(x)\equiv 0$, $B:=\mathrm{Bs}|L|=\{y_0=y_1=0\}\simeq \boldsymbol{P}^1$. Let D and D' be members of $|L|$ defined by

$$D: s=\sum_{i=0}^{2} s_i(x)y_i=0, \quad D': s'=\sum_{i=0}^{2} s_i'(x)y_i=0.$$

The intersection $l:=D\cap D'$ is the locus of $s=s'=0$. If four elements s_i and s_j' $(i, j=0, 1)$ have common zeroes $w\in \boldsymbol{P}^1$, then l contains a surface $\pi^{-1}(w)$. Assume that D and D' have no irreducible component in common. Let $\varDelta:=s_0s_1'-s_1s_0'\in H^0(O_{P1}(n+2))$. For any zero w of $\varDelta$, we have a line $C_w:=(\{s=s'=0\}\cap \pi^{-1}(w))_{\mathrm{red}}$ in $\pi^{-1}(w)$ with multiplicity $\mathrm{ord}_w \varDelta\geq 1$. Hence we have

$$l\simeq B+\sum_{w\in \mathrm{zero}(\Delta)}\mathrm{ord}_w(\Delta)\cdot C_w$$

where $\deg \Delta=n+2$. It is clear that $(LC_w)_X=1$. Hence $(LB)_X=(L^3)_X-(n+2)=-n$. Let $f: Y\to X$ be the blowing-up of X with B center, E the total transform of B, and $N:=f^*L-E$. We see also that $N_{B/X}\simeq O_B(-a)\oplus O_B(-b)$ and $E\simeq \boldsymbol{F}_{a-b}(a-b\geq 0)$.

Let $N_E:=N\otimes O_E$, *and* e_0 (resp. e_∞ or f_0) a section (resp. a section or a fiber) of $f_{|E}: E\to B$ with $(e_0^2)_E=a-b$ (resp. $(e_\infty^2)_E=-a+b$). Then we see

$$(f^*L)_E\simeq f^*(L_B)\simeq -nf_0,\quad N_E\simeq e_0+(b-n)f_0,\quad E_E\simeq -e_0-bf_0,$$

$$(N_E^2)_E=n+2,\ h^0(Y,N)=n+4,\ \mathrm{Bs}|N|=\phi,\ H^0(Y,N)\simeq H^0(E,N_E).$$

Let $\hat{C}_w$ be a proper transform of a line C_w in $F(\simeq \boldsymbol{P}^2)$ by f. Since C_w intersects B transversally at one point, $(E\hat{C}_w)_Y=1$ and $(N\hat{C}_w)_Y=0$. Hence the morphism $g: Y\to \boldsymbol{P}^{n+3}$ associated with $|N|$ has an image $g(Y)\simeq g(E)$. Since $(N_E^2)_E=n+2$ and $h^0(E,N_E)=n+4\geq 5$, the image $g(E)$ is a cone over a smooth variety of minimal degree. In fact, if $b>n$, then $g(E)\simeq E\simeq \boldsymbol{F}_{a-b}$ and Y is a $\boldsymbol{P}^1$-bundle over $g(E)$. If $b=n$, then $g_{|E}$ contracts e_∞ so that $g(E)$ is a cone over a smooth rational curve $g(e_0)$ of degree $n+2$ with $g(e_\infty)$ its vertex.

Case 3. In this case, $h^0(X,L)=a-n+1(\geq n+4)$, $B:=\mathrm{Bs}|L|\simeq \boldsymbol{P}(0\oplus O_{P1}(b)\oplus O_{P1})\simeq \boldsymbol{F}_b$ and $|L|=|(a-n)F|+B$. The image of a morphism h_0 associated with $|L|$ is $\boldsymbol{P}^1$. The natural projection π is the same as that associated with $|F|$.

Note. Some $\boldsymbol{P}^2$-bundles $\boldsymbol{P}(\mathcal{F})$ over a curve C of genus $g\geq 1$ can satisfy $c_1(X)=3c_1(L)$, where $\mathcal{F}$ is a locally free sheaf of rank 3 on C. However we have $h^1(X,O_X)=g\geq 1$.

§2. Threefolds with $K_X=-3L$.

In the sections 2-4 we work over an algebraically closed field k of any characteristic.

(2.1) PROPOSITION [N1, (A.1)]. (char $k\geq 0$) *Let X be an algebraic (or a Moishezon) 3-fold defined over k and L a line bundle on X. Assume that $h^1(X,O_X)=0$, $K_X=-aL$ for some integer $a\geq 4$ and $h^0(X,L)\geq 2$. Assume moreover that there is an irreducible reduced member in $|L|$. Then $X\simeq \boldsymbol{P}^3$.*

PROOF. We have $h^1(X, -pL)=0$ for $p\geq 0$ by the assumption. See (2.3). Hence $h^2(X, -pL)=h^1(X, -(a-p)L)=0$ for $p\leq a$. Therefore $h^q(X, -pL)=0$ for $1\leq p\leq a-1$. Since $\chi(X, -pL)$ is a cubic polynomial in $p, \chi(X, -pL)$ is identically zero if $a\geq 5$, which contradicts $\chi(X, O_X)=1$. Hence $a=4$ and we have $\chi(X, -pL)=(1-p)(2-p)(3-p)/6$, whence $L^3=1$. Since $h^2(X, L)=h^1(X, -5L)=0$, we have $h^0(X, L)\geq 4$ by Riemann-Roch theorem. Let l be a complete intersection of two general members of $|L|$. Then by the same argument as in [N1, (A.1)], we see that $l\simeq \boldsymbol{P}^1$. It follows that $X\simeq \boldsymbol{P}^3$. q.e.d.

See also (5.1). By (2.1) it is natural to study the case of index three. The sections 2-4 are devoted to proving

(2.2) THEOREM. (char $k\geq 0$) *Let X be a Moishezon 3-fold or an algebraic 3-fold defined over k (we assume projective if* char $k>0$) *and L a line bundle on X. Assume that $h^1(X, O_X)=0$, $K_X=-3L$ and $h^0(X, L)\geq 2$. Assume moreover that there is an irreducible reduced member in $|L|$. Then $X\simeq \boldsymbol{Q}^3$ or $\boldsymbol{P}(\mathcal{F}(a, b, 0))$ $(a\geq b\geq n\geq 0,\ a+b=3n+2)$.*

We note that some of $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ may have no irreducible reduced members in $|c_1/3|$ even when c_1 is divisible by three. See § 1. See also (5.2). The assumption of projectivity for char $k>0$ in (2.2) is necessary in the proof of (4.1) because Kodaira-Enriques classification of surfaces in char $k>0$ is available only in the projective case [Mu]. In what follows in the sections 2-4, we consider the 3-fold X satisfying the conditions in (2.2) Our proof of (2.2) is completed in (4.2).

(2.3) LEMMA. *Let $l:=D\cap D'$ for distinct members $D, D'\in |L|$. Assume that D and D' have no irreducible components in common. Then*

(2.3.1) $h^1(X, -pL)=0$ *for* $p\geq 1$,

(2.3.2) $h^2(X, -pL)=0$ *for* $0\leq p\leq 3$,

(2.3.3) $h^0(l, O_l)=1,\ h^1(l, O_l)=0$.

(2.3.4) $L^3=Ll=2,\ \chi(X, -pL)=\dfrac{1}{6}(1-p)(2-p)(3-2p)$.

PROOF. By the assumption a general member of $|L|$ is connected reduced. Choose a general member D of $|L|$. Then $h^0(O_D)=1$. We also see that $h^0(O_{D_1+\cdots+D_m})=1$ for general $D_i\in |L|$ and any positive integer m. Then the above assertions (2.3.1)-(2.3.3) follow easily. See [N1, (1.4) and (1.6)]. By (2.3.1) aud (2.3.2) we have $\chi(X, -pL)=0$ for $p=1, 2$, while $\chi(X, -pL)=1$ (resp. -1) for $p=0$ (resp. $p=3$). Since $\chi(X, -pL)$ is a cubic polynomial in p, we have (2.3.4). q.e.d.

(2.4) LEMMA [N1]. *Let* $l:=D\cap D'$ *as in* (2.3) and $B:=\mathrm{Bs}|L|$.

(2.4.1) *Any irreducible component of* l *is a smooth rational curve* C. *For the component* C, *one of the following is true if* C *is not contained in* B.

(2.4.1.1) $LC=2$, $N_{C/X}\simeq O_C(2)^{\oplus 2}$.
(2.4.1.2) $LC=1$, $N_{C/X}\simeq O_C\oplus O_C(1)$.
(2.4.1.3) $LC=0$, $N_{C/X}\simeq O_C\oplus O_C(-2)$ *or* $O_C(-1)^{\oplus 2}$.

(2.4.2) *For any pair* C, C' *of irreducible components of* l, C *and* C' *intersect nowhere or transversally at a point, while no triple of irreducible components of* l *meet at any point.*

(2.4.3) l *is one of the following.*

(2.4.3.1) $l\simeq C$ *and* $LC=2$.
(2.4.3.2) $l\simeq 2C$, $l_{\mathrm{red}}\simeq C$ *and* $LC=1$.
(2.4.3.3) $l\simeq 2C+C'$, $l_{\mathrm{red}}\simeq C+C'$, $LC=1$, $LC'=0$ *where* C *and* C' *intersect.*

(2.4.3.4) $l\simeq C_0+C_1+\cdots+C_m$ *with* $LC_0=LC_m=1$, $LC_i=0$ $(1\leq i\leq m-1)$ *where* C_{i-1} *and* C_i $(1\leq i\leq m)$ *intersect, while* $C_i\cap C_j=\phi$ *(otherwise).*

(2.4.3.5) $l\simeq m_1C_1+\cdots+m_rC_r+B$, $LC_i=1$, $LB=-(m_1+\cdots+m_r)+2<0$ *where* $B\simeq \boldsymbol{P}^1$, $C_i\cap C_j=\phi$ $(i\neq j)$, *while* C_i *and* B *intersect transversally at a point* p_i.

See [N1, (3.1)-(3.5)] for the precise structures of l. The assertions follow from the proofs of [N1, (3.1)-(3.5) and § 8]. Compare § 1.

(2.5) COROLLARY. $h^0(X, L)=5$ (resp. $-LB+4\geq 5$) *and* $B:=\mathrm{Bs}|L|=\phi$ (resp. $\boldsymbol{P}^1$).

PROOF: See [N1, (3.7) and (8.8)]. Note $B:=\mathrm{Bs}|L|=\phi$ except (2.4.3.5). q.e.d.

§ 3. The case where $B=\phi$.

(3.1) LEMMA. *Assume* $B=\phi$. *Let* $h: X\to \boldsymbol{P}^4$ *be a morphism associated with* $|L|$, $W:=h(X)$ *and* $d:=\deg W$. *Then* $\dim W=3$, $d=2$, *and* h *is a birational morphism of* X *onto an irreducible quadric hypersurface* W.

PROOF: Since $B=\phi$, we have $h^0(X, L)=5$ by (2.5). Hence we have a morphism $h: X\to \boldsymbol{P}^4$ associated with $|L|$. Then we have $d\geq 5-\dim W$.

If dim $W=1$, then $d=1$ by the irreducibility of a member of $|L|$, a contradiction. If dim $W=2$, then $d\geq 3$, which shows that a general complete intersection $l:=D\cap D'(D, D'\in|L|)$ has 3 movable irreducible components C_i with $LC_i\geq 1$. However there are no such cases by (2.4). Hence $\dim W=3$, $d=L^3=Ll=2$ by (2.5). q.e.d.

(3.2) LEMMA. *If W is smooth, then $X\simeq W\simeq \boldsymbol{Q}^3$.*

PROOF: By the assumptions, $K_X=-3L\simeq -3h^*H$ for a hyperplane H of W. Let ω be a meromorphic 3-form on W with its polar divisor $3H$. Then the pull back $h^*(\omega)$ is a meromorphic 3-form on X with $(h^*\omega)=K_X \simeq -3h^*H$. Hence $h^*\omega$ has no new zeroes caused by the vanishing of the Jacobian of h, whence h is unramified. Consequently $X\simeq W$ by (3.1). q.e.d.

The following lemma is more or less well-known.

(3.3) LEMMA. *If $W=\{(x_i)\in \boldsymbol{P}^4; f(x)=0\}$ is a singular quadric hypersurface in $\boldsymbol{P}^4$, then $f(x)=x_1^2-x_0x_2$ or $x_0x_3-x_1x_2$ by choosing suitable homogeneous coordinates of $\boldsymbol{P}^4$.*

PROOF: A singular quadric hypersurface W is a cone over either a smooth conic in $\boldsymbol{P}^2$ or a smooth quadric in $\boldsymbol{P}^3$. Since a smooth conic in $\boldsymbol{P}^2$ is a rational curve, its normal form is given by $f(x)=x_1^2-x_0x_2$. Therefore it suffices to prove that a smooth quadric surface Q in $\boldsymbol{P}^3$ is defined by $f(x):=x_0x_3-x_1x_2=0$. Let p be a point of Q, T_pQ the tangent plane of Q at p. We may normalize $p:(x_i)=(1,0,0,0)$ and $T_pQ: x_3=0$ by choosing suitable homogeneous coordinates x_i. Then the equation $f(x)$ defining Q is of the form

$$f(x)=(x_0+g(x_1, x_2, x_3))x_3+h(x_1, x_2).$$

Therefore by taking $x_0+g(x)$ instead of x_0 and by redefining x_1 and x_2 suitably, we have $f(x)=x_0x_3-x_1x_2$ as desired. q.e.d.

(3.4) LEMMA. *Assume that $B=\phi$ and* Sing $W\neq\phi$. *Then*

(3.4.1) *No divisor on X is contracted to a point by h.*

(3.4.2) *There is an irreducible curve C on X with $LC=0$. If $LC=0$ for an irreducible curve C on X, then it is a smooth rational curve with $N_{C/X}\simeq O_C(-1)^{\oplus 2}$ $O_C\oplus O_C(-2)$.*

(3.4.3) *W is the unique normal algebraic variety with the properties that*

(3.4.3.1) $X\backslash h^{-1}(\mathrm{Sing}\, W)\simeq W\backslash \mathrm{Sing}\, W$,

(3.4.3.2) $h^{-1}(q)$ *is a smooth rational curve* C *with* $LC=0$ *for any* $q\in \mathrm{Sing}\, W$.

PROOF: First we prove (3.4.1). Let H be a hyperplane of W. Assume that there is an irreducible divisor E on X such that $h(E)$ is a smooth point of W, say q. Let p be a point of E, (s, t, u) a coordinate system at p such that E is given by $s=0$. If q is a smooth point of W, then we choose a local coordinate y_i at q and write a local generator $\omega=dy_1\wedge dy_2\wedge dy_3$ of the dualising sheaf ω_W. Since $h(E)=\{p\}$, we may assume that $h^*y_i=s^{e_i}f_i(s, t, u)$ for some integer $e_i\geq 1$ and regular f_i. Let the divisor $(h^*\omega)$ be $eE+\cdots$. Then $e\geq e_1+e_2+e_3-1\geq 2$. However since $K_X=-3L=-3h^*H$, we have $(h^*\omega)=0$, which is a contradiction.

We can derive a contradiction in a similar manner if q is a singular point of W. For instance, assume that $(W, q)\simeq\{(y_i)\in(A^4, o);\ y_0y_3=y_1y_2\}$ with suitable coordinates y_i. Then a local generator of the dualising sheaf of W at q is given by $\omega:=(1/y_3)dy_1\wedge dy_2\wedge dy_3$. Let $h^*y_i=s^{e_i}f_i(s, t, u)$ for some $e_i\geq 1$. Then we have $(h^*\omega)=eE+\cdots$, $e\geq e_1+e_2-1\geq 1$, a contradiction. If $(W, q)\simeq\{(y_i)\in(A^4, o);\ y_1^2=y_0y_2\}$, then $\omega:=(1/y_2)dy_1\wedge dy_2\wedge dy_3$, $(h^*\omega)=eE+\cdots$ and $e\geq e_1+e_3-1\geq 1$. Thus we derive a contradiction in any case.

Next we prove (3.4.2). If $LC<0$, then C is contained in $B(=\phi)$. Hence $LC\geq 0$ for any irreducible curve C on X. Since W is singular, the birational morphism h contracts some curves. Hence there is an irreducible curve C with $LC=0$. Let $q:=h(C)$. Choose two general hyperplane sections H and H' of W which pass through q. Then their pull-backs $D:=g^*H$ and $D':=g^*H'$ are irreducible by (3.4.1) if H and H' are general enough. Therefore the structure of $l:=D\cap D'$ is given by (2.4), where C is a component of l. By (2.4) C is a smooth rational curve with $N_{C/X}\simeq O_C(-1)^{\oplus 2}$ or $O_C\oplus O_C(-2)$.

Finally we prove (3.4.3). The Jacobian of h is nonvanishing on $X\backslash h^{-1}(\mathrm{Sing}\, W)$ by the proof of (3.2). Hence h is unramified and birational there so that it is an isomorphism there. Let q be a singular point of W, p any point of $h^{-1}(q)$. If $\dim h^{-1}(q)=0$, then $(X, p)\simeq(W, q)$ by Zariski main theorem, which contradicts that (W, q) is singular. Hence $\dim h^{-1}(q)=1$ by (3.4.1). Any general member of $|O_W(1)|$ passing through the point q of W is reduced, possibly reducible. Therefore any general member D of $|L|$ passing through p is reduced by (3.4.1). For general D and $D'\in|L|$ passing through p, a complete intersection $l:=D\cap D'$ is of type (2.4.3.4), say, $l \simeq C_0 + \cdots + C_m$ with the notation there. Since $LC_0 = LC_m = 1$, C_0 and C_m are mapped onto lines on W respectively. Any pair of lines on W passing through the point q are algebraically equivalent to each other (as lines passing through q), so that C_0 and C_m meet the same component C_1 for general D and D', whence $m=2$. Therefore $h^{-1}(q)_{\mathrm{red}} = C_1$.

It remains to prove the uniqueness of W. Assume that we are given an algebraic veriety V and a morphism $g: X \to V$ with the property (3.4.3). $\Sigma := h^{-1}(\mathrm{Sing}\, W)$ is the union of all rational curves C on X with $LC=0$. Then $W \backslash \mathrm{Sing}\, W \simeq X \backslash \Sigma \simeq V \backslash \mathrm{Sing}\, V$ by (3.4.3.1). By (3.4.3.2) we have a natural bijection from W onto V, which induces an isomorphism of $W \backslash \mathrm{Sing}\, W$ onto $V \backslash \mathrm{Sing}\, V$. Since both W and V are normal, we have $O_W \simeq i_*(O_{W \setminus \mathrm{Sing}\, W}) \simeq j_*(O_{V \setminus \mathrm{Sing}\, V}) \simeq O_V$, where $i: W \backslash \mathrm{Sing}\, W \to W$ and $j: V \backslash \mathrm{Sing}\, V \to V$ are inclusion maps. Hence W and V are isomorphic. q.e.d.

(3.5) LEMMA. *If* $B = \phi$ *and if* $\dim \mathrm{Sing}\, W = 0$, *then* $X \simeq \boldsymbol{P}(\mathcal{F}(1,1,0))$.

PROOF: *Step* 1. Assume $\dim \mathrm{Sing}\, W = 0$. This means that W has isolated singularities. Then by (3.3) we may assume $W = \{(x_k) \in \boldsymbol{P}^4;\ x_0x_3 - x_1x_2 = 0\}$. Let q be a unique singular point of W. Let $W_{ij} = W_{ji} := \{(x_k) \in \boldsymbol{P}^4;\ x_i = x_j = 0\} (\simeq \boldsymbol{P}^2)$, where $(i,j) = (0,1), (0,2), (3,1), (3,2)$. Then $H_i := W_{ij} + W_{ik} \in |O_W(1)|$ for $j \neq k$, whence $D_i := h^*(H_i) \in |L|$. Since $\dim h^{-1}(q) = 1$ by (3.4.2), D_i has two irreducible components Z_{ij} and $Z_{ik} (j \neq k)$ with $W_{ij} = h(Z_{ij})$, $W_{ik} = h(Z_{ik})$. We define $Z_{ji} := Z_{ij}$. Each Z_{ij} is nonsingular outside $h^{-1}(q)$ by (3.4.1). We see that Z_{01} (resp. Z_{02}) is linearly equivalent to Z_{32} (resp. Z_{31}). Let $C := h^{-1}(q)$, $\bar{C}_i := W_{ij} \cap W_{ik} \simeq \boldsymbol{P}^1$, and let C_i be the proper transform of $\bar{C}_i$. Since $C_i \cap C \neq \phi$, we set $p_i := C_i \cap C$.

Step 2. Since D_i contains C, either Z_{ij} or Z_{ik} contains C. Assume $C \subset Z_{ij}$. Then we prove $C \not\subset Z_{ik}$. We may assume $(i,j,k) = (0,1,2)$ without loss of generality. So we assume $C \subset Z_{01}$. A complete intersection $l := D_0 \cap D_3$ contains both C_1 and C_2 as well as C. Since $LC_i = 1$ for any i, l is of type (2.4.3.4) with $m=2$. By (2.4.3) or [N1, (3.5)],

$$I_{l,p} = I_{C,p} = (x, y) \quad (p \neq p_1, p_2).$$

In particular, both D_0 and D_3 are smooth along $C \backslash \{p_1, p_2\}$, whence $C \not\subset Z_{02}$. Similarly it follows that $C \subset Z_{32}$, while $C \not\subset Z_{31}$.

Step 3. By STEP 2, we (may) assume from now on that $C \subset Z_{01}$. Now we prove $p_0 = p_2$, $p_1 = p_3$ and $Z_{02} \cap Z_{31} = \phi$. Since D_0 is singular along C_0, it is singular at p_0. Hence $p_0 = p_1$ or $p_0 = p_2$, because D_0 is smooth along

$C\setminus\{p_1, p_2\}$ by STEP 2. If $p_0=p_1$, then $Z_{02}\cap Z_{31}\neq\phi$, whence $Z_{02}\cap Z_{31}$ contains a curve. As $W_{02}\cap W_{31}=\{q\}$, we have $Z_{02}\cap Z_{31}=h^{-1}(q)$, contradicting STEP 2. Therefore $p_0=p_2$. Similarly $p_1=p_3$.

Step 4. We prove that $Z_{01}\simeq Z_{32}\simeq \boldsymbol{F}_1$ and $Z_{02}\simeq Z_{31}\simeq \boldsymbol{P}^2$. Since $W_{ij}\simeq \boldsymbol{P}^2$, it is sufficient to prove that Z_{ij} is smooth. Consider $l=D_0\cap D_3$ at $p_0(=p_2)$. By [N1, (3.5)],

$$I_{l,p_0}=I_{C,p_0}=(x, yz).$$

As we saw above, D_0 and D_3 are smooth along $C\setminus\{p_0, p_1\}$. Since $D_0=Z_{01}+Z_{02}$ is singular along C_0, we may assume by the form of the ideal I_{l,p_0} that $I_{D_0,p_0}=(yz)$ and $I_{D_3,p_0}=(x)$, whence Z_{01}, Z_{02} and Z_{32} are smooth at p_0. Similarly we also see that Z_{01}, Z_{32}, and Z_{31} are smooth at $p_1(=p_3)$. Consequently Z_{ij} is smooth everywhere.

Step 5. Let $F:=O_X(Z_{02})\simeq O_X(Z_{31})\in \mathrm{Pic}\, X$ by STEP 1. Since $Z_{02}\cap Z_{31}=\phi$ by STEP 3, $h^0(X, F)=2$ and $\mathrm{Bs}|F|=\phi$. Therefore we have a morphism $\pi: X\to \boldsymbol{P}^1$ with general fiber $F\simeq \boldsymbol{P}^2$, where we may view $F=Z_{02}$ or Z_{31}. The morphism π is given by the rational function $h^*(x_0/x_1)=h^*(x_2/x_3)$. Then $\pi_*(L)$ is a torsion free sheaf of rank 3 because $L_{Z_{02}}\simeq h^*H_{W_{02}}\simeq O_{P^2}(1)$. Therefore by a theorem of Grothendieck we have $\pi_*(L)\simeq \mathcal{F}(a, b, c)$ for some $a\geq b\geq c$ under the notation in § 1. As $h^0(X, L)=5$ and $\mathrm{Bs}|L|=\phi$ by (2.5), we have $a+b=2$, $a\geq b\geq 0$, $c=0$. It follows from $H^0(X, L)\simeq H^0(W, O_W(1))$ that $a=b=1$ and that $h^*(x_0)$ and $h^*(x_1)$ (resp. $h^*(x_2)$ and $h^*(x_3)$) are bases of $H^0(O_{P^1}(a)\oplus 0\oplus 0)$ (resp. $H^0(0\oplus O_{P^1}(b)\oplus 0)$). Thus we have a birational morphism $g: X\to \boldsymbol{P}(\mathcal{F}(1, 1, 0))$ $(=: \boldsymbol{P}(\mathcal{F}))$. Since $K_X\simeq -3L\simeq -3g^*h_0^*(O_W(1))\simeq g^*K_{P(\mathcal{F})}$ by § 1, g is unramified. Hence $X\simeq \boldsymbol{P}(\mathcal{F})$. q.e.d.

(3.6) LEMMA. *If* $B=\phi$ *and if* $\dim \mathrm{Sing}\, W=1$, *then* $X=\boldsymbol{P}(\mathcal{F}(2, 0, 0))$.

PROOF: By (3.3) we may assume $W=\{(x_k)\in \boldsymbol{P}^4;\ x_1^2-x_0x_2=0\}$. Let $W_{ij}=W_{ji}:=\{(x_k)\in \boldsymbol{P}^4;\ x_i=x_j=0\}$ $(\simeq \boldsymbol{P}^2)$, and let Z_{ij} be the proper transform of W_{ij} by h where $(i, j)=(0, 1), (1, 2)$. Let $\Sigma:=\mathrm{Sing}\, W$, $E:=h^{-1}(\Sigma)_{\mathrm{red}}$ and $e_{ij}:=Z_{ij}\cap E$. The Cartier divisor (x_1) of W is $W_{01}+W_{12}$, while the divisor (x_0) (resp. (x_2)) of W is $2W_{01}$ (resp. $2W_{12}$). Hence W_{01} is linearly equivalent to W_{12}. We have

$$D_i:=(h^*x_i)=2Z_{i1}+a_iE\ \ (i=0, 2),\ \ D_1:=(h^*x_1)=Z_{01}+a_1E+Z_{12}$$

for some positive integers a_i. Let $W_i:=(x_i)=h(D_i)$, and let $\bar{D}$ be a general member of $|O_W(1)|$ which does not contain Σ, $D:=h^*\bar{D}$. Since Σ is a line in $\boldsymbol{P}^4$, the intersection $\Sigma \cap \bar{D}$ is a single point q. Meanwhile W_i $(i=0, 2)$ is a double plane, whereas W_1 is a union of two copies of $\boldsymbol{P}^2$. Hence the intersection $W_i \cap \bar{D}$ is a double line $(i=0, 2)$, or a union of two lines $(i=1)$. By (2.4) and by the same argument as (3.5) STEP 2, we have

$$l_i := D_i \cap D \simeq 2C_i + C_1 \ (i=0, 2),\ l_1 := D_1 \cap D \simeq C_0 + C_1 + C_2$$

where $C_1 := h^{-1}(q)$, and $LC_0 = LC_2 = 1$, $LC_1 = 0$.

Since $D_i \cap D = 2Z_{i1} \cap D + a_i E \cap D$, we have $C_i \simeq Z_{i1} \cap D$, $a_i = 1$ $(i=0, 2)$ and $C_1 \simeq E \cap D$. Since C_i is smooth, Z_{ij} is smooth along C_i $(i=0, 2)$. Similarly E is smooth along C_1. We also have $D_1 \cap D = Z_{01} \cap D + a_1 E \cap D + Z_{12} \cap D \simeq C_0 + C_1 + C_2$, whence $a_1 = 1$. We note that $C_1 \not\subset Z_{ij}$, whence $C_1 \not\subset e_{ij}$. In other words, $h^{-1}(q) \not\subset e_{ij}$ for any $q \in \Sigma$. Since $C_0 \simeq Z_{01} \cap D$ intersects C_1 transversally, we have $(e_{01} C_1)_E = (Z_{01} C_1)_X = 1$. Similarly $(e_{12} C_1)_E = (Z_{12} C_1)_X = 1$, whence e_{ij} is bijectively mapped onto Σ. Since $(e_{ij} D)_X = (Z_{ij} ED)_X = (Z_{ij} C_1)_X = 1$, Z_{ij} intersects E transversally along e_{ij}, and e_{ij} is smooth so that $e_{ij} \simeq \Sigma \simeq \boldsymbol{P}^1$. Therefore Z_{ij} and E are smooth along e_{ij}, whence Z_{ij} is smooth everywhere. Since $W_{01} \cap W_{12} \simeq \Sigma$ and $C_0 \cap C_2 = \phi$, we have $Z_{01} \cap Z_{12} = \phi$ and $e_{01} \cap e_{12} = \phi$. Thus we see $Z_{ij} \simeq W_{ij} \simeq \boldsymbol{P}^2$ and $(e_{ij}^2)_{Z_{ij}} = 1$. As a Cartier divisor $D \cap E \simeq C_1$ of E is smooth for any $q \in \Sigma$, so is E everywhere too. We have

$$(e_{ij}^2)_E = (Z_{ij}^2 E)_X = ((D_1 - E - Z_{ik}) E Z_{ij})_X = (L e_{ij})_X - (e_{ij}^2)_{Z_{ij}} = 0,$$

whence $E \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

Since $a_i = 1$, Z_{01} is linearly equivalent to Z_{12}. Let $Z = Z_{01}$ and $F := O_X(Z_{01}) \simeq O_X(Z_{12}) \in \operatorname{Pic} X$. As $O_Z(F) \simeq O_Z(Z_{12}) \simeq O_Z$, we have $h^0(X, F) = 2$ and $\operatorname{Bs}|F| = \phi$. Thus we have a morphism $\pi : X \to \boldsymbol{P}^1$ associated with $|F|$, where π is given explicitly by a rational function $h^*(x_1/x_0) = h^*(x_2/x_1)$ on X. Let $\mathcal{F} := \pi_* L$. As $\mathcal{F}$ is a torsion free sheaf of rank 3 on $\boldsymbol{P}^1$ by $L_Z \simeq O_{P^2}(1)$, we have $\mathcal{F} \simeq \mathcal{F}(a, b, c)$ for some $a \geq b \geq c$ under the notation in § 1. Then since $h^0(X, L) = 5$ and $\operatorname{Bs}|L| = \phi$ by (2.5), we have $a + b = 2$ and $c = 0$. Since $h^*(x_1)^2 = h^*(x_0) h^*(x_2)$, we have $a = 2$ and $b = 0$, whence we have a birational morphism $g : X \to \boldsymbol{P}(\mathcal{F})$. Since $K_X \simeq -3L \simeq -3g^* h_0^*(O_W(1)) \simeq g^* K_{P(\mathcal{F})}$ by § 1, g is unramified. Therefore $X \simeq \boldsymbol{P}(\mathcal{F}(2, 0, 0))$. q.e.d.

(3.7) REMARK. (char $k \neq 2$) Assume that $B = \phi$. If there is a smooth rational curve C on X with $LC = 0$ and $N_{C/X} \simeq O_C(-1)^{\oplus 2}$ (resp. $N_{C/X} \simeq O_C \oplus O_C(-2)$), then Hessian-rank $W = 4$ (resp. 3), or equivalently $\dim \operatorname{Sing} W = 0$ (resp. 1).

PROOF: (3.7) is easily proved by applying (3.2), (3.5) and (3.6). Here is however a direct proof as in [Mo, (3.23)]. We consider the normal variety W with the property (3.4.3). Let $h: X \to W$ be the morphism in (3.4). Let q be a singular point of $W, C := h^{-1}(q)_{\mathrm{red}} \simeq \boldsymbol{P}^1$.

Case 1. First consider the case where $N_{C/X} \simeq O_C(-1)^{\oplus 2}$. We prove

$$\hat{O}_{W,q} \simeq k[[u_0, u_1, u_2, u_3]]/(u_0u_3 - u_1u_2).$$

Let I_C be the ideal sheaf of O_X defining C, m_q the maximal ideal sheaf of O_W defining the point q. Note that $\dim m_q/m_q^2 = 4$. Then we have a diagram of exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_C^n/I_C^{n+1} & \longrightarrow & O_X/I_C^{n+1} & \longrightarrow & O_X/I_C^n & \longrightarrow & 0 \\
0 & \longrightarrow & h_*(I_C^n/I_C^{n+1}) & \longrightarrow & h_*(O_X/I_C^{n+1}) & \longrightarrow & h_*(O_X/I_C^n) & \longrightarrow & 0 \\
 & & \downarrow \phi_n & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \\
0 & \longrightarrow & m_q^n/m_q^{n+1} & \longrightarrow & O_W/m_q^{n+1} & \longrightarrow & O_W/m_q^n & \longrightarrow & 0
\end{array}$$

Since $h_*O_X \simeq O_W$ by (3.4.3), the composite of natural homomorphisms $h_*O_X \to O_W \to O_W/m_q^{n+1}$ is surjective, whence so is ϕ_n for any n. As $h^0(h_*(O_X/I_C)) = h^0(O_X/I_C) = 1$ and $h^0(I_C/I_C^2) = 4$, the homomorphisms ϕ_0, ϕ_1 and ψ_1 are isomorphisms. $S^n(H^0(I_C/I_C^2))$ (resp. $S^n(m_q/m_n^2)$) are naturally mapped onto $H^0(I_C^n/I_C^{n+1})$ (resp. m_q^n/m_q^{n+1}), so that ψ_n is surjective. Therefore ψ_n is surjective. Hence we have an epimorphism

$$\hat{\phi} : \lim_{\leftarrow} h_*(O_X/I_C^n) \to O_{W,q}.$$

Now look at ϕ_i and ψ_i $(i=1,2)$. We can choose generators y_0, y_1, y_2 and y_3 of $H^0(I_C/I_C^2) \simeq H^0(O_C(1))^{\oplus 2}$ such that y_0 and y_1 (resp. y_2 and y_3) generate the first (resp. the second) factor, satisfying the relation $y_0y_3 = y_1y_2$ in I_C^2/I_C^3.

Since $h^0(I_C^2/I_C^3) = 9$ and ψ_2 is surjective, there is a unique quadratic relation among y_i, which is just the above one. Therefore we can choose $x_i \in O_{W,q}$ such that $\psi_1(y_i) = x_i \bmod m_q^2$ and $x_0x_3 - x_1x_2 = 0 \bmod m_q^3$. It is easy to see that there is a formal solution $\hat{x}_i \in O_{W,q}$ such that $\hat{x}_i = x_i \bmod m_q^2$, $\hat{x}_0\hat{x}_3 - \hat{x}_1\hat{x}_2 = 0$ in $O_{W,q}$. Let $\hat{R} := k[[u_0, u_1, u_2, u_3]]/(u_0u_3 - u_1u_2)$. Then we define an epimorphism $\rho : \hat{R} \to \hat{O}_{W,q}$ by $\rho(u_i) = \hat{x}_i$. Since Krull-dim $\hat{R} = 3$, it follows that ρ is (whence $\hat{\phi}$ is also) an isomorphism. Since W is a quadric hypersurface in $\boldsymbol{P}^4$, this also shows that Hessian-rank $W = 4$.

Case 2. Next we consider the case where $N_{C/X} \simeq O_C \oplus O_C(-2)$. We prove

$$\hat{O}_{W,q} \simeq k[[u_0, u_1, u_2, u_3]]/(u_1^2 - u_0 u_2).$$

With the same notation as in *Case* 1, we see that ϕ_n and ψ_n are surjective. It follows that we have an epimorphism

$$\hat{\phi} : \varprojlim h_*(O_X/I_C^n) \to \hat{O}_{W,q}.$$

We can choose generators y_0, y_1, y_2 and y_3 of $H^0(I_C/I_C^2) \simeq H^0(O_C(2)) \oplus H^0(O_C)$ such that y_0, y_1 and y_2 (resp. y_3) generate the first (resp. the second) factor, satisfying the relation $y_1^2 = y_0 y_2$ in I_C^2/I_C^3. Since char $k \neq 2$, we can choose $x_i \in O_{W,q}$ such that $\psi_1(y_i) = x_i \mod m_q^2$ and $x_1^2 - x_0 x_2 = 0 \mod m_q^3$. It is easy to see that there is a formal solution $\hat{x}_i \in O_{W,q}$ such that either $\hat{x}_1^2 - \hat{x}_0 \hat{x}_2 = 0$ or $\hat{x}_1^2 - \hat{x}_0 \hat{x}_2 - \hat{x}_3^m = 0$ for some $m \geq 3$.

Let $\hat{R} := k[[u_0, u_1, u_2, u_3]]/(u_1^2 - u_0 u_2)$ or $\hat{R} := k[[u_0, u_1, u_2, u_3]]/(u_1^2 - u_0 u_2 - u_3^m)$. Then $\hat{R} \simeq O_{W,q}$ by the same argument as in *Case* 1. Since W is a quadric hypersurface in $\boldsymbol{P}^5$, the second case is impossible and Hessian-rank $W = 3$. q.e.d.

§ 4. The case where $B \neq \phi$.

(4.1) LEMMA. *If* $B \neq \phi$ *and if* $\dim B \leq 1$, *then* $X \simeq \boldsymbol{P}(\mathcal{F}(a, b, 0))$ $(a \geq b \geq n \geq 1,\ a + b = 3n + 2)$.

PROOF: By (2.4), $B \simeq \boldsymbol{P}^1$. Let the normal bundle $B_{B/X} \simeq O_B(-a) \oplus O_B(-b)$ $(a \geq b)$ and $n := -LB$. By [N1, (8.8)+(8.10)], we have $a \geq b \geq n \geq 1$, while $a + b = 3n + 2$ by the relation $c_1(N_{B/X}) = (c_1(X)B)_X - c_1(B) = 3(LB)_X - 2$. Let $f : Y \to X$ be the blowing-up of X with B center, E the total tranform of B and $N := f^*L - E$. Then by [N1, § 8], we see that with the notation in [ibid.], (compare also § 1)

$$E \simeq \boldsymbol{F}_{a-b},\ (f^*L)_E \simeq f^*(L_B) \simeq -n f_0,\ N_E \simeq e_0 + (b-n) f_0,$$

$$E_E \simeq -e_0 - b f_0,\ (N_E^2)_E = n + 2,\ (f^*L^2E)_Y = 0,$$

$$(f^*LE^2)_Y = ((f^*L)_E E_E)_E = n,\ (E^3)_Y = 3n + 2,\ (N^3)_Y = 0,$$

$$h^0(Y, N) = n + 4,\ \mathrm{Bs}|N| = \phi,\ H^0(Y, N) \simeq H^0(E, N_E).$$

Let W be the image of Y by the morphism $g : Y \to \boldsymbol{P}^{n+3}$. Then $W \simeq E$ if $b > n$, while if $b = n$, then W is a normal surface obtained from E by contracting a unique smooth rational curve e_∞ with $(e_\infty^2)_E=-c=-(n+2)$. Note that $a=2n+2$ if $b=n$.

Now we prove that there is a surjective morphism $\pi: X\to \boldsymbol{P}^1$ whose general fibers are $\boldsymbol{P}^2$. First we consider the case where $b>n$. Then $g(f_0)$ is a line of $\boldsymbol{P}^{n+3}$. We can choose a hyperplane section H of W containing exactly $(a-n)$ distinct lines because $g^*(H)_E\equiv N_E\equiv e_\infty+(a-n)f_0$. Let σ be a general line $g(f_0)$ on W and $\tilde{F}:=g^{-1}(\sigma)$. Let $\tilde{l}_q=g^{-1}(q)(q\in\sigma)$ be a fiber of $g_{|\tilde{F}}$ and $\tau:=E\cap\tilde{F}$. Then $\tilde{l}_q\simeq \boldsymbol{P}^1$ for general q and the divisor $\tilde{F}$ is irreducible. Since $g_{|E}$ is an isomorphism, $\tau\in|f_0|$ on E. The curves τ and $\tilde{l}_q$ on $\tilde{F}$ intersect at a unique point transversally. Therefore we have

$$(E_{\tilde{F}}^2)_{\tilde{F}}=(E^2\tilde{F})_Y=(E_E\tilde{F}_E)_E=-((e_0+bf_0)f_0)_E=-1$$

$$(f^*(L)_{\tilde{F}}^2)_{\tilde{F}}=((E+N)_{\tilde{F}}^2)_{\tilde{F}}=((E_{\tilde{F}}+\tilde{l}_q)^2)_{\tilde{F}}=-1+2(\tau\tilde{l}_q)_{\tilde{F}}=1\,,$$

whence $\kappa(\tilde{F},f^*(L))=2$. Note that the above intersection numbers on $\tilde{F}$ make sense because $\tilde{F}$ is smooth along $\tilde{l}_q$ for general $q\in\sigma$ and $E, h^*(L)$ are Cartier divisors on Y.

Let $F=f(\tilde{F})$, and $h: S\to F$ the minimal resolution of the normalisation of F. Then there exists an effective divisor P on S such that the canonical bundle of S is given by

$$K_S=h^*(K_X+F)-P.$$

See [N2, (2.A)]. By the choice of $\tilde{F}$, we have an effective divisor $\tilde{Q}$ of Y such that $N=(a-n)\tilde{F}+\tilde{Q}$. Hence $L=(a-n)F+Q$ where $Q=f_*(\tilde{Q})$. Therefore we have

$$K_S=-(3a-3n-1)h^*(F)-3h^*(Q)-P.$$

Therefore S is either $\boldsymbol{P}^2$ or ruled because S is Moishezon or projective by the assumption. If S has a pencil f_t of smooth rational curves with $(f_t^2)_S=0$, then we have

$$2=-(K_Sf_t)_S=(3a-3n+1)(h^*(F)f_t)_S+3(h^*(Q)f_t)_S+(Pf_t)_S.$$

Since $a\geq b$ and $a+b=3n+2$, we see $3a-3n-1\geq a+1\geq n+3$. Hence $(h^*(F)f_t)_S=(h^*(Q)f_t)_S=0$, whence $(h^*(L)f_t)_S=0$. This implies that $\kappa(F, L)=\kappa(S, h^*(L))\leq 1$, a contradiction. It follows that $S\simeq \boldsymbol{P}^2$ and that $P=h^*(F)=0$. Since $P=0$, F has only isolated singularities. (This is true in arbitrary characteristic because F is a Cartier divisor of X.) This also implies that F is normal. Since P^2 has no curves with negative self-intersection numbers, $F\simeq P^2$. Therefore $O_F(F)\simeq O_F$. Thus we have a morphism $\pi: X\to P^1$ associated with the linear system $|F|$.

We consider next the case where $b=n$. Then W has a unique isolated singular point v_0. A general hyperplane section of W passing through v_0 is the union of mutually distinct $(n+2)$ lines in P^{n+3}, any of which passes through v_0. Let σ be one of the these lines, $\tilde{F}$ the unique irreducible component of the divisor $g^{-1}(\sigma)$ on Y which is mapped onto σ by g. Let $F=f(\tilde{F})$. We define $\tilde{l}_q=g^{-1}(q)(q\in\sigma)$ to be a fiber of $g_{|\tilde{F}}$ and $\tau:=E\cap\tilde{F}$. Then $\tilde{l}_q\simeq P^1$ for general q. Since $g_{|E}$ is an isomorphism outside e_∞, the curves τ and $\tilde{l}_q$ meet at a unique point transversally, while $\tilde{F}$ is smooth along $\tilde{l}_q$ for general $q\in\sigma$. Therefore in the same manner as above, we have $(f^*(L)^2_{\tilde{F}})_{\tilde{F}}=1$, whence $\kappa(\tilde{F},f^*(L))=2$. Then $\kappa(F,L)=2$. Moreover by the choice of $\tilde{F}$, we have an effective divisor $\tilde{Q}$ of Y such that $N=(n+2)\tilde{F}+\tilde{Q}$. Hence $L=(n+2)F+Q$ where $Q=f_*(\tilde{Q})$. Then by the same argument as above we see that $F\simeq P^2$ and that $O_F(F)\simeq O_F$.

Thus in either case we have a surjective morphism $\pi: X\to P^1$ associated with $|F|$.

Next we prove that π is a P^2-bundle. Let $F'=\sum_{i=0}^{r} m_iF_i$ be any fiber of π, F_i irreducible components of F'. We prove that $F'\simeq P^2$. By the upper semi-continuity, we have for any positive integer m,

$$h^0(F', mL_{F'})\geq h^0(P^2, O_{P^2}(3m)),$$

whence there is an irreducible component F_0 of F' such that $\kappa(F_0, L_{F_0})=2$.

Let $h: S\to F_0$ the minimal resolution of the normalization of F_0. Then the canonical bundle of S is given by $K_S=h^*(K_X+F_0)-P$ for some effective divisor P of S. Hence we have

$$m_0K_S=-3m_0h^*(L)-\sum_{i\neq 0} m_ih^*(F_i)-m_0P.$$

Therefore S is either P^2 or a ruled surface. If S has a pencil of smooth rational curves f_t with $(f_t^2)_S=0$, then we have

$$2=-(K_Sf_t)_S\geq 3(h^*(L)f_t)_S\ ,$$

whence $(h^*(L)f_t)_S=0$. This contradicts $\kappa(S, h^*(L))=\kappa(F_0, L_{F_0})=2$. Hence $S\simeq P^2$.

Since $S\simeq P^2$, we have $P=0, h^*(F_i)=0$ $(i\neq 0)$, $h^*(L)\simeq O_{P^2}(1)$. Hence $F'=m_0F_0$ because F' is connected. Since $P=0, F_0$ has only isolated singularities. Since F_0 is a divisor of X, this implies that F_0 is normal. Hence $F_0 \simeq S \simeq \boldsymbol{P}^2$. Since $O_{F_0}(m_0F_0) \simeq O_{F_0}$, we have $O_{F_0}(F_0) \simeq O_{F_0}$ and $h^q(F_0, O_{F_0}(kF_0))=0$ for any $q, k \geq 1$. It is easy to see that $h^0(F', O_{F'})=m_0$ and $h^q(F', O_{F'})=0$ for $q \geq 1$, whence

$$m_0=\chi(F', O_{F'})=\chi(F, O_F)=\chi(\boldsymbol{P}^2, O_{P^2})=1.$$

This implies that F' is reduced. Therefore $F' \simeq \boldsymbol{P}^2$.

The direct image sheaf $\pi_*(L)$ of L is a torsion free (hence a locally free) O_{P^1}-module of rank 3. It is clear that X is isomorphic to $\boldsymbol{P}(\pi_*(L))$ and $\pi_*(L) \simeq \mathcal{F}(a', b', c')$ for some a', b' and c' with the notation in § 1. Since Pic $X(\simeq \boldsymbol{Z}^{\oplus 2})$ has no torsions, L is uniquely determined by $K_X \simeq -3L$, whence B is also uniquely determined. Hence as $N_{B/X} \simeq O_B(-a) \oplus O_B(-b)$, we have $a=a'-c'$ and $b=b'-c'$ by § 1. We have $\chi(X, O_X)=1$ by (2.3.2) by $h^3(X, O_X)=0$, whence

$$c_2L=c_1c_2/3=8\chi(X, O_X)=8,$$

$$\chi(X, L)=\chi(X, O_X)+(c_1^2+c_2)L/12+c_1L^2/4+L^3/6=5.$$

It follows from $\chi(X, L)=\chi(\boldsymbol{P}^1, \mathcal{F})$ that $a'+b'+c'=2$. Since $a \geq b \geq n \geq 1$ and $a+b=3n+2$, we have $a'=a-n$, $b'=b-n$ and $c'=-n$. Thus $X \simeq \boldsymbol{P}(\mathcal{F}(a, b, 0))$. q.e.d.

(4.2) COMPLETION OF THE PROOF OF (2.2). If $B=\phi$, then $X \simeq \boldsymbol{Q}^3$ or $\boldsymbol{P}(\mathcal{F}(1, 1, 0))$ or $\boldsymbol{P}(\mathcal{F}(2, 0, 0))$ by (3.2), (3.5) and (3.6). If $B \neq \phi$ and if dim $B \leq 1$, then by (4.1) $X \simeq \boldsymbol{P}(\mathcal{F}(a, b, 0))$ $(a \geq b \geq n \geq 1,\ a+b=3n+2)$. This completes the proof of (2.2).

§ 5. Theorems.

In the present section we work over an algebraically closed field k of characteristic zero.

[N2, (3.3)] contains a gap in the proof. Here we correct it.

(5.1) THEOREM [N2, (3.3)]. (char $k=0$) *Let X be an algebraic (or a Moishezon) 3-fold defined over k and L a line bundle on X. Assume that $c_1(X)=ac_1(L)$ for some integer $a \geq 4$ and $h^0(X, L) \geq 2$. Then* $(X, L) \simeq (\boldsymbol{P}^3, O_{P^3}(1))$.

PROOF: Let $\rho:=\rho_L$ be the rational map associated with $|L|$. Let

F be the fixed components of $|L|$, d the number of movable irreducible Z_i of a general member $D\in|L|$. Then we have $c_1(X)=adc_1(N)+ac_1(F)$ where $N:=Z_1$. Let $Z:=Z_1$. Then by Bertini's theorem Z is smooth outside $\mathrm{Bs}|L|$. Let $h:S\to Z$ be the minimal resolution of the normalisation of Z. For simplicity we assume $K_X=-adN-aF$. Then we have

$$K_S=h^*(K_X+Z)-E-G=-(ad-1)h^*N-ah^*(F)-E-G$$

where E and G are some effective divisors of S with $\mathrm{Supp}(E+G)\subset h^{-1}(\mathrm{Supp}\,B)$ which measures the singularities of Z and the intersection of Z with the other irreducible components Z_j. See [N2, (2.A)].

If $h^*(N+F)\neq 0$, then either $S\simeq \boldsymbol{P}^2$ or S has a movable rational curve f with $(f^2)_S=0$. In the second case $(K_Sf)_S=-2$, whence $(h^*(N)f)_S\geq 1$ or $(h^*(F)f)_S\geq 1$. However

$$2=(ad-1)(h^*(N)f)_S+a(h^*(F)f)_S+((E+G)f)_S\geq a-1\geq 3,$$

a contradiction. Hence $S\simeq \boldsymbol{P}^2$. Then $X\simeq \boldsymbol{P}^3$ by the same argument as in [N2, (3.3)].

Next we consider the case where $h^*(N)=h^*(F)=0$. (The proof of [N2, (3.3)] ignores this case.) Then we have $E=G=0$. Since Z_j, E and G are effective divisors, this implies that $Z_j\cap Z=F\cap Z=\phi$, $E=G=0$, whence $B=\phi$. By Bertini's theorem Z is smooth. Therefore $S\simeq Z$ and S is an algebraic surface with $K_S=0$. Moreover ρ is a morphism of X onto an algebraic curve $\overline{W}$.

By blowing up X suitably we have a projective 3-fold $\hat{X}$. Let $\phi:\hat{X}\to X$ be the natural morphism, $\pi:=\rho\cdot\phi, g$ the genus of $\overline{W}$, and $\omega_{\hat{X}/\overline{W}}$ the relative dualising sheaf of π. By Fujita [F2, (2.7)], $\deg\pi_*(\omega_{\hat{X}/\overline{W}})\geq 0$. Therefore we have

$$\begin{aligned} h^0(X, K_X-\rho^*(\omega_{\overline{W}})+gN) &= h^0(K_{\hat{X}}-\pi^*(\omega_{\overline{W}})+g\phi^*(N)) \\ &= h^0(\overline{W}, \rho_*(\omega_{\hat{X}/\overline{W}})+gp)\geq 1 \end{aligned}$$

where $p:=\rho(N)$. However $K_X-\rho^*(\omega_{\overline{W}})+gN$ is algebraically equivalent to a (strictly negative) Cartier divisor $-(ad+g-2)N-aF$. Since X is Moishezon, we have $h^0(K_X-\rho^*(\omega_{\overline{W}})+gN)=0$, a contradiction.

Now we assume $c_1(X)=ac_1(L)$ for some $a\geq 4$ instead of $K_X=-aL$. Then we can argue as above so as to prove $X\simeq \boldsymbol{P}^3$ in the first case where $h^*(N+F)\neq 0$. In the second case where $h^*(N)=h^*(F)=0$, we see $c_1(S)=0$, whence S is either an abelian surface, or an algebraic K3 surface or a hyperelliptic surface. Hence by Kawamata [K, Theorem 1] deg $\pi_*(12\omega_{\hat{X}/W})$ ≥ 0, whence we derive a contradiction in the same manner as above. This completes the proof of (5.1). q.e.d.

(5.2) THEOREM. (char $k=0$) *Let X be an algebraic (or a Moishezon) 3-fold defined over k and L a line bundle on X. Assume that $h^1(X, O_X)$ $=0$, $c_1(X)=3c_1(L)$, $h^0(X, L)\geq 2$, and that $|L|$ has no fixed components. Then $X\simeq \boldsymbol{Q}^3$ or $\boldsymbol{P}(\mathcal{F}(a, b, 0))$ $(a\geq b\geq n\geq 0,\ a+b=3n+2)$.*

PROOF: In view of (2.2) it suffices to prove that any general member of $|L|$ is irreducible. Assume the contrary. Let $D=D_1+\cdots+D_r$ $(r\geq 2)$ be a general member of $|L|$, smooth outside B by Bertini's theorem. We note that any D_i is linearly equivalent to each other by $h^1(X, O_X)=0$. Let $Z=D_1$, and let $\nu: \hat{Z}\to Z$ be the normalization, $\tau: S\to\hat{Z}$ the minimal resolution and $\sigma:=\nu\cdot\tau$. Then we have $K_S=\sigma^*(K_X+Z)-E-G$ for some effective divisors E and G as in [N2, (3.3)], whence $c_1(S)=(3r-1)c_1(\sigma^*D_i)$ $+c_1(E+G)$. If $\sigma^*D_i=0$, then we can derive a contradiction in the same manner as in (5.1). Hence $D_{i|Z}$ is nonzero effective on Z, so that S is either $\boldsymbol{P}^2$ or ruled.

We prove that both the cases are impossible. In fact, if $S\simeq\boldsymbol{P}^2$, then $D_i\cap Z=\phi$ for $i\geq 2$ because $r\geq 2$. However $0=r^3Z^2D_i=L^3\geq 1$, a contradiction. If S is ruled, then there is a pencil of rational curves F on S with $F^2=0$. Hence we have

$$2=-K_SF=(3r-1)\sigma^*(D_i)F+(E+G)F.$$

It follows that $\sigma^*(D_i)F=0$ and $(E+G)F=2$. However since $E_{\mathrm{red}}+G_{\mathrm{red}}\subset$ $\sigma^*(D_i)$ for general D_i, we have $EF=GF=0$, a contradiction. Thus $r\geq 2$ is impossible. q.e.d.

(5.3) MOISHEZON-FANO THREEFOLDS OF INDEX 3. We call a Moishezon 3-fold X a *Moishezon-Fano 3-fold* of index 3 if $h^1(O_X)=0$ and if X has a line bundle L such that $c_1(X)=3c_1(L)$, $\kappa(X, L)\geq 1$. It is natural to exclude those threefolds with $\kappa(X, L)\leq 0$ because there are examples far from being Fano threefolds. In the present article we studied Moishezon-Fano 3-folds of index three under the condition that $h^0(X, L)\geq 2$ and $|L|$ has no fixed components. In [N4] we study those 3-folds in the fifth class of the table (5.4) under some stronger conditions, which are satisfied by any global deformation of $\boldsymbol{P}(\mathcal{F}(1,1,0))$ or $\boldsymbol{P}(\mathcal{F}(2,0,0))$.

REMARK. $\kappa(X, L)\geq 1$ is equivalent to the condition that $h^0(X, mL)\geq 2$

for some positive integer m.

(5.4) *Table.* Threefolds with $h^1(X, O_X)=0$, $c_1(X)=3c_1(L)$, $h^0(X, L)\geq 2$

	Bs$\|L\|$	$C(P\simeq^1)$ with $LC=0$	dim W*	Sing W	X
1	ϕ	none	3	ϕ	Q^3
2	ϕ	$N_{C/X}\simeq O_C(-1)^{\oplus 2}$	3	one point	$P(\mathcal{F}(1,1,0))$
3	ϕ	$N_{C/X}\simeq O_C\oplus O_C(-2)$	3	P^1	$P(\mathcal{F}(2,0,0))$
4	curve	none	2	at most one point	$P(\mathcal{F}(a,b,0))$ $a\geq b\geq n\geq 1$, $a+b=3n+2$
5	surface	?	?	?	?**

* W is the image of the rational map $h:X\to P^m$ associated with $|L|$, $m=h^0(X,L)-1$.
** The only known examples are those in Section 1 Case 3.

(5.5) THEOREM. (char $k=0$) *Let X be an algebraic (or a Moishezon) 3-fold defined over k and L a line bundle on X. Assume that $h^1(X, O_X)\geq 1$, $c_1(X)=3c_1(L)$ and $\kappa(X, L)\geq 1$. Then X is isomorphic to a P^2-bundle over a smooth algebraic curve Δ of genus $h^1(X, O_X)$.*

PROOF: First we prove $h^0(X, mK_X)=0$ for any $m\geq 1$. Otherwise, we have a nonzero effective divisor D of X which is algebraically equivalent to zero. Since X is Moishezon, we have movable curves C_t on X intersecting D properly. Hence $DC_t\geq 1$, which contradicts that D is algebraically equivalent to zero.

Let $alb: X\to Alb(X)$ be the Albanese mapping, T the image of alb, and Δ the normalization of T. We have a morphism $\pi: X\to\Delta$. If dim $\Delta=3$, then $h^0(X, K_X)=h^0(X, \Omega_X^3)\geq 1$, a contradiction. If dim $\Delta=2$, and if the genus of a general fiber is positive, then $\kappa(X):=\kappa(X, K_X)\geq 0$ by Viehweg [V], a contradiction. Hence any general fiber of π is the disjoint union of smooth rational curves. Let C be an irreducible component of a general fiber of π. Then $\omega_C\simeq K_X\otimes O_C\simeq -3L_C$, which contradicts $C\simeq P^1$. It follows that dim $\Delta=1$.

Let F be a general fiber of π. Then we prove that F is an irreducible smooth algebraic surface. We have a morphism $\hat{\pi}: X\to\hat{\Delta}$ by the Stein factorization of π. Let $\hat{g}$ (resp. g) be the genus of $\hat{\Delta}$ (resp. Δ). If $g\geq 2$ and if $\hat{\Delta}$ is not isomorphic to Δ, then $\hat{g}\geq g+1$, which contradicts $h^1(X, O_X)=\dim Alb(X)=g$. Hence $\hat{\Delta}\simeq\Delta$. If $g=1$, then $Alb(X)\simeq T\simeq\Delta$. In this case if $\hat{g}\geq 2$, then we have a contradiction in the same manner.

If $\hat{g}=1$, then $\hat{\Delta}\simeq Alb(X)\simeq\Delta$. It follows that $\hat{\Delta}\simeq\Delta$. Therefore in either case any fiber of π is connected. In particular, any general fiber F of π is an irreducible smooth algebraic surface. We note that $O_F(F)\simeq O_F$.

We see $c_1(\omega_F)=c_1((K_X+F)_F)=-3c_1(L_F)$. If some multiple of L_F is zero, then F is either a minimal abelian surface, or a minimal K3 surface or a minimal hyperelliptic surface. In either case we have deg $\pi_*(\omega_{X/\Delta})\geq 1$ by [K, Theorem 1], whence $h^0(X, 12K_X)\geq 1$ because Δ is an algebraic curve of genus ≥ 1. This contradicts $h^0(X, mK_X)=0$. Hence some positive multiple of L_F is nonzero effective so that $(F, L_F)\simeq(\boldsymbol{P}^2, O_{\boldsymbol{P}^2}(1))$.

We can prove in the same manner as in (4.1) that any fiber of π is isomorphic to $\boldsymbol{P}^2$. The direct image sheaf $\pi_*(L)$ of L is a torsion free (hence a locally free) O_Δ-module of rank 3. It is clear that X is isomorphic to $\boldsymbol{P}(\pi_*(L))$. q.e.d.

Bibliography

[F1] Fujita, T., Classification of polarized varieties, London Math. Soc. Lecture Notes Series, Cambridge Univ. Press **155** (1990).

[F2] Fujita, T., On Kähler fiber spaces over curves, J. Math. Soc. Japan **30** (1978), 779-794.

[Ii] Iitaka, S., On D-dimensions of algebraic varieties, J. Math. Soc. Japan **23** (1971), 356-373.

[Is] Iskovskikh, V.A., Fano 3-folds I., Math. USSR-Izv. **11** (1977), 485-527.

[K] Kawamata, Y., Kodaira dimension of algebraic fiber spaces over curves, Invent. Math. **66** (1982), 57-71.

[Mo] Mori, S., Threefolds whose canonical bundles are not numerically effective, Ann. of Math. **116** (1982), 133-176.

[Mu] Mumford, D., Enriques's classification of surfaces in char p:I, Global Analysis, Papers in Honor of K. Kodaira, Univ. of Tokyo Press and Princeton Univ. Press (1969), 325-339.

[N1] Nakamura, I., Threefolds homeomorphic to a hyperquadric in $\boldsymbol{P}^4$, Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya, Tokyo Japan (1987), 379-404.

[N2] Nakamura, I., On Moishezon manifolds homeomorphic to $\boldsymbol{P}_C^n$, J. Math. Soc. Japan **44** (1992), 667-694.

[N3] Nakamura, I., Moishezon fourfolds homeomorphic to $\boldsymbol{Q}_C^4$, Proc. Japan Acad. **67A** (1991), 329-332.

[N4] Nakamura, I., Global deformations of $\boldsymbol{P}_C^2$-bundles over $\boldsymbol{P}_C^1$, preprint (1992).

[V] Viehweg, E., Klassifikationstheorie algebraischer Varietäten der Dimension drei Compositio Math. **41** (1980), 361-400.

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Department of Mathematics
Hokkaido University
Sapporo
060 Japan